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On lifts of linear tensor fields to Weil bundles

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Abstract: In this paper, we generalize for an arbitrary double vector bundle, some results on linear tensor fields. Moreover we study some properties of their lifts with respect to a product preserving bundle functor.

Key words: Double vector bundle, local triviality, duality, linear section, Weil bundle functor, lift.

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1. Introduction

The theory of double vector bundles was developped by J. Pradines in [20] with the concept of maximal atlas of double vector bundle charts of a surjective map π from a set D to a smooth manifold M. A *double vector chart* of (D, M, π) is a system $c = (U, \varphi, E_0, E_1, E_2)$ where U is an open set of M, E_i, $i = 0, 1, 2$ are Banach spaces and $\varphi : \pi^{-1}(U) \to U \times E_1 \times E_2 \times E_0$ is a bijective map such that $pr_1 \circ \varphi = \pi \mid_{\pi^{-1}(U)}$ (i.e., $\varphi(D_x) = \{x\} \times E_1 \times E_2 \times E_0$, where D_x denotes the fiber over $x \in U$). Two double vector charts $c = (U, \varphi, E_0, E_1, E_2)$, $c' = (U', \varphi', E'_0, E'_1, E'_2)$ of (D, M, π) are said *compatible* if the transition bijection

$$\varphi' \circ \varphi^{-1} : (U \cap U') \times E_1 \times E_2 \times E_0 \longrightarrow (U \cap U') \times E'_1 \times E'_2 \times E'_0$$

is of the form

$$(x, X, Y, Z) \longmapsto (x, u_1(x) \cdot X, u_2(x) \cdot Y, u_0(x) \cdot Z + \omega(x) \cdot (X, Y)), \qquad (1.1)$$

where $u_i : U \cap U' \to \mathcal{L}(E_i, E'_i)$, $i = 0, 1, 2$ and $\omega : U \cap U' \to \mathcal{L}_2(E_1, E_2; E'_0)$ are smooth. Definitions of atlas and structure of double vector bundle are similar to those for vector bundles. The following result gives some basic results for double vector bundles.

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THEOREM 1.1. ([20]) *Let (D, M, π) be a double vector bundle with the associated maximal atlas $\overline{\mathcal{A}}_0$.*

(a) *There is only one structure of smooth manifold on D for which vector charts of $\overline{\mathcal{A}}_0$ are diffeomorphisms. With respect to this structure, $\overline{\mathcal{A}}_0$ is an atlas of local trivializations for the fibration (D, M, π).*

(b) *The sets*

$$\begin{cases} \overline{A} = \coprod\limits_{(U,\varphi,E_0,E_1,E_2)\in\overline{\mathcal{A}}_0} \varphi^{-1}\big(U \times E_1 \times \{0_{E_2}\} \times \{0_{E_0}\}\big) \\ \overline{B} = \coprod\limits_{(U,\varphi,E_0,E_1,E_2)\in\overline{\mathcal{A}}_0} \varphi^{-1}\big(U \times \{0_{E_1}\} \times E_2 \times \{0_{E_0}\}\big) \\ \overline{C} = \coprod\limits_{(U,\varphi,E_0,E_1,E_2)\in\overline{\mathcal{A}}_0} \varphi^{-1}\big(U \times \{0_{E_1}\} \times \{0_{E_2}\} \times E_0\big), \end{cases} \tag{1.2}$$

endowed with restrictions $q_{\overline{A}}$, $q_{\overline{B}}$, $q_{\overline{C}}$ of π to $\overline{A}$, $\overline{B}$, $\overline{C}$ respectively, are vector bundles over M.

(c) *If we denote $q^D_{\overline{A}} : D \to \overline{A}$, $q^D_{\overline{B}} : D \to \overline{B}$ the maps with local expressions*

$$(x, X, Y, Z) \mapsto (x, X, 0_{E_2}, 0_{E_0}), \qquad (x, X, Y, Z) \mapsto (x, 0_{E_1}, Y, 0_{E_0})$$

on suitable vector charts, hence $(D, \overline{A}, q^D_{\overline{A}})$, $(D, \overline{B}, q^D_{\overline{B}})$ are vector bundles such that

$$\begin{array}{ccc} D & \xrightarrow{q^D_{\overline{B}}} & \overline{B} \\ {\scriptstyle q^D_{\overline{A}}}\downarrow & & \downarrow{\scriptstyle q_{\overline{B}}} \\ \overline{A} & \xrightarrow{q_{\overline{A}}} & M \end{array}$$

is a commutative diagram of vector bundle morphisms.

(d) *$\overline{A}$, $\overline{B}$ are vector subbundles*[1] *of $(D, \overline{B}, q^D_{\overline{B}})$, $(D, \overline{A}, q^D_{\overline{A}})$ respectively.*

(e) *The addition and the scalar multiplication of each vector bundle structure on D is a vector bundle morphism with respect to the other structure.*

The proof of this result is clear by definitions and the gluing theorem for vector bundles in [7]. In [14] and [17], an axiomatic for double vector bundles is presented without indicating how to deduce double vector charts. For this reason, some structures of vector bundles are given without proof of the local

[1] In the more general sense

triviality condition for vector bundles (ex. duals of a double vector bundle). In [6] (see also [10]), the author gives a proof of the existence of local linear splittings that are equivalent to the existence of double vector bundle charts.

As an application of double vector bundles, let us recall their importance in the modern formulation of the concept of linear Poisson structures. For a smooth manifold M, a *Poisson structure* on M (see [17]) is a bracket of smooth functions $\{\ ,\ \}: C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ with respect to which $C^\infty(M)$ is an $\mathbb{R}$-Lie algebra, and such that for all $u, v, w \in C^\infty(M)$,

$$\{u, vw\} = v\,\{u, w\} + w\,\{u, v\}\,;$$

the bivector $\pi : M \to \bigwedge^2 TM$ given by $\pi(du, dv) := \{u, v\}$ is called the *Poisson bivector* and the induced morphism of vector bundles $\pi^\sharp : T^*M \to TM$ is called the *Poisson morphism* associated to the given Poisson structure. A *Lie algebroid* on M is a vector bundle (A, M, q) on which the module $\Gamma(A)$ of smooth sections of A is endowed with a Lie algebra structure and there is a base-preserving morphism of vector bundles $\rho : A \to TM$, called the *anchor* of A, such that:

(a) $\forall\, s_1, s_2 \in \Gamma(A)$, $\forall f \in C^\infty(M)$, $[s_1, f \cdot s_2] = f[s_1, s_2] + (\rho(s_1) \cdot f)s_2$.

(b) The induced map $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism.

Lie groupoids generalize Lie groups (see [17]) and the infinitesimal counterpart of a Lie groupoid $G \rightrightarrows M$ is a Lie algebroid called the Lie algebroid of $G \rightrightarrows M$. In [4], it was established that the dual $A^* \to M$ of the Lie algebroid $A \to M$ of a Lie groupoid $G \rightrightarrows M$ is endowed with a Poisson structure such that the following conditions on brackets of functions hold:

- the bracket of two linear functions is linear;
- the bracket of a linear function and a function constant on fibres is constant on fibres;
- the bracket of two functions constant on fibres is zero.

This Poisson structure is called the *linear Poisson structure* on A^* associated to A. In [5], the equivalence between abstract Lie algebroid structures and linear Poisson structures on their duals was established. An arbitrary vector bundle (E, M, q) endowed with a linear Poisson structure is called a *Poisson vector bundle*. For a bivector $\pi : E \to \bigwedge^2 TE$, the pair (E, π) is a Poisson vector bundle if and only if the Poisson morphism $\pi^\sharp : T^*E \to TE$ is a morphism of double vector bundles over a map $a : E^* \to TM$ (see [17]).

This result was used in [11] to lift linear symplectic forms and linear Poisson structures.

Product preserving bundle functors on manifolds still called Weil functors were classified by [8]. Indeed this author has shown in particular that the set of equivalence classes of such functors are in bijection with the set of equivalence classes of Weil algebras. These functors were used by many authors (see, e.g., [3, 9, 12, 18]) to present some lifts of various geometric objects (smooth functions, tensor fields, linear connections on manifolds,...).

In the first part of this paper, we give another proof of the existence of local connections on each double vector bundle by applying Lemma 4.5. When M is paracompact, it is well-known that there are global connections on each vector bundle $E \to M$ (see [7, Theorem 17.16.7] or [13]); one can then deduce that each double vector bundle (D, M, π) admits global connections. Moreover, when F denotes a Weil functor, we consider the F-prolongation of a double vector bundle (Proposition 5.2). We finally give new proofs of some known results (Theorem 6.1, Proposition 6.4 and Theorem 6.6) on duals of a double vector bundle.

In the second part, we recall lifts of linear sections (with respect to a Weil functor) presented in [9] and study some of their additional properties. In particular we extend to an arbitrary double vector bundle the characterization of linear k-forms on a tangent double vector bundle from [1, 2, 14] (see Theorem 9.1, Theorem 10.2 and Corollary 10.4).

2. Weil functors

2.1. Weil algebra. A *Weil algebra* is a finite-dimensional quotient of the algebra of germs $\mathcal{E}_p = C_0^\infty(\mathbb{R}^p, \mathbb{R})$ $(p \in \mathbb{N}^*)$. For other equivalent definitions of Weil algebras, one can refer to [13].

Let us denote $\mathcal{M}_p \subset \mathcal{E}_p$ the ideal of germs vanishing at 0; hence $\mathcal{M}_p$ is the maximal ideal of the local algebra $\mathcal{E}_p$.

It is clear that $\mathbb{R} = \mathcal{E}_p/\mathcal{M}_p$ and the algebra of jets $J_0^r(\mathbb{R}^p, \mathbb{R}) = \mathcal{E}_p/\mathcal{M}_p^{r+1}$ are examples of Weil algebras.

2.2. Covariant description of a Weil functor $T^A : \mathcal{M}f \to \mathcal{FM}$. We write $\mathcal{M}f$ for the category of finite dimensional smooth manifolds and mappings of class C^∞; furthermore, $\mathcal{FM}$ is the category of fibered manifolds and fibered manifolds morphisms.

Let $A = \mathcal{E}_p/I$ be a Weil algebra and consider a manifold M. In the set of

$\varphi \in C^\infty(\mathbb{R}^p, M)$ such that $\varphi(0) = x$, define an equivalence relation $\sim_x$ by:

$$\varphi \sim_x \psi \quad \text{if and only if} \quad [h]_x \circ [\psi]_0 - [h]_x \circ [\varphi]_0 \in I,$$

for all $[h]_x \in C^\infty_x(M, \mathbb{R})$. The equivalence class of φ is denoted by $j^A\varphi$ and is called the $A-$*velocity* of φ at 0; the class $j^A\varphi$ depends only on the germ of φ at 0. The quotient set is denoted by $(T^AM)_x$ and the disjoint union of $(T^AM)_x$, $x \in M$ by T^AM.

The mapping $\pi_{A,M} : T^AM \to M, j^A\varphi \mapsto \varphi(0)$, defines a bundle structure on T^AM and for all smooth map $f : M \to N$, one defines a bundle morphism $T^Af : T^AM \to T^AN$, (over f) by, $T^Af(j^A\varphi) = j^A(f \circ \varphi)$.

The correspondence $T^A : \mathcal{M}f \to \mathcal{FM}$ is a well-defined product-preserving bundle functor called the *Weil functor* associated to A ([13]).

When $A = \mathcal{E}_p/\mathcal{M}_p^{r+1}$, then T^A is equivalent to the bundle functor T^r_p of (p, r)-velocities and when A is the algebra of dual numbers $\mathbb{D} = \mathcal{E}_1/\mathcal{M}_1^2$, then $T^A = T$ is the tangent functor.

2.3. The canonical flow natural equivalence $\kappa : T^A \circ T \to T \circ T^A$. Given two Weil functors T^A, T^B with $A = \mathcal{E}_p/I$, $B = \mathcal{E}_q/J$; let M be a manifold. For any $\zeta = j^A\varphi \in T^AT^BM$, there is a differentiable mapping $\Phi : \mathbb{R}^p \times \mathbb{R}^q \to M$ such that $\varphi(z) = j^B\Phi_z$, in a neighbourhood of $0 \in \mathbb{R}^p$ (see [3] for bundle functors of $(p, 1)$-velocities or [13] for Weil functors). By this result, one can define a natural equivalence

$$\kappa_{A,B} : T^A \circ T^B \longrightarrow T^B \circ T^A$$

by : $(\kappa_{A,B})_M(\zeta) = j_B\eta$, where $\eta : \mathbb{R}^q \to T^AM$, $t \mapsto j^A\Phi^t$. In particular, for $T^B = T$, we obtain the canonical flow natural equivalence

$$\kappa : T^A \circ T \longrightarrow T \circ T^A \tag{2.1}$$

associated to the bundle functor T^A, i.e., the following diagram commutes for every manifold M and every vector field X on M:

$$\begin{array}{ccc} T^AM & \xrightarrow{\mathcal{F}_MX} & TT^AM \\ {\scriptstyle T^AX}\big\downarrow & \overset{\kappa_M}{\nearrow} & \big\downarrow{\scriptstyle \pi_{T^AM}} \\ T^ATM & \xrightarrow[T^A\pi_M]{} & T^AM \end{array}$$

with $\mathcal{F}_M X$ the vector field on $T^A M$ given by:

$$\mathcal{F}_M X(u) = \frac{\partial}{\partial t} T^A(Fl_t^X)(u)|_{t=0} \in T_u T^A M,$$

and $Fl^X : \mathbb{R} \times M \supseteq \Omega_X \to M$ the global flow of X. $X^c := \mathcal{F}_M X$ is called the complete lift of X to $T^A M$ and $\mathcal{F} : T \rightsquigarrow TT^A$ is called the flow operator of T^A (see [13]).

Remarks 2.1. (a) Given a product preserving bundle functor F, $A^F = F(\mathbb{R})$ is a real associative, commutative, unital and finite dimensional algebra. The fiber $N = F_0(\mathbb{R})$ over 0 is the ideal of nilpotent elements of A^F and we have $A^F = \mathbb{R}\cdot 1 \oplus N$; moreover, there is a canonical natural equivalence $\Theta : F \to T^{A^F}$ (see [13]). The algebra A^F is called the *Weil algebra* of F.

(b) Weil functors $T^A : \mathcal{M}f \to \mathcal{FM}$ preserve immersions, embeddings, submersions, surjective submersions, transversal maps,... In particular let (Y, M, q) a fibered manifold and a smooth map $f : N \to M$; the canonical isomorphism

$$\big(T^A pr_1, T^A pr_2\big) : T^A\big(N \times_{(f,q)} Y\big) \longrightarrow T^A N \times T^A Y$$

induces an isomorphism of fibered manifolds

$$T^A\big(N \times_{(f,q)} Y\big) \longrightarrow T^A N \times_{(T^A f, T^A q)} T^A Y$$

over $T^A N$, which can be written $T^A(f^*(Y)) = (T^A f)^*(T^A Y)$.

(c) For a smooth manifold M, one can consider the vector bundles

$$\big(T^A TM, T^A M, T^A \pi_M\big) \qquad \text{and} \qquad \big(TT^A M, T^A M, \pi_{T^A M}\big);$$

let $0_{TM} : M \to TM$ be the zero vector field; hence the zero section of $T^A TM \to T^A M$ is just

$$T^A(0_{TM}) : T^A M \longrightarrow T^A TM \qquad \text{and} \qquad \kappa_M : T^A TM \longrightarrow TT^A M$$

is an isomorphism of vector bundles over $id_{T^A M}$.

(d) For a vector bundle (E, M, q), $\big(T^A E, T^A M, T^A q\big)$ is a vector bundle with the addition $T^A(+) : T^A E \times_{T^A M} T^A E \to T^A E$ and the multiplication $(t, \widetilde{e}) \mapsto T^A(m_t^E)(\widetilde{e})$, where m^E denotes the multiplication on E. If $(g, \underline{g})$ is a morphism of vector bundles, then $\big(T^A g, T^A \underline{g}\big)$ is also a morphism of vector bundles.

For the sake of simplicity, in the rest of the document $F : \mathcal{M}f \to \mathcal{FM}$ is a product preserving bundle functor with the associated Weil algebra A^F. We will often write DVB for "double vector bundle".

3. Axiomatic of double vector bundles

3.1. Double vector bundles.

Definition 3.1. ([17]) A *double vector bundle* is a system $(D; A, B; M)$ of four vector bundle structures

$$\begin{array}{ccc} D & \xrightarrow{q_B^D} & B \\ {\scriptstyle q_A^D}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow{q_A} & M \end{array} \tag{3.1}$$

where D is a vector bundle on bases A and B, which are themselves vector bundles on M, such that each of the four structure maps of each vector bundle structure on D (projection, addition, scalar multiplication and the zero section) is a vector bundle morphism with respect to the other structure.

Let $(D; A, B; M)$ be a double vector bundle.

The vector bundle $D \overset{q_A^D}{\to} A$ is called the *vertical bundle structure* on D and $D \overset{q_B^D}{\to} B$ is called *the horizontal bundle structure* on D.

Notation 3.2. ([17]) Vector bundles $D \overset{q_A^D}{\to} A$ and $D \overset{q_B^D}{\to} B$ are usually denoted $\widetilde{D}_A$ and $\widetilde{D}_B$ respectively; the zero sections of vector bundles A, B, $\widetilde{D}_A$, $\widetilde{D}_B$ are respectively denoted 0^A, 0^B, $\widetilde{0}^A$, $\widetilde{0}^B$; the additions and scalar multiplications of $D \overset{q_A^D}{\to} A$ and $D \overset{q_B^D}{\to} B$ are denoted $\underset{A}{+}$, $\underset{A}{\cdot}$ and $\underset{B}{+}$, $\underset{B}{\cdot}$ respectively. We denote D_a the fiber of $\widetilde{D}_A$ over $a \in A$ and D_b the fiber of $\widetilde{D}_B$ over $b \in B$.

Remarks 3.3. (1) We have $D_a \cap D_b \neq \emptyset$ if and only if $q_A(a) = q_B(b)$ and for d in $D_a \cap D_b$, $D_a \cap D_b = d \underset{A}{+} \ker(q_B^D)_a = d \underset{B}{+} \ker(q_A^D)_b$ is an affine subspace of both D_a and D_b. Hence $D_a \cap D_b$ is a vector subspace of both D_a and D_b if and only if $\ker(q_B^D)_a = \ker(q_A^D)_b$, i.e., $a = 0^A(x)$, $b = 0^B(x)$, i.e., the vector bundles $\widetilde{D}_A$ and $\widetilde{D}_B$ coincide on the set $C := \bigcup_{x \in M} D_{0^A(x)} \cap D_{0^B(x)}$.

(2) When we say that the addition $\underset{A}{+} : \widetilde{D}_A \oplus \widetilde{D}_A \to \widetilde{D}_A$ is a morphism of vector bundles over the addition $B \oplus B \to B$ we implicitly admit that $\widetilde{D}_A \oplus \widetilde{D}_A \to B \oplus B$ is a vector bundle. In fact $\widetilde{D}_A \oplus \widetilde{D}_A$ is a subbundle of the

restriction $L := \widetilde{D}_B \times \widetilde{D}_B \mid_{B\oplus B}$ since (q_A^D, q_B) and (q_B^D, q_A) are morphisms of vector bundles. Indeed $\widetilde{D}_A \oplus \widetilde{D}_A \subset L$ and for all (b, b') in $B_x \oplus B_x$,

$$\begin{aligned}\widetilde{D}_A \oplus \widetilde{D}_A \cap L_{(b,b')} &= \Big(\bigcup_{a\in A_x} D_a \times D_a\Big) \cap D_b \times D_{b'} \\ &= \left[\left(q_A^D\right)_b \times \left(q_A^D\right)_{b'}\right]^{-1} (\Delta_{A_x\times A_x})\end{aligned}$$

is a vector subspace of $L_{(b,b')} = D_b \times D_{b'}$, where $\Delta_{A_x\times A_x}$ denotes the diagonal of $A_x \times A_x$. Moreover, $\left(q_A^D \times q_A^D\right)(L) = A\oplus A$ and the induced map $q_A^D \oplus_B q_A^D : L \to A \oplus A$ is a fibrewise surjective morphism of vector bundles over the projection $q_{B\oplus B}$, hence $\widetilde{D}_A \oplus \widetilde{D}_A = \left[q_A^D \oplus_B q_A^D\right]^{-1} (\Delta_{A\times A})$ is a subbundle of L, since $\Delta_{A\times A} \subset A \oplus A$ is a subbundle.

(3) Using the fact that a continuous map $\theta : V \to W$ between two real vector spaces is linear if and only if $\theta(v+v') = \theta(v)+\theta(v')$, for all v, v'in V, a commutative diagram (3.1) of four vector bundle structures is a double vector bundle if and only if we have

$$q_A^D(d \underset{B}{+} d') = q_A^D(d) + q_A^D(d'), \text{ for all } d, d' \text{ in } \widetilde{D}_B\,, \tag{3.2}$$

$$q_B^D(d \underset{A}{+} d') = q_B^D(d) + q_B(d'), \text{ for all } d, d' \text{ in } \widetilde{D}_A\,, \tag{3.3}$$

$$(d_1 \underset{A}{+} d_2) \underset{B}{+} (d_3 \underset{A}{+} d_4) = (d_1 \underset{B}{+} d_3) \underset{A}{+} (d_2 \underset{B}{+} d_4)\,, \tag{3.4}$$

for all (d_1, d_2), (d_3, d_4) in $\widetilde{D}_A \oplus \widetilde{D}_A$ such that $q_B^D(d_1) = q_B^D(d_3)$ and $q_B^D(d_2) = q_B^D(d_4)$.

Examples 3.4. (1) Given three vector bundles A, B, C over the same base M, hence $(A \oplus B \oplus C; A, B; M)$ is a DVB. Indeed let us consider the vector bundles $A \oplus B \oplus C \to A$ as $q_A^*(B \oplus C)$ and $A \oplus B \oplus C \to B$ as $q_B^*(A \oplus C)$; these data satisfy obviously (3.2), (3.3) and (3.4). In particular, $A\oplus B \cong A\oplus B\oplus 0^C(M)$ is a DVB $(A\oplus B\oplus C; A, B; M)$ called a *decomposed double vector bundle.*

(2) Let $(D; A, B; M)$ be a DVB and $U \subset M$ a non empty open set. There is a structure of DVB on $D_U := (q_A \circ q_A^D)^{-1}(U) = (q_B \circ q_B^D)^{-1}(U)$. Indeed let $D_U \to q_A^{-1}(U)$, $D_U \to q_B^{-1}(U)$ and $q_A^{-1}(U) \to U$, $q_B^{-1}(U) \to U$ be the restrictions $\widetilde{D}_A \mid_{q_A^{-1}(U)}$, $\widetilde{D}_B \mid_{q_B^{-1}(U)}$ and $A \mid_U$, $B \mid_U$ respectively; hence the

induced commutative diagram

$$\begin{array}{ccc} D_U & \xrightarrow{q_B^D} & B\mid_U \\ {\scriptstyle q_A^D}\downarrow & & \downarrow{\scriptstyle q_B} \\ A\mid_U & \xrightarrow{q_A} & M \end{array}$$

of vector bundle structures obviously satisfies (3.2), (3.3) and (3.4). This DVB denoted $D\mid_U$ is called the *restriction* of D to U.

3.2. Core and exact sequences. Let us recall that given a morphism of vector bundles $f : E \to E'$ over $\underline{f} : M \to M'$, one can associate a morphism of vector bundles $f^! : E \to \underline{f}^*E'$ over M ([17]) defined by $f^!(e) = (q_E(e), f(e))$. One has $\ker f^! = \ker f$ and $f^!$ is fibrewise surjective if f is fibrewise surjective.

Let $(D; A, B; M)$ be a double vector bundle.

In Remark 3.3, we noticed that the vector bundles $\widetilde{D}_A$ and $\widetilde{D}_B$ coincide on the set $C := \bigcup_{x\in M} D_{0^A(x)} \cap D_{0^B(x)}$, intersection of kernels $\ker q_A^D$ and $\ker q_B^D$. For x in M,

$$C_x := D_{0^A(x)} \cap D_{0^B(x)} = \ker(q_B^D)_{0^A(x)} = \ker(q_A^D)_{0^B(x)}$$

is a vector subspace of both $D_{0^A(x)}$ and $D_{0^B(x)}$; but $\widetilde{0}^A \circ 0^A(x) = \widetilde{0}^B \circ 0^B(x)$ and $c \underset{A}{+} c' = c \underset{B}{+} c'$ by (3.4), hence the induced structures of vector space on C_x coincide. Now, $C = \ker(q_B^D)\mid_{0^A(M)} = \ker(q_A^D)\mid_{0^B(M)}$ is a vector bundle over $0^A(M)$ and $0^B(M)$ diffeomorphic to M, hence there are two structures of vector bundle on C over M with the same projection $q_C := q_A \circ q_A^D \mid_C = q_B \circ q_B^D \mid_C$ and the same structure of vector space on each fibre, so (C, M, q_C) is a vector bundle called the *core* of $(D; A, B; M)$.

Moreover, the maps

$$\begin{array}{rcl} \tau_A : q_A^*C & \longrightarrow & \widetilde{D}_A \\ (a, c) & \longmapsto & \widetilde{0}^A(a) \underset{B}{+} c \end{array} \quad \text{and} \quad \begin{array}{rcl} \tau_B : q_B^*C & \longrightarrow & \widetilde{D}_B \\ (b, c) & \longmapsto & \widetilde{0}^B(b) \underset{A}{+} c \end{array}$$

are fibrewise injective morphisms of vector bundles (such that $\operatorname{Im}\tau_A = \ker q_B^D$, $\operatorname{Im}\tau_B = \ker q_A^D$) called *translations* over A and B respectively. Hence there are short exact sequences of morphisms of vector bundles

$$0 \longrightarrow q_A^*C \xrightarrow{\tau_A} \widetilde{D}_A \xrightarrow{(q_B^D)^!} q_A^*B \longrightarrow 0\,, \tag{3.5}$$

$$0 \longrightarrow q_B^* C \xrightarrow{\tau_B} \widetilde{D}_B \xrightarrow{(q_A^D)^!} q_B^* A \longrightarrow 0\,, \tag{3.6}$$

called *core sequences* over A and B respectively.

Remarks 3.5. (1) It is clear by (3.4) that

$$d \underset{A}{+} \tau_A(a,c) = d \underset{B}{+} \tau_B(b,c), \tag{3.7}$$

for all $(a,b,c) \in A \oplus B \oplus C$ and $d \in D_a \cap D_b$.

(2) Given a smooth section $\gamma \in \Gamma(C)$, the image of $q_A^*\gamma \in \Gamma(q_A^* C)$ by τ_A is a section of $\ker q_B^D \to A$ denoted γ^A and called in [17] the core section corresponding to γ with respect to (3.5). Clearly, $\gamma^A(a) = \tau_A(a, \gamma(q_A(a)))$ and since τ_A induces an isomorphism of vector bundles $q_A^* C \to \ker q_B^D$ over A, for a local frame (γ_l) of C, (γ_l^A) is a local frame of $\ker q_B^D$.

3.3. Morphisms of double vector bundles. Let $(D; A, B; M)$ and $(D'; A', B'; M')$ be two double vector bundles with cores C and C'.

Definition 3.6. [17] A *morphism* from $(D; A, B; M)$ to $(D'; A', B'; M')$ is a system $(\varphi; \varphi_A, \varphi_B; \varphi_M)$ of smooth maps $\varphi : D \to D'$, $\varphi_A : A \to A'$, $\varphi_B : B \to B'$, $\varphi_M : M \to M'$ such that (φ, φ_A), (φ, φ_B), (φ_A, φ_M), (φ_B, φ_M) are morphisms of vector bundles. In fact,

$$\varphi_A = q_{A'}^{D'} \circ \varphi \circ \widetilde{0}^A, \qquad \varphi_B = q_{B'}^{D'} \circ \varphi \circ \widetilde{0}^B,$$

$$\begin{aligned} \varphi_M &= q_{B'} \circ \varphi_B \circ 0^B = q_{A'} \circ \varphi_A \circ 0^A \\ &= q_{B'} \circ q_{B'}^{D'} \circ \varphi \circ \widetilde{0}^B \circ 0^B = q_{A'} \circ q_{A'}^{D'} \circ \varphi \circ \widetilde{0}^A \circ 0^A. \end{aligned}$$

If $M = M'$ and $\varphi_M = id_M$, φ is said *over* M; if $A = A'$ and $\varphi_A = id_A$, we say that φ *preserves* A ; if $A = A'$, $B = B'$ and $\varphi_A = id_A$, $\varphi_B = id_B$, we say φ *preserves the side bundles*.

Remark 3.7. Let $(\varphi; \varphi_A, \varphi_B, ; \varphi_M)$ be a morphism from $(D; A, B; M)$ to $(D'; A', B'; M')$. It is clear that $\varphi(\ker q_A^D) \subset \ker q_{A'}^{D'}$ and $\varphi(\ker q_B^D) \subset \ker q_{B'}^{D'}$; in particular, $\varphi(C) \subset C'$ and the induced map $\varphi_C : C \to C'$ is a morphism of vector bundles over φ_M called the core morphism of $(\varphi; \varphi_A, \varphi_B; \varphi_M)$.

4. Linear sections, splittings, vertical and horizontal lifts ([14, 17, 6])

Let $(D; A, B; M)$ be a double vector bundle with core C.

4.1. Linear sections and splittings.

Definition 4.1. A *linear section* of D with respect to its vertical vector bundle structure $D \to A$ is a pair $(\sigma, \underline{\sigma})$ of sections where $\sigma \in \Gamma_A(D)$, $\underline{\sigma} \in \Gamma(B)$ and σ is a morphism of vector bundles

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & D \\ {\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle q_B^D} \\ M & \xrightarrow{\underline{\sigma}} & B \end{array} . \tag{4.1}$$

The set of linear sections with respect to $D \to A$ is denoted $\Gamma_A^{\ell in}(D)$; this is a $C^\infty(M)$-module where the multiplication is given by $f \cdot \sigma := f \circ q_A \cdot \sigma$.

One can also define in the same way a linear section of D with respect to its horizontal vector bundle structure $D \to B$.

Definition 4.2. A *linear splitting* of the exact sequence (3.5) is a right inverse of the surjective morphism of vector bundles $(q_B^D)^! : \widetilde{D}_A \to q_A^* B$, i.e., a morphism of vector bundles $\psi : q_A^* B \to \widetilde{D}_A$ over A such that $(q_B^D)^! \circ \psi = id_{q_A^* B}$. One can define in the same way a linear splitting of the exact sequence (3.6).

A *linear splitting* (or a *linear connection*) of D is a linear splitting $\psi : A \oplus B \to D$ of (3.5) and (3.6).

Remark 4.3. Each of the exact sequences (3.5) and (3.6) admits local linear splittings. Indeed let U be an open paracompact set of M over which A and B are trivializable; since $q_A^{-1}(U)$ and $q_B^{-1}(U)$ are paracompact manifolds, all subbundles of vector bundles $\widetilde{D}_A \mid_{q_A^{-1}(U)}$ and $\widetilde{D}_B \mid_{q_B^{-1}(U)}$ admit complements (see [7, 16.17.3]), i.e., there are subbundles $K \subseteq \widetilde{D}_A \mid_{q_A^{-1}(U)}$, $L \subseteq \widetilde{D}_B \mid_{q_B^{-1}(U)}$ such that

$$\widetilde{D}_A \mid_{q_A^{-1}(U)} = \ker q_B^D \mid_{q_A^{-1}(U)} \oplus K \,, \qquad \widetilde{D}_B \mid_{q_B^{-1}(U)} = \ker q_A^D \mid_{q_B^{-1}(U)} \oplus L \,.$$

Hence

$$\psi : q_A^* B \mid_{q_A^{-1}(U)} \xrightarrow{((q_B^D)^! |_K)^{-1}} K \hookrightarrow \widetilde{D}_A \,,$$
$$\eta : q_B^* A \mid_{q_B^{-1}(U)} \xrightarrow{((q_A^D)^! |_L)^{-1}} L \hookrightarrow \widetilde{D}_B$$

are respectively local splittings of (3.5) and (3.6).

Definition 4.4. When $\eta : q_B^* A \to \widetilde{D}_B$ is a linear splitting of (3.6), the horizontal *lift* (with respect to η) of a section $\underline{\beta} \in \Gamma(B)$ is the linear section $\beta \in \Gamma_A^{\ell in}(D)$ defined by $\beta(a) = \eta(\underline{\beta}(q_A(a), a)$.

When $\psi : q_A^* B \to \widetilde{D}_A$ is a linear splitting of (3.5), the *vertical lift* (with respect to ψ) of a section $\underline{\alpha} \in \Gamma(A)$ is the linear section $\alpha \in \Gamma_B^{\ell in}(D)$ defined by $\alpha(b) = \psi(\underline{\alpha}(q_B(b)), b)$.

Lemma 4.5. *For a linear function $h : A \to \mathbb{R}$ and a linear section $\alpha \in \Gamma_B^{\ell in}(D)$, the map $\alpha_h : A \oplus B \to D$, given by $\alpha_h(a,b) = h(a) \underset{B}{\cdot} \alpha(b)$, is a morphism of double vector bundles $(\alpha_h; h \cdot \underline{\alpha} \circ q_A, id_B; id_M)$.*

Proposition 4.6. ([6]) *$(D; A, B; M)$ admits local linear splittings.*

Proof. Let $\psi : q_A^{-1}(U) \oplus q_B^{-1}(U) \to \widetilde{D}_A$ a local linear splitting of (3.5); consider $(\underline{\alpha}_i)_{1 \le i \le n_1}$ a local frame of A on U and $\alpha_i : q_B^{-1}(U) \to \widetilde{D}_A$, $1 \le i \le n_1$ their associated horizontal lifts. Let $h_i : q_A^{-1}(U) \to \mathbb{R}$, $1 \le i \le n_1$ be the linear functions given by $h_i(a) = \left\langle \underline{\alpha}^i((q_A(a)), a \right\rangle$, where $(\underline{\alpha}^i)_{1 \le i \le n_1}$ is the dual frame of $(\underline{\alpha}_i)_{1 \le i \le n_1}$; hence (by the previous lemma) the map $\widetilde{\psi} := \sum_B h_i \underset{B}{\cdot} \alpha_i$ is a morphism of double vector bundles over $\sum_{i=1}^{n_1} h_i \underline{\alpha} \circ q_A = id_{q_A^{-1}(U)}$ and $id_{q_B^{-1}(U)}$, i.e., a local linear splitting of D. ∎

Remark 4.7. Each linear splitting $\sigma : A \oplus B \to D$ of $(D; A, B; M)$ is equivalent to an isomorphism of double vector bundles $\phi : D \longrightarrow A \oplus B \oplus C$ over A and B such that $\phi_C = id_C$. Indeed the map

$$\begin{array}{rcl} \theta : A \oplus B \oplus C & \longrightarrow & D \\ (a,b,c) & \longmapsto & \sigma(a,b) \underset{A}{+} \tau_A(a,c) = \sigma(a,b) \underset{B}{+} \tau_B(b,c)\,, \end{array}$$

is an isomorphism of DVB over A and B such that $\theta_C = id_C$, hence $\phi := \theta^{-1}$ is an isomorphism of DVB over A and B such that $\phi_C = id_C$. Conversely given such an isomorphism of double vector bundles, a linear splitting is defined by $\sigma(a,b) = \phi^{-1}(a, b, 0^C(x))$. Such an isomorphism is called a *decomposition* of D.

Given a local decomposition $\phi : D\mid_U \to A \oplus B \oplus C \mid_U$ of D such that A, B, C are trivializable over U, one can associate a double vector chart $\varphi := (\varphi_A \oplus \varphi_B \oplus \varphi_C) \circ \phi$ of D, where $\varphi_A, \varphi_B, \varphi_C$ are local trivializations of A, B, C and

$$\varphi_A \oplus \varphi_B \oplus \varphi_C : q_{A \oplus B \oplus C}^{-1}(U) \longrightarrow U \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_0}$$

is the isomorphism of DVB (5.1).

Conversely a double vector chart $\varphi : D \mid_U \to U \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_0}$ is an isomorphism of DVB over local trivializations φ_A, φ_B; if φ_C is its core morphism, hence $\phi := (\varphi_A^{-1} \oplus \varphi_B^{-1} \oplus \varphi_C^{-1}) \circ \varphi$ is a local decomposition of D. This shows (by the previous result) that the structure of D is entirely described by local decompositions.

LEMMA 4.8. *Let $\emptyset \neq U \subset M$ be a domain of chart over which A and B are trivializable, $f : U \to \mathbb{R}$ a smooth function with a compact support and a local linear splitting $\psi : A \oplus B \mid_U \to D$ of D. Hence the map $\widetilde{\psi} : A \oplus B \to D$, given by*

$$\begin{cases} \widetilde{\psi} := f \circ q_{A\oplus B} \underset{B}{\cdot} \psi & \text{on } q_{A\oplus B}^{-1}(U), \\ \widetilde{\psi} := \widetilde{0}^B \circ p_2 & \text{on } A \oplus B \backslash q_{A\oplus B}^{-1}(U), \end{cases}$$

is a morphism of DVB over $\overline{f} id_A$ and id_B, where $\overline{f} \in C^\infty(M)$ is equal to f on U and 0 on $M \backslash U$.

Proof. It is sufficient to show that $\widetilde{\psi}$ is smooth. Indeed $\widetilde{\psi}$ is smooth on $q_{A\oplus B}^{-1}(U)$ and $A \oplus B \backslash \overline{q_{A\oplus B}^{-1}(U)}$; moreover for (a, b) in $\overline{q_{A\oplus B}^{-1}(U)} \backslash q_{A\oplus B}^{-1}(U)$, $q_{A\oplus B}(a, b) \in \overline{U} \backslash U \subset M \backslash \operatorname{Supp}(f) = V$, hence $q_{A\oplus B}^{-1}(V)$ is an open neighborhood of (a, b) such that $\widetilde{\psi} \mid_{q_{A\oplus B}^{-1}(V)} = \widetilde{0}^B \circ p_2 \mid_{q_{A\oplus B}^{-1}(V)}$, so $\widetilde{\psi}$ is smooth. ∎

This is sufficient to generalize [7, Theorem 17.16.7].

THEOREM 4.9. ([20]) *If M is paracompact, $(D; A, B; M)$ admits (global) connections. In particular $(D; A, B; M)$ admits global decompositions.*

Proof. Let $(U_i)_{i\in I}$ be a locally finite atlas of M such that A and B are trivializable over each U_i. Let $\psi_i : A \oplus B \mid_{U_i} \to D$, $i \in I$ be a local linear splitting (Proposition 4.6) and let $(f_i)_{i\in I}$ be a partition of unity subordinate to this open cover. By the previous lemma, the maps given by

$$\begin{cases} \widetilde{\psi}_i := f_i \circ q_{A\oplus B} \underset{B}{\cdot} \psi_i & \text{on } q_{A\oplus B}^{-1}(U_i), \\ \widetilde{\psi}_i := \widetilde{0}^B \circ p_2 & \text{on } A \oplus B \backslash q_{A\oplus B}^{-1}(U_i), \end{cases} \qquad i \in I$$

are morphisms of DVB over $f_i id_A$ and id_B. Hence $\widetilde{\psi} := \sum_B \widetilde{\psi}_i$ is a linear splitting of D. ∎

4.2. LOCAL COORDINATE SYSTEMS. Let $\phi : D \mid U \to A \oplus B \oplus C \mid U$ be a local decomposition of a double vector bundle $(D; A, B; M)$ where A, B, C are trivializable over U. For local frames $\big(\underline{\alpha}_k\big)_{1\leq k\leq n_1}$, $\big(\underline{\beta}_j\big)_{1\leq j\leq n_2}$, $\big(\underline{\gamma}_l\big)_{1\leq l\leq n_0}$ of A, B, C on U, it is clear by definitions of horizontal lifts, core sections and the previous remark that

$$\begin{cases} \alpha_k(b) = \phi^{-1}\big(\underline{\alpha}_k(q_B(b)), b, 0^C(q_B(b))\big), & 1 \leq k \leq n_1, \\ \gamma'_l(b) := \gamma_l^B(b) = \phi^{-1}\big(0^A(q_B(b)), b, \underline{\gamma}_l(q_B(b))\big), & 1 \leq l \leq n_0, \\ \beta_j(a) = \phi^{-1}\big(a, \underline{\beta}_j(q_A(a)), 0^C(q_A(a))\big), & 1 \leq j \leq n_2, \\ \gamma_l(a) := \gamma_l^A(a) = \phi^{-1}\big(a, 0^B(q_A(a)), \underline{\gamma}_l(q_A(a))\big), & 1 \leq l \leq n_0, \end{cases}$$

define local frames

$$(\alpha_k, \gamma'_l) \text{ and } (\beta_j, \gamma_l) \tag{4.2}$$

of $\widetilde{D}_B$ and $\widetilde{D}_A$ respectively. Moreover, considering local trivializations φ_A, φ_B, φ_C associated to $(\alpha_k)_{1\leq k\leq n_1}$, $(\beta_j)_{1\leq j\leq n_2}$, $(\gamma_l)_{1\leq l\leq n_0}$, one can associate by the previous remark a double vector chart $(U, \varphi, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \mathbb{R}^{n_0})$ such that

$$\begin{cases} \beta_j(a) = \varphi^{-1}\big(\varphi_A(a), e_j^2, 0\big), & \gamma_l(a) = \varphi^{-1}\big(\varphi_A(a), 0, e_l^0\big), \\ \alpha_k(b) = \varphi^{-1}\big(x, e_k^1, (\varphi_B)_x(b), 0), & \gamma'_l(b) = \varphi^{-1}\big(x, 0, (\varphi_B)_x(b), e_l^0\big), \end{cases} \tag{4.3}$$

where $x = q_B(b)$ and $\big(e_k^1\big)_{1\leq k\leq n_1}$, $\big(e_j^2\big)_{1\leq j\leq n_2}$, $\big(e_l^0\big)_{1\leq l\leq n_0}$ are bases of $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$, $\mathbb{R}^{n_0}$ respectively.

When U is a domain of chart (U, u), there are adapted local coordinate systems (x^i, z^k), (x^i, r^j) of A, B on $q_A^{-1}(U)$, $q_B^{-1}(U)$ and an adapted local coordinate system (x^i, a^k, b^j, c^l) of $D \to M$ on $\pi^{-1}(U)$ such that

$$\begin{cases} x^i = u^i \circ \pi \mid_{\pi^{-1}(U)}, & 1 \leq i \leq m, \\ a^k = z^k \circ q_A^D \mid_{\pi^{-1}(U)}, & 1 \leq k \leq n_1, \\ b^j = r^j \circ q_B^D \mid_{\pi^{-1}(U)}, & 1 \leq j \leq n_2, \end{cases} \tag{4.4}$$

and functions c^l, $1 \leq l \leq n_0$ are linear on fibers of both $\widetilde{D}_A|_{q_A^{-1}(U)}$ and $\widetilde{D}_B|_{q_B^{-1}(U)}$.

5. Examples of double vector bundles

5.1. The decomposed double vector bundle. Let A, B, C be three structures of vector bundles over the same base M. The *decomposed double vector bundle* also called *trivial DVB* in [17],

$$\begin{array}{ccc} A\oplus B\oplus C & \xrightarrow{p_2} & B \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle q_2} \\ M & \xrightarrow{q_1} & B \end{array}$$

associated to A, B, C is already defined and denoted $\{A, B; C\}$ in [20]. The double vector chart corresponding to vector charts $(U, \varphi_A, \mathbb{R}^{n_1})$, $(U, \varphi_B, \mathbb{R}^{n_2})$, $(U, \varphi_C, \mathbb{R}^{n_0})$ of A, B, C is the map

$$\begin{array}{rcl} q^{-1}_{A\oplus B\oplus C}(U) & \xrightarrow{\varphi_A\oplus\varphi_B\oplus\varphi_C} & U\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_0} \\ A_x\oplus B_x\oplus C_x\ni(a,b,c) & \longmapsto & (x,(\varphi_A)_x(a),(\varphi_B)_x(b),(\varphi_C)_x(c)). \end{array} \tag{5.1}$$

Remark 5.1. One can consider the following particular cases: $A = 0^A(M)$ and $B = 0^B(M)$, $C = 0^C(M)$ isomorphic to $B \oplus C$ and A respectively.

Since each real vector space V is a vector bundle over a one point manifold, hence $V = V \oplus \{0\} \oplus \{0\}$ is a double vector bundle and a linear map $\varphi : V \to W$ a is morphism of double vector bundles.

5.2. Double vector bundles $\ker q^D_A$, $\ker q^D_B$, C. Given a double vector bundle $(D; A, B; M)$ with core C, hence $\ker q^D_A$, $\ker q^D_B$ and C are double vector bundles respectively isomorphic to $\{0^A(M), B; C\}$, $\{A, 0^B(M); C\}$ and $\{0^A(M), 0^B(M); C\}$. The (global) linear splittings associated to these double vector bundles are respectively $\widetilde{0}^A$, $\widetilde{0}^B$, $\widetilde{0}^A \circ 0^A = \widetilde{0}^B \circ 0^B$.

5.3. The tangent double vector bundle of a vector bundle. Let (E, M, q) be a vector bundle. Consider a chart $c = (U, u, m)$ of M such that E is trivializable over U and let $\varphi : q^{-1}(U) \to u(U) \times \mathbb{R}^n$ be a fibered chart of E. $T\varphi$ is an isomorphism of vector bundles over Tu and φ; moreover if $\tau_{u(U)\times\mathbb{R}^n} : T(u(U)\times\mathbb{R}^n) \to u(U)\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n$ is the canonical isomorphism, hence

$$Tq^{-1}(U) \xrightarrow{\Phi:=(u^{-1}\times id_{\mathbb{R}^n}\times id_{\mathbb{R}^m}\times id_{\mathbb{R}^n})\circ\tau_{u(U)\times\mathbb{R}^n}\circ T\varphi} U\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n$$

is a double vector chart of (TE, M, π), where $\pi = \pi_M \circ Tq = q \circ \pi_E$. Indeed if

$$Tq^{-1}(U_1) \xrightarrow{\Phi_1 := (u_1^{-1} \times id_{\mathbb{R}^n} \times id_{\mathbb{R}^m} \times id_{\mathbb{R}^n}) \circ \tau_{u(U_1) \times \mathbb{R}^n} \circ T\varphi_1} U_1 \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$$

is another double vector chart with $\emptyset \neq Tq^{-1}(U) \cap Tq^{-1}(U_1) = Tq^{-1}(U \cap U_1)$ and

$$\begin{array}{rcl} \varphi_1 \circ \varphi^{-1} : u(U \cap U_1) \times \mathbb{R}^n & \longrightarrow & u(U \cap U_1) \times \mathbb{R}^n \\ (z, k) & \longmapsto & (u_1 \circ u^{-1}(z), a(z) \cdot k) \end{array}$$

with $a : u(U \cap U_1) \to Gl(\mathbb{R}^n)$ a smooth map, hence $\Phi_1 \circ \Phi^{-1}$ is the map

$$\begin{array}{rcl} U \cap U_1 \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n & \longrightarrow & U \cap U_1 \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \\ (x, k, h, t) & \longmapsto & (x, \alpha(x) \cdot k, \beta(x) \cdot h, \eta(x) \cdot t + \omega(x) \cdot (k, h)), \end{array}$$

where $\alpha = \eta = a \circ u : U \cap U_1 \to Gl(\mathbb{R}^n)$, $\beta = D(u_1 \circ u^{-1}) \circ u : U \cap U_1 \to Gl(\mathbb{R}^m)$ and $\omega : U \cap U_1 \to \mathcal{L}_2(\mathbb{R}^n, \mathbb{R}^m; \mathbb{R}^n)$, given by $\omega(x) \cdot (k, h) = (Da(u(x)) \cdot h) \cdot k$. These charts constitute an atlas for a structure of double vector bundle

$$\begin{array}{ccc} TE & \xrightarrow{Tq} & TM \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ E & \xrightarrow{q} & M \end{array} . \tag{5.2}$$

Its core $\bigcup_{x \in M} T_{0^E(x)} E_x$ is isomorphic to E.

The local coordinate system (4.4) of $(TE; E, TM; M)$ associated to (x^i, y^k) and $(x^i, \dot{x}^j)$ is exactly the usual local coordinate system $(x^i, y^k, \dot{x}^j, \dot{y}^l)$.

5.4. The F-prolongation of a double vector bundle. Consider a product preserving bundle functor. $F : \mathcal{M}f \to \mathcal{FM}$.

Proposition 5.2. *For a double vector bundle $(D; A, B; M)$,*

$$\begin{array}{ccc} FD & \xrightarrow{Fq_B^D} & FB \\ {\scriptstyle Fq_A^D}\downarrow & & \downarrow{\scriptstyle Fq_B} \\ FA & \xrightarrow{Fq_A} & FM \end{array} \tag{5.3}$$

is a double vector bundle.

Proof. Indeed if $(U, \varphi, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \mathbb{R}^{n_0})$ is a double vector chart of $(D; A, B; M)$ then $(FU, F\varphi, F\mathbb{R}^{n_1}, F\mathbb{R}^{n_2}, F\mathbb{R}^{n_0})$ is a double vector chart of $(FD; FA, FB; FM)$; to see it consider another double vector chart $(U, \varphi', \mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \mathbb{R}^{n_0})$ of $(D; A, B; M)$ such that

$$\varphi' \circ \varphi^{-1}(x, X, Y, Z) = (x, u_1(x) \cdot X, u_2(x) \cdot Y, u_0(x) \cdot Z + \omega(x)(X, Y)),$$

with $u_i : U \to GL(\mathbb{R}^{n_i})$, $i = 0, 1, 2$ and $\omega : U \to \mathcal{L}_2(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}; \mathbb{R}^{n_0})$ smooth maps; hence $F\varphi' \circ F\varphi^{-1}(\widetilde{x}, \widetilde{X}, \widetilde{Y}, \widetilde{Z})$ is equal to

$$(\widetilde{x}, j_{\mathbb{R}^{n_1}}(Fu_1(\widetilde{x})) \cdot \widetilde{X}, j_{\mathbb{R}^{n_2}}(Fu_2(\widetilde{x})) \cdot \widetilde{Y}, j_{\mathbb{R}^{n_0}}(Fu_0(\widetilde{x})) \cdot \widetilde{Z} + \ell_{1,2}(F\omega(\widetilde{x}))(\widetilde{X}, \widetilde{Y})),$$

where $j_{\mathbb{R}^{n_i}} : FGL(\mathbb{R}^i) \to GL(F\mathbb{R}^i)$, $i = 0, 1, 2$ are canonical representations and $\ell_{1,2} : F\mathcal{L}_2(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}; \mathbb{R}^{n_0}) \to \mathcal{L}_2(F\mathbb{R}^{n_1}, F\mathbb{R}^{n_2}; F\mathbb{R}^{n_0})$, the canonical map. This proves that double vector charts $(FU, F\varphi, F\mathbb{R}^{n_1}, F\mathbb{R}^{n_2}, F\mathbb{R}^{n_0})$ constitute an atlas for a structure of double vector bundle on FD. ∎

Definition 5.3. We call it the *F-prolongation* of $(D; A, B; M)$.

6. Duality on double vector bundles

Let us recall that given a morphism of vector bundles $\varphi : E \to E'$ over a diffeomorphism $\underline{\varphi} : M \to M'$, its transpose morphism is the morphism of vector bundles $\varphi^t : E'^* \to E^*$ over the diffeomorphism $\underline{\varphi}^{-1} : M' \to M$, defined on fibres by $(\varphi^t)_{x'} := \left[\varphi_{\underline{\varphi}^{-1}(x')}\right]^t$.

6.1. Duals of a double vector bundle. Let $(D; A, B; M)$ be a double vector bundle. Let $\left(D^*A, A, (q_A^D)^*\right)$ denotes the dual bundle of $\left(D, A, q_A^D\right)$. The transpose of the exact sequence (3.5) is the exact sequence

$$0 \to q_A^* B^* \overset{\left[(q_B^D)^!\right]^t}{\rightarrowtail} D^*A \xrightarrow{(\tau_A)^t} q_A^* C^* \to 0 \tag{6.1}$$

of morphisms of vector bundles over A; the composition of vector bundles morphisms

$$\begin{array}{ccccc} D^*A & \xrightarrow{(\tau_A)^t} & q_A^* C^* & \xrightarrow{p_2} & C^* \\ {\scriptstyle (q_A^D)^*}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle q_{C^*}} \\ A & \xrightarrow{id_A} & A & \xrightarrow{q_a} & M \end{array}$$

is a surjective submersion $q_{C^*}^{*A} : D^*A \to C^*$.

THEOREM 6.1. ([17]) *The commutative diagram of surjective submersions*

$$\begin{array}{ccc} D^*A & \xrightarrow{q^{*A}_{C^*}} & C^* \\ {\scriptstyle (q^D_A)^*}\big\downarrow & & \big\downarrow{\scriptstyle q_{C^*}} \\ A & \xrightarrow{q_A} & M \end{array} \tag{6.2}$$

is a double vector bundle $(D^*A; A, C^*; M)$ *with* (6.1) *as a core sequence.*

Proof. Let us denote as $\pi^* : D^*A \to M$ the surjective submersion $q_A \circ (q_A^D)^* = q_{C^*} \circ q^{*A}_{C^*}$. Consider a double vector chart $(U, \varphi, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \mathbb{R}^{n_0})$ of $(D; A, B; M)$ over local trivializations

$$\varphi_A : q_A^{-1}(U) \longrightarrow U \times \mathbb{R}^{n_1}, \qquad \varphi_B : q_B^{-1}(U) \longrightarrow U \times \mathbb{R}^{n_2}$$

of A, B; the transpose morphism $(\varphi^*, \varphi_A) := ([\varphi^{-1}]^t, \varphi_A)$ of $(\varphi^{-1}, \varphi_A^{-1})$ is an isomorphism of vector bundles given by

$$\begin{array}{rcl} \varphi^* : (\pi^*)^{-1}(U) & \longrightarrow & U \times \mathbb{R}^{n_1} \times (\mathbb{R}^{n_2})^* \times (\mathbb{R}^{n_0})^* \\ D_a^* \ni \Phi & \longmapsto & (\varphi_A(a), \Phi \circ \varphi_a^{-1}(\cdot, 0), \Phi \circ \varphi_a^{-1}(0, \cdot)). \end{array}$$

If $(U, \varphi', \mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \mathbb{R}^{n_0})$ is another double vector chart of $(D; A, B; M)$ such that

$$\varphi' \circ \varphi^{-1}(x, X, Y, Z) = \big(x, u_1(x) \cdot X, u_2(x) \cdot Y, u_0(x) \cdot Z + \omega(x)(X, Y)\big),$$

with $u_i : U \to GL(\mathbb{R}^{n_i})$, $i = 0, 1, 2$ and $\omega : U \to \mathcal{L}_2(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}; \mathbb{R}^{n_0})$ smooth maps, hence $\varphi'^* \circ (\varphi^*)^{-1}(x, X, Y^*, Z^*)$ is equal to

$$\big(x, u_1(x)^{-1} \cdot X, [u_2(x)]^t\,(Y^*) + [\omega(x)(X, \cdot)]^t\,(Z^*), [u_0(x)]^t \cdot Z^*\big). \tag{6.3}$$

So double vector charts $(U, \varphi^*, \mathbb{R}^{n_1}, (\mathbb{R}^{n_0})^*, (\mathbb{R}^{n_2})^*)$ constitute an atlas for a structure of double vector bundle on D^*A with left and right sides $D^*A \xrightarrow{(q_A^D)^*} A$ and $D^*A \xrightarrow{q^{*A}_{C^*}} C^*$. ∎

DEFINITION 6.2. [17] The double vector bundle (6.2) is called the vertical dual (or dual over A) of (3.1). Its core is obviously isomorphic to the dual B^* of B.

Remark 6.3. The transpose

$$0 \to q_B^* A^* \xrightarrow{[(q_A^D)^!]^t} D^*B \xrightarrow{(\tau_B)^t} q_B^* C^* \to 0 \tag{6.4}$$

of the core sequence (3.6) allows in the same way to consider the horizontal dual

$$\begin{array}{ccc} D^*B & \xrightarrow{(q^D_B)^*} & B \\ {\scriptstyle q^{*B}_{C^*}}\big\downarrow & & \big\downarrow{\scriptstyle q_B} \\ C^* & \xrightarrow{q_{C^*}} & M \end{array} \tag{6.5}$$

of (3.1) with the core isomorphic to A^*.

Hence the following properties are clear.

Proposition 6.4. ([17]) *Let $\theta \in C^*$.*

(i) *For all Φ, Φ' in $(D^*A)_\theta$ with $(q^D_A)^*(\Phi) = a$, $(q^D_A)^*(\Phi') = a'$ we have:*

$$(\Phi \underset{C^*}{+} \Phi')(d'') = \Phi(d) + \Phi'(d'). \tag{6.6}$$

where $d'' \in D_{a+a'}$, $d \in D_a$, $d' \in D_{a'}$ and *$d'' = d \underset{B}{+} d'$.*

(ii) *For all Φ in $(D^*A)_\theta$ with $(q^D_A)^*(\Phi) = a$, we have:*

$$(s \underset{C^*}{\cdot} \Phi)(s \underset{B}{\cdot} d) = s\Phi(d), \tag{6.7}$$

for all $s \in \mathbb{R}$.

Proof. Let us consider a double vector chart $(U, \varphi, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \mathbb{R}^{n_0})$ of $(D; A, B; M)$ such that $x := q_{C^*}(\theta) \in U$.

(i) For all Φ, Φ' in $(D^*A)_\theta$ with $(q^D_A)^*(\Phi) = a$, $(q^D_A)^*(\Phi') = a'$ and $d'' \in D_{a+a'}$, $d \in D_a$, $d' \in D_{a'}$ such that $d'' = d \underset{B}{+} d'$, let $d = \varphi^{-1}(x, X, Y, Z)$, $d' = \varphi^{-1}(x, X', Y, Z')$, $\Phi = (\varphi^*)^{-1}(x, X, Y^*, Z^*)$, $\Phi' = (\varphi^*)^{-1}(x, X', Y'^*, Z^*)$; since $\Phi \underset{C^*}{+} \Phi' := (\varphi^*)^{-1}(\varphi^*(\Phi) \underset{(\mathbb{R}^{n_0})^*}{+} \varphi^*(\Phi'))$ by definition, we have $\Phi \underset{C^*}{+} \Phi' = (\varphi^*)^{-1}(x, X + X', Y^* + Y'^*, Z^*)$, hence

$$\begin{aligned} \Phi \underset{C^*}{+} \Phi'(d'') &= \left\langle Y, Y^* + Y'^* \right\rangle + \left\langle Z + Z', Z^* \right\rangle \\ &= \langle Y, Y^* \rangle + \langle Z, Z^* \rangle + \left\langle Y, Y'^* \right\rangle + \langle Z, Z^* \rangle = \Phi(d) + \Phi'(d'). \end{aligned}$$

(ii) Similarly, since $s \underset{C^*}{\cdot} \Phi := (\varphi^*)^{-1}\Big(s \underset{(\mathbb{R}^{n_0})^*}{\cdot} \varphi^*(\Phi)\Big)$, we have

$$s \underset{C^*}{\cdot} \Phi = (\varphi^*)^{-1}(x, sX, sY^*, Z^*)$$

hence

$$(s \underset{C^*}{\cdot} \Phi)(s \underset{B}{\cdot} d) = \langle Y, sY^* \rangle + \langle sZ, Z^* \rangle = s(\langle Y, Y^* \rangle + \langle Z, Z^* \rangle) = s\Phi(d).$$

∎

Remark 6.5. The dual frame (β^j, γ^l) of the local frame (β_j, γ_l) of $\widetilde{D}_A|_{q_A^{-1}(U)}$ defined by (4.3) is clearly the local frame of $D^*A\,|_{q_A^{-1}(U)}$ induced by the double vector chart $(U, \varphi^*, \mathbb{R}^{n_1}, (\mathbb{R}^{n_0})^*, (\mathbb{R}^{n_2})^*)$, i.e.,

$$\begin{cases} \beta^j(a) = (\varphi^*)^{-1}(\varphi_A(a), (e_j^2)^*, 0)\,, & 1 \leq j \leq n_2\,, \\ \gamma^l(a) = (\varphi^*)^{-1}(\varphi_A(a), 0, (e_l^0)^*)\,, & 1 \leq l \leq n_0\,. \end{cases} \tag{6.8}$$

Likewise, we have,

$$q_{C^*}^{*A}(\beta^j) = 0_{C^*_{q_A(a)}} \quad \text{and} \quad q_{C^*}^{*A}(\gamma^l) = \underline{\gamma}_l^*(q_A(a)),\ \ 1 \leq j \leq n_2,\ \ 1 \leq l \leq n_0. \tag{6.9}$$

In the case of the vertical dual

$$\begin{array}{ccc} T^*E & \xrightarrow{\ r_E\ } & E^* \\ {\scriptstyle \pi_E^*}\big\downarrow & & \big\downarrow{\scriptstyle q_{E^*}} \\ E & \xrightarrow{\ q_E\ } & M \end{array}$$

of (5.2), we have for x in E_x,

$$\begin{cases} r_E(dx^i(e)) = 0_{E_x^*}\,, & 1 \leq i \leq m\,, \\ r_E(dy^j(e)) = y^j\,|_{E_x}\,, & 1 \leq j \leq n\,. \end{cases} \tag{6.10}$$

6.2. Canonical pairing. We give a new proof of the following result.

Theorem 6.6. ([14, 16]) *There is a natural duality between D^*A and D^*B over C^* given by*

$$|\ \Phi, \Psi\ | = \langle d, \Phi \rangle_A - \langle d, \Psi \rangle_B\,, \tag{6.11}$$

where $q_A^D(d) = (q_A^D)^*(\Phi)$ and $q_B^D(d) = (q_B^D)^*(\Psi)$.

Proof. Let $\theta \in C_x^*$; consider a double vector chart $(U, \varphi, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \mathbb{R}^{n_0})$ of $(D; A, B; M)$ such that $x \in U$. For all $(\Phi, \Psi) \in (D^*A)_\theta \oplus (D^*B)_\theta$, let:

$$\Phi = (\varphi^*)^{-1}(x, X, Y^*, Z^*) \qquad \text{and} \qquad \Psi = (\widetilde{\varphi}^*)^{-1}(x, X^*, Y, Z^*)$$

with $\theta = \varphi_C^{*-1}(x, Z^*)$; for all $d = \varphi^{-1}(x, X, Y, Z) \in D_{(q_A^D)^*(\Phi)} \cap D_{(q_B^D)^*(\psi)}$, we have

$$\Phi(d) = \langle Y, Y^* \rangle + \langle Z, Z^* \rangle \qquad \text{and} \qquad \Psi(d) = \langle X, X^* \rangle + \langle Z, Z^* \rangle \, ;$$

let us show that the real number

$$\Phi(d) - \Psi(d) := \langle Y, Y^* \rangle - \langle X, X^* \rangle$$

does not depends on $(U, \varphi, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \mathbb{R}^{n_0})$. Indeed let $(U', \varphi', \mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \mathbb{R}^{n_0})$ be another double vector chart such that $x \in U'$ and

$$\varphi' \circ \varphi^{-1}(x, X, Y, Z) = \big(x, u_1(x) \cdot X, u_2(x) \cdot Y, u_0(x) \cdot Z + \omega(x)(X, Y)\big),$$

with $u_1 : U \cap U' \to GL(\mathbb{R}^{n_1})$, $u_2 : U \cap U' \to GL(\mathbb{R}^{n_2})$, $u_0 : U \cap U' \to GL(\mathbb{R}^{n_0})$, $\omega : U \cap U' \to \mathcal{L}_2(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}; \mathbb{R}^{n_0})$ of class C^∞, hence

$$\Phi = (\varphi'^*)^{-1}(x, X', Y'^*, Z'^*) \qquad \text{and} \qquad \Psi = (\widetilde{\varphi}'^*)^{-1}(x, X'^*, Y', Z'^*)$$

imply

$$\begin{cases} X = u_1(x)^{-1} \cdot X' \\ X^* = [u_1(x)]^t \cdot X'^* + [\omega(x)(\cdot, Y)]^t \, (Z'^*) \\ Y = u_2(x)^{-1} \cdot Y' \\ Y^* = [u_2(x)]^t \cdot Y'^* + [\omega(x)(X, \cdot)]^t \, (Z'^*) \end{cases}$$

and

$$\begin{cases} \langle Y, Y^* \rangle = \langle Y', Y'^* \rangle + \langle \omega(x)(X, Y), Z'^* \rangle \\ \langle X, X^* \rangle = \langle X', X'^* \rangle + \langle \omega(x)(X, Y), Z'^* \rangle \, , \end{cases}$$

so $\langle Y, Y^* \rangle - \langle X, X^* \rangle = \langle Y', Y'^* \rangle - \langle X', X'^* \rangle$, i.e., (6.11) is well-defined. Finally, since the map

$$\begin{array}{rcl} \mathbb{R}^{n_0} \times (\mathbb{R}^{n_2})^* \times (\mathbb{R}^{n_0})^* \times \mathbb{R}^{n_2} & \longrightarrow & \mathbb{R}, ((X, Y^*), \\ (X^*, Y)) & \longmapsto & \langle Y, Y^* \rangle - \langle X, X^* \rangle \end{array}$$

is bilinear and nondegenerate, (6.11) defines a pairing. ∎

7. Some natural morphisms of double vector bundles

In this section, we generalize for DVB some natural morphisms of vector bundles attached to a Weil functor $F : \mathcal{M}f \longrightarrow \mathcal{FM}$ (see [13] or [9]).

7.1. Canonical isomorphisms $\kappa_E : FTE \to TFE$. Let us consider the canonical flow natural equivalence (2.1) $\kappa : FT \to TF$ associated to F. For a vector bundle (E, M, q), one can consider the F-prolongation

$$\begin{array}{ccc} FTE & \xrightarrow{F(Tq)} & FTM \\ {\scriptstyle F(\pi_E)}\big\downarrow & & \big\downarrow{\scriptstyle F(\pi_M)} \\ FE & \xrightarrow{Fq} & FM \end{array}$$

of $(TE; E, TM; M)$ and the tangent double vector bundle

$$\begin{array}{ccc} TFE & \xrightarrow{T(Fq)} & TFM \\ {\scriptstyle \pi_{FE}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{FM}} \\ FE & \xrightarrow{Fq} & FM \end{array}$$

of (FE, FM, Fq). It is clear to see that

$$\kappa_E : FTE \longrightarrow TFE \tag{7.1}$$

is a double vector bundle isomorphism $(\kappa_E; id_{FE}, \kappa_M; id_{FM})$.

7.2. Natural transformations $\overline{Q}(a) : F \to F$. For a vector bundle (E, M, q), the fibered multiplication $\mu_E : \mathbb{R} \times E \to E$ is a morphism of vector bundles over the projection $\mathbb{R} \times M \to M$; hence $F\mu_E : A^F \times FE \to FE$ is a morphism of vector bundles over the projection $A^F \times FM \to FM$, i.e., the partial maps $F\mu_E(a, \cdot) : FE \longrightarrow FE$ ($a \in A^F$) are morphisms of vector bundles over id_{FM}. One deduces (for a in A^F) a natural transformation

$$\overline{Q}(a) : F \longrightarrow F \tag{7.2}$$

by $\overline{Q}(a)_E := F\mu_E(a, \cdot)$ that for all $s, t \in \mathbb{R}$ and $a, b \in A^F$ satisfies

$$\begin{aligned} \overline{Q}(1_{A^F}) &= id_{FE}\,, \\ \overline{Q}(sa + tb) &= s\overline{Q}(a) + t\overline{Q}(b)\,, \\ \overline{Q}(ab) &= \overline{Q}(a) \circ \overline{Q}(b)\,. \end{aligned} \tag{7.3}$$

Now, let $(D; A, B; M)$ be a double vector bundle. Hence for all a in A^F, $\overline{Q}(a)_{\widetilde{D}_A} : FD \to FD$ is a morphism of double vector bundles $\big(\overline{Q}(a)_{\widetilde{D}_A}; id_{FA}, \overline{Q}(a)_B; id_{FM}\big)$, since the fibered multiplication $\mathbb{R} \times \widetilde{D}_A \to \widetilde{D}_A$ is a morphism of vector bundles over the projection $\mathbb{R} \times A \to A$ and the multiplication on B. Similarly, for all a in A^F, $\overline{Q}(a)_{\widetilde{D}_B} : FD \to FD$ is a morphism of double vector bundles $\big(\overline{Q}(a)_{\widetilde{D}_B}; id_{FB}, \overline{Q}(a)_A; id_{FM}\big)$.

7.3. Natural transformations $Q(a) : TF \to TF$. For all smooth manifold M, let $Q(a)_M := \kappa_M \circ \overline{Q}(a)_{TM} \circ \kappa_M^{-1}$; one defines in this way a natural transformation

$$Q(a) : TF \longrightarrow TF \tag{7.4}$$

between Weil functors satisfying (7.3).

Finally, for a vector bundle (E, M, q), $\mu_{TE} : \mathbb{R} \times TE \to TE$ is a morphism of double vector bundles from

$$\begin{array}{ccc} \mathbb{R} \times TE & \xrightarrow{\mathrm{id}_{\mathbb{R}} \times Tq} & \mathbb{R} \times TM \\ {\scriptstyle \mathrm{id}_{\mathbb{R}} \times \pi_E}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{R}} \times \pi_M} \\ \mathbb{R} \times E & \xrightarrow{\mathrm{id}_{\mathbb{R}} \times q} & \mathbb{R} \times M \end{array} \quad \text{to} \quad \begin{array}{ccc} TE & \xrightarrow{Tq} & TM \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ E & \xrightarrow{q} & M \end{array}$$

over the projection $\mathbb{R} \times E \to E$ and $\mu_{TM} : \mathbb{R} \times TM \to TM$. So, the maps

$$Q(a)_E : TFE \longrightarrow TFE, \ (a \in A^F) \tag{7.5}$$

become morphisms of double vector bundles $(Q(a)_E; \mathrm{id}_{FE}, Q(a)_M; \mathrm{id}_{FM})$, as composition of morphisms of double vector bundles.

8. On lifts of linear functions and sections

Let us recall these tools developed in [9, 17] (and in [19] with a product preserving gauge bundle functor on vector bundles).

8.1. Lifts of linear functions. Let (E, M, q) be a vector bundle and $\{pt\}$ a one point manifold.

A smooth function $h : E \to \mathbb{R}$ is *linear on fibres* (or a linear function) if

$$\begin{array}{ccc} E & \xrightarrow{h} & \mathbb{R} \\ {\scriptstyle q}\downarrow & & \downarrow \\ M & \longrightarrow & \{pt\} \end{array}$$

is a morphism of vector bundles over a constant map.

One denotes by $C^\infty_{\ell in}(E, \mathbb{R})$ the set all smooth linear functions on E. This is a module over $C^\infty(M, \mathbb{R})$ isomorphic to the module $\Gamma(E^*)$ of smooth sections of the dual of E, since the map $\ell : \Gamma(E^*) \to C^\infty_{\ell in}(E, \mathbb{R})$ given by $(\ell_\sigma)_x = \sigma(x)$, is an isomorphism of modules.

Each h in $C^{\infty}_{\ell in}(E,\mathbb{R})$ is a morphism of double vector bundles, hence for a linear function $\lambda : A^F \to \mathbb{R}$, $h^{(\lambda)} := \lambda \circ Fh$, belongs to $C^{\infty}_{\ell in}(FE,\mathbb{R})$ (as composition of morphisms of double vector bundles) and is called the *λ-lift* of h to FE.

8.2. Lifts of linear sections. Let $(D; A, B; M)$ be a double vector bundle. For a linear section

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & D \\ {\scriptstyle q_A}\big\downarrow & & \big\downarrow{\scriptstyle q^D_B} \\ M & \xrightarrow{\underline{\sigma}} & B \end{array}$$

of the vertical bundle structure, its local expression in a local frame (β_j, γ_l) of $\widetilde{D}_A$ (see subsection 4.2) is

$$\sigma \mid_{q_A^{-1}(U)} = \sum_{j=1}^{n_2} \underline{\sigma}^j \circ q_A \beta_j + \sum_{l=1}^{n_0} \sigma^l \gamma_l, \tag{8.1}$$

where $\underline{\sigma} \mid_U = \sum_{l=1}^{n_0} \underline{\sigma}^j \overline{\beta}_j$ and $\sigma^l : q_A^{-1}(U) \to \mathbb{R}$, $1 \le k \le n_0$, linear functions.

Moreover for $a \in A^F$, let

$$\underline{\sigma}^{(a)} := \overline{Q}(a)_B \circ F\underline{\sigma}\,, \qquad \sigma^{(a)} := \overline{Q}(a)_{\widetilde{D}_A} \circ F\sigma\,. \tag{8.2}$$

Proposition 8.1. *$(\sigma^{(a)}, \underline{\sigma}^{(a)})$ is a smooth linear section of $FD \to FA$.*

Proof. $\sigma^{(a)}$ is a section of $FD \to FA$ since $\overline{Q}(a)_{\widetilde{D}_A}$ is a morphism of vector bundles over id_{FA}. Moreover $\sigma^{(a)}$ is the composition of two morphisms of vector bundles, namely

$$\begin{array}{ccc} FA & \xrightarrow{F\sigma} & FD \\ {\scriptstyle Fq_A}\big\downarrow & & \big\downarrow{\scriptstyle F(q^D_B)} \\ FM & \xrightarrow{F\underline{\sigma}} & FB \end{array} \quad \text{and} \quad \begin{array}{ccc} FD & \xrightarrow{\overline{Q}(a)_D} & FD \\ {\scriptstyle F(q^D_B)}\big\downarrow & & \big\downarrow{\scriptstyle F(q^D_B)} \\ FB & \xrightarrow{\overline{Q}(a)_B} & FB \end{array},$$

hence $\sigma^{(a)}$ is a linear section over $\underline{\sigma}^{(a)}$. ∎

Definition 8.2. $(\sigma^{(a)}, \underline{\sigma}^{(a)})$ is called the a-lift of $(\sigma, \underline{\sigma},)$ to FA.

One may define in the same way the a-lift $(\beta^{(a)}, \underline{\beta}^{(a)})$ of a linear section

$$\begin{array}{ccc} B & \xrightarrow{\beta} & D \\ {\scriptstyle q_B}\downarrow & & \downarrow{\scriptstyle q_A^D} \\ M & \xrightarrow{\underline{\beta}} & A \end{array}$$

of the horizontal bundle structure $D \to B$.

8.3. Application to lifts of linear vector fields. In this subsection, we set $(D; A, B; M) = (TE; E, TM; M)$ the tangent prolongation double vector bundle of a vector bundle (E, M, q).

Given a linear vector field (ξ, x) and a in A^F; by [9], we have some vector fields $x^{(a)} \in \mathfrak{X}(FM)$, $\xi^{(a)} \in \mathfrak{X}(FE)$ given by

$$x^{(a)} = Q(a)_M \circ \mathcal{F}_M x\,, \qquad \xi^{(a)} = Q(a)_E \circ \mathcal{F}_E \xi\,. \tag{8.3}$$

Since $(\xi^{(a)}, x^{(a)})$ is the composition of (κ_E, κ_M) with the a-lift (8.2) of (ξ, x) to FE, $(\xi^{(a)}, x^{(a)})$ is a linear vector field .

Definition 8.3. $(\xi^{(a)}, x^{(a)})$ or $\xi^{(a)}$ is called the a-lift of (ξ, x) related to F.

Remark 8.4. Some properties of lifts of vector fields and functions on manifolds can be found in [3, 9, 12, 18]. For some additional properties of the particular case of linear vector fields, one can refer to [19].

9. On lifts of linear sections on duals of a double vector bundle

Let $(D; A, B; M)$ be a double vector bundle with core C and (6.2) its vertical dual.

9.1. Linear sections on duals of a double vector bundle. Let us recall that a *linear section* $\omega \in \Gamma_A^{\ell in}(D^*A)$ of (6.2) with respect to its vertical vector bundle structure is a morphism of vector bundles

$$\begin{array}{ccc} A & \xrightarrow{\omega} & D^*A \\ {\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle q_{C^*}^{*A}} \\ M & \xrightarrow{\underline{\omega}} & C^* \end{array} \;. \tag{9.1}$$

where $\underline{\omega} \in \Gamma(C^*)$.

A *linear* 1-*form* on E is a linear section of the vertical dual $(T^*E; E, E^*; M)$ of $(TE; E, TM; M)$ with respect to its vertical vector bundle structure, i.e., a smooth 1-form on E that is a morphism of vector bundles

$$\begin{array}{ccc} E & \xrightarrow{\omega} & T^*E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r_E} \\ M & \xrightarrow{\underline{\omega}} & E^* \end{array} \tag{9.2}$$

over a smooth section of E^*.

THEOREM 9.1. *Let $\omega \in \Gamma_A(D^*A)$ and denote $\widetilde{\omega} \in C^\infty_{\ell in}(\widetilde{D}_A, \mathbb{R})$ the corresponding function on D. The following assertions are equivalent:*

(1) *$(\omega, \underline{\omega})$ is a linear section.*
(2) *$\widetilde{\omega} \in C^\infty_{\ell in}(\widetilde{D}_B, \mathbb{R})$.*

Proof. (1) implies (2): We have to prove that

$$\begin{array}{ccc} D & \xrightarrow{\widetilde{\omega}} & \mathbb{R} \\ {\scriptstyle q^D_B}\downarrow & & \downarrow \\ B & \longrightarrow & \{pt\} \end{array}$$

is a vector bundle morphism. Let $d, d' \in D$ such that $q^D_B(d) = q^D_B(d')$ with $a = q^D_A(d)$, $a' = q^D_A(d')$; we have

$$\begin{aligned} \widetilde{\omega}(d \underset{B}{+} d') &= \omega(a + a')(d \underset{B}{+} d') \\ &= (\omega(a) \underset{C^*}{+} \omega(a'))(d \underset{B}{+} d') \quad \text{(by (1))} \\ &= \omega(a)(d) + \omega(a')(d') \quad \text{(by (6.6))} \\ &= \widetilde{\omega}(d) + \widetilde{\omega}(d'), \end{aligned}$$

hence the result follows by Remark 3.3 (3).

(2) implies (1): Let $a, a' \in A$ such that $q_A(a) = q_A(a') = x$; we have $q^{*A}_{C^*}(\omega_a) = q^{*A}_{C^*}(\omega_{a'})$. Indeed, for all c in C_x, $d = \tau_A(a, -c)$ and $d' = \tau_A(a', +c)$ satisfy $d \underset{B}{+} d' = \widetilde{0}^A(a + a')$, hence,

$$0 = \widetilde{\omega}(d \underset{B}{+} d'') = \widetilde{\omega}(d) + \widetilde{\omega}(d') = \omega_a \circ \tau_A(a, -c) + \omega_{a'} \circ \tau_A(a', c)$$

i.e., $q^{*A}_{C^*}(\omega_a) = \omega_a \circ \tau_A(a, \cdot) = \omega_{a'} \circ \tau_A(a', \cdot) = q^{*A}_{C^*}(\omega_{a'})$. Hence, the relations $\left\langle d \underset{B}{+} d', \widetilde{\omega} \right\rangle = \langle d, \widetilde{\omega}\rangle + \langle d', \widetilde{\omega}\rangle$, for every (d, d') in $D_a \times D_{a'}$ imply that $\omega(a + a') = \omega(a) \underset{C^*}{+} \omega(a')$; whence $\omega_{sa} = s \underset{C^*}{\cdot} \omega_a$, for all (s, a) in $\mathbb{R} \times A$, by continuity. Finally, for all a in A, $\omega(a) \in (D^*A)_{\underline{\omega}(q_A(a))}$, where $\underline{\omega} : M \to C^*$ is a map and since $q^{*A}_{C^*} \circ \omega = \underline{\omega} \circ q_A$, $\underline{\omega}$ is a smooth section of C^*, hence $(\omega, \underline{\omega})$ is a linear section. ∎

Remark 9.2. Some similar characterizations for linear 1-forms may be found in [17], [14] or [15].

Let $(\omega, \underline{\omega})$ be a linear section of (6.2) with respect to its vertical vector bundle structure; hence by Remark 6.5 we have

$$\omega \mid_{q_A^{-1}(U)} = \sum\nolimits_{l=1}^{n_0} \underline{\omega}_l \gamma^l + \sum\nolimits_{j=1}^{n_2} \omega_j \beta^j, \tag{9.3}$$

where $\underline{\omega} \mid_U = \sum_{l=1}^{n_0} \underline{\omega}_l \underline{\gamma}^l$ and $\omega_j : q_A^{-1}(U) \to \mathbb{R}$, $1 \leq j \leq n_2$ linear functions. In the particular case of a linear 1-form $(\omega, \underline{\omega})$ on E, one can write

$$\omega \mid_{q^{-1}(U)} = \sum\nolimits_{l=1}^{n} \underline{\omega}_l dy^l + \sum\nolimits_{j=1}^{m} \omega_j dx^j, \tag{9.4}$$

where $\underline{\omega} \mid_U = \sum_{l=1}^{n} \underline{\omega}_l \varepsilon^l$ and $\omega_j : q^{-1}(U) \to \mathbb{R}$, $1 \leq j \leq m$ linear functions.

9.2. Lifts of linear sections. Now, given a linear section of (6.2) $(\omega, \underline{\omega})$ and a linear map $\lambda : A^F \to \mathbb{R}$, let

$$\widetilde{\underline{\omega}^{(\lambda)}} = \lambda \circ F\widetilde{\underline{\omega}} : FM \longrightarrow \mathbb{R}\,, \qquad \widetilde{\omega^{(\lambda)}} = \lambda \circ F\widetilde{\omega} : FA \longrightarrow \mathbb{R}\,. \tag{9.5}$$

Proposition 9.3. *Hence* $(\omega^{(\lambda)}, \underline{\omega}^{(\lambda)})$ *is a linear section of the vertical dual of* (5.3) *(with respect to its vertical side bundle structure) such that*

$$\omega^{(\lambda)}(\sigma^{(a)}) = (\omega(\sigma))^{(\lambda_a)}, \tag{9.6}$$

for all linear section $(\sigma, \underline{\sigma})$ *in* $\Gamma^{lin}_A(D)$, $\lambda : A^F \to \mathbb{R}$ *a linear map,* $a \in A^F$ *and* $\lambda_a : A^F \to \mathbb{R}$ *given by* $\lambda_a(x) = \lambda(ax)$.

Proof. Indeed

$$\begin{aligned} \omega^{(\lambda)}(\sigma^{(a)}) &= \widetilde{\omega^{(\lambda)}} \circ \sigma^{(a)} \;=\; \lambda \circ F\widetilde{\omega} \circ \overline{Q}(a)_D \circ F\sigma \quad \text{(by (8.2))} \\ &= \lambda \circ \overline{Q}(a)_{\mathbb{R}} \circ F\widetilde{\omega} \circ F\sigma \qquad \text{(by (7.2))} \\ &= \lambda_a \circ F(\omega(\sigma)) \;=\; (\omega(\sigma))^{(\lambda_a)} \end{aligned}$$

and since $F\widetilde{\omega} : FD \to A^F$ is a morphism of double vector bundles over $FA \to \{pt\}$ and $FB \to \{pt\}$, the result follows. ∎

Definition 9.4. The pair $(\omega^{(\lambda)}, \underline{\omega}^{(\lambda)})$ is called the λ-lift of $(\omega, \underline{\omega})$ to FA.

9.3. Application to lifts of linear 1-forms. Given a vector bundle (E, M, q), let

$$\begin{array}{ccc} T^*E & \xrightarrow{r_E} & E^* \\ {\scriptstyle \pi_E^*}\downarrow & & \downarrow{\scriptstyle q^*} \\ E & \xrightarrow{q} & M \end{array}$$

be the vertical dual of the tangent double vector bundle $(TE; E, TM; M)$.

Let us give another proof of this result of [14].

For a linear function $f \in C^\infty_{\ell in}(E)$, (df, f) is a linear 1-form on E. Indeed let $\xi \in T_eE$, $\xi' \in T_{e'}E$ such that $Tq(\xi) = Tq(\xi')$ and $g, h \in C^\infty(\mathbb{R}, E)$, $q(g_t) = q(h_t)$ in a neighborhood of 0 with $\xi = \frac{dg_t}{dt}\Big|_{t=0}$, $\xi' = \frac{dh_t}{dt}\big|_{t=0}$; we have

$$\begin{aligned} \widetilde{df}\Big(\xi \underset{TM}{+} \xi'\Big) &= \widetilde{df}\bigg(\frac{d(g_t+h_t)}{dt}\Big|_{t=0}\bigg) = \frac{d}{dt} f(g_t + h_t)\big|_{t=0} \\ &= \frac{d}{dt} f(g_t)\big|_{t=0} + \frac{d}{dt} f(h_t)\big|_{t=0} = \widetilde{df}(\xi) + df(e')(\xi') \end{aligned}$$

and the result follows by continuity (see Remark 3.3 (3)).

Now, given a linear 1-form $(\omega, \underline{\omega})$ on E and a linear map $\lambda : A^F \to \mathbb{R}$, by [9], we have some 1-forms $\underline{\omega}^{(\lambda)} \in \Omega^1(FM)$, $\omega^{(\lambda)} \in \Omega^1(FE)$ by

$$\widetilde{\underline{\omega}^{(\lambda)}} = \lambda \circ F\widetilde{\underline{\omega}} \circ \kappa_M^{-1} : TFM \longrightarrow \mathbb{R}, \qquad \widetilde{\omega^{(\lambda)}} = \lambda \circ F\widetilde{\omega} \circ \kappa_E^{-1} : TFE \longrightarrow \mathbb{R}.$$

Corollary 9.5. *$(\omega^{(\lambda)}, \underline{\omega}^{(\lambda)})$ is a linear 1-form on FE such that*

$$\omega^{(\lambda)}(\xi^{(a)}) = (\omega(\xi))^{(\lambda_a)}. \tag{9.7}$$

for all linear vector field (x, ξ) on TE, linear map $\lambda : A^F \to \mathbb{R}$ and $a \in A^F$.

Proof. (9.7) comes from [9] and $(\omega^{(\lambda)}, \underline{\omega}^{(\lambda)})$ is the composition of the transpose morphism $((\kappa_E)^t, (\kappa_M)^t)$ with the λ-lift (9.5) of $(\omega, \underline{\omega})$ to FE, hence $(\omega^{(\lambda)}, \underline{\omega}^{(\lambda)})$ is a linear 1-form. ∎

Definition 9.6. [9]$(\omega^{(\lambda)}, \underline{\omega}^{(\lambda)})$ is called the λ-lift of $(\omega, \underline{\omega})$ to FE.

10. On lifts of linear covariant tensor fields on double vector bundles

10.1. Linear covariant tensor fields. Let $(D^*A; A, C^*; M)$ be the vertical dual of a double vector bundle $(D; A, B; M)$. Let us set $\bigoplus^0 \widetilde{D}_A = A$ and $\bigoplus^0 A = M$.

Definition 10.1. A covariant tensor field $\omega : A \to \bigotimes^k D^*A$ $(k \geq 1)$ on $\widetilde{D}_A$ is said *linear* if the associated multilinear morphism over A,

$$\begin{array}{rcl} \bigoplus^{k-1}\widetilde{D}_A & \xrightarrow{\omega^\flat} & D^*A \\ \bigoplus^{k-1}D_a \ni (d_1, \ldots, d_{k-1}) & \longmapsto & \omega(a)(d_1, \ldots, d_{k-1}, \cdot) \end{array} \tag{10.1}$$

is a morphism of vector bundles

$$\begin{array}{ccc} \bigoplus^{k-1}\widetilde{D}_A & \xrightarrow{\omega^\flat} & D^*A \\ {\scriptstyle (\oplus^{k-1} q_B^D)}\downarrow & & \downarrow{\scriptstyle q_{C^*}^{*A}} \\ \bigoplus^{k-1}B & \xrightarrow{\underline{\omega}} & C^* \end{array} \tag{10.2}$$

over a smooth map $\underline{\omega}$. In this case, $\underline{\omega}$ is in fact a multilinear morphism of vector bundles over M.

When $k = 0$, a linear tensor field is just a linear function $A \to \mathbb{R}$, while for $k = 1$, a linear tensor field is a linear section $A \to D^*A$ of (6.2).

One defines in the same way the concept of linear covariant tensor fields on $\widetilde{D}_B$.

Theorem 10.2. *Let $\omega : A \to \bigotimes^k D^*A$ $(k \geq 1)$ be a covariant tensor field on $\widetilde{D}_A$. The following assertions are equivalent:*

(1) *ω is a linear tensor field.*

(2) *The associated multilinear function,*

$$\begin{array}{rcl} \bigoplus^{k}\widetilde{D}_A & \xrightarrow{\widetilde{\omega}} & \mathbb{R} \\ \bigoplus^{k}D_a \ni (d_1, \ldots, d_k) & \longmapsto & \omega(a)(d_1, \ldots, d_k) \end{array}$$

is a morphism of vector bundles

$$\begin{array}{ccc} \bigoplus^{k}\widetilde{D}_A & \xrightarrow{\widetilde{\omega}} & \mathbb{R} \\ {\scriptstyle (\oplus^{k} q_B^D)}\downarrow & & \downarrow \\ \bigoplus^{k}B & \longrightarrow & \{pt\} \end{array} .$$

Proof. The result is already proved for $k=1$. Let $k \geq 2$.

(1)⇒(2) Let $(b_1, \ldots, b_k)$ in $\bigoplus^k B_x$; for $(d_i)_{1\leq i\leq k}$, $(d'_i)_{1\leq i\leq k}$ in

$$\left(\bigoplus^k \widetilde{D}_A\right)_{(b_i)_{1\leq i\leq k}} = \bigcup_{i=1} \left[\bigoplus_{i=1}^k D_a \cap D_{b_i}\right]$$

such that $d_i \in D_a \cap D_{b_i}$ and $d'_i \in D_{a'} \cap D_{b_i}$,

$$\begin{aligned}
\widetilde{\omega}((d_i)_{1\leq i\leq k} + (d'_i)_{1\leq i\leq k}) &= \widetilde{\omega}((d_i \underset{B}{+} d'_i)_{1\leq i\leq k}) \\
&= \omega^\flat((d_i \underset{B}{+} d'_i)_{1\leq i\leq k-1})(d_k \underset{B}{+} d'_k) \\
&= \omega^\flat((d_i)_{1\leq i\leq k-1} + (d'_i)_{1\leq i\leq k-1})(d_k \underset{B}{+} d'_k) \\
&= \left[\omega^\flat((d_i)_{1\leq i\leq k-1}) \underset{C^*}{+} \omega^\flat((d'_i)_{1\leq i\leq k-1})\right](d_k \underset{B}{+} d'_k) \quad \text{(by (1))} \\
&= \omega^\flat((d_i)_{1\leq i\leq k-1})(d_k) + \omega^\flat((d'_i)_{1\leq i\leq k-1})(d'_k) \quad \text{(by (6.6))} \\
&= \widetilde{\omega}((d_i)_{1\leq i\leq k}) + \widetilde{\omega}((d'_i)_{1\leq i\leq k}).
\end{aligned}$$

Moreover, for $t \in \mathbb{R}$ and $(d_i)_{1\leq i\leq k} \in \bigoplus_{i=1}^k D_a \cap D_{b_i}$, the equality $\widetilde{\omega}(t\cdot(d_i)_{1\leq i\leq k}) = t\widetilde{\omega}((d_i)_{1\leq i\leq k})$ holds by continuity.

(2)⇒(1) • Let $(d_i)_{1\leq i\leq k-1}$, $(d'_i)_{1\leq i\leq k-1}$ in $(\bigoplus^{k-1}\widetilde{D}_A)_{(b_i)_{1\leq i\leq k-1}}$, $x = q_B(b_i)$ such that $d_i \in D_a \cap D_{b_i}$ and $d'_i \in D_{a'} \cap D_{b_i}$; we have $q^{*A}_{C^*}(\omega^\flat((d_i)_{1\leq i\leq k-1})) = q^{*A}_{C^*}(\omega^\flat((d'_i)_{1\leq i\leq k-1}))$. Indeed, for all c in C_x, $d_k = \tau_A(a, -c)$ and $d'_k = \tau_A(a', +c)$ satisfy $d_k \underset{B}{+} d'_k = \widetilde{0}^A(a+a')$, hence,

$$\begin{aligned}
0 &= \widetilde{\omega}((d_i)_{1\leq i\leq k} + (d'_i)_{1\leq i\leq k}) = \widetilde{\omega}((d_i)_{1\leq i\leq k}) + \widetilde{\omega}((d'_i)_{1\leq i\leq k}) \\
&= \omega^\flat((d_i)_{1\leq i\leq k-1}) \circ \tau_A(a, -c) + \omega^\flat((d'_i)_{1\leq i\leq k-1}) \circ \tau_A(a', c),
\end{aligned}$$

i.e., $q^{*A}_{C^*}(\omega^\flat((d_i)_{1\leq i\leq k-1})) = q^{*A}_{C^*}(\omega^\flat((d'_i)_{1\leq i\leq k-1}))$. So, for all $(d_i)_{1\leq i\leq k-1}$ in $(\bigoplus^{k-1}\widetilde{D}_A)_{(b_i)_{1\leq i\leq k-1}}$, one has

$$\omega^\flat((d_i)_{1\leq i\leq k-1}) \in (D^*A)_{\underline{\omega}((b_i)_{1\leq i\leq k-1})},$$

where $\underline{\omega} : \bigoplus^{k-1} B \to C^*$ is a map; since $q^{*A}_{C^*} \circ \omega = \underline{\omega} \circ (\oplus^{k-1} q^D_B)$, $\underline{\omega}$ is a smooth fibered map over M.

• Moreover, the equalities

$$\widetilde{\omega}((d_i)_{1\leq i\leq k} + (d'_i)_{1\leq i\leq k}) = \widetilde{\omega}((d_i)_{1\leq i\leq k}) + \widetilde{\omega}((d'_i)_{1\leq i\leq k}),$$

for all $(d_i)_{1\leq i\leq k}$, $(d'_i)_{1\leq i\leq k}$ in $(\bigoplus^k \widetilde{D}_A)_{(b_i)_{1\leq i\leq k}}$, such that $d_i \in D_a$ and $d'_i \in D_{a'}$ imply

$$\omega^\flat((d_i)_{1\leq i\leq k-1} + (d'_i)_{1\leq i\leq k-1}) = \omega^\flat((d_i)_{1\leq i\leq k-1}) \underset{C^*}{+} \omega^\flat((d'_i)_{1\leq i\leq k-1}),$$

for all $(d_i)_{1\leq i\leq k-1}$, $(d'_i)_{1\leq i\leq k-1}$ in $(\bigoplus^{k-1} \widetilde{D}_A)_{(b_i)_{1\leq i\leq k-1}}$, such that $d_i \in D_a$ and $d'_i \in D_{a'}$; by continuity,

$$\omega^\flat((s \underset{B}{\cdot} d_i)_{1\leq i\leq k-1}) = s \underset{C^*}{\cdot} \omega^\flat((d_i)_{1\leq i\leq k-1}),$$

for all $(s, (d_i)_{1\leq i\leq k-1})$ in $\mathbb{R} \times \bigoplus^k D_a$.

Therefore, $(\omega, \underline{\omega})$ is a linear covariant tensor field. ∎

Remark 10.3. Let $\omega : A \to \bigotimes^k D^*A$ $(k \geq 2)$ be a linear covariant tensor field on $\widetilde{D}_A$; the associated multilinear morphism (10.1) over id_A is a morphism of vector bundles over id_A if and only if $k = 2$. In this case $\underline{\omega} : B \to C^*$ is a morphism of vector bundles.

Corollary 10.4. *Let $\omega : A \to \bigotimes^k D^*A$ $(k \geq 2)$ be a linear covariant tensor field on $\widetilde{D}_A$ and (β^j, γ^l) the dual frame of the local frame (β_j, γ_l) of $\widetilde{D}_A\mid_{q_A^{-1}(U)}$ defined by* (4.2)*. Hence $\omega\mid_{q_A^{-1}(U)}$ equals*

$$\sum_{\alpha=1}^{k} \omega_{j_1\cdots j_{\alpha-1} l j_{\alpha+1}\cdots j_k} \circ q_A \mid_{q_A^{-1}(U)} \beta^{j_1} \otimes \ldots \otimes \beta^{j_{\alpha-1}} \otimes \gamma^l \otimes \beta^{j_{\alpha+1}} \otimes \ldots \otimes \beta^{j_k} \\ + \omega_{j_1\cdots j_k} \beta^{j_1} \otimes \ldots \otimes \beta^{j_k}, \tag{10.3}$$

where $\omega_{j_1\cdots j_{\alpha-1} l j_{\alpha+1}\cdots j_k} : U \to \mathbb{R}$ are smooth, $\omega_{j_1\cdots j_{k-1}h} : q_A^{-1}(U) \to \mathbb{R}$ are linear functions and $\omega_{j_1\cdots j_{k-1}l} = \underline{\omega}_{j_1\cdots j_{k-1}l}$ are given by the relations $\underline{\omega}(\overline{\beta}_{j_1} \ldots \overline{\beta}_{j_{k-1}}) = \underline{\omega}_{j_1\cdots j_{k-1}l}\overline{\gamma}^l$.

Proof. It is clear by (6.8) that each term of the right hand of (10.3) is a linear tensor field on $\widetilde{D}_A\mid_{q_A^{-1}(U)}$. Conversely, since $q^{*A}_{C^*}(\omega^\flat(d_1, \ldots, d_{k-1})) = 0$ when one of these vectors belongs to the kernel of q^D_B, all terms of $\omega\mid_{q_A^{-1}(U)}$ of the form $fX^1 \otimes \ldots \otimes X^{k-1} \otimes \gamma^{l_k}$ with $\{X^1, \ldots, X^{k-1}\} \cap \{\gamma^1, \ldots, \gamma^{n_0}\} \neq \emptyset$ vanish by (6.9).

Likewise, all terms of $\omega\mid_{q_A^{-1}(U)}$ of the form $fX^1 \otimes \ldots \otimes X^{k-1} \otimes \beta^{j_k}$ vanish when the cardinality s of the intersection $\{X^1, \ldots, X^{k-1}\} \cap \{\gamma^1, \ldots, \gamma^{n_0}\}$

is greater than 1; indeed let $(d_i)_{1\leq i\leq k}$ in $\bigoplus^k \widetilde{D}_A$ such that $d_k = \beta_{j_k}(a)$ and $d_1,\ldots,d_{k-1}$ belong to the union $\{\beta_1(a),\ldots,\beta_{n_2}(a)\}\cup\{\gamma_1(a),\ldots,\gamma_{n_0}(a)\}$; the equalities $\widetilde{\omega}(t\cdot(d_i)_{1\leq i\leq k}) = t\widetilde{\omega}((d_i)_{1\leq i\leq k})$, for all real number t write, $t^s\omega_{r_1\ldots r_{k-1}j_k}(ta) = t\omega_{r_1\ldots r_{k-1}j_k}(a)$ hence $\omega_{r_1\ldots r_{k-1}j_k}(a) = 0$.

Moreover, $\underline{\omega}\circ(\oplus^{k-1}q_B^D)(d) = q_{C^*}^{*A}\circ\omega^\flat(d)$ with $d = (\beta_{j_1}(a),\ldots,\beta_{j_{k-1}}(a))$ gives by (6.9),

$$\underline{\omega}_{j_1\ldots j_{k-1}l}(q_A(a))\overline{\gamma}^l(q_A(a)) = \omega_{j_1\ldots j_{k-1}l}(a)\overline{\gamma}^l(q_A(a)),$$

i.e., $\omega_{j_1\ldots j_{k-1}l} = \underline{\omega}_{j_1\ldots j_{k-1}l}\mid_{q_A^{-1}(U)}$.

Applying the linearity of $\widetilde{\omega}$ on $d = (\beta_{j_\alpha}(a))_{1\leq\alpha\leq k}$, $d' = (\beta_{j_\alpha}(a'))_{1\leq\alpha\leq k}$ and $t\in\mathbb{R}$, the linearity of $\omega_{j_1\ldots j_k}$ follows. Thus $\omega\mid_{q_A^{-1}(U)}$ is of the form (10.3). ∎

Remark 10.5. In the case of a skew-symmetric tensor field $\omega : A \to \bigwedge^k D^*A$ $(k\geq 2)$, we have

$$\omega\mid_{q_A^{-1}(U)} = \frac{1}{(k-1)!}\underline{\omega}_{j_1\ldots j_{k-1}l}\circ q_A\mid_{q_A^{-1}(U)}\beta^{j_1}\wedge\ldots\otimes\beta^{j_{\alpha-1}} + \omega_{j_1\ldots j_k}\beta^{j_1}\wedge\ldots\wedge\beta^{j_k},$$

where $\omega_{j_1\ldots j_k} : q_A^{-1}(U)\to\mathbb{R}$ are linear functions and $\omega_{j_1\ldots j_{k-1}l} = \underline{\omega}_{j_1\ldots j_{k-1}l}$ are given by the relations $\underline{\omega}(\overline{\beta}_{j_1}\wedge\ldots\wedge\overline{\beta}_{j_{k-1}}) = \underline{\omega}_{j_1\ldots j_{k-1}l}\overline{\gamma}^l$. In the particular case of a linear k-form on a vector bundle (E, M, q), we have

$$\begin{aligned}\omega\mid_{q^{-1}(U)} = \frac{1}{(k-1)}&\underline{\omega}_{i_1\ldots i_{k-1}j}dx^{i_1}\wedge\ldots\wedge dx^{i_{k-1}}\wedge dy^j \\ &+ \frac{1}{k!}\omega_{i_1\ldots i_k j}y^j dx^{i_1}\wedge\ldots\wedge dx^{i_k},\end{aligned} \tag{10.4}$$

as in [15].

10.2. Lifts of linear covariant tensor fields. Now, let a linear covariant tensor field $\omega\in\Gamma_A^{\ell in}(\bigotimes^k D^*A)$ given by its morphism of DVB

$$\begin{array}{ccc}\bigoplus^k\widetilde{D}_A & \xrightarrow{\widetilde{\omega}} & \mathbb{R} \\ {\scriptstyle\oplus^k q_B^D}\downarrow & & \downarrow \\ \bigoplus^k B & \longrightarrow & \{pt\}\end{array} \qquad (k\geq 2).$$

One defines a covariant tensor field $\omega^{(\mu)} : FA \to \bigotimes^k (FD)^*FA$ by

$$\widetilde{\omega^{(\mu)}} = \mu\circ F\widetilde{\omega} : \bigoplus^k\big(\widetilde{FD}\big)_{FA}\longrightarrow\mathbb{R},$$

where $\mu : A^F\to\mathbb{R}$ is a linear map.

Definition 10.6. $\omega^{(\mu)}$ is called the μ-lift of ω.

Proposition 10.7. *Then $\omega^{(\mu)}$ is a linear covariant tensor field on $(\widetilde{FD})_{FA}$ such that*

$$\omega^{(\mu)}\big(\sigma_1^{(a_1)},\dots,\sigma_k^{(a_k)}\big) = (\omega(\sigma_1,\dots,\sigma_k))^{(\mu_{a_1\dots a_k})}, \tag{10.5}$$

for all linear sections $(\sigma_i,\underline{\sigma}_i)_{1\le i\le k}$ in $\Gamma_A^{lin}(D)$, $\mu : A^F \to \mathbb{R}$ a smooth linear map, $a_1,\dots,a_k \in A^F$ and $\mu_{a_1\dots a_k} : A^F \to \mathbb{R}$ given by $\mu_{a_1\dots a_k}(x) = \mu(a_1\dots a_k x)$.

Proof. Since $F\omega : F(\bigoplus^k \widetilde{D}_A) \cong \bigoplus^k (\widetilde{FD})_{FA} \to A^F$ is a morphism of vector bundles over $F(\bigoplus^k B) \cong \bigoplus^k FB \to \{pt\}$ and $FA \to \{pt\}$, $\omega^{(\mu)}$ is a linear covariant tensor field on $(\widetilde{FD})_{FA}$. Moreover

$$\begin{aligned}
\omega^{(\mu)}(\sigma_1^{(a_1)},\dots,\sigma_k^{(a_k)}) &= \mu\circ F\widetilde{\omega}\circ\Big(\sigma_1^{(a_1)}\oplus\dots\oplus\sigma_k^{(a_k)}\Big)\\
&= \mu\circ F\widetilde{\omega}\circ\Big(\textstyle\bigoplus_{i=1}^k \overline{Q}(a_i)_D\Big)\circ\Big(\textstyle\bigoplus_{i=1}^k F\sigma_i\Big)\\
&= \mu\circ\Big(\textstyle\bigoplus_{i=1}^k \overline{Q}(a_i)_{\mathbb{R}}\Big)\circ F\widetilde{\omega}\circ F\Big(\textstyle\bigoplus_{i=1}^k \sigma_i\Big)\\
&= \mu_{a_1\dots a_k}\circ F\Big(\widetilde{\omega}\circ\textstyle\bigoplus_{i=1}^k \sigma_i\Big)\\
&= \mu_{a_1\dots a_k}\circ F\big((\omega(\sigma_1,\dots,\sigma_k))\big)\\
&= \big(\omega(\sigma_1,\dots,\sigma_k)\big)^{(\mu_{a_1\dots a_k})}.
\end{aligned}$$

∎

Remark 10.8. (10.5) is in fact a modification of a result of [9]. All materials developped in this section are valid for linear contravariant tensor fields.

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