EXTRACTA MATHEMATICAE
Vol. **39**, Num. 2 (2024), 255 – 271

doi:10.17398/2605-5686.39.2.255
Available online August 05, 2024

Some operators on finite-dimensional non-Archimedean normed spaces

M. Babahmed [1]

[1] *Department of Mathematics, University of Moulay Ismail, Faculty of Sciences, Meknes, Morocco.*

m.babahmed@umi.ac.ma

Received January 17, 2024
Accepted April 27, 2024

Presented by J. Navarro

Abstract: In this paper, we are interested in the study of certain operators in non-Archimedean normed spaces of finite dimension. We introduce the notion of p-delta function, then we characterize the simple operators, the similarities and the expansions. We show if E has an orthogonal basis, then each injective operator on E is the composition of an isometry and an expansion.

Key words: Non-Archimedean normed space; t-Orthogonal basis; p-delta function; Simple operator; Similarity; Isometry; Expansion.

MSC (2020): 46S10; 46B20.

1. Introduction

We consider the family of n-dimensional non-Archimedean normed spaces over a non-Archimedean valued field $\mathbb{K}$. We discuss some geometrical aspects of these spaces related to the notions of orthogonality and t-orthogonality. Then, we study some particular and important operators on these spaces. It is very known that all n-dimensional non-Archimedean normed spaces over $\mathbb{K}$ are linearly homeomorphic to $\mathbb{K}^n$. And one of the fundamental facts in the theory of non-Archimedean normed spaces states that every finite-dimensional normed space over a spherically complete valued field $\mathbb{K}$ has an orthogonal basis. But, if $\mathbb{K}$ is not spherically complete, finite-dimensional normed spaces over $\mathbb{K}$ without any orthogonal basis exist [5, p. 68]. However, every finite-dimensional normed space E over $\mathbb{K}$ admits a t-orthogonal basis (Theorem 2.5). Therefore, using these geometrical aspects of finite-dimensional normed space E over a non-Archimedean valued field $\mathbb{K}$, we will introduce the notion of p-delta function similarly to volume function, which is analogous to the natural volume function in a real Hilbert space, introduced by van Rooij in [6]. Then, we characterize three families of operators defined on E; namely,

ISSN: 0213-8743 (print), 2605-5686 (online)

similarities, isometries and expansions.

For more details in non-Archimedean normed spaces, we refer to [3] , [4] and [5]. And for more information on the subject, we refer to [1] and [2].

2. Preliminaries

Throughout the present paper, $\mathbb{K}$ will denote a non-Archimedean complete valued field with a non-trivial absolute value. $\mathbb{K}$ is said to be spherically complete if every shrinking sequence of closed balls in $\mathbb{K}$ has a non-empty intersection. Clearly the spherical completion implies completion, but the converse is not true in general [4]. Normed spaces over $\mathbb{K}$ are defined in a natural way. We say that a norm $\|\cdot\|$ on a $\mathbb{K}$-vector space E is non-Archimedean if it satisfies the strong triangle inequality: $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in E$.

We say that a normed space is non-Archimedean if its topology is defined by a non-Archimedean norm.

Let E be a non-Archimedean normed space. E is spherically complete if every shrinking sequence of closed balls in E has a non-empty intersection. For any subset A of E, $[A]$ will denote the linear hull of A in E.

Let $t \in]0,1]$, nonzero elements x and y of E are called t-orthogonal if $d(x,[y]) \geq t \cdot \|x\|$, where $d(x,[y]) = \inf\{\|x-z\| \,:\, z \in [y]\}$ is the distance of x to $[y]$. We write $x \perp_t y$. If $t = 1$, we say that x and y are orthogonal, and we write $x \perp y$. We check easily that $x \perp_t y$ if, and only if, $\|\alpha x + \beta y\| \geq t \cdot \max\{\|\alpha x\|, \|\beta y\|\}$ for each $\alpha, \beta \in K$.

We say that a family of nonzero elements $(x_i)_{i\in I}$ of E is t-orthogonal if for each $i \in I$, $x_i \perp_t x_j$ for all $j \in I\backslash\{i\}$.

Clearly, $(x_i)_{i\in I}$ is t-orthogonal if, and only if, for each distinct $i_1, \ldots, i_n \in I$, and for each $\lambda_1, \ldots, \lambda_n \in K$, $\| \sum_{k=1}^{n} \lambda_k x_{i_k} \| \geq t \cdot \max_{1\leq k \leq n} \|\lambda_k x_{i_k}\|$. If, in addition, $E = \overline{[x_i \,:\, i \in I]}$, we say that $(x_i)_{i\in I}$ is a t-orthogonal basis of E.

If $(x_i)_{i\in I}$ is a t-orthogonal basis of E, for every $x \in E$, there is a unique family $(\lambda_i)_{i\in I} \in K^I$ such that: $x = \sum_{i\in I} \lambda_i x_i$ and $\|x\| \geq t \cdot sup_{i\in I} \|\lambda_i x_i\|$.

We note that if $(x_i)_{i\in I}$ is a t-orthogonal family in E, then $(x_i)_{i\in I}$ is a linearly independent family; and if $(\lambda_i)_{i\in I}$ is a family of nonzero elements of $\mathbb{K}$, then $(\lambda_i x_i)_{i\in I}$ is also a t-orthogonal family in E. In particular, if we take $\pi \in K$ with $0 < |\pi| < 1$, then we can choose $(\lambda_i)_{i\in I}$ such that $|\pi| \leq \|\lambda_i x_i\| \leq 1$ for all $i \in I$.

As a consequence, if $(x_i)_{i\in I}$ is a t-orthogonal basis of E, without loss of generality, we can suppose that $(x_i)_{i\in I}$ satisfies $|\pi| \leq \|x_i\| \leq 1$ for all $i \in I$.

From now on, $(E, \|.\|)$ will be a non-Archimedean normed space of dimension $n \geq 2$.

It is well known that:

(1) E is linearly homeomorphic to $\mathbb{K}^n$;

(2) E is a Banach space;

(3) All linear functionals $f : E \to K$ are continuous;

(4) All subspaces are closed.

An operator on E is a linear function $T : E \to E$. All operators on E are bounded.

We note S_m the set of all permutations σ of $\{1, \ldots, m\}$ for each $m \geq 2$.

THEOREM 2.1. (PRINCIPLE OF VAN ROOIJ) *Let $t \in]0, 1]$, let x, y be elements of E such that $\|x + y\| \geq t \cdot \|x\|$. Then $\|x + y\| \geq t \cdot \|y\|$.*

Proof. See [4, Theorem 2.2.1]. ∎

THEOREM 2.2. *Let $t \in]0, 1]$, let $e_1, \ldots, e_n$ be distinct non-zero vectors. The following are equivalent:*

(i) *$\{e_1, \ldots, e_n\}$ is a t-orthogonal system;*

(ii) *For all $\lambda_1, \ldots, \lambda_n \in K$, $\| \sum_{k=1}^{n} \lambda_k e_k\| \geq t \cdot \max_{1\leq k\leq n} \|\lambda_k e_k\|$;*

(iii) *For all $j \in \{1, \ldots, n-1\}$, $e_{j+1} \perp_t [e_1, \ldots, e_j]$.*

Proof. Analogous to the proof of [4, Theorem 2.2.7]. ∎

LEMMA 2.3. *Let F be a closed subspace of E and $t \in]0, 1[$. For each $a \in E \backslash F$, there exists $e \in E$ such that: $[a] + F = [e] + F$ and $e \perp_t F$.*

Proof. Let $r = d(a, F)$. Since F is closed, $r > 0$. Let $z \in F$ such that $\|a - z\| \leq \frac{r}{t}$. Let $e = a - z$, then $[a] + F = [e] + F$. On the other hand, $d(e, F) = d(a - z, F) = r \geq t.\|e\|$.

For all $x \in F$ and $\lambda \in K$, $\|\lambda e + x\| \geq d(\lambda e, F) = |\lambda| d(e, F) \geq t \cdot \|\lambda e\|$. Then, by the van Rooij principle (Theorem 2.1), $\|\lambda e+x\| \geq t\cdot \max\{\|\lambda e\|, \|x\|\}$. Hence, $e \perp_t F$. ∎

Remark 2.4. (1) If e satisfies the conditions of the Lemma 2.3, it is the same for all λe with $\lambda \in K \backslash \{0\}$. Then, for all $\alpha \in]0, 1[$, we can choose e such that $\alpha \leq \|e\| \leq 1$.

(2) If F is spherically complete, we can choose e such that $e \perp F$.

Using this lemma, we prove the following interesting theorem, see [4, Theorem 2.3.7] for another proof.

THEOREM 2.5. *For each $t \in]0,1[$, there exists $\{e_1, \dots, e_n\}$ a t-orthogonal basis of E.*

Proof. Let $e_1 \in E \backslash \{0\}$, and set $F_1 = [e_1]$. Let $a \in E \backslash F_1$. By Lemma 2.3, there exists $e_2 \in E$ such that $[a] + F_1 = [e_2] + F_1$ and $d(e_2, F_1) \geq t \cdot \|e_2\|$. Then, $[a, e_1] = [e_1, e_2]$ and $e_2 \perp_t F_1$. Hence, $\{e_1, e_2\}$ is a t-orthogonal system in E. Now set $F_2 = [e_1, e_2]$. If $\dim(E) > 2$, there is $b \in E \backslash F_2$, and by Lemma 2.3, there exists $e_3 \in E$ such that $[b] + F_2 = [e_3] + F_2$ and $d(e_3, F_2) \geq t \cdot \|e_3\|$. Then, $[b, e_1, e_2] = [e_1, e_2, e_3]$ and $e_3 \perp_t F_2$. Hence, $\{e_1, e_2, e_3\}$ is a t-orthogonal system in E. Continuing like this we construct $\{e_1, \dots, e_n\}$ a t-orthogonal system in E. And the result follows since each t-orthogonal system in E is linearly independent. ∎

THEOREM 2.6. *Let $t \in]0,1[$, then each one-dimensional subspace of E is t-orthocomplemented in E.*

Proof. Let $F = [x_1]$ be a one-dimensional subspace of E. Using Lemma 2.3 as in the proof of Theorem 2.5, we construct $\{x_1, \dots, x_n\}$ a t-orthogonal basis of E. Then, $G = [x_2, \dots, x_n]$ is an orthocomplement subspace of F in E. ∎

THEOREM 2.7. *If E has an orthogonal basis, then each one-dimensional subspace of E is orthocomplemented in E.*

Proof. Let $\{e_1, \dots, e_n\}$ be an orthogonal basis of E and $F = [x]$ a one-dimensional subspace of E.

Let $x = \sum_{i=1}^{n} \lambda_i e_i$ with $\|x\| = \max_{1 \leq i \leq n} \|\lambda_i e_i\| = \|\lambda_j e_j\|$ $(1 \leq j \leq n)$.

Let $G = [e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_n]$. For each $y = \sum_{i=1, i \neq j}^{n} \lambda_i e_i \in G$ we have:

$$\begin{aligned}\|x - y\| &= \|(\lambda_1 - \alpha_1)e_1 + \cdots + (\lambda_{j-1} - \alpha_{j-1})e_{j-1} + \lambda_j e_j \\ &\qquad + (\lambda_{j+1} - \alpha_{j+1})e_{j+1} + \cdots + (\lambda_n - \alpha_n)e_n\| \\ &= \max\big\{\|(\lambda_1 - \alpha_1)e_1\|, \dots, \|(\lambda_{j-1} - \alpha_{j-1})e_{j-1}\|, \|\lambda_j e_j\|, \\ &\qquad \|(\lambda_{j+1} - \alpha_{j+1})e_{j+1}\|, \dots, \|(\lambda_n - \alpha_n)e_n\|\big\} \geq \|\lambda_j e_j\| = \|x\|\end{aligned}$$

Then, $x \perp G$. And

$$[x] + G = [\lambda_1 e_1 + \ldots + \lambda_n e_n] + G = [e_j] + [e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n] = E.$$

Therefore, G is an orthocomplemented subspace of F in E. ∎

LEMMA 2.8. *Let $F \subset G$ be two subspaces of E and $\pi : E \to E/F$ the canonical surjection. On E/F we consider the non-Archimedean norm defined as follows: $\|\pi(x)\|_q = d(x, F)$. Then, $d(x, G) = d(\pi(x), \pi(G)) \quad \forall x \in E$.*

Proof. Let $x \in E$. It's about showing:

$$\alpha = \inf_{z \in G} \|x + z\| = \inf_{z \in G} \|\pi(x + z)\|_q = \beta.$$

Let $z \in G$, $\|\pi(x+z)\|_q = d(x+z, F) \leq \|x+z\|$. Then, $\beta \leq \alpha$. To show $\alpha \leq \beta$ it suffices to check that for each $r > 0$, $\beta < r \Rightarrow \alpha \leq r$. Let $r > 0$ such that $\beta < r$. And let $\epsilon > 0$ such that $\beta = \inf_{z \in G} \|\pi(x + z)\|_q = r - \epsilon$. Then, there exists $z_\epsilon \in G$ such that $\|\pi(x + z_\epsilon)\|_q < \beta + \epsilon = r$. So, $d(x + z_\epsilon, F) < r$. Hence, there exists $y_\epsilon \in F$ such that $\|x + z_\epsilon + y_\epsilon\| < r$. Since $z_\epsilon + y_\epsilon \in G$, $d(x, G) \leq r$. Then, $\alpha \leq r$, and the result follows. ∎

An operator T on E is said simple if there exists a linear functional $\varphi : E \to \mathbb{K}$ and a vector $z \in E$ such that: $Tx = x + \varphi(x) \cdot z$ for all $x \in E$. Then, we say that T is a (φ, z)-simple operator.

PROPOSITION 2.9. *Let T be a (φ, z)-simple operator on E. Then,* $\det(T) = 1 + \varphi(z)$. *Hence, T is a bijection if, and only if, $\varphi(z) \neq -1$.*

Proof. We have $Tx = x + \varphi(x).z$ for all $x \in E$. If $z = 0$, then $T = id_E$ and $\det(T) = 1$.

If $z \neq 0$, let $B = (z, z_2, \ldots, z_n)$ be a basis of E. The matrix of T in the basis B is:

$$\begin{pmatrix} 1 + \varphi(z) & \varphi(z_2) & \varphi(z_3) & \ldots & \ldots & \varphi(z_n) \\ 0 & 1 & 0 & \ldots & \ldots & 0 \\ \vdots & 0 & \ddots & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 & 0 \\ 0 & 0 & \ldots & \ldots & 0 & 1 \end{pmatrix}$$

Therefore, $\det(T) = 1 + \varphi(z)$. ∎

We note that if T is not a bijective (φ, z)-simple operator on E, then $z \in Ker(T)$.

Observe that elementary operations (on a fixed basis) are particular cases of simple operators (in fact, simple operators have elementary matrices in some basis). Moreover, it is trivial that any bijective operator in a finite-dimensional vector space E is the composition of elementary operations (as any invertible matrix can be reduced to the identity by multiplying by elementary matrices, and the inverses of these elementary matrices are again elementary). Therefore, it is evident that any bijective operator on E is the composition of simple operators.

3. p-delta functions

For each $p \geq 2$ we define the p-delta function as follows:

$$\delta_E^p(x_1, \dots, x_p) = \prod_{i=1}^{p-1} d(x_i, [x_{i+1}, \dots, x_p]) \cdot \|x_p\| \quad \forall\, x_1, \dots, x_p \in E.$$

We easily verify that for each $\lambda \in K$, $i \in \{1, \dots, p\}$ and $x_1, \dots, x_p \in E$

$$\delta_E^p(x_1, \dots, x_{i-1}, \lambda x_i, x_{i+1}, \dots, x_p) = |\lambda|\, \delta_E^p(x_1, \dots, x_p)$$

and

$$\delta_E^p(x_1, \dots, x_p) \leq \prod_{i=1}^{p} \|x_i\|.$$

Proposition 3.1. *Let $x_1, \dots, x_p \in E$, then we have:*

(1) $\delta_E^p(x_1, \dots, x_p) = 0$ *if, and only if, $\{x_1, \dots, x_p\}$ is linearly dependent.*

(2) $\delta_E^p(x_1, \dots, x_p) = \prod_{i=1}^p \|x_i\|$ *if, and only if, $\{x_1, \dots, x_p\}$ is an orthogonal system.*

Proof. (1) $\delta_E^p(x_1, \dots, x_p) = 0 \Leftrightarrow \|x_p\| = 0$ or there exists $i \in \{1, \dots, p-1\}$ such that $d(x_i, [x_{i+1}, \dots, x_p]) = 0 \Leftrightarrow x_p = 0$ or there exists $i \in \{1, \dots, p-1\}$ such that $x_i \in [x_{i+1}, \dots, x_p] \Leftrightarrow \{x_1, \dots, x_p\}$ is linearly dependent.

(2) Suppose that $\delta_E^p(x_1, \dots, x_p) = \prod_{i=1}^p \|x_i\|$. Since $d(x_i, [x_{i+1}, \dots, x_p]) \leq \|x_i\|$ for all $i \in \{1, \dots, p-1\}$, we must have $d(x_i, [x_{i+1}, \dots, x_p]) = \|x_i\|$ for all $i \in \{1, \dots, p-1\}$. Then, $x_i \perp [x_{i+1}, \dots, x_p]$ for all $i \in \{1, \dots, p-1\}$. Therefore, $\{x_1, \dots, x_p\}$ is an orthogonal system.

Conversely, if $\{x_1, \ldots, x_p\}$ is an orthogonal system , then

$$d(x_i, [x_{i+1}, \ldots, x_p]) = \|x_i\| \qquad \forall\, i \in \{1, \ldots, p-1\}.$$

Therefore, $\delta_E^p(x_1, \ldots, x_p) = \prod_{i=1}^p \|x_i\|$. ∎

As a consequence of Proposition 3.1, we have $\delta_E^p = 0$ for all $p > n$.

PROPOSITION 3.2. *Let $t \in]0,1[$. If $\{e_1, \ldots, e_p\}$ is a t-orthogonal system in E, then:*

$$\delta_E^p(e_1, \ldots, e_p) \geq t^{p-1} \prod_{i=1}^p \|e_i\|.$$

Proof. $\delta_E^p(e_1, \ldots, e_p) = \prod_{i=1}^{p-1} d(e_i, [e_{i+1}, \ldots, e_p]) \cdot \|e\|_p$. For each $i = 1, \ldots, p-1$ we have:

$$d(e_i, [e_{i+1}, \ldots, e_p]) = \inf\left\{ \left\| e_i + \sum\nolimits_{j=i+1}^p \lambda_j e_j \right\| \,:\, \lambda_{i+1}, \ldots, \lambda_p \in \mathbb{K} \right\}.$$

Since $\{e_1, \ldots, e_p\}$ is a t-orthogonal system, $e_i \perp_t [e_{i+1}, \ldots, e_p]$ for all $i = 1, \ldots, p-1$. Then,

$$\left\| e_i + \sum\nolimits_{j=i+1}^p \lambda_j e_j \right\| \geq t \cdot \max\left\{ \|e_i\| \left\| \sum\nolimits_{j=i+1}^p \lambda_j e_j \right\| \right\} \geq t \cdot \|e_i\|.$$

Therefore, $\delta_E^p(e_1, \ldots, e_p) \geq t^{p-1} \prod_{i=1}^p \|e_i\|$. ∎

LEMMA 3.3. *Let $x, y \in E$, then $\delta_E^2(x,y) = \delta_E^2(y,x)$.*

Proof. Let $\Phi(u,v) = \frac{d(u,[v])}{\|u\|}$ for all $u, v \in E\backslash\{0\}$. It's about showing $\Phi(x,y) = \Phi(y,x)$.

For this, it is enough to show

$$\Phi(x,y) = \inf\left\{ \frac{\|\alpha x + \beta y\|}{\max\{\|\alpha x\|, \|\beta y\|\}} \,:\, \alpha, \beta \in \mathbb{K}\backslash\{0\} \right\}.$$

For each $\alpha, \beta \in \mathbb{K}\backslash\{0\}\}$, we have:

$$\|\alpha x + \beta y\| \geq d(\alpha x, [\beta y]) = |\alpha|\, d(x,[y]) = |\alpha|\, \|x\| \Phi(x,y) = \|\alpha x\| \Phi(x,y).$$

Since $\Phi(x,y) \in]0,1]$, by the van Rooij principle (Theorem 2.1),

$$\|\alpha x + \beta y\| \geq \Phi(x,y) \cdot \max\{\|\alpha x\|, \|\beta y\|\}.$$

Then, $\Phi(x,y) \leq \frac{\|\alpha x+\beta y\|}{\max\{\|\alpha x\|,\|\beta y\|\}}$. Therefore,

$$\Phi(x,y) \leq \inf\left\{\frac{\|\alpha x+\beta y\|}{\max\{\|\alpha x\|,\|\beta y\|\}} \,:\, \alpha,\beta\in\mathbb{K}\backslash\{0\}\right\}.$$

On the other hand,

$$\begin{aligned}\Phi(x,y) = \frac{d(x,[y])}{\|x\|} &= \inf\left\{\frac{\|x+\beta y\|}{\|x\|} \,:\, \beta\in\mathbb{K}\backslash\{0\}\right\}\\ &\geq \inf\left\{\frac{\|x+\beta y\|}{\max\{\|\alpha x\|,\|\beta y\|\}} \,:\, \alpha,\beta\in\mathbb{K}\backslash\{0\}\right\}\\ &\geq \inf\left\{\frac{\|\alpha x+\beta y\|}{\max\{\|\alpha x\|,\|\beta y\|\}} \,:\, \alpha,\beta\in\mathbb{K}\backslash\{0\}\right\}.\end{aligned}$$

Then, $\Phi(x,y) = \inf\big\{\frac{\|\alpha x+\beta y\|}{\max\{\|\alpha x\|,\|\beta y\|\}} : \alpha,\beta\in\mathbb{K}\backslash\{0\}\big\}$. And the result follows. ∎

Proposition 3.4. *Let $x_1,\ldots,x_p\in E$, then for each $i\in\{2,\ldots,p-1\}$ we have:*

$$\delta_E^p(x_1,\ldots,x_{i-1},x_{i+1},x_i,x_{i+2},\ldots,x_p) = \delta_E^p(x_1,\ldots,x_p).$$

Proof. If one of the vectors $x_1,\ldots,x_p$ is null, the result is trivial. Then, suppose all these vectors are nonzero. For $i=p-1$, we have:

$$\begin{aligned}\delta_E^p(x_1,\ldots,x_{p-2},x_p,x_{p-1}) = \|x_{p-1}\|\big\{&d(x_1,[x_2,\ldots,x_p])\cdot d(x_2,[x_3,\ldots,x_p])\\ &\cdot\ldots\cdot d(x_{p-2},[x_p,x_{p-1}])\cdot d(x_p,[x_{p-1}])\big\}.\end{aligned}$$

By Lemma 3.3, $\delta_E^2(x_{p-1},x_p) = \delta_E^2(x_p,x_{p-1})$, so

$$\|x_{p-1}\|\cdot d(x_p,[x_{p-1}]) = \|x_p\|\cdot d(x_{p-1},[x_p]).$$

Then,

$$\begin{aligned}\delta_E^p(x_1,\ldots,x_{p-2},x_p,x_{p-1}) &= \|x_p\|\big\{d(x_1,[x_2,\ldots,x_p])\\ &\qquad\cdot\ldots\cdot d(x_{p-2},[x_{p-1},x_p])\cdot d(x_{p-1},[x_p])\big\}\\ &= \delta_E^p(x_1,\ldots,x_p).\end{aligned}$$

Now let $i\in\{2,\ldots,p-2\}$:

$$\begin{aligned}&\delta_E^p(x_1,\ldots,x_{i-1},x_{i+1},x_i,x_{i+2},\ldots,x_p)\\ &\quad= \|x_p\| d(x_1,[x_2,\ldots,x_p])\cdot\ldots\cdot d(x_{i-1},[x_i,\ldots,x_p])\\ &\qquad\cdot d(x_{i+1},[x_i,x_{i+2},\ldots,x_p])\cdot d(x_i,[x_{i+2},\ldots,x_p])\\ &\qquad\cdot d(x_{i+2},[x_{i+3},\ldots,x_p])\cdot\ldots\cdot d(x_{p-1},[x_p]).\end{aligned}$$

Then it's enough to show

$$d(x_{i+1},[x_i,x_{i+2},\dots,x_p])\cdot d(x_i,[x_{i+2},\dots,x_p]) \\ = d(x_i,[x_{i+1},\dots,x_p])\cdot d(x_{i+1},[x_{i+2},\dots,x_p]).$$

Let $F=[x_{i+2},\dots,x_p]$ and consider the canonical surjection $\pi : E \to E/F$:

$$\begin{aligned} d(x_{i+1},[x_i,&x_{i+2},\dots,x_p])\cdot d(x_i,[x_{i+2},\dots,x_p]) \\ &= d(x_{i+1},[x_i]+F)\cdot d(x_i,F) \\ &= d(\pi(x_{i+1}),\pi([x_i]+F))\cdot d(\pi(x_i),\pi(F)) \quad \text{(Lemma 2.8)} \\ &= d(\pi(x_{i+1}),[\pi(x_i)])\cdot d(\pi(x_i),\pi(F)) \\ &= [\delta^2_{E/F}(\pi(x_{i+1}),\pi(x_i))\cdot\|\pi(x_{i+1})\|_q]\cdot\|\pi(x_i)\|_q \\ &= [\delta^2_{E/F}(\pi(x_i),\pi(x_{i+1}))\cdot\|\pi(x_i)\|_q]\cdot\|\pi(x_{i+1})\|_q \\ &= d(\pi(x_i),[\pi(x_{i+1})])\cdot d(\pi(x_{i+1}),\pi(F)) \\ &= d(\pi(x_i),[\pi([x_{i+1}]+F)])\cdot d(\pi(x_{i+1}),\pi(F)) \\ &= d(x_i,[x_{i+1}]+F)\cdot d(x_{i+1},F) \quad \text{(Lemma 2.8)} \\ &= d(x_i,[x_{i+1},\dots,x_p])\cdot d(x_{i+1},[x_{i+2},\dots,x_p]). \end{aligned}$$

And the result follows. ∎

Remark 3.5. For each nonzero vectors $x_1,\dots,x_p\in E$, we have also:

$$\delta^p_E(x_2,x_1,x_3,\dots,x_p)=\delta^p_E(x_1,x_2,\dots,x_p).$$

It's enough to show:

$$d(x_2,[x_1,x_3,\dots,x_p])\cdot d(x_1,[x_3,\dots,x_p]) \\ = d(x_1,[x_2,\dots,x_p])\cdot d(x_2,[x_3,\dots,x_p]).$$

We set $F=[x_3,\dots,x_p]$, and we follow the same approach of the proof of Proposition 3.4.

COROLLARY 3.6. *For each nonzero vectors $x_1,\dots,x_p\in E$ and for each permutation $\sigma\in S_p$, we have:*

$$\delta^p_E(x_{\sigma(1)},\dots,x_{\sigma(p)})=\delta^p_E(x_1,\dots,x_p).$$

Proposition 3.7. *Let $x_1, \dots, x_p \in E$ and $\lambda_1, \dots, \lambda_p \in K$, then for each $i \in \{1, \dots, p\}$ we have:*

$$\delta_E^p\Big(x_1, \dots, x_{i-1}, \sum\nolimits_{j=1}^{p} \lambda_j x_j, x_{i+1}, \dots, x_p\Big) = |\lambda_i|\, \delta_E^p(x_1, \dots, x_p).$$

Proof. Let $i \in \{1, \dots, p\}$, then, by the definition of the p-delta function, we have:

$$\begin{aligned}
&\delta_E^p\Big(x_1, \dots, x_{i-1}, \sum\nolimits_{j=1}^{p} \lambda_j x_j, x_{i+1}, \dots, x_p\Big)\\
&\quad = \delta_E^p\Big(x_1, \dots, x_{i-1}, \sum\nolimits_{j=1}^{i} \lambda_j x_j, x_{i+1}, \dots, x_p\Big)\\
&\quad = \delta_E^p\Big(x_1, \dots, x_{i-2}, \sum\nolimits_{j=1}^{i} \lambda_j x_j, x_{i-1}, x_{i+1}, \dots, x_p\Big) \quad \text{(Proposition 3.4)}\\
&\quad = \delta_E^p\Big(x_1, \dots, x_{i-2}, \lambda_i x_i + \sum\nolimits_{j=1}^{i-1} \lambda_j x_j, x_{i-1}, x_{i+1}, \dots, x_p\Big)\\
&\quad = \delta_E^p\Big(x_1, \dots, x_{i-2}, \lambda_i x_i + \sum\nolimits_{j=1}^{i-2} \lambda_j x_j, x_{i-1}, x_{i+1}, \dots, x_p\Big) \quad (\text{def. of } \delta_E^p)\\
&\quad = \delta_E^p\Big(x_1, \dots, x_{i-3}, \lambda_i x_i + \sum\nolimits_{j=1}^{i-2} \lambda_j x_j, x_{i-2}, x_{i-1}, x_{i+1}, \dots\Big) \quad \text{(Prop. 3.4)}\\
&\quad = \delta_E^p\Big(x_1, \dots, x_{i-3}, \lambda_i x_i + \sum\nolimits_{j=1}^{i-3} \lambda_j x_j, x_{i-2}, x_{i-1}, x_{i+1}, \dots, x_p\Big) \ (\text{def. of } \delta_E^p)\\
&\quad = \delta_E^p\Big(x_1, \dots, x_{i-1}, \lambda_i x_i, \sum\nolimits_{j=1}^{p} \lambda_j x_j, x_{i+1}, x_{i+2}, \dots, x_p\Big)\\
&\quad = \delta_E^p\Big(x_1, \lambda_i x_i + \sum\nolimits_{j=1}^{2} \lambda_j x_j, x_2, x_3, \dots, x_{i-1}, x_{i+1}, \dots, x_p\Big)\\
&\quad = \delta_E^p(x_1, \lambda_i x_i + \lambda_1 x_1, x_2, x_3, \dots, x_{i-1}, x_{i+1}, \dots, x_p)\\
&\quad = \delta_E^p(\lambda_i x_i, x_1, x_2, x_3, \dots, x_{i-1}, x_{i+1}, \dots, x_p)\\
&\quad = |\lambda_i|\, \delta_E^p(x_i, x_1, x_2, x_3, \dots, x_{i-1}, x_{i+1}, \dots, x_p) = |\lambda_i|\, \delta_E^p(x_1, \dots, x_p).
\end{aligned}$$

∎

Proposition 3.8. *Let T be a $(\varphi, z)-$simple operator, then:*

$$\delta_E^n(Tx_1, \dots, Tx_n) = |\det(T)|\, \delta_E^n(x_1, \dots, x_n) \qquad \forall\, x_1, \dots, x_n \in E.$$

Proof. $Tx = x + \varphi(x) \cdot z$ for all $x \in E$. Let $x_1, \dots, x_n \in E$. If $\{x_1, \dots, x_n\}$ is linearly dependent, then so is $\{Tx_1, \dots, Tx_n\}$ and the result follows easily. So, we suppose that $\{x_1, \dots, x_n\}$ is linearly independent, so it is a basis for E. If T

is not bijective, then $\det(T) = 1 + \varphi(z) = 0$, and there exists $x = \sum_{i=1}^n \lambda_i x_i \in E\backslash\{0\}$, $(\lambda_1, \ldots, \lambda_n \in \mathbb{K})$, such that $Tx = 0$. Then $\sum_{i=1}^n \lambda_i Tx_i = 0$, and $\{Tx_1, \ldots, Tx_n\}$ is linearly dependent. Therefore, $\delta_E^n(Tx_1, \ldots, Tx_n) = 0$. Now suppose that T is bijective, then $\det(T) = 1 + \varphi(z) \neq 0$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ such that $z = \sum_{i=1}^n \alpha_i x_i$. For each $i \in \{1, \ldots, n\}$ let $\beta_i = \varphi(x_i)$;

$$\det(T) = 1 + \varphi(z) = 1 + \sum\nolimits_{i=1}^n \alpha_i \varphi(x_i) = 1 + \sum\nolimits_{i=1}^n \alpha_i \beta_i.$$

For each $i \in \{1, \ldots, n\}, Tx_i = x_i + \varphi(x_i) \cdot z = x_i + \beta_i \cdot z$. Then, it is about showing that:

$$\delta_E^n(x_1 + \beta_1 z, \ldots, x_n + \beta_n z) = \Big|1 + \sum\nolimits_{i=1}^n \alpha_i \beta_i\Big| \, \delta_E^n(x_1, \ldots, x_n).$$

We can suppose, without loss of generality, that $\alpha_1 \neq 0$ (otherwise, we can make a permutation of the vectors $x_1, \ldots, x_n$);

$$\Big(1 + \sum\nolimits_{i=1}^n \alpha_i \beta_i\Big) z = \sum\nolimits_{i=1}^n \alpha_i x_i + \sum\nolimits_{i=1}^n \alpha_i \beta_i z = \sum\nolimits_{i=1}^n \alpha_i (x_i + \beta_i z).$$

Then, we have:

$$\begin{aligned}
\delta_E^n(Tx_1, \ldots, Tx_n) &= \delta_E^n(x_1 + \beta_1 z, \ldots, x_n + \beta_n z) \\
&= \frac{1}{|\alpha_1|} \, \delta_E^n\Big(\sum\nolimits_{i=1}^n \alpha_i (x_i + \beta_i z), x_2 + \beta_2 z, \ldots, x_n + \beta_n z\Big) \\
&= \frac{1}{|\alpha_1|} \, \delta_E^n\Big((1 + \sum\nolimits_{i=1}^n \alpha_i \beta_i) z, x_2 + \beta_2 z, \ldots, x_n + \beta_n z\Big) \\
&= \frac{\big|1 + \sum_{i=1}^n \alpha_i \beta_i\big|}{|\alpha_1|} \, \delta_E^n(z, x_2 + \beta_2 z, \ldots, x_n + \beta_n z) \\
&= \frac{|\det(T)|}{|\alpha_1|} \, \delta_E^n(z, x_2, \ldots, x_n) \\
&= \frac{|\det(T)|}{|\alpha_1|} \, \delta_E^n\Big(\sum\nolimits_{i=1}^n \alpha_i x_i, x_2, \ldots, x_n\Big) \\
&= \frac{|\det(T)|}{|\alpha_1|} \, |\alpha_1| \, \delta_E^n(x_1, x_2, \ldots, x_n) \\
&= |\det(T)| \, \delta_E^n(x_1, x_2, \ldots, x_n).
\end{aligned}$$

∎

THEOREM 3.9. *Let T be an operator on E, then:*

$$\delta_E^n(Tx_1, \ldots, Tx_n) = |\det(T)| \delta_E^n(x_1, \ldots, x_n) \qquad \forall x_1, \ldots, x_n \in E.$$

Proof. Let $x_1, \dots, x_n \in E$. We can assume that the operator T is bijective (otherwise, $\{Tx_1, \dots, Tx_n\}$ is always linearly dependent and hence $\delta_E^n(Tx_1, \dots, Tx_n)$ is zero). There exist simple operators $T_1, \dots, T_m$ such that $T = T_m T_{m-1} \cdots T_1$. Then

$$\delta_E^n(Tx_1, \dots, Tx_n) = \delta_E^n(T_m T_{m-1} \cdots T_1 x_1, \dots, T_m T_{m-1} \cdots T_1 x_n).$$

So, by Proposition 3.8, we have:

$$\delta_E^n(Tx_1, \dots, Tx_n) = |\det(T_m)|\, \delta_E^n(T_{m-1} T_{m-2} \cdots T_1 x_1, \dots, T_{m-1} T_{m-2} \cdots T_1 x_n).$$

And applying the same result over and over we will get:

$$\begin{aligned} \delta_E^n(Tx_1, \dots, Tx_n) &= |\det(T_m)| \cdots |\det(T_1)|\, \delta_E^n(x_1, \dots, x_n) \\ &= |\det(T)|\, \delta_E^n(x_1, \dots, x_n). \end{aligned}$$

∎

4. Similarities

An operator T on E is said a similarity if there exist $r > 0$ such that: $\|Tx\| = r\|x\|$ for all $x \in E$. Then, we say that T is a r-similarity. An isometry on E is a 1-similarity. It is immediate that any similarity is a bijective operator. If T is an r-silimarity, then $\|T\| = r$ and $\|Tx\| = \|T\| \cdot \|x\|$ for all $x \in E$.

Proposition 4.1. *Let $t \in]0, 1]$ and T a similarity on E. If $\{e_1, \dots, e_n\}$ is a t-orthogonal basis in E, then so is $\{Te_1, \dots, Te_n\}$.*

Proof. Let $\lambda_1, \dots, \lambda_n \in K$:

$$\begin{aligned} \Big\| \sum_{i=1}^n \lambda_i T e_i \Big\| &= \Big\| T\Big(\sum_{i=1}^n \lambda_i e_i \Big) \Big\| = \|T\| \Big\| \sum_{i=1}^n \lambda_i e_i \Big\| \\ &\geq \|T\|\, t \cdot \max_{1 \leq i \leq n} \|\lambda_i e_i\| = t \cdot \max_{1 \leq i \leq n} \|T\| \|\lambda_i e_i\| \\ &= t \cdot \max_{1 \leq i \leq n} \|T(\lambda_i e_i)\| = t \cdot \max_{1 \leq i \leq n} \|\lambda_i T e_i\|. \end{aligned}$$

Then, $\{Te_1, \dots, Te_n\}$ is a t-orthogonal system in E. Hence, it is a t-orthogonal basis in E. ∎

THEOREM 4.2. *Let T be a bijective operator on E. Then, we have:*

(1) $|\det(T)| \leq \|T\|^n$;

(2) $|\det(T)| = \|T\|^n$ *if, and only if, T is a similarity.*

Proof. (1) Let $t \in]0,1[$. There exists $\{e_1, \ldots, e_n\}$ a t-orthogonal basis in E (Theorem 2.5). By Proposition 3.2, $\delta_E^n(e_1, \ldots, e_n) \geq t^{n-1} \prod_{i=1}^n \|e_i\|$. Then,

$$\begin{aligned} t^{n-1}\,|\det(T)| \prod_{i=1}^n \|e_i\| &\leq |\det(T)|\, \delta_E^n(e_1, \ldots, e_n) \\ &= \delta_E^n(Te_1, \ldots, Te_n) \\ &\leq \prod_{i=1}^n \|Te_i\| \leq \|T\|^n \prod_{i=1}^n \|e_i\|. \end{aligned}$$

Then, $t^{n-1}|\det(T)| \leq \|T\|^n$ for each $t \in]0,1[$. Hence, $|\det(T)| \leq \|T\|^n$.

(2) Assume that $|\det(T)| = \|T\|^n$. If T is not a similarity, there exists $e_1 \in E\backslash\{0\}$ such that $\|Te_1\| < \|T\|\|e_1\|$.

Let $t \in]0,1[$ such that $\|Te_1\| < t^{n-1} \cdot \|T\| \cdot \|e_1\|$. Complete e_1 to obtain a t-orthogonal basis $\{e_1, \ldots, e_n\}$ in E. Then, we have:

$$\begin{aligned} \delta_E^n(Te_1, \ldots, Te_n) &\leq \prod_{i=1}^n \|Te_i\| = \|Te_1\| \prod_{i=2}^n \|Te_i\| \\ &< \left(t^{n-1} \cdot \|T\|\|e_1\|\right) \prod_{i=2}^n \|Te_i\| \\ &= t^{n-1}\|T\|^n \cdot \prod_{i=1}^n \|e_i\| \leq \|T\|^n \delta_E^n(e_1, \ldots, e_n). \end{aligned}$$

Then, $\delta_E^n(Te_1, \ldots, Te_n) < |\det(T)|\delta_E^n(e_1, \ldots, e_n)$, which is a contradiction.

Reciprocally, assume that T is a similarity. Let $t \in]0,1[$, and consider a t-orthogonal basis $\{e_1, \ldots, e_n\}$ in E. By Proposition 3.2,

$$\begin{aligned} &\delta_E^n(Te_1, \ldots, Te_n) \geq t^{n-1} \prod_{i=1}^n \|Te_i\| = t^{n-1} \cdot \|T\|^n \prod_{i=1}^n \|e_i\| \\ &\quad\Rightarrow \quad |\det(T)|\, \delta_E^n(e_1, \ldots, e_n) \geq t^{n-1}\|T\|^n \prod_{i=1}^n \|e_i\| \\ &\quad\Rightarrow \quad |\det(T)| \geq (t^{n-1}\|T\|^n) \frac{\prod_{i=1}^n \|e_i\|}{\delta_E^n(e_1, \ldots, e_n)} \Rightarrow |\det(T)| \geq t^{n-1}\|T\|^n, \end{aligned}$$

this being for all $t \in]0,1[$, then $|\det(T)| \geq \|T\|^n$. Therefore, $|\det(T)| = \|T\|^n$. ∎

Theorem 4.3. *Let T be an operator on E. Then, T is an isometry if, and only if,* $\|T\| = |\det(T)| = 1$.

Proof. Suppose that T is an isometry, then by Theorem 4.2, $|\det(T)| = \|T\|^n = 1$. Then, $\|T\| = |\det(T)| = 1$.

Reciprocally, assume that $\|T\| = |\det(T)| = 1$. It is about showing that $\|Tx\| = \|x\|$ for all $x \in E$. Suppose that there exists $e_1 \in E\backslash\{0\}$ such that $\|Te_1\| < \|e_1\|$. Let $t \in]0,1[$ such that $\|Te_1\| < t^{n-1}\|e_1\|$, and complete e_1 to obtain a t-orthogonal basis $\{e_1, \ldots, e_n\}$ in E. Then, we have:

$$\begin{aligned} \delta_E^n(e_1,\ldots,e_n) &= \frac{1}{|\det(T)|}\,\delta_E^n(Te_1,\ldots,Te_n) \\ &= \delta_E^n(Te_1,\ldots,Te_n) \le \prod_{i=1}^n \|Te_i\| \\ &= \|Te_1\| \prod_{i=2}^n \|Te_i\| < t^{n-1}\|e_1\| \prod_{i=2}^n \|T\|\|e_i\| \\ &= t^{n-1} \prod_{i=1}^n \|e_i\| \le \delta_E^n(e_1,\ldots,e_n), \end{aligned}$$

which is a contradiction. ∎

5. Expansions

An operator T on E is said an expansion if there exists a basis $\{e_1, \ldots, e_n\}$ in E and $\lambda_1, \ldots, \lambda_n \in K$ such that: $Te_i = \lambda_i e_i$ for all $i = 1, \ldots, n$.

Theorem 5.1. *Let $t \in]0,1]$, E and F be two non-Archimedean normed spaces of dimension $n \ge 2$ each having a t-orthogonal basis, and $T : E \to F$ a nonzero operator. Then, there exist a nonzero vector e in E and a subspace G of E such that:*

1. $t\|T\|\|e\| \le \|Te\| \le \|T\|\|e\|$;
2. $e \perp_{t^2} G$;
3. $Te \perp_t TG$.

Proof. Let $t \in]0,1[$, and consider $\{x_1, \ldots, x_n\}$ a t-orthogonal basis in E and $\{y_1, \ldots, y_n\}$ a t-orthogonal basis in F. Set $\delta = \max_{1\le i\le n} \frac{\|Tx_i\|}{\|x_i\|} = \frac{\|Tx_k\|}{\|x_k\|}$ $(k \in \{1, \ldots, n\})$; $\delta \le \|T\|$.

Let $e = x_k$. For each $\lambda_1, \dots, \lambda_n \in K$, we have:

$$\Big\|T\Big(\sum_i^n \lambda_i x_i\Big)\Big\| \leq \max_{1\leq i\leq n} \big(|\lambda_i|\, \|Tx_i\|\big)$$
$$\leq \delta \max_{1\leq i\leq n} \big(|\lambda_i|\, \|x_i\|\big) \leq \frac{\delta}{t} \Big\| \sum_{i=1}^n \lambda_i x_i\Big\|.$$

Then, $\|T\| \leq \frac{\delta}{t}$, and $\delta \leq \|T\| \leq \frac{\delta}{t}$. Hence, $t\|T\|\|e\| \leq \|Te\| \leq \|T\|\|e\|$.

It is clear that $\|Te\| \neq 0$, otherwise $\|Tx_i\| = 0$ for all $i = 1, \dots, n$, and $T = 0$. By Theorem 2.6, the subspace $[Te]$ admits a t-orthogonal complement in F. Let H be this t-orthogonal complement, $F = [Te] \oplus_t H$. Set $G = T^{-1}(H)$. Then, G is a subspace of E and $Te \perp_t TG$.

For each $y \in G$, $Ty \in H$ and we have:

$$\|e + y\| \geq \frac{\|Te + Ty\|}{\|T\|} \geq \frac{t\, \max(\|Te\|, \|Ty\|)}{\|T\|} \geq t\frac{\|Te\|}{\|T\|} \geq t\frac{t\|T\|\|e\|}{\|T\|} = t^2\|e\|.$$

Hence, $e \perp_{t^2} G$.

For $t = 1$, we apply the Theorem 2.7 for orthogonal bases with the same reasoning. Then, we have:

(1) $\|Te\| = \|T\|\|e\|$;

(2) $e \perp G$;

(3) $Te \perp TG$. ∎

Remark 5.2. If the operator T is bijective, then the subspace G is of dimension $n - 1$.

THEOREM 5.3. *Let $t \in]0, 1]$, E and F be two non-Archimedean normed spaces of dimension $n \geq 2$ each having a t-orthogonal basis, and $T : E \to F$ an injective operator. Then, there exists $\{e_1, \dots, e_n\}$ a t-orthogonal basis of E such that $\{Te_1, \dots, Te_n\}$ is a $\sqrt{t}$-orthogonal basis of F.*

Proof. Let $t \in]0, 1[$. By Theorem 5.1, there is $e_n \in E\backslash\{0\}$ and a subspace G_{n-1} of E such that:

$$\sqrt{t}\|T\|\|e_n\| \leq \|Te_n\| \leq \|T\|\|e_n\|, \quad e_n \perp_t G_{n-1}, \quad Te_n \perp_{\sqrt{t}} TG_{n-1}$$

and $\dim(G_{n-1}) = n - 1$. By applying the theorem again to $T_{n-1} = T_{|G_{n-1}}$, there exist $e_{n-1} \in G_{n-1}\backslash\{0\}$ and a subspace G_{n-2} of G_{n-1} such that:

$$\sqrt{t}\|T_{n-1}\|\|e_{n-1}\| \leq \|T_{n-1}e_{n-1}\| \leq \|T_{n-1}\|\|e_{n-1}\|,$$
$$e_{n-1} \perp_t G_{n-2}, \quad T_{n-1}e_{n-1} \perp_{\sqrt{t}} T_{n-1}G_{n-2}$$

and $\dim(G_{n-2}) = n - 2$. And by continuing in this way, we will have the existence of a sequence of subspaces $E = G_n \supset \cdots \supset G_1$, $\dim(G_k) = k$ $(1 \leq k \leq n)$, and $e_k \in G_k \backslash \{0\}$, $(2 \leq k \leq n)$ such that:

$$\sqrt{t}\|T_k\|\|e_k\| \leq \|T_k e_k\| \leq \|T_k\|\|e_k\|, \quad e_k \perp_t G_{k-1}, \quad T_k e_k \perp_{\sqrt{t}} T_k G_{k-1},$$

with $T_k = T_{|G_k}$ $(2 \leq k \leq n-1)$. Let $e_1 \in G_1 \backslash \{0\}$, then $G_1 = [e_1]$. By Theorem 2.2, $\{e_1, \ldots, e_n\}$ is a t-orthogonal basis of E and $\{Te_1, \ldots, Te_n\}$ is a $\sqrt{t}$-orthogonal basis of F.

For $t = 1$, the same reasoning gives us the existence of an orthogonal basis $\{e_1, \ldots, e_n\}$ of E such that $\{Te_1, \ldots, Te_n\}$ is an orthogonal basis of F. ∎

THEOREM 5.4. *If E has an orthogonal basis and $\|E\| \subset |\mathbb{K}|$, then each injective operator on E is the composition of an isometry and an expansion.*

Proof. By Theorem 5.3, there exists an orthogonal basis $\{e_1, \ldots, e_n\}$ of E such that $\{Te_1, \ldots, Te_n\}$ is an orthogonal basis of E. For each $i \in \{1, \ldots, n\}$, let $\lambda_i \in \mathbb{K}$ such that $\|Te_i\| = |\lambda_i|$, and set $z_i = \frac{1}{\lambda_i} Te_i$; $\{z_1, \ldots, z_n\}$ is an orthogonal basis of E. Let U and V be the operators on E defined by:

$$Ue_i = z_i \quad \text{and} \quad Vz_i = \lambda_i z_i \qquad \text{for all } i = 1, \ldots, n.$$

It is clear that V is an expansion. And for each $x = \sum_{i=1}^n \alpha_i e_i \in E$, we have:

$$\|Ux\| = \Big\| \sum_{i=1}^n \alpha_i U e_i \Big\| = \Big\| \sum_{i=1}^n \alpha_i z_i \Big\| = \max_{1 \leq i \leq n} |\alpha_i| = \|x\|.$$

Then, U is an isometry.

$$\begin{aligned} VU(x) &= V\Big(\sum_{i=1}^n \alpha_i U e_i\Big) = V\Big(\sum_{i=1}^n \alpha_i z_i\Big) = \sum_{i=1}^n \alpha_i V z_i \\ &= \sum_{i=1}^n \alpha_i \lambda_i z_i = \sum_{i=1}^n \alpha_i T e_i = T\Big(\sum_{i=1}^n \alpha_i e_i\Big) = T(x). \end{aligned}$$

Then, $T = VU$. ∎

ACKNOWLEDGEMENTS

The author especially thanks the referee for his valuable remarks and suggestions in preparing this paper.

References

[1] J. Cabello Sánchez, J. Navarro Garmendia, Isometries of ultrametric normed spaces, *Ann. Funct. Anal.* **12** (2021), 11 pp., DOI: 10.1007/s43034-021-00144-7.

[2] A. Kubzdela, Isometries, Mazur–Ulam theorem and Aleksandrov problem for non-Archimedean normed spaces, *Nonlinear Analysis: Theory, Methods and Applications*, **75**, pp. 2060-2068 (2012) DOI: 10.1016/j.na.2011.10.006.

[3] A.F. Monna, Analyse Non-archimédienne, Berlin-Springer (1970).

[4] C. Perez-García, W.H. Schikkof, Locally Convex Spaces over non-Archimedean valued fields, Cambridge Studies in Advanced Mathematics 119, (2010).

[5] A.C.M. van Rooij, Non-Archimedean Functional analysis, New York, Dekker, (1978).

[6] A.C.M. van Rooij, Notes on p-Adic Banach Spaces, Report 7633, Math. Inst. Kathol. Univ. Nijmegen, (1976).