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The easiest polynomial differential systems in $\mathbb{R}^3$ having an invariant cylinder

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Abstract: This paper answers the following two questions: *What are the easiest polynomial differential systems in* $\mathbb{R}^3$ *having an invariant hyperbolic, parabolic or elliptic cylinder*?, and *for such polynomial differential systems what are their phase portraits on such invariant cylinders*?

Key words: Polynomial differential systems in $\mathbb{R}^3$, hyperbolic cylinder, parabolic cylinder, elliptic cylinder.

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1. Introduction and statement of the main results

When an orbit of a differential system having a point in a set S is completely contained in S, we say that the set S is *invariant* for such a differential system. The set S for a differential system in $\mathbb{R}^3$ usually is a surface. Darboux in [4] was one of first mathematicians to study invariant sets of the polynomial differential systems. In fact he used the existence of invariant sets for studying the integrability of the polynomial differential systems.

After Darboux, many authors have studied different differential systems having many distinct kinds of invariant sets, including invariant circles [1, 10, 9], invariant spheres [2, 12, 13], and invariant tori [11, 15, 16]. In this paper we want to study the simplest polynomial differential systems in the space $\mathbb{R}^3$ having some invariant cylinder.

A polynomial differential system in $\mathbb{R}^3$ is a differential system of the form

$$\begin{aligned} \dot{x} &= P(x,y,z), \\ \dot{y} &= Q(x,y,z), \\ \dot{z} &= R(x,y,z), \end{aligned} \tag{1}$$

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where P, Q and R are real polynomials in the variables x, y and z, and the dot denotes derivative with respect to the time t. The degree of the polynomial differential system (1) is the maximum of the degrees of the polynomials P, Q and R.

Here we shall work with the following three surfaces in $\mathbb{R}^3$, the hyperbolic cylinder $x^2-y^2=1$, the parabolic cylinder $x^2-y=1$, and the elliptic cylinder $x^2+y^2=1$. We say that one of these cylinders is *invariant* under the flow of the differential system (1) if for every orbit $(x(t),y(t),z(t))$ of the differential system (1) having a point on that cylinder the whole orbit is contained in it.

Two natural questions about the invariant cylinders of polynomial differential systems (1) are: *What are the easiest polynomial differential systems* (1) *in* $\mathbb{R}^3$ *having an invariant cylinder*, and *for such polynomial differential systems what are their phase portraits on the invariant cylinder*? The objective of this paper is to give an answer to these two questions.

Here to look for the easiest polynomial differential systems in $\mathbb{R}^3$ having an invariant cylinder, means that we are looking for the polynomial differential systems in $\mathbb{R}^3$ with the smallest degree having some of the mentioned three kind of cylinders as an invariant surface. We shall see that such polynomial differential systems will have degree one and two.

Let U be an open and dense set in $\mathbb{R}^3$. We recall that a C^1 function $H\colon U\to\mathbb{R}$ which is non-locally constant is a *first integral* of the differential system (1) if H is constant on all the solutions $(x(t),y(t),z(t))$ contained in U. In other words, on the solution $(x(t),y(t),z(t))$ we have that

$$\frac{dH}{dt}=\frac{\partial H}{\partial x}P+\frac{\partial H}{\partial y}Q+\frac{\partial H}{\partial z}R=0. \tag{2}$$

Let $f(x,y,z)$ be a real polynomial. The algebraic surface $f(x,y,z)=0$ is *invariant* for the polynomial differential system (1) if there exists a real polynomial $k(x,y,z)$ satisfying the equality

$$\frac{\partial f}{\partial x}P+\frac{\partial f}{\partial y}Q+\frac{\partial f}{\partial z}R=kf. \tag{3}$$

The polynomial k is called the *cofactor* of the invariant algebraic surface $f(x,y,z)=0$, and from (3) it follows that the degree of the polynomial k is at most the degree of the polynomial differential system (1) minus one.

From (3) it follows that the gradient $(\partial f/\partial x, \partial f/\partial x, \partial f/\partial z)$ of f is orthogonal to the vector field (P,Q,R) of the polynomial differential system (1) on the points of the invariant algebraic surface $f(x,y,z)=0$. Therefore the vector field (P,Q,R) is contained in the tangent plane to the surface $f(x,y,z)=0$

at every point of the invariant algebraic surface $f(x,y,z)=0$. Hence the surface $f(x,y,z)=0$ is formed by orbits of the vector field (P,Q,R), in other words if an orbit has a point on the invariant surface $f(x,y,z)=0$, then the whole orbit is contained in it. This justifies the name "invariant algebraic surface" for the surface $f(x,y,z)=0$. For more details on first integrals and invariant algebraic surfaces, see [5, Chapter 8].

The existence of invariant surfaces in the polynomial differential systems of $\mathbb{R}^3$ many times force the existence of first integrals, this phenomenon was studied by the Darboux theory of integrability, see for instance [4, 5, 7, 8]. Here we shall see that for the linear differential systems the existence of an invariant cylinder it is sufficient for the existence of a first integral.

The *phase portrait* of the differential system (1) on an invariant cylinder is the decomposition of the cylinder as union of all its orbits. The best qualitative result for a differential system is to provide its *phase portrait*, i.e., the decomposition of the domain of definition of the differential system as union of all its orbits.

Certainly the most easiest polynomial differential systems in $\mathbb{R}^3$ are the linear ones:

$$\begin{aligned} \dot{x} &= a_0 + a_1x + a_2y + a_3z, \\ \dot{y} &= b_0 + b_1x + b_2y + b_3z, \\ \dot{z} &= c_0 + c_1x + c_2y + c_3z, \end{aligned} \tag{4}$$

i.e., the polynomial differential systems of degree one. After these linear differential systems, without loss of generality, the polynomial differential systems of the form

$$\begin{aligned} \dot{x} &= a_0 + a_1x + a_2y + a_3z, \\ \dot{y} &= b_0 + b_1x + b_2y + b_3z, \\ \dot{z} &= c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5xy + c_6xz + c_7y^2 + c_8yz + c_9z^2, \end{aligned} \tag{5}$$

are the easiest ones.

In the next theorems we characterize the linear differential systems (4) and the polynomial differential systems (5) having an invariant either hyperbolic, or parabolic, or elliptic cylinder, and we describe the dynamics of these differential systems on those invariant cylinders.

THEOREM 1. *The linear differential systems (4) in $\mathbb{R}^3$ for which the hyperbolic cylinder $x^2 - y^2 = 1$ is invariant are*

$$\begin{aligned} \dot{x} &= a_2 y, \\ \dot{y} &= a_2 x, \\ \dot{z} &= c_0 + c_1 x + c_2 y + c_3 z. \end{aligned} \tag{6}$$

These differential systems have the first integral $H = H(x, y, z) = x^2 - y^2$, so $\mathbb{R}^3$ is foliated by the invariant hyperbolic cylinders $H = h$ when $h \neq 0$ and by one invariant surface $H = 0$, product of two planes.

(a) *Assume $a_2 = 0$. Then every straight line parallel to the z-axis is invariant by the flow of system (6).*

(a.1) *If $c_3 \neq 0$ every one of these invariant straight lines $x = x_0$, $y = y_0$, is formed by three orbits, one of them is the equilibrium point*

$$p = -\frac{c_0 + c_1 x_0 + c_2 y_0}{c_3},$$

and the other two either start at infinity and ends at the equilibrium point p, or start at the equilibrium point and end at infinity.

(a.2) *If $c_3 = 0$ then every one of these invariant straight lines $x = x_0$, $y = y_0$ is formed by a unique orbit starting and ending at infinity if $c_0 + c_1 x_0 + c_2 y_0 \neq 0$. If $c_0 + c_1 x_0 + c_2 y_0 = 0$ the invariant straight line $x = x_0$, $y = y_0$ is filled with equilibria.*

(b) *Assume $a_2 > 0$ (otherwise we change the time of sign).*

(b.1) *Asssume $h \neq 0$ and $c_3 \neq 0$. Then all orbits on the hyperbolic cylinders start and end at infinity.*

(b.2) *Asssume $h = 0$ and $c_3 \neq 0$. Then the differential system (6) has a unique equilibrium point $p = (0, 0, -c_0/c_3)$. The local phase portrait of this equilibrium point on the invariant plane $x - y = 0$ is a hyperbolic saddle if $c_3 < 0$, or a hyperbolic unstable node if $c_3 > 0$. While on the plane $x + y = 0$ the equilibrium p is a hyperbolic saddle if $c_3 > 0$, or a hyperbolic stable node if $c_3 < 0$.*

(b.3) *Asssume $c_3 = 0$ and $c_0 \neq 0$. Then on the invariant hyperbolic cylinders and on the invariant two planes every orbit starts and ends at infinity.*

(b.4) *Asssume $h \neq 0$ and $c_3 = c_0 = 0$. Then on the invariant hyperbolic cylinders every orbit starts and ends at infinity.*

(b.5) *Asssume $h = 0$ and $c_3 = c_0 = 0$. The straight line of the intersection of the two invariant planes $x - y = 0$ and $x + y = 0$ is filled with equilibria, and at each one of this equilibria either arrives two orbits, or exit two orbits.*

We note that when we say that an orbit "starts" on an equilibrium point or at infinity we are saying that the α-limit of this orbit is an equilibrium point or the infinity. For a definition of α-limit of an orbit see for instance [5]. In a similar way when we say that an orbit "ends" on an equilibrium point or at infinity we are saying that the ω-limit of this orbit is an equilibrium point or the infinity.

THEOREM 2. *The polynomial differential systems* (5) *in* $\mathbb{R}^3$ *for which the hyperbolic cylinder $x^2 - y^2 = 1$ is invariant are*

$$\begin{aligned}\dot{x} &= a_2 y,\\ \dot{y} &= a_2 x,\\ \dot{z} &= c_0 + c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 xy + c_6 xz + c_7 y^2 + c_8 yz + c_9 z^2.\end{aligned} \tag{7}$$

These differential systems have the first integral $H = H(x, y, z) = x^2 - y^2$, so $\mathbb{R}^3$ is foliated by the invariant hyperbolic cylinders $H = h$ when $h \neq 0$, and by one invariant surface $H = 0$ product of two planes. These differential systems have no periodic orbits.
Let $A = (c_3 + c_6 x + c_8 y)^2 - 4c_9\left(c_0 + c_1 x + c_2 y + c_4 x^2 + c_5 xy + c_7 y^2\right)$ be.

(a) *Assume $a_2 = 0$. Then all the orbits live on paralell straight lines to the z-axis.*

(a.1) *If $c_9 \neq 0$ and $A > 0$, then every straight line parallel to the z-axis, $x = x_0$ and $y = y_0$, is invariant containing two equilibria:*

$$p_1 = -\frac{c_3 + c_6 x_0 + c_8 y_0 + \sqrt{A}}{2c_9}, \qquad p_2 = -\frac{c_3 + c_6 x_0 + c_8 y_0 - \sqrt{A}}{2c_9}.$$

Every invariant straight line is formed by five orbits, two of these orbits are the equilibria p_1 and p_2. These five orbits are: one orbit starts at infinity and ends at the equilibrium point p_i, the equilibrium point p_i, another orbit starts at the equilibrium point p_j with $j \neq i$ and ends at the equilibrium point p_i, the equilibrium point p_j, and one orbit that starts in p_j and ends at infinity.

(a.2) *If* $c_9 \neq 0$ *and* $A = 0$, *then the invariant straight line* $x = x_0$ *and* $y = y_0$ *contains the equilibrium point*

$$p = -\frac{c_3 + c_6 x_0 + c_8 y_0}{2c_9}.$$

This invariant straight line is formed by three orbits, one orbit starting at infinity and ending at the equilibrium point p, *the equilibrium point* p, *and one orbit starting at the equilibrium pont* p *and ending at infinity.*

(a.3) *If* $c_9 \neq 0$ *and* $A < 0$, *then every one of these invariant straight lines is formed by a unique orbit starting and ending at infinity.*

(a.4) *If* $c_9 = 0$, *then every invariant straight line parallel to the* z*-axis,* $x = x_0$ *and* $y = y_0$, *with* $c_3 + c_6 x_0 + c_8 y_0 \neq 0$ *contains one equilibrium*

$$q = -\frac{c_0 + c_1 x_0 + c_2 y_0 + c_4 x_0^2 + c_5 x_0 y_0 + c_7 y_0^2}{c_3 + c_6 x_0 + c_8 y_0}.$$

This invariant straight line is formed by three orbits, one of them is the equilibrium point q *and the other two either start at infinity and end at the equilibrium point* q, *or start at the equilibrium point and end at infinity. If* $c_3 + c_6 x_0 + c_8 y_0 = 0$ *and* $c_0 + c_1 x_0 + c_2 y_0 + c_4 x_0^2 + c_5 x_0 y_0 + c_7 y_0^2 \neq 0$, *then on the invariant straight line there is a unique orbit that starts and ends at infinity. If* $c_3 + c_6 x_0 + c_8 y_0 = 0$ *and* $c_0 + c_1 x_0 + c_2 y_0 + c_4 x_0^2 + c_5 x_0 y_0 + c_7 y_0^2 = 0$, *then the invariant straight line is filled with equilibria.*

(b) *Assume* $a_2 > 0$ *(otherwise we can change the time of sign).*

(b.1) *If* $h = 0$, $c_9 \neq 0$ *and* $c_3^2 - 4c_0 c_9 > 0$, *then the differential system* (7) *has two equilibria*

$$P_1 = \left(0, 0, \frac{-c_3 - \sqrt{c_3^2 - 4c_0 c_9}}{2c_9}\right)$$

and

$$P_2 = \left(0, 0, \frac{-c_3 + \sqrt{c_3^2 - 4c_0 c_9}}{2c_9}\right),$$

contained in the straight line intersection of the two invariant planes $x - y = 0$ *and* $x + y = 0$. *The local phase portrait at these equilibria* *on the invariant plane $x - y = 0$ is a hyperbolic saddle for P_1 and a hyperbolic unstable node for P_2, while on the invariant plane $x + y = 0$ is a hyperbolic stable node for P_1 and a hyperbolic saddle for P_2.*

(b.2) *If $h = 0$, $c_9 \neq 0$ and $c_3^2 - 4c_0c_9 = 0$, then the differential system (7) has only one equilibrium point $P = (0, 0, -c_3/2c_9)$ contained in the straight line intersection of the two invariant planes. The local phase portrait of this equilibrium on both invariant planes $x - y = 0$ and $x + y = 0$ is a semi-hyperbolic saddle-node.*

(b.3) *If $h = 0$, $c_9 \neq 0$ and $c_3^2 - 4c_0c_9 < 0$, then the differential system (7) does not present equilibria and all the orbits start and end at infinity.*

(b.4) *Assume $h = 0$ and $c_9 = 0$. Then if $c_3 \neq 0$ the differential system (7) has the equilibrium point $Q = (0, 0, -c_0/c_3)$ in $H = 0$. The local phase portrait on the invariant plane $x - y = 0$ at this equilibrium is a hyperbolic unstable node when $c_3 > 0$, and a hyperbolic saddle when $c_3 < 0$. On the invariant plane $x + y = 0$ Q is a hyperbolic saddle when $c_3 > 0$, and a hyperbolic stable node when $c_3 < 0$. If $c_3 = 0$ then the system does not have equilibria and all orbits in $H = 0$ start and end at infinity.*

(b.5) *Assume $h \neq 0$. Then all orbits in $H = h$ start and end at infinity.*

THEOREM 3. *The linear differential systems (4) in $\mathbb{R}^3$ for which the parabolic cylinder $x^2 - y = 1$ is invariant are*

$$\begin{aligned} \dot{x} &= a_0 + a_1 x, \\ \dot{y} &= 2a_1 + 2a_0 x + 2a_1 y, \\ \dot{z} &= c_0 + c_1 x + c_2 y + c_3 z. \end{aligned} \tag{8}$$

These differential systems have the first integral $H(x, y, z) = H = x^2 - y$ when $a_1 = 0$, then $\mathbb{R}^3$ is foliated by the invariant parabolic cylinders $H = h$, where $h \in \mathbb{R}$.

(a) *If $a_1 = 0$ and $a_0 = 0$ then system (8) coincides with system (6) with $a_2 = 0$, and consequently the statements (a.k) for $k = 1, 2$ of Theorem 1 hold for system (8).*

(b) *If $a_1 = 0$ and $a_0 \neq 0$, then the differential system (8) has no equilibria and every one of the orbits start and end at infinity.*

(c) *Assume* $a_1 > 0$ *(otherwise we can change the time of the sign).*

(c.1) *If* $c_3 \neq 0$ *then the differential system* (8) *has a unique equilibrium point*

$$p = \left(-\frac{a_0}{a_1}, \frac{a_0^2 - a_1^2}{a_1^2}, -\frac{a_0 a_1 c_1 - a_0^2 c_2 + a_1^2 (c_2 - c_0)}{a_1^2 c_3} \right),$$

that lives on the parabolic cylinder $x^2 - y = 1$. *Its local phase portrait on this cylinder is a hyperbolic saddle when* $c_3 < 0$, *and a hyperbolic unstable node when* $c_3 > 0$.

(c.2) *If* $c_3 = 0$ *then the differential system* (8) *has no equilibria and the orbits start and end at infinity.*

Theorem 4. *The polynomial differential systems* (5) *in* $\mathbb{R}^3$ *for which the parabolic cylinder* $x^2 - y = 1$ *is invariant are*

$$\begin{aligned} \dot{x} &= a_0 + \frac{b_2 x}{2}, \\ \dot{y} &= b_2 + 2a_0 x + b_2 y, \\ \dot{z} &= c_0 + c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 xy + c_6 xz + c_7 y^2 + c_8 yz + c_9 z^2. \end{aligned} \tag{9}$$

Let A, p_1, p_2 *and* p *be the constant and the equilibria defined in the statement of Theorem* 2.

(a) *Assume* $b_2 = 0$. *Then the function* $H(x, y, z) = H = x^2 - y$ *is a first integral of the differential system* (9).

(a.1) *If* $a_0 \neq 0$ *then the differential system* (9) *has no equilibria and all the orbits start and end at infinity.*

(a.2) *If* $a_0 = 0$ *then system system* (9) *coincides with system* (7) *with* $a_2 = 0$ *and consequently all statements* (a.k) *for* $k = 1, 2, 3, 4$ *of Theorem* 2 *hold for system* (9).

(b) *Assume* $b_2 > 0$ *(otherwise we change the time sign). We define*

$$\begin{aligned} A &= -b_2^4 c_3 + 2a_0 b_2^3 c_6 - 4a_0^2 b_2^2 c_8 + b_2^4 c_8, \\ B &= A^2 - 4b_2^4 c_9 C, \\ C &= -b_2^4 c_0 - 2a_0 b_2^3 c_1 + 4a_0^2 b_2^2 c_2 - b_2^4 c_2 + 4a_0^2 b_2^2 c_4 - 8a_0^3 b_2 c_5 \\ &\qquad + 2a_0 b_2^3 c_5 + 16 a_0^4 c_7 - 8a_0^2 b_2^2 c_7 + b_2^4 c_7, \\ D &= b_2^2 c_3 - 2a_0 b_2 c_6 + 4a_0^2 c_8 - b_2^2 c_8. \end{aligned}$$

(b.1) *If $c_9 \neq 0$ and $B > 0$, then the differential system (9) has on the parabolic cylinder $x^2 - y - 1 = 0$ has no periodic orbits and two equilibria*

$$P_1 = \left(\frac{-2a_0}{b_2}, \frac{4a_0^2}{b_2^2} - 1, \frac{A - \sqrt{B}}{2b_2^4 c_9} \right)$$

and

$$P_2 = \left(\frac{-2a_0}{b_2}, \frac{4a_0^2}{b_2^2} - 1, \frac{A + \sqrt{B}}{2b_2^4 c_9} \right).$$

P_1 is a hyperbolic saddle and P_2 a hyperbolic unstable node.

(b.2) *If $c_9 \neq 0$ and $B = 0$, then on the parabolic cylinder $x^2 - y = 1$ there are no periodic orbits and one equilibrium point,*

$$P = \left(\frac{-2a_0}{b_2}, \frac{4a_0^2}{b_2^2} - 1, \frac{A}{2b_2^4 c_9} \right),$$

that is a semi-hyperbolic saddle-node.

(b.3) *If $c_9 \neq 0$ and $B < 0$, then the differential system (9) on the parabolic cylinder $x^2 - y = 1$ has no equilibria and the orbits start and end at infinity.*

(b.4) *Assume $c_9 = 0$. Then, if $D \neq 0$ the differential system (9) has no periodic orbits and a unique equilibrium point*

$$Q = \left(\frac{-2a_0}{b_2}, \frac{4a_0^2}{b_2^2} - 1, \frac{C}{b_2^2 D} \right),$$

in the parabolic cylinder $x^2 - y = 1$. Q is a hyperbolic unstable node if $D > 0$, a hyperbolic saddle. If $D = 0$ the system has no equilibria and all orbits on $H = 0$ start and end at infinity.

THEOREM 5. *The linear differential systems (4) in $\mathbb{R}^3$ for which the elliptic cylinder $x^2 + y^2 = 1$ is invariant are*

$$\begin{aligned} \dot{x} &= a_2 y, \\ \dot{y} &= -a_2 x, \\ \dot{z} &= c_0 + c_1 x + c_2 y + c_3 z. \end{aligned} \tag{10}$$

These differential systems have the first integral $H = H(x, y, z) = x^2 + y^2$, so $\mathbb{R}^3$ is foliated by the invariant elliptic cylinders $H = h > 0$, and by the invariant z-axis when $H = 0$.

(a) *If $a_2 = 0$ then system* (10) *coincides with system* (6) *with $a_2 = 0$, and consequently the statements (a.k) for $k = 1, 2$ of Theorem 1 hold for system* (10).

(b) *Assume $a_2 \neq 0$. Then the dynamics on the invariant cylinders $H = x^2 + y^2 = h > 0$ and on the z-axis $H = x^2 + y^2 = h = 0$ are described in what follows.*

(b.1) *Assume $h > 0$ and $c_3 \neq 0$. Then on the cylinder $H = h$ there is a unique periodic orbit γ_h, and the orbits on the cylinder distinct to this periodic orbit either start in γ_h and end at infinity, or start at infinity and end at γ_h.*

(b.2) *Assume $h > 0$, $c_3 = 0$ and $c_0 \neq 0$. Then on the invariant cylinders $H = h$ all orbits start and end at infinity.*

(b.3) *Assume $h > 0$ and $c_3 = c_0 \neq 0$. Then on the invariant cylinders $H = h$ all orbits are periodic.*

(b.4) *Assume $h = 0$ and $c_3 \neq 0$. Then on the invariant z-axis there is the equilibrium point $p = (0, 0, -c_0/c_3)$, and two orbits that either start at p and end at infinity, or start at infinity and end at p.*

(b.5) *Assume $h = 0$, $c_3 = 0$ and $c_0 \neq= 0$. Then on the invariant z-axis there is a unique orbit that starts and ends at infinity.*

(b.6) *Assume $h = 0$, $c_3 = c_0 = 0$. Then the invariant z-axis if filled with equilibria.*

THEOREM 6. *The polynomial differential systems* (5) *in $\mathbb{R}^3$ for which the elliptic cylinder $x^2 + y^2 = 1$ is invariant are*

$$\begin{aligned} \dot{x} &= a_2 y, \\ \dot{y} &= -a_2 x, \\ \dot{z} &= c_0 + c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 xy + c_6 xz + c_7 y^2 + c_8 yz + c_9 z^2. \end{aligned} \tag{11}$$

These differential systems have the first integral $H = H(x, y, z) = x^2 + y^2$. So $\mathbb{R}^3$ is foliated by the invariant elliptic cylinders $H = h > 0$ and by the invariant straight line $H = 0$ (the z-axis).

(a) *Assume $a_2 = 0$. Then all statements of* (*a*) *of Theorem* 2 *hold for system* (11).

(b) *Assume $a_2 > 0$ (otherwise we can change the time of sign).*

(b.1) *If $h = 0$, $c_9 \neq 0$ and $c_3^2 - 4c_0c_9 > 0$, then the differential system* (11) *has the two equilibria*

$$P_1 = \left(0, 0, \frac{-c_3 - \sqrt{c_3^2 - 4c_0c_9}}{2c_9}\right)$$

and

$$P_2 = \left(0, 0, \frac{-c_3 + \sqrt{c_3^2 - 4c_0c_9}}{2c_9}\right),$$

contained in $H = 0$. The local phase portrait of these equilibria on the z-axis is formed by five orbits, two of these orbits are the equilibria P_1 and P_2. These five orbits are: one orbit starts at infinity and ends at the equilibrium point P_i, the equilibrium point P_i, another orbit starts at the equilibrium point P_j with $j \neq i$ and ends at the equilibrium point P_i, the equilibrium point P_j, and one orbit that starts in P_j and ends at infinity.

(b.2) *If $h = 0$, $c_9 \neq 0$ and $c_3^2 - 4c_0c_9 = 0$, then the differential system* (11) *has only one equilibrium point $p = (0, 0, -c_3/2c_9)$ contained in $H = 0$. This invariant straight line is formed by three orbits, one orbit starting at infinity and ending at the equilibrium point p, the equilibrium point p, and one orbit starting at the equilibrium pont p and ending at infinity.*

(b.3) *If $h = 0$, $c_9 \neq 0$ and $c_3^2 - 4c_0c_9 < 0$, the differential system* (11) *has no equilibria and all orbits start and end at infinity.*

(b.4) *If $h = 0$, $c_9 = 0$ and $c_3 \neq 0$, then the differential system* (11) *has the equilibrium point $q = (0, 0, -c_0/c_3)$ in $H = 0$. This invariant straight line is formed by three orbits, one of them is the equilibrium point q and the other two either start at infinity and end at the equilibrium point q, or start at the equilibrium point and end at infinity.*

(b.5) *If $h = 0$, $c_9 = c_3 = 0$, then the invariant straight line $H = 0$ is filled with equilibria.*

(b.6) *If $h > 0$ then all the orbits start and end at infinity.*

2. Proof of the theorems

Proof of Theorem 1. A linear differential system (4) having the invariant hyperbolic cylinder $f = f(x, y, z) = x^2 - y^2 - 1 = 0$ must satisfy the equality

(3), i.e.,

$$2x\,(a_0 + a_1x + a_2y + a_3z) - 2y\,(b_0 + b_1x + b_2y + b_3z) = k(x^2 - y^2 - 1),$$

where $k \in \mathbb{R}$ is the cofactor, because the differential system (4) has degree one. From the previous equality we obtain

$$a_0 = a_1 = a_3 = b_0 = b_2 = b_3 = k_0 = 0, \qquad b_1 = a_2.$$

Substituting these values in the differential system (4), we obtain the differential system (6) of the statement of Theorem 1. Using (2) we obtain that $H = H(x, y, z) = x^2 - y^2$ is a first integral of the differential system (6). So all hyperbolic cylinders $H = h$ with $h \neq 0$ are invariant, and also are invariant the two planes $H = 0$.

Assume $a_2 = 0$ and $c_3 \neq 0$. Then the planes $x = x_0$ and $y = y_0$, are invariant under the flow of the differential system (6), and consequently all the straight lines parallel to the z-axis are formed by orbits of the differential system (6). Every one of these invariant straight lines are formed by three orbits, one starting at infinity and ending in the equilibrium point $p = (x_0, y_0, -(c_0 + c_1x_0 + c_2y_0)/c_3)$, this equilibrium point p, and one starting at the equilibrium point p and ending at infinity. Statement (a.1) is proved.

Assume $a_2 = c_3 \neq 0$. Then every one of these invariant straight lines $x = x_0$, $y = y_0$ is formed by a unique orbit starting and ending at infinity if $c_0 + c_1x_0 + c_2y_0 \neq 0$. If $c_0 + c_1x_0 + c_2y_0 = 0$ then the invariant straight lines $x = x_0$, $y = y_0$ is filled with equilibria. Statement (a.2) is proved.

Assume $a_2 > 0$, $h \neq 0$ and $c_3 \neq 0$. Since on the invariant hyperbolic cylinders there are no equilibrium points, also there are no periodic orbits because if a periodic orbit exists on one of these invariant surfaces, in the region bounded by this periodic orbit must be an equilibrium point, see [5, Theorem 1.31]. Since the orbits of any differential system are either equilibrium points, or periodic orbits, or homeomorphic to a straight line, it follows that on the hyperbolic cylinders all orbits are homeomorphic to a straight line. These orbits $\gamma(t) = (x(t), y(t), z(t))$ homeomorphic to a straight line are defined for all time in $\mathbb{R}$ because are solutions of a linear differential system, see [14, Chapter 3]. Then, by [5, Theorem 1.2] it follows that $\gamma(t)$ tends to the boundary of the hyperbolic cylinder or to the boundary of the invariant planes when t tends to $\pm\infty$. In other words, all orbits on the hyperbolic cylinders start and end at infinity. This proves statement (b.1).

Assume $a_2 > 0$, $h = 0$ and $c_3 \neq 0$. Then the differential system (6) has a unique equilibrium point $p = (0, 0, -c_0/c_3)$. This equilibrium point lives in the intersection of the two planes $H = 0$. On the invariant plane $x - y = 0$, the equilibrium p has eigenvalues a_2 and c_3, so by [5, Theorem 2.15] p is a hyperbolic saddle if $c_3 < 0$, or a hyperbolic unstable node if $c_3 > 0$. On the invariant plane $x + y = 0$, the equilibrium p has eigenvalues $-a_2$ and c_3, so by [5, Theorem 2.15] p is a hyperbolic saddle if $c_3 > 0$, or a hyperbolic stable node if $c_3 < 0$. Since these two planes are invariant we have two linear differential systems with saddles or nodes, their phases portraits are very well known. Statement (b.2) is proved.

Assume $a_2 > 0$, $c_3 = 0$ and $c_0 \neq 0$. Since on the invariant hyperbolic cylinders or in the invariant planes there are no equilibrium points, the same arguments than in the case $a_2 > 0$, $h \neq 0$ and $c_0 \neq 0$, it follows that all orbits on the hyperbolic cylinders or on the two invariant planes the start and end at infinity. This proves statement (b.3).

Assume $a_2 > 0$, $h \neq 0$ and $c_3 = c_0 \neq 0$. Since on the invariant hyperbolic cylinders there are no equilibrium points, using again the same arguments than in the case $a_2 > 0$, $h \neq 0$ and $c_0 \neq 0$, it follows that all orbits on the hyperbolic cylinders start and end at infinity. Statement (b.4) is proved.

Assume $a_2 > 0$, $h = 0$ and $c_3 = c_0 \neq 0$. Then is easy to verify that the straight line L of the intersection of the invariant planes $x - y = 0$ and $x + y = 0$ is filled of equilibria. Since the eigenvalues of the Jacobian matrix of the differential system (6) are $\pm a_2$ and 0, the straight line is a normally hyperbolic manifold and consequently to each equilibrium point of this line either arrives or exit two orbits, for more details see [6]. This proves statement (b.5). Hence the theorem is proved. ∎

Proof of Theorem 2. The polynomial differential system (5) having the invariant hyperbolic cylinder $f = f(x, y, z) = x^2 - y^2 - 1 = 0$ must satisfy the equality (3), i.e.,

$$\begin{aligned} 2x(a_0 + a_1x + a_2y + a_3z) - 2y\,(b_0 + b_1x + b_2y + b_3z) \\ = (k_0 + k_1x + k_2y + k_3z)\left(x^2 - y^2 - 1\right), \end{aligned}$$

where $k_0 + k_1x + k_2y + k_3z$, with $k_0, k_1, k_2, k_3 \in \mathbb{R}$ is the cofactor, because the differential system (5) has degree two. From the above equality, we get

$$a_0 = a_1 = a_3 = b_0 = b_2 = b_3 = k_0 = k_1 = k_2 = k_3 = 0, \qquad b_1 = a_2.$$

Substituting these values into the differential system (5), we obtain the differential system (7) of the statement of Theorem 2.

Using (2) we obtain that $H = H(x, y, z) = x^2 - y^2$ is a first integral of the differential system (7). So all hyperbolic cylinders $H = h$ with $h \neq 0$ are invariant, and also are invariant the two planes $H = 0$.

If $a_2 = 0$ then the planes $x = x_0$ and $y = y_0$ are invariant for the differential system (7), and consequently all the straight lines paralell to the z-axis are formed by orbits of the differential system (7). Moreover, since the restriction of the differential system (7) on the invariant surfaces $H = h$ are linear or quadratic polynomial differential systems that as we shall see have neither foci nor centers, these systems can not have periodic orbits, because the linear differential systems distinct from a center have no periodic orbits, and the quadratic systems without centers or without focus also do not have periodic orbits, see [3].

If $a_2 = 0$, $c_9 \neq 0$ and $A > 0$ (where the constant A is defined in the statement of the theorem), every one of these invariant straight lines is formed by five orbits, two of them are the equilibria p_1 and p_2 (defined in the statement of the theorem). This five orbits are: one orbit starting at infinity and ending in p_i, the orbit p_i, the orbit starting in p_j with $j \neq i$ and ending in p_i, the orbit p_j, and the orbit starting in p_j and ending at infinity. Statement (a.1) is proved.

If $a_2 = 0$, $c_9 \neq 0$ and $A = 0$, then every one of these invariant straight lines is formed by three orbits, one of them is the equilibrium p (defined in the statement of the theorem). This three orbits are: one orbit starting at infinity and ending in p, the equilibrium p, and the orbit starting in p and ending at infinity. So statement (a.2) follows.

If $a_2 = 0$, $c_9 \neq 0$ and and $A < 0$, then every one of these invariant straight lines is formed by a unique orbit starting and ending at infinity. So statement (a.3) is proved.

If $a_2 = 0$ and $c_9 = 0$, then the invariant straight line $x = x_0$ and $y = y_0$ with $c_3 + c_6 x_0 + c_8 y_0 \neq 0$ contains the equilibrium q (defined in the statement of the theorem). Then, clearly this invariant straight line is formed by three orbits, one of them is the equilibrium point q and the other two either start at infinity and end at the equilibrium point q, or start at the equilibrium point and end at infinity. The rest of the statement (a.4) follows easily.

Assume $a_2 > 0$, $h = 0$, $c_9 \neq 0$ and $c_3^2 - 4c_0c_9 > 0$. Then the differential system (7) has the two equilibria P_1 and P_2 (defined in the statment of the theorem). These two equilibria live in the intersection of the two planes $H = 0$. On the invariant plane $x - y = 0$ the equilibrium P_1 has eigenvalues a_2 and $-\sqrt{c_3^2 - 4c_0c_9}$, so by [5, Theorem 2.15] P_1 is a hyperbolic saddle. Since P_2 has eigenvalues a_2 and $\sqrt{c_3^2 - 4c_0c_9}$, so by [5, Theorem 2.15] P_2 is a hyperbolic unstable node. On the invariant plane $x + y = 0$ the equilibrium P_1 has eigenvalues $-a_2$ and $-\sqrt{c_3^2 - 4c_0c_9}$ and so it is a hyperbolic stable node and P_2 has the eigenvalues $-a_2$ and $\sqrt{c_3^2 - 4c_0c_9}$, hence it is a hyperbolic saddle. Statement (b.1) is proved.

Assume $a_2 > 0$, $h = 0$, $c_9 \neq 0$ and $c_3^2 - 4c_0c_9 = 0$. Then we have only one equilibrium point P. This equilibrium point lives in the intersection of the two planes $H = 0$. On the invariant plane $x - y = 0$, the equilibrium P has eigenvalues 0 and a_2, so by [5, Theorem 2.19] P is a semi-hyperbolic saddle-node. On the invariant plane $x + y = 0$, the equilibrium P has eigenvalues 0 and $-a_2$, again by [5, Theorem 2.19] p is a semi-hyperbolic saddle-node. Hence statement (b.2) follows.

Assume $a_2 > 0$, $h = 0$, $c_9 \neq 0$ and $c_3^2 - 4c_0c_9 < 0$. Then the system has no equilibria and so all orbits start and end at infinity, as explained in the proof of statement (b.1) of Theorem 1. This completes the proof of statement (b.3).

Assume $h = 0$ when $c_9 = 0$. Then the differential system (7) presents only one equilibrium point Q (defined in the statement of the theorem) that lives in the intersection of the two planes $H = 0$, when $c_3 \neq 0$. On the invariant plane $x - y = 0$ the equilibrium Q has eigenvalues a_2 and c_3, so when $c_3 > 0$ p is a hyperbolic unstable node, and when $c_3 < 0$ then Q is a hyperbolic saddle. On the invariant plane $x + y = 0$ the equilibrium Q has eigenvalues $-a_2$ and c_3, which characterizes Q as a hyperbolic saddle when $c_3 > 0$ and a hyperbolic stable node when $c_3 < 0$. If we have $c_3 = 0$, the system does not have equilibria and so all orbits on $H = 0$ start and end at infinity, as it was explained in the proof of statement (b.1) of Theorem 1. Statement (b.4) is proved.

Assume $h \neq 0$. Sine on the the hyperbolic cylinders $H = h$ there are no equilibria, by the arguments of the proof of statement (b.1) of Theorem 1 all orbits start and end at infinity. Statement (b.5) is proved. Hence the proof of the theorem is done. ∎

Proof of Theorem 3. A linear differential system (4) having the invariant parabolic cylinder $f = f(x, y, z) = x^2 - y - 1 = 0$, must satisfy the equality (3), i.e.,

$$2x(a_0 + a_1x + a_2y + a_3z) - (b_0 + b_1x + b_2y + b_3z) = k\left(x^2 - y - 1\right),$$

where $k \in \mathbb{R}$ is the cofactor. From the previous equality we obtain

$$a_0 = a_3 = b_0 = b_3 = 0, \quad b_0 = 2a_1, \quad b_1 = 2a_0, \quad b_2 = 2a_1, \quad k = 2a_1.$$

If $a_1 = 0$ by direct computations the function $H(x, y, z) = H = x^2 - y$ is a first integral of system (8).

The proof of statement (a) is given in the proofs of statements (a) of Theorem 1.

Assume $a_1 = 0$ and $a_0 \neq 0$. Then the differential system (8) does not have equilibria and all orbits are homeomorphic to a straight line paralell to the z-axis, as explained it has been explaned in the proof of statement (b.1) of Theorem 1. So statement (b) follows. Assume that $a_1 > 0$ and $c_3 \neq 0$. Then the differential system (8) has the unique equilibrium point p (defined in the statement of the theorem). The equilibrium p lives on the invariant parabolic cylinder $x^2 - y = 1$. The differential system (8) restricted to the this cylinder becomes

$$\begin{aligned} \dot{x} &= a_0 + a_1 x, \\ \dot{z} &= c_0 + c_1 x + c_2(x^2 - 1) + c_3 z, \end{aligned} \tag{12}$$

after subtituting $y = x^2 - 1$. The Jacobian matrix of the differential system (12) at the equilibrium p has eigenvalues a_1 and c_3. So statement (c.1) follows.

The proof of statement (c.2) is the same than the proof of statement (b.1) of Theorem 1. This completes the proof of the theorem. ∎

Proof of Theorem 4. The polynomial differential system (5) having the invariant parabolic cylinder $f = x^2 - y - 1 = 0$, must satisfy the equality (3), i.e.,

$$\begin{aligned} 2x(a_0 + a_1 x + a_2 y + a_3 z) &- (b_0 + b_1 x + b_2 y + b_3 z) \\ &= (k_0 + k_1 x + k_2 y + k_3 z)\left(x^2 - y - 1\right), \end{aligned}$$

where $k_0 + k_1 x + k_2 y + k_3 z$, with $k_0, k_1, k_2, k_3 \in \mathbb{R}$ is the cofactor. From the previous equality we obtain

$$a_2 = a_3 = b_3 = k_1 = k_2 = k_3 = 0, \quad b_1 = 2a_0, \quad b_2 = 2a_1, \quad b_0 = b_2, \quad k_0 = b_2.$$

Therefore the differential system (9) is obtained.

If $b_2 = 0$, the function $H(x, y, z) = H = x^2 - y$ is a first integral of the system (9), because the cofactor of $f = 0$ is zero.

If $b_2 = 0$ and $a_0 \neq 0$ this system does not have equilibria and all the orbits start and end at infinity, using the arguments in the proof of statement (b.1) of Theorem 1. Statement (a.1) is proved.

The proofs of statement (a.2) are given in the proof of Theorem 2 with $a_2 = 0$. If $b_2 > 0$, $c_9 \neq 0$ and $B > 0$, then on the parabolic cylinder $x^2 - y - 1 = 0$ there are the two equilibria P_1 and P_2 defined in the statement of the theorem. The eigenvalues of P_1 are $b_2/2$ and $-\sqrt{B}/b_2^4$, so P_1 is a hyperbolic saddle. The eigenvalues of P_2 are $b_2/2$ and $\sqrt{B}/b_2^4$, hence P_2 is a hyperbolic unstable node. On this parabolic cylinder we can apply the arguments of the proof of Theorem 2 for proving the non-existence of periodic orbits. So statement (b.1) follows. If $b_2 > 0$, $c_9 \neq 0$ and $B = 0$, then system (9) has on the parabolic cylinder $H = 0$ only the equiilibrium point P. This equilibrium point P has eigenvalues 0 and $b_2/2$, i.e., P is a semi-hyperbolic equilibrium point. By [5, Theorem 2.19] P is a saddle-node. Since the topological index of a saddle-node is zero, the differential system restricted to the invariant parabolic cylinder can not have periodic orbits, because a periodic orbit must surround equilibria with a total index equal to one, for instance this result is a corollary of [5, Proposition 6.26]. Hence statement (b.2) is proved.

If $b_2 > 0$, $c_9 \neq 0$ and $B < 0$, then system (9) has no equilibria on $x^2 - y = 1$, and all orbits on $x^2 - y = 1$ start and end at infinity, using previous arguments. This completes the proof of statement (b.3). If $b_2 > 0$ and $c_9 = 0$, then on the parabolic cylinder $x^2 - y = 1 = 0$ there is a unique equilibrium point Q. The eigenvalues of Q are $b_2/2$ and D/b_2^2, consequently Q is a saddle if $D > 0$ and an unstable node if $D > 0$. Using the arguments of the proof of the case $b_2 > 0$, $c_9 \neq 0$ and $B > 0$ on the hyperbolic cylinder there are no periodic orbits. Hence statement (b.4) follows. This completes the proof of the theorem. ∎

Proof of Theorem 5. A linear differential system (4) having the invariant elliptic cylinder $f = f(x, y, z) = x^2 + y^2 + 1 = 0$ must satisfy the equality (3), i.e.,

$$2x(a_0 + a_1x + a_2y + a_3z) + 2y(b_0 + b_1x + b_2y + b_3z) = k\left(x^2 + y^2 + 1\right),$$

where $k \in \mathbb{R}$ is the cofactor, because the differential system (4) has degree one. From the previous equality we obtain

$$a_0 = a_1 = a_3 = b_0 = b_2 = b_3 = k = 0, \qquad b_1 = -a_2.$$

Substituting these values in the differential system (4), we obtain the differential system (10) of the statement of Theorem 5. Using (2) we obtain that $H = H(x, y, z) = x^2 + y^2$ is a first integral of the system (10). So all elliptic cylinders $H = h$ with $h > 0$ are invariant, and also is invariant the z-axis when $H = 0$. The proof of statement (a) is given in the proofs of statements (a) of Theorem 1.

We write the differential systems (10) in cylindrical coordinates (r, θ, z) where $x = r\cos\theta$, $y = r\sin\theta$, and we obtain the differential system

$$\begin{aligned}\dot{r} &= 0,\\ \dot{\theta} &= -a_2,\\ \dot{z} &= c_0 + c_1 r\cos\theta + c_2 r\sin\theta + c_3 z.\end{aligned}$$

The restriction of this differential system on $H = h$ taking θ as the new independent variable becomes the differential equation

$$\frac{dz}{d\theta} = c_0 + c_1\sqrt{h}\cos\theta + c_2\sqrt{h}\sin\theta + c_3 z. \tag{13}$$

Assume $h > 0$ and $c_3 \neq 0$. Then the solution $z(\theta, z_0)$ of the differential equation (13) such that $z(0, z_0) = z_0$ is

$$\begin{aligned}z(\theta, z_0) = {} & \frac{e^{-\frac{c_3\theta}{a_2}}\left(a_2^2c_0 - a_2c_2c_3\sqrt{h} + c_0c_3^2 + c_1c_3^2\sqrt{h} + (a_2^2c_3 + c_3^3)z_0\right)}{c_3(a_2^2+c_3^2)} c_0\left(a_2^2 + c_3^2\right)\\ & - c_3\sqrt{h}\big((a_2c_1 + c_2c_3)\sin\theta + (c_1c_3 - a_2c_2)\cos\theta\big).\end{aligned}$$

Therefore on the cyclinder $H = h$ there is a unique periodic orbit $z(\theta, z_0)$ when

$$z_0 = -\frac{a_2^2c_0 - a_2c_2c_3\sqrt{h} + c_0c_3^2 + c_1c_3^2\sqrt{h}}{a_2^2c_3 + c_3^3}.$$

The rest of statement (b.1) follows.

Assume $h > 0$, $c_3 = 0$ and $c_0 \neq 0$. Then the solution $z(\theta, z_0)$ of the differential equation (13) such that $z(0, z_0) = z_0$ is

$$z(\theta, z_0) = \frac{-c_2\sqrt{h} + a_2z_0 - c_0\theta + \sqrt{h}(c_2\cos\theta - c_1\sin\theta)}{a_2}. \tag{14}$$

So, clearly the orbits on the cylinder $H = h$ start end end at infinity. Statement (b.2) is proved.

Assume $h > 0$ and $c_3 = c_0 \neq 0$. From (14) it follows that all the orbits on the cylinder $H = h$ are periodic. Statement (b.3) is proved.

Assume $h = 0$ and $c_3 \neq 0$. Then the solution $z(\theta, z_0)$ of the differential equation (13) such that $z(0, z_0) = z_0$ is

$$z(\theta, z_0) = \frac{e^{-c_3\theta/a_2}(c_0 + c_3z_0) - c_0}{c_3}.$$

Therefore on the invariant z-axis there is the equilibrium point $p = (0, 0, -c_0/c_3)$, and two orbits that either start at p and end at infinity, or start at infinity and end at p. Statement (b.4) follows.

Assume $h = 0$, $c_3 = 0$ and $c_0 \neq 0$. Then the solution $z(\theta, z_0)$ of the differential equation (13) such that $z(0, z_0) = z_0$ is

$$z(\theta, z_0) = z_0 - \frac{c_0\theta}{a_2}. \tag{15}$$

Hence on the invariant z-axis there is a unique solution that starts and ends at infinity. So statement (b.5) is proved.

Assume $h = 0$, $c_3 = c_0 = 0$. Then form (15) the invariant z-axis if filled with equilibria. Statement (b.6) is proved. This completes the proof of the theorem. ∎

Proof of Theorem 6. The polynomial differential system (11) having the invariant elliptic cylinder $f = f(x, y, z) = x^2 + y^2 - 1 = 0$ must satisfy the equality (3), i.e.,

$$\begin{aligned} 2x(a_0 + a_1x + a_2y + a_3z) &+ 2y(b_0 + b_1x + b_2y + b_3z) \\ &= (k_0 + k_1x + k_2y + k_3z)\left(x^2 + y^2 - 1\right), \end{aligned}$$

where $k_0 + k_1x + k_2y + k_3z$, with $k_0, k_1, k_2, k_3 \in \mathbb{R}$ is the cofactor, because the differential system (11) has degree two. From the above equality, we get

$$a_0 = a_1 = a_3 = b_0 = b_2 = b_3 = k_0 = k_1 = k_2 = k_3 = 0, \qquad b_1 = -a_2.$$

Substituting these values into the differential system (5), we obtain the differential system (11) of the statement of Theorem 6.

Using (2) we obtain that $H = H(x, y, z) = x^2 + y^2$ is a first integral of the differential system (11). So all elliptic cylinders $H = h > 0$ are invariant, and also are invariant the z-axis $H = 0$.

If $a_2 = 0$ then the differential system (11) coincides with the differential system (7), so statement (a) is proved.

If $a_2 > 0$, $h = 0$, $c_9 \neq 0$ and $c_3^2 - 4c_0c_9 > 0$, every one of the invariant z-axis is formed by five orbits, two of them are the equilibria P_1 and P_2 (defined in the statement of the theorem). This five orbits are: one orbit starting at infinity and ending in P_i, the orbit P_i, the orbit starting in P_j with $j \neq i$ and ending in P_i, the orbit P_j, and the orbit starting in P_j and ending at infinity. Statement (b.1) is proved.

If $a_2 > 0$, $h = 0$, $c_9 \neq 0$ and $c_3^2 - 4c_0c_9 = 0$, then the invariant z-axis is formed by three orbits, one of them is the equilibrium p (defined in the statement of the theorem). This three orbits are: one orbit starting at infinity and ending in p, the equilibrium p, and the orbit starting in p and ending at infinity. So statement (b.2) follows.

If $a_2 > 0$, $h = 0$, $c_9 \neq 0$ and $c_3^2 - 4c_0c_9 < 0$, then the invariant z-axis is formed by a unique orbit starting and ending at infinity. So statement (b.3) is proved.

If $a_2 > 0$, $h = 0$, $c_9 = 0$ and $c_3 \neq 0$, then on the invariant z-axis there is the equilibrium q (defined in the statement of the theorem). Then, clearly this invariant straight line is formed by three orbits, one of them is the equilibrium point q and the other two either start at infinity and end at the equilibrium point q, or start at the equilibrium point and end at infinity. Statement (b.4) follows.

Assume $a_2 > 0$, $h = 0$, $c_9 = c_3 = 0$. Then it is easy to verify that the differential system (11) has the z-axis filled with equilibria. So statement (b.5) is proved.

Assume $h > 0$. Then on the elliptic cylinders $H = h$ there are no equilibria. Therefore using the arguments of the proof of statement (b.1) of Theorem 1 statement (b.5) follows. Hence the proof of the theorem is done. ∎

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