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A spectral theorem for a non-Archimedean valued field whose residue field is formally real

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Abstract: In this paper, we will prove a spectral theorem for self-adjoint compactoid operators. Also, we will study the condition on which the coefficient field must be imposed. In order to get the theorems, we will use the Fredholm theory for compactoid operators. Moreover, the property of maximal complete field is important for our study. These facts will allow us to find that the spectral theorem depends only on the residue class field, and is independent of the valuation group of the coefficient field. As a result, we can settle the problem of the spectral theorem in the case where the residue class field is formally real.

Key words: self-adjoint operators, spectral theorem.

MSC (2020): Primary 46S10; Secondary 12J25.

1. Introduction and preliminaries

1.1. Introduction The spectral theory on non-Archimedean functional analysis has been studied by many researchers. In this paper, we will prove the spectral theorem of self adjoint compactoid operators in the case where the residue class field is formally real (Theorem 3.5, Theorem 3.6, Corollary 3.7). This claim was proposed in [2, Theorem 4.3], but the proof makes mistakes and the claim must be modified. We will give a correct proof and the exact condition in Section 3.

For the study of the spectral theorem of compactoid operators, the Fredholm theory of compactoid operators (see [10]) will play an important role. Schikhof proved that if the coefficient field is algebraically closed, a compactoid operator is a spectral operator ([10, Definition 6.5]).

In [10, section 6], the coefficient field is assumed to be algebraically closed, but the assumption seems too strong for some results. Therefore, we will modify this theory to remove the assumption that the coefficient field is algebraically closed. In Section 5, we summarize the discussion as an appendix.

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As a result, we can apply the method of operator analysis to the spectral theory if the coefficient field K satisfies the condition $(H)_K$ (see Section 3), which is the condition on the diagonalization of a symmetric matrix.

In Section 4, we will study the condition $(H)_K$. Keller and Ochsenius found that a symmetric matrix over $\mathbb{R}((t))$ can be diagonalized by an orthogonal matrix (see [6]). In this paper, we will extend this fact (Theorem 4.2), and get Corollary 4.4. For the proof, we will use the spherical completion (c.f. [12]), and the property of maximally complete field ([5]).

1.2. Preliminaries In this paper, K is a non-archimedean non-trivially valued field which is complete under the metric induced by the valuation $|\cdot| : K \to [0, \infty)$. A unit ball of K is denoted by $B_K := \{x \in K : |x| \leq 1\}$. We denote by k the residue class field of K.

Throughout, $(E, \|\cdot\|)$ is a non-archimedean Banach space over K. Let $a \in E$, $r > 0$, we write $B_E(a, r)$ for the closed ball with radius r about a, that is, $B_E(a, r) := \{x \in E : \|x - a\| \leq r\}$. For a subset $X \subseteq E$, we denote by $[X]$ the K-vector space generated by X. Let $t \in (0, 1]$. A sequence $(x_n)_{1 \leq n \leq N} \subseteq E \setminus \{0\}$, $N \in \mathbb{N} \cup \{\infty\}$, is said to be a t-orthogonal (orthogonal for $t = 1$) if for each sequence $(\lambda_n)_{1 \leq n \leq N} \subseteq K$, the inequality

$$t \cdot \max_{1 \leq i \leq N} \|\lambda_i x_i\| \leq \left\| \sum\nolimits_{i=1}^{N} \lambda_i x_i \right\|$$

holds.

A subset A of E is said to be a compactoid if for every $r > 0$, there exist finite elements $a_1, \ldots, a_n$ of E such that $A \subseteq B_E(0, r) + B_K a_1 + \cdots + B_K a_n$.

Let $(F, \|\cdot\|)$ be a non-archimedean Banach space, we denote by $\mathcal{L}(E, F)$ the Banach space consisting of all continuous maps from E to F with the usual operator norm. If $(E, \|\cdot\|) = (F, \|\cdot\|)$, we write $\mathcal{L}(E) := \mathcal{L}(E, F)$. An operator $T \in \mathcal{L}(E, F)$ is said to be a compactoid operator if $T(B_E(0, 1))$ is a compactoid. For details of compactoid operators, see [9, 10, 12].

For $T \in \mathcal{L}(E)$, we define a spectrum of T as

$$\sigma(T) := \{\lambda \in K \,:\, \lambda I - T \text{ is not invertible}\},$$

where $I \in \mathcal{L}(E)$ is the identical operator on E, and we write

$$\sigma_p(T) := \{\lambda \in K \,:\, \operatorname{Ker}(\lambda I - T) \neq 0\}$$

for eigenvalues of T. Also, we set

$$U_T := \{\lambda \in K \,:\, I - \lambda T \text{ is invertible}\},$$

and

$$D_T := \{r \in |K| : r \neq 0, B_K(0,r) \subseteq U_T\}.$$

Let $r \in |K|$, $r \neq 0$. A function $f : B_K(0,r) \to E$ is said to be analytic if there exists a sequence $a_0, a_1, a_2, \ldots \in E$ such that $\lim_{n\to\infty} \|a_n\| r^n = 0$, and f can be represented by

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \qquad (\lambda \in B_K(0,r)).$$

2. Non-Archimedean inner product on c_0

Let $(c_0, \|\cdot\|)$ be the Banach space of all null sequences $x = (x_n)_{n\in\mathbb{N}}$ in K, and $\|x\| := \sup_{n\in\mathbb{N}} |x_n|$. There exists a symmetric bilinear form $\langle\cdot,\cdot\rangle$ on c_0 defined by

$$\langle x, y\rangle := \sum_{n\in\mathbb{N}} x_n y_n$$

where $x = (x_n)$, $y = (y_n) \in c_0$.

We denote by $e_1, e_2, \ldots \in c_0$ the canonical unit vectors. Then, an operator $T \in \mathcal{L}(c_0)$ can be written as a pointwise convergent sum

$$T = \sum_{i,j} a_{i,j} \cdot (e'_j \otimes e_i)$$

where $e'_j \otimes e_i(x) := \langle e_j, x\rangle e_i$. Applying this representation, we can characterize compactoid operators.

Theorem 2.1. (c.f. [9, Theorem 8.1.9]) *Let $T = \sum_{i,j} a_{i,j} \cdot (e'_j \otimes e_i) \in \mathcal{L}(c_0)$. Then T is a compactoid operator if and only if $\lim_{i\to\infty} \sup_j |a_{i,j}| = 0$.*

Definition 2.2. We say that $T \in \mathcal{L}(c_0)$ admits an adjoint operator $S \in \mathcal{L}(c_0)$ if for each $x, y \in c_0$, S satisfies

$$\langle T(x), y\rangle = \langle x, S(y)\rangle.$$

If T admits an adjoint operator S, then, since S is uniquely determined by T, we write $T^* := S$.

It is easy to see that $T = \sum_{i,j} a_{i,j} \cdot (e'_j \otimes e_i)$ admits an adjoint operator if and only if for each $i \in \mathbb{N}$, we have $\lim_j a_{i,j} = 0$. If T admits an adjoint operator T^*, then T^* can be represented by

$$T^* = \sum_{i,j} a_{j,i} \cdot (e'_j \otimes e_i),$$

and $\|T^*\| = \|T\|$. From Theorem 2.1, we have the following theorem.

THEOREM 2.3. *Let* $T = \sum_{i,j} a_{i,j} \cdot (e'_j \otimes e_i) \in \mathcal{L}(c_0)$. *Then, T is a compactoid operator which admits an adjoint operator if and only if*

$$\lim_{n\to\infty} \sup_{n<i,j} |a_{i,j}| = 0.$$

In general, the symmetric bilinear form $\langle\cdot,\cdot\rangle$ on c_0 does not satisfy the equality $\|x\|^2 = |\langle x, x\rangle|$. On the other hand, if the residue class field k of K is formally real, $\langle\cdot,\cdot\rangle$ induces the norm $\|\cdot\|$ on c_0 (see [7]).

DEFINITION 2.4. A field F is called formally real if for any finite subset $(a_i)_{1\le i\le n} \subseteq F$, $\sum_{1\le i\le n} a_i^2 = 0$ implies $a_i = 0$ for each i.

THEOREM 2.5. ([7, THEOREM 6.1]) *Suppose the residue class field k of K is formally real. Then, we have $\|x\|^2 = |\langle x, x\rangle|$ for each $x \in c_0$.*

From now on, in this section, we suppose that the residue class field k of K is *formally real.*

DEFINITION 2.6. A subset $X \subseteq c_0$ is called orthonormal if for each distinct pair $x, y \in X$, we have $\langle x, y\rangle = 0$.

THEOREM 2.7. ([7, THEOREM 3.1]) *Suppose that the residue class field k of K is formally real. Then, an orthonormal subset $X \subseteq c_0$ is orthogonal, that is, for any finite distinct elements $x_1, x_2, \ldots, x_n \in X$, the equality*

$$\max_{1\le i\le n} \|\lambda_i x_i\| = \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| \qquad (\lambda_1, \lambda_2, \ldots, \lambda_n \in K)$$

holds.

By the Gram-Schmidt procedure, we have the following theorem.

THEOREM 2.8. ([7, SECTION 7]) *Let $M \subseteq c_0$ be a finite-dimensional subspace. Then, there exists a basis $\{x_1, \ldots, x_n\} \subseteq M$ as a K-vector space such that it is an orthonormal set.*

DEFINITION 2.9. Let $X \subseteq c_0$. We denote by $X^\perp := \{y \in c_0 : \langle x, y\rangle = 0$ for each $x \in X\}$ the normal complement of X. A closed subspace $M \subseteq c_0$ is called normally complemented if $M \oplus M^\perp = c_0$.

Even if k is formally real, there exists a closed subspace $M \subseteq c_0$ which is not normally complemented ([7, Remark 9.1]). On the other hand, if M is finite-dimensional, it is normally complemented.

THEOREM 2.10. ([7, COROLLARY 8.2]) *Let $M \subseteq c_0$ be a finite-dimensional subspace. Then, M is normally complemented.*

We introduce a normal projection to characterize whether a closed subspace is normally complemented.

DEFINITION 2.11. ([1, DEFINITION 6]) An operator $P \in \mathcal{L}(E)$ is called a normal projection if $P^2 = P$ and $P^* = P$.

THEOREM 2.12. ([1, COROLLARY 3]) *Let $M \subseteq c_0$ be a closed subspace. Then, M is normally complemented if and only if there exists a normal projection P onto M.*

THEOREM 2.13. *Let $M \subseteq c_0$ be a finite-dimensional subspace, and let $\{x_1, \ldots, x_n\} \subseteq M$ be an orthonormal basis. Then, the normal projection P onto M can be represented by*

$$P(x) = \sum_{i=1}^{n} \frac{\langle x, x_i \rangle}{\langle x_i, x_i \rangle} x_i.$$

Proof. For each j $(1 \leq j \leq n)$, we have

$$P(x_j) = \sum_{i=1}^{n} \frac{\langle x_j, x_i \rangle}{\langle x_i, x_i \rangle} x_i = x_j.$$

Therefore, we get $P^2 = P$. Moreover, for each $x, y \in c_0$, we have

$$\langle x, P(y) \rangle = \langle x, \sum_{i=1}^{n} \frac{\langle y, x_i \rangle}{\langle x_i, x_i \rangle} x_i \rangle = \sum_{i=1}^{n} \frac{\langle x, x_i \rangle \cdot \langle y, x_i \rangle}{\langle x_i, x_i \rangle} = \langle P(x), y \rangle,$$

which implies $P^* = P$. Thus, the proof is complete. ∎

3. THE SPECTRAL THEOREM

In this section, suppose that the residue class field k of K is *formally real*. We say that an operator $T \in \mathcal{L}(c_0)$ is self-adjoint if T admits an adjoint operator T^*, and $T = T^*$. We can prove the following propositions by the classical way.

PROPOSITION 3.1. *Let $T \in \mathcal{L}(c_0)$ be a self-adjoint operator. Then, $\|T^2\| = \|T\|^2$. In particular, the equality $\lim_{n\to\infty} \|T^n\|^{1/n} = \|T\|$ holds.*

Proof. The inequality $\|T^2\| \leq \|T\|^2$ is clear. On the other hand, we have

$$\|T\|^2 = \sup_{\|x\|\leq 1} \|T(x)\|^2 = \sup_{\|x\|\leq 1} |\langle T(x), T(x)\rangle| = \sup_{\|x\|\leq 1} |\langle T^*T(x), x\rangle|$$
$$\leq \sup_{\|x\|,\|y\|\leq 1} |\langle T^*T(x), y\rangle| = \|T^*T\| = \|T^2\|.$$

Therefore, we get the equality $\|T^2\| = \|T\|^2$. Moreover, we have

$$\lim_{n\to\infty} \|T^n\|^{1/n} = \lim_{n\to\infty} \left\|T^{2^n}\right\|^{1/2^n} = \|T\|,$$

which completes the proof. ∎

Proposition 3.2. *Let $T \in \mathcal{L}(c_0)$ be a self-adjoint operator, and $M \subseteq c_0$ be a closed subspace that is normally complemented. Then, $T(M) \subseteq M$ if and only if $TP = PT$ where P is a normal projection onto M. In particular, $T(M) \subseteq M$ implies $T(M^\perp) \subseteq M^\perp$.*

Proof. Let P be a normal projection onto M. If $TP = PT$, then we have

$$T(x) = TP(x) = PT(x) \in M \qquad \text{for each } x \in M,$$

hence $T(M) \subseteq M$. Conversely, suppose $T(M) \subseteq M$. Then, for each $x \in M^\perp$, $y \in M$, we have $\langle y, T(x)\rangle = \langle T(y), x\rangle = 0$. Since $y \in M$ is arbitrary, we obtain $T(x) \in M^\perp$, hence $T(M^\perp) \subseteq M^\perp$. For each $z \in c_0$, we have the trivial equality

$$PT(z) + (I-P)T(z) = T(z) = TP(z) + T(I-P)(z).$$

Now, from $T(M) \subseteq M$ and $T(M^\perp) \subseteq M^\perp$, it follows that $TP(z) = PT(z)$. ∎

Proposition 3.3. *Let $T \in \mathcal{L}(c_0)$ be a self-adjoint operator, and let λ_1, $\lambda_2 \in \sigma_p(T)$ be distinct elements. Then, for each $x_1 \in \mathrm{Ker}(\lambda_1 I - T)$, $x_2 \in \mathrm{Ker}(\lambda_2 I - T)$, we have $\langle x_1, x_2\rangle = 0$.*

For a formally real field F, we consider the condition $(H)_F$:

$(H)_F$ For each $n \in \mathbb{N}$ and each symmetric matrix $A \in \mathcal{M}_n(F)$, A is diagonalizable over F,

where $\mathcal{M}_n(F)$ is the set of all n-dimensional square matrices over F.

Before proving the main theorems, we recall Fredholm's Alternative for compactoid operators presented in [10].

PROPOSITION 3.4. ([10, COROLLARY 3.3, THEOREM 5.6]) *Let* $T \in \mathcal{L}(E)$ *be a compactoid operator. Then, we have the following:*

(1) *If* $\lambda \in \sigma(T)$, $\lambda \neq 0$ *then* $\lambda \in \sigma_p(T)$ *and* $\mathrm{Ker}(\lambda I - T)$ *is finite-dimensional.*

(2) *If* $\lambda_1, \lambda_2, \ldots \in \sigma(T)$ *are distinct, then* $\lim_{n\to\infty} \lambda_n = 0$.

In addition to the above proposition, we use the results of Section 5 for the proof of the main theorems. For details, see Section 5.

THEOREM 3.5. *Suppose that the residue class field* k *of* K *is formally real, and* K *satisfies the condition* $(H)_K$. *Let* $T \in \mathcal{L}(c_0)$ *be a self-adjoint compactoid operator. Then, we have the following:*

(1) *If* K *is densely valued, then we have*

$$\|T\| = \max\{\, |\lambda| \,:\, \lambda \in \sigma_p(T)\}.$$

(2) *If* K *is discretely valued, then we have*

$$\|T\| \leq |\pi|^{-1} \max\{\, |\lambda| \,:\, \lambda \in \sigma_p(T)\}$$

where $\pi \in B_K$ *is a generating element of a maximal ideal of* B_K.

Proof. We write $T := \sum_{i,j} a_{i,j} \cdot (e'_j \otimes e_i)$, and let $T_n := \sum_{1\leq i,j\leq n} a_{i,j} \cdot (e'_j \otimes e_i)$. Then by Theorem 2.3, we have $\lim_{n\to\infty} \|T_n - T\| = 0$. Moreover, it follows from the condition $(H)_K$ that for each $n \in \mathbb{N}$ and each $r \in D_{T_n}$, the function

$$\lambda \longmapsto (I - \lambda T_n)^{-1}$$

is analytic in $B_K(0, r)$. Therefore, combining these facts with Theorem 5.9, we have that $T_1, T_2, \ldots$ and T satisfy the assumptions of Theorem 5.3. Hence, by Corollary 5.4, we have the following:

(1) If K is densely valued, then $\lim_{n\to\infty} \|T^n\|^{1/n} = \sup_{\lambda\in\sigma(T)} |\lambda|$.

(2) If K is discretely valued, then $\lim_{n\to\infty} \|T^n\|^{1/n} \leq |\pi|^{-1} \sup_{\lambda\in\sigma(T)} |\lambda|$.

Moreover, it follows from Proposition 3.1 that $\lim_{n\to\infty} \|T^n\|^{1/n}$ is equal to $\|T\|$, and by Proposition 3.4, we have $\sup_{\lambda\in\sigma(T)} |\lambda| = \max_{\lambda\in\sigma_p(T)} |\lambda|$. This completes the proof. ∎

THEOREM 3.6. *With the same assumptions as those of Theorem 3.5, there exist an orthonormal sequence $x_1, x_2, \ldots \in c_0$ and $(\lambda_n) \in c_0$ such that*

$$T(x) = \sum_{n=1}^{\infty} \lambda_n \frac{\langle x, x_n \rangle}{\langle x_n, x_n \rangle} x_n.$$

Proof. We may assume $T \neq 0$. Then by Theorem 3.5, we have $\sigma_p(T) \setminus \{0\} \neq \emptyset$. By Proposition 3.4, there exists a decreasing sequence $(r_n)_{1 \le n \le N}$ ($N \in \mathbb{N} \cup \{\infty\}$) of positive numbers such that

$$\{\, |\lambda| \,:\, \lambda \in \sigma_p(T) \setminus \{0\}\} = \{r_n \,:\, 1 \le n \le N\}.$$

Moreover, we have $\lim_{n\to\infty} r_n = 0$ if $N = \infty$.

For each $n \in \mathbb{N}$, we put $\{\lambda_{n1}, \ldots, \lambda_{nm_n}\} = \{\lambda \in \sigma_p(T) \,:\, |\lambda| = r_n\}$ and

$$N_n = \sum_{1 \le l \le n} \sum_{1 \le k \le m_l} \mathrm{Ker}(\lambda_{lk} I - T).$$

Then, we easily have $T(N_n) \subseteq N_n$. We shall prove the theorem in the case $N = \infty$ (If $N < \infty$, the same discussion works). By Proposition 3.4, N_n is finite-dimensional and therefore, it follows from Theorem 2.13 that there exists a normal projection P_n onto N_n.

For each $n \in \mathbb{N}$, by Proposition 3.2 and Proposition 3.3, we have

$$\sigma_p(T) = \sigma_p(TQ_n) \cup \sigma_p(TP_n), \qquad \sigma_p(TQ_n) = \sigma_P(T) \setminus \left(\bigcup_{1 \le l \le n} \bigcup_{1 \le k \le m_l} \{\lambda_{lk}\} \right)$$

where $Q_n := I - P_n$ is a normal projection onto $N_n^{\perp}$. Since PQ_n is a self-adjoint compactoid operator, we have

$$\|TQ_n\| \le C \cdot \max_{\lambda \in \sigma_p(TQ_n)} |\lambda| \le C r_{n+1}$$

by Theorem 3.5 where C is a suitable constant independent of n. In particular, we obtain $\lim_{n\to\infty} \|TQ_n\| = 0$ and therefore, we have $T(x) = \lim_{n\to\infty} TP_n(x)$ for each $x \in c_0$.

Finally, for each $l \in \mathbb{N}$, $1 \le k \le m_l$, let $\{x_{lkj} \,:\, 1 \le j \le p_{lk}\}$ be an orthonormal basis of $\mathrm{Ker}(\lambda_{lk} I - T)$. Then by Theorem 2.13 and Proposition 3.3, $P_n(x)$ can be represented by

$$P_n(x) = \sum_{1 \le l \le n} \sum_{1 \le k \le m_l} \sum_{1 \le j \le p_{lk}} \frac{\langle x, x_{lkj} \rangle}{\langle x_{lkj}, x_{lkj} \rangle} x_{lkj}.$$

Hence, we have

$$T(x) = \sum_{l=1}^{\infty} \sum_{1 \le k \le m_l} \sum_{1 \le j \le p_{lk}} \lambda_{lk} \frac{\langle x, x_{lkj} \rangle}{\langle x_{lkj}, x_{lkj} \rangle} x_{lkj},$$

which completes the proof. ∎

Theorem 3.6 implies the following corollary which refines Theorem 3.5.

COROLLARY 3.7. *With the same assumptions as those of Theorem* 3.5, *if K is discretely valued, then we have*

$$\|T\| = \max_{\lambda \in \sigma_p(T)} |\lambda|.$$

Remark 3.8. Theorem 3.6 is the modified result of [2, Theorem 4.3]. The condition $(H)_K$ is necessary for Theorem 3.6. Indeed, if K does not satisfy the condition $(H)_K$, there exist $n \in \mathbb{N}$ and a symmetric matrix $A = (a_{i,j})_{i,j} \in \mathcal{M}_n(K)$ such that A is not diagonalizable over K. Let us define

$$T = \sum\nolimits_{i,j} b_{i,j} \cdot (e'_j \otimes e_i) \in \mathcal{L}(c_0)$$

by

$$b_{i,j} := \begin{cases} a_{i,j} & \text{if } 1 \le i, j \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then, T is a self-adjoint compactoid operator but does not satisfy the conclusion in Theorem 3.6.

Despite the counterexample above, the proof of [2, Theorem 4.3] is independent of the condition $(H)_K$. Hence, the proof makes a mistake. Specifically, in the fifth step of the proof, it makes a fatal mistake. Similarly, the proof of [3, Theorem 10] is wrong. On the other hand, we can apply a similar method to that of this paper to [3, Theorem 10].

4. THE CONDITION $(H)_K$

In this section, we study the condition $(H)_F$. Let F be a formally real field (see Definition 2.6). A matrix $U \in \mathcal{M}_n(F)$ is called an orthogonal matrix if its transpose U^* is equal to the inverse U^{-1}. In addition to the condition $(H)_F$, we consider the condition $(H')_F$:

$(H')_F$ For each $n \in \mathbb{N}$ and each symmetric matrix $A \in \mathcal{M}_n(F)$, A can be diagonalized by an orthogonal matrix over F.

PROPOSITION 4.1. *Let F be a formally real field. Then, F satisfies the condition $(H)_F$ if and only if F satisfies the condition $(H')_F$.*

Proof. Suppose that F satisfies the condition $(H)_F$. Then, for each $a, b \in F$, a symmetric matrix

$$\begin{pmatrix} 0 & \frac{b}{4} \\ \frac{b}{4} & a \end{pmatrix}$$

is diagonalizable over F. Hence, we have $\sqrt{a^2+b^2} \in F$, and by induction, for any finite subset $\{a_1, \ldots, a_n\} \subseteq F$, we have $\sqrt{a_1^2 + \cdots + a_n^2} \in F$. Let $A \in \mathcal{M}_n(F)$ be a symmetric matrix. Then by the hypothesis, there exists a subset $\{x_1, \ldots, x_n\} \subseteq F^n$ whose linear span is equal to F^n such that each x_i is an eigenvector of A. Since A is symmetric, using the Gram-Schmidt procedure, we can choose $x_1, \ldots, x_n$ satisfying that a matrix $U := (x_1, \ldots, x_n)$ is an orthogonal matrix. ∎

Let F be a formally real field, and let $(\Gamma, \leq)$ be a totally ordered abelian group. A subset $\{c_{\alpha.\beta}\}_{(\alpha,\beta)\in\Gamma\times\Gamma} \subseteq F^*$ indexed by $\Gamma \times \Gamma$ is called a factor set if it satisfies

- $c_{0,0} = c_{0,\gamma} = c_{\gamma,0} = 1$,
- $c_{\alpha,\beta} = c_{\beta,\alpha}$,
- $c_{\alpha,\beta}c_{\alpha+\beta,\gamma} = c_{\alpha,\beta+\gamma}c_{\beta,\gamma}$

for each $\alpha, \beta, \gamma \in \Gamma$. We denote by $F((\Gamma, c_{\alpha,\beta}))$ the Hahn-field defined by a factor set $\{c_{\alpha,\beta}\}$:

$$F((\Gamma, c_{\alpha,\beta})) := \{f : \Gamma \to F \,:\, \operatorname{supp} f \text{ is a well-ordered set}\},$$
$$f \cdot g(\gamma) := \sum_{\alpha+\beta=\gamma} f(\alpha)g(\beta)c_{\alpha,\beta} \qquad \big(f, g \in F((\Gamma, c_{\alpha,\beta}))\big),$$

where $\operatorname{supp} f := \{\gamma \in \Gamma \,:\, f(\gamma) \neq 0\}$.

The Hahn-field $F((\Gamma, c_{\alpha,\beta}))$ is maximally complete with respect to a general valuation $V(f) := \min \operatorname{supp} f$, $f \in F((\Gamma, c_{\alpha,\beta}))$ (c.f. [11]). The next theorem is an extension of [6, Theorem 1].

THEOREM 4.2. *Put $L := F((\Gamma, c_{\alpha,\beta}))$, and suppose that F satisfies the condition $(H')_F$. Then, L satisfies the condition $(H')_L$.*

Proof. We write $f = \sum_\gamma f(\gamma)t^\gamma$ for an element $f \in L$. Let $n \geq 2$, and let $\mathcal{A} \in \mathcal{M}_n(L)$ be a symmetric matrix. Then, $\mathcal{A}$ can be represented by

$$\mathcal{A} = \sum\nolimits_{\gamma \in S} A_\gamma \, t^\gamma$$

where $S \subseteq \Gamma$ is a well-ordered set, and $A_\gamma \in \mathcal{M}_n(F)$ is a symmetric matrix for each $\gamma \in S$. To prove the theorem, we may assume that the expansion of $\mathcal{A}$ is started from 0,

$$\mathcal{A} = A_0 + \cdots, \qquad S \subseteq \{\gamma \in \Gamma \,:\, \gamma \geq 0\},$$

and A_0 is diagonal matrix, but not a multiple of the unit matrix I. Moreover, after conjugating by some permutation matrix, we may assume that there exists an r, $1 \leq r < n$, such that A_0 is of the form

$$\begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix}$$

where

$$a_{ii} = a_{11} \;\text{ for } 1 \leq i \leq r \qquad \text{and} \qquad a_{ii} \neq a_{11} \;\text{ for }\; r+1 \leq i \leq n.$$

We shall prove that there exists an orthogonal matrix $\mathcal{U} \in \mathcal{M}_n(L)$ such that $\mathcal{U}^* \mathcal{A} \mathcal{U}$ is of the form

$$\begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix}$$

where $\mathcal{A}_1 \in \mathcal{M}_r(L)$, $\mathcal{A}_2 \in \mathcal{M}_{n-r}(L)$, then by an induction on size n, we complete the proof. In general, we call an n-square matrix $(r, n-r)$-blockdiagonal if it has the shape

$$\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

where B is an r-square matrix and C is an $(n-r)$-square matrix.

Let $T := \{\gamma_1 + \cdots + \gamma_n : n \in \mathbb{N}, \gamma_1, \ldots \gamma_n \in S\}$ be the semigroup generated by S. Then by [8, Theorem 3.4], T is a well-ordered set. By the transfinite construction, we will construct a sequence $U_0, \ldots, U_\gamma, \ldots \in \mathcal{M}_n$ indexed by $\gamma \in T$ such that

(1) $U_0^* U_0 = I$,

(2) $\sum_{\substack{\alpha+\beta=\gamma\\ \alpha,\beta\in T}} U_\alpha^* U_\beta c_{\alpha,\beta} = 0$ for each $\gamma \in T$, $\gamma > 0$,

(3) $V_\gamma := \sum_{\substack{\alpha+\beta+\eta=\gamma\\ \alpha,\beta,\eta\in T}} U_\alpha^* A_\beta U_\eta c_{\alpha,\beta,\eta}$ is $(r, n-r)$-blockdiagonal for each $\gamma \in T$,

where $c_{\alpha,\beta,\eta} := c_{\alpha,\beta} c_{\alpha+\beta,\eta} = c_{\alpha,\beta+\eta} c_{\beta,\eta}$, hence $c_{\alpha,\beta,\eta} = c_{\eta,\beta,\alpha}$. Then, $\mathcal{U} := \sum_\gamma U_\gamma t^\gamma$ is the desired orthogonal matrix.

For $\gamma = 0$, we put $U_0 = I$. Let $\delta \in T$, and suppose we have determined $U_0, \ldots, U_\gamma, \ldots$, $\gamma < \delta$, satisfying (1) – (3). Consider the condition (2) with $\gamma = \delta$. Since $U_0 = I$, we can rewrite this condition as

$$U_\delta^* + U_\delta + \sum_{\substack{\alpha+\beta=\delta\\ \alpha,\beta\neq\delta}} U_\alpha^* U_\beta c_{\alpha,\beta} = 0.$$

Put

$$S_\delta := \sum_{\substack{\alpha+\beta=\delta\\ \alpha,\beta\neq\delta}} U_\alpha^* U_\beta c_{\alpha,\beta},$$

then it follows from $c_{\alpha,\beta} = c_{\beta,\alpha}$ that S_δ is a symmetric matrix. Hence (2) holds if and only if U_δ is of the form

$$U_\delta = -\frac{1}{2} S_\delta + Q_\delta$$

where Q_δ is any antisymmetric matrix. Therefore, the task is to choose an antisymmetric matrix Q_δ such that $U_\delta = -(1/2)S_\delta + Q_\delta$ satisfies (3) with $\gamma = \delta$.

Now, we can rewrite V_δ as

$$\begin{aligned} V_\delta &= U_\delta^* A_0 + A_0 U_\delta + \sum_{\substack{\alpha+\beta+\eta=\gamma\\ \alpha,\eta\neq 0}} U_\alpha^* A_\beta U_\eta c_{\alpha,\beta,\eta} \\ &= -Q_\delta A_0 + A_0 Q_\delta + T_\delta \end{aligned}$$

where

$$T_\delta := -\frac{1}{2}(S_\delta A_0 + A_0 S_\delta) + \sum_{\substack{\alpha+\beta+\eta=\gamma\\ \alpha,\eta\neq 0}} U_\alpha^* A_\beta U_\eta c_{\alpha,\beta,\eta}.$$

Since S_δ and all the A_γ's are symmetric, combining $c_{\alpha,\beta,\eta} = c_{\eta,\beta,\alpha}$, it follows that T_δ is symmetric. Notice that T_δ is expressed in terms of matrices already determined.

Write

$$V_\delta = \begin{pmatrix} v_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & v_{nn} \end{pmatrix}, \quad Q_\delta = \begin{pmatrix} q_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q_{nn} \end{pmatrix}, \quad T_\delta = \begin{pmatrix} t_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_{nn} \end{pmatrix}.$$

Then, we have

$$v_{ij} = -q_{ij}a_{jj} + a_{ii}q_{ij} + t_{ij} = -q_{ij}(a_{jj} - a_{ii}) + t_{ij}$$

for all $1 \leq i, j \leq n$. If either $1 \leq i \leq r < j \leq n$ or $1 \leq j \leq r < i \leq n$, then by choosing a_{ii}, we have $a_{ii} \neq a_{jj}$. Finally, we put

$$q_{ij} := \begin{cases} \frac{t_{ij}}{a_{jj} - a_{ii}} & (1 \leq i \leq r < j \leq n \ \text{ or } \ 1 \leq j \leq r < i \leq n), \\ 0 & \text{otherwise.} \end{cases}$$

Then, we can check that Q_δ is antisymmetric and V_δ is $(r, n-r)$-blockdiagonal. This completes the proof. ∎

By using the above theorem, we can characterize the condition for which K satisfies the condition $(H)_K$.

THEOREM 4.3. *Suppose that the residue class field k of K is formally real. Then, K satisfies the condition $(H')_K$ if and only if k satisfies the condition $(H')_k$.*

Proof. The sufficiency is easy to prove by the reduction to the residue class field. Conversely, suppose that k satisfies the condition $(H')_k$. Let L be an immediate extension of K which is maximally complete (c.f. [12, Theorem 4.49]). Then by the well-known result (c.f. [4, Chapter 3, Corollary to Theorem 10]), K is algebraically closed in L. Therefore, if L satisfies the condition $(H)_L$, then K satisfies the condition $(H)_K$, hence the condition $(H')_K$ by Proposition 4.1. On the other hand, by [5, Theorem 6], L is analytically isomorphic to the Hahn-field $k((G, c_{\alpha,\beta}))$ where G is the valuation group of K and $\{c_{\alpha,\beta}\} \subseteq k^*$ is a factor set. Hence, by Theorem 4.2, L satisfies the condition $(H')_L$, which completes the proof. ∎

By Proposition 4.1, we have the next corollary. Surprisingly, the condition $(H)_K$ is independent of the valuation group of K.

COROLLARY 4.4. *Suppose that the residue class field k of K is formally real. Then, K satisfies the condition $(H)_K$ if and only if k satisfies the condition $(H)_k$.*

Combining Theorem 3.6 with Corollary 4.4, we can say that the spectral theorem of self-adjoint compactoid operators holds if the residue class field k satisfies the condition $(H)_k$. The condition is independent of the valuation group of K.

THEOREM 4.5. *Suppose that the residue class field k of K is formally real, and satisfies the condition $(H)_k$. Let $T \in \mathcal{L}(c_0)$ be a self-adjoint compactoid operator. Then, there exist an orthonormal sequence $x_1, x_2, \ldots \in c_0$ and $(\lambda_n) \in c_0$ such that $\langle x_n, x_n \rangle = 1$ for each $n \in \mathbb{N}$, and*

$$T(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, x_n \rangle x_n.$$

Proof. From Theorem 3.6 and Corollary 4.4, it suffices to prove that for each $x \in c_0$, $x \neq 0$, we have $\sqrt{\langle x, x \rangle} \in K$. By the proof of Proposition 4.1, we have $\sqrt{a_1^2 + \ldots a_n^2} \in k$ for each finite subset $\{a_1, \ldots, a_n\} \subseteq k$. Therefore, applying Hensel's lemma, we have the claim. ∎

5. APPENDIX

In this appendix, we summarize the results of [10, Section 6]. In [10, Section 6], the coefficient field K is assumed to be algebraically closed. On the other hand, in this appendix, we give no condition on K. Hence, it can be perhaps discretely valued.

PROPOSITION 5.1. ([10, PROPOSITION 6.2]) *Suppose K is densely valued or the residue class field k of K is an infinite field. Let $r \in |K|$, $r \neq 0$, and $f : B_K(0, r) \to E$, $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ be an analytic function. Then we have*

$$\sup_{\lambda \in B_K(0,r)} \|f(\lambda)\| = \max_n \|a_n\| r^n.$$

Proof. In the case $E = K$, the conclusion of Proposition 5.1 is well-known. Hence, the same proof as that of [10, Proposition 6.2] works. ∎

Corollary 5.2. ([10, Corollary 6.3]) *With the same assumptions as those of Proposition 5.1, the set of analytic functions* $B_K(0,r) \to E$ *is uniformly closed.*

Theorem 5.3. ([10, Lemma 6.9]) *Let* $T_1, T_2, \ldots \in \mathcal{L}(E)$, *and let* $T = \lim_{n\to\infty} T_n$ *in the sense of the operator norm. Suppose that*

(1) *for each* $n \in \mathbb{N}$ *and each* $r \in D_{T_n}$, $(I - \lambda T_n)^{-1}$ *is analytic in* $B_K(0,r)$,
(2) *for each* $r \in D_T$, $M_r := \sup_{|\lambda|\le r} \|(I-\lambda T)^{-1}\| < \infty$, *and*
(3) K *is densely valued or the residue class field* k *of* K *is an infinite field.*

Then, $(I - \lambda T)^{-1}$ *is analytic in* $B_K(0,r)$ *for each* $r \in D_T$.

Proof. We can apply the same proof as that of [10, Lemma 6.9]. ∎

Corollary 5.4. *With the same assumptions as those of Theorem 5.3, we have the following:*

(1) *If* K *is densely valued, then we have*

$$\lim_{n\to\infty} \|T^n\|^{1/n} = \sup\{\, |\lambda| \,:\, \lambda \in \sigma(T)\}.$$

(2) *If* K *is discretely valued and the residue class field* k *of* K *is an infinite field, then we have*

$$\lim_{n\to\infty} \|T^n\|^{1/n} \le |\pi|^{-1} \sup\{\, |\lambda| \,:\, \lambda \in \sigma(T)\}$$

where $\pi \in B_K$ *is a generating element of a maximal ideal of* B_K.

Proof. For a sufficiently small $r > 0$, $(I - \lambda T)^{-1}$ is of the form $\sum_n (\lambda T)^n$ in $B_K(0,r)$. Therefore, by Proposition 5.1 and Theorem 5.3, we have $(I - \lambda T)^{-1} = \sum_n (\lambda T)^n$ in $B_K(0,r)$ for each $r \in D_T$. Hence, we derive (1), (2). ∎

For $x_1, \ldots, x_n \in E$, we define the volume function of $x_1, \ldots, x_n \in E$ by

$$\mathrm{Vol}(x_1, \ldots, x_n) := \prod_{i=i}^{n} \mathrm{dist}\,\bigl(x_i, [x_j \,:\, j < i]\bigr).$$

These properties can be found in [13, Chapter 1].

From now on, when K is discretely valued, we assume that a Banach space $(E, \|\cdot\|)$ satisfies $\|E\| \subseteq |K|$.

Definition 5.5. ([10, Definition 6.10]) Let E be infinite-dimensional, let $T \in \mathcal{L}(E)$. For $n \in \mathbb{N}$, we set

$$\Delta_n(T) := \sup \left\{ \frac{\mathrm{Vol}(T(x_1), \dots, T(x_n))}{\mathrm{Vol}(x_1, \dots, x_n)} \ : \ x_1, \dots, x_n \ \text{linearly independent} \right\},$$
$$\Delta_-(T) := \liminf_{n\to\infty} (\Delta_n(T))^{1/n},$$
$$\Delta_+(T) := \limsup_{n\to\infty} (\Delta_n(T))^{1/n}.$$

By [13, Corollary 1.5], if $[x_1, \dots, x_n] = [y_1, \dots, y_n]$, then we have

$$\mathrm{Vol}(x_1, \dots, x_n) = \mathrm{Vol}(y_1, \dots, y_n).$$

Thus, we obtain

$$\Delta_n(T) = \sup \big\{ \mathrm{Vol}(T(x_1), \dots, T(x_n)) \ : \ \|x_i\| \le 1 \text{ for each } i \big\}.$$

Proposition 5.6. ([10, Proposition 6.11]) *Let $T \in \mathcal{L}(E)$ be a compactoid operator. Then, we have $\Delta_+(T) = 0$.*

Proof. See the proof of [10, Proposition 6.11]. ∎

Lemma 5.7. ([10, Lemma 6.13]) *Let $x_1, \dots, x_n \in E$, $\|x_i\| \le 1$ for each i and $0 < \varepsilon < \mathrm{Vol}(x_1, \dots, x_n)$. If $y_1, \dots, y_n \in E$, $\|y_i - x_i\| < \varepsilon$ for each i, then we have $\mathrm{Vol}(x_1, \dots, x_n) = \mathrm{Vol}(y_1, \dots, y_n)$.*

Proof. See the proof of [10, Lemma 6.13]. ∎

The next proposition is proved in [10], but the proof makes a little mistake. We shall give a modified proof.

Proposition 5.8. ([10, Proposition 6.12]) *Let $T \in \mathcal{L}(E)$ be such that*

$$M_s = \sup_{|\lambda| \le s} \|(I - \lambda T)^{-1}\| = \infty$$

for some $s \in D_T$, then we have $\Delta_-(T) > 0$.

Proof. By assumption, there exists a sequence $\lambda_1, \lambda_2, \dots \in B_K(0, s)$ satisfying that $\|(I - \lambda_n T)^{-1}\|$ tends to ∞. Thus, there exists a sequence $y_1, y_2, \dots \in E$ tending to 0 such that for

$$x_n := (I - \lambda_n T)^{-1} y_n,$$

we have $\inf_n \|x_n\| > 0$ and $\sup_n \|x_n\| < \infty$. It follows from the same reason of part I of the proof of [10, Proposition 6.12] that $\lambda_1, \lambda_2, \ldots$ does not have a convergent subsequence (part I of the proof of [10, Proposition 6.12,] is correct). Thus, by taking a suitable subsequence, we may assume $\inf_{n\neq m} |\lambda_n - \lambda_m| \neq 0$ and $\inf_n |\lambda_n| > 0$. By replacing a norm $\|\cdot\|$ with a suitable norm equivalent to $\|\cdot\|$, we may assume that $\|x_n\| = 1$ for each $n \in \mathbb{N}$. Also, it is easy to see that we may assume $\|T\| < 1$.

Put $\mu_n := \lambda_n^{-1}$ for each n, then we have

- $\|T\| < 1$,
- $|\lambda_n| \leq s$, $|\mu_n| \leq C$ for each n,
- $0 < \rho < \inf_{n\neq m} |\mu_n - \mu_m|$,
- $\|x_n\| = 1$ for each n,
- $\lim_{n\to\infty}(x_n - \lambda_n T(x_n)) = 0$, $\lim_{n\to\infty}(\mu_n x_n - T(x_n)) = 0$,

where $\rho < 1$ and $C > 1$ are suitable constants. We claim that for each $n \in \mathbb{N}$, there exists a positive number $r(n) \leq 1$ such that

$$r(n) \leq \mathrm{Vol}(x_{k+1}, x_{k+2}, \ldots, x_{k+n})$$

for all but finitely many $k \in \mathbb{N}$. We prove the claim by the induction on n. For $n = 1$, we can take $r(1) = 1$. Suppose that $r(1), r(2), \ldots, r(n-1)$ have been determined. Then, there exists a natural number $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$, we have

$$r(l) \leq \mathrm{Vol}(x_{k+1}, x_{k+2}, \ldots, x_{k+l}) \qquad (1 \leq l \leq n-1),$$

and

$$\|\mu_k x_k - T(x_k)\| < \varepsilon$$

where $0 < \varepsilon < \rho \cdot t^2$, $t := \prod_{i=1}^{n-1} r(i)$. In particular, for each $k \geq k_0$, $x_{k+1}, \ldots, x_{k+n-1}$ is t-orthogonal. Put $r(n) := C^{-1}\rho \cdot r(n-1) \cdot t \leq 1$, and we shall prove that $r(n)$ is the desired constant. In fact, by the induction hypothesis, we have

$$\begin{aligned}
&\mathrm{Vol}(x_{m+1}, x_{m+2}, \ldots, x_{m+n}) \\
&\quad = \mathrm{dist}\big(x_{m+n}, [x_{m+1}, \ldots, x_{m+n-1}]\big) \cdot \mathrm{Vol}(x_{m+1}, x_{m+2}, \ldots, x_{m+n-1}) \\
&\quad \geq r(n-1) \cdot \mathrm{dist}\big(x_{m+n}, [x_{m+1}, \ldots, x_{m+n-1}]\big)
\end{aligned}$$

for each $m \geq k_0$. Thus, we have to show that for each choice of $\xi_1, \dots, \xi_{n-1} \in K$,

$$y := x_{m+n} - (\xi_1 x_{m+1} + \cdots + \xi_{n-1} x_{m+n-1})$$

has norm $\geq C^{-1}\rho t$. Since $C^{-1}\rho t \leq 1$, we may assume $1 = \|x_{m+n}\| = \|\xi_1 x_{m+1} + \cdots + \xi_{n-1} x_{m+n-1}\|$. Then, we have

$$\begin{aligned}
&\Big\| \sum\nolimits_{i=1}^{n-1} \xi_i (\mu_{m+n} - \mu_{m+i}) x_{m+i} \Big\| \\
&\quad = \Big\| \mu_{m+n} \cdot \Big(\sum\nolimits_{i=1}^{n-1} \xi_i x_{m+i} - x_{m+n} \Big) + \big(\mu_{m+n} x_{m+n} - T(x_{m+n})\big) \\
&\qquad + T\Big(x_{m+n} - \sum\nolimits_{i=1}^{n-1} \xi_i x_{m+i} \Big) + \sum\nolimits_{i=1}^{n-1} \xi_i \cdot \big(T(x_{m+i}) - \mu_{m+i} x_{m+i} \big) \Big\| \\
&\quad \leq C\|y\| \vee \varepsilon \vee \|y\| \vee \Big(\varepsilon \cdot \max_{1 \leq i \leq n-1} |\xi_i| \Big) \\
&\quad = C\|y\| \vee \varepsilon \vee \Big(\varepsilon \cdot \max_{1 \leq i \leq n-1} |\xi_i| \Big).
\end{aligned}$$

On the other hand, by t-orthogonality of $x_{m+1}, \dots, x_{m+n-1}$, we obtain

$$1 = \|\xi_1 x_{m+1} + \cdots + \xi_{n-1} x_{m+n-1}\| \geq t \cdot \max_{1 \leq i \leq n-1} |\xi_i|$$

and

$$\begin{aligned}
\Big\| \sum\nolimits_{i=1}^{n-1} \xi_i (\mu_{m+n} - \mu_{m+i}) x_{m+i} \Big\| &\geq t \cdot \max_{1 \leq i \leq n-1} |\xi_i| \cdot |\mu_{m+n} - \mu_{m+i}| \\
&\geq t\rho \cdot \max_{1 \leq i \leq n-1} |\xi_i| \\
&\geq t\rho \cdot \|\xi_1 x_{m+1} + \cdots + \xi_{n-1} x_{m+n-1}\| = t\rho.
\end{aligned}$$

Consequently, we have

$$t\rho \leq C\|y\| \vee \varepsilon t^{-1}.$$

By our choice $\varepsilon < \rho t^2$, we must have

$$C^{-1}\rho t \leq \|y\|,$$

which proves the claim.

Finally, we prove $\Delta_-(T) > 0$. Let $n \in \mathbb{N}$. Choose a positive number ε' with $0 < \varepsilon' < r(n)$, and choose a natural number $k_0 \in \mathbb{N}$ such that

$\mathrm{Vol}(x_{k+1}, \dots, x_{k+n}) \geq r(n)$ and $\|x_k - \lambda_k T(x_k)\| < \varepsilon'$ for all $k \geq k_0$. By Lemma 5.7, we have

$$\begin{aligned}\mathrm{Vol}(x_{k_0+1}, \dots, x_{k_0+n}) &= \mathrm{Vol}\left(\lambda_{k_0+1} T(x_{k_0+1}), \dots, \lambda_{k_0+n} T(x_{k_0+n})\right) \\ &= |\lambda_{k_0+1} \dots \lambda_{k_0+n}| \; \mathrm{Vol}\left(T(x_{k_0+1}), \dots, T(x_{k_0+n})\right) \\ &\leq |\lambda_{k_0+1} \dots \lambda_{k_0+n}| \, \Delta_n(T) \; \mathrm{Vol}(x_{k_0+1}, \dots, x_{k_0+n}).\end{aligned}$$

Therefore, we obtain

$$\Delta_n(T) \geq |\lambda_{k_0+1} \dots \lambda_{k_0+n}|^{-1} \geq s^{-n}.$$

As a consequence, we have the desired inequality $\Delta_-(T) \geq s^{-1} > 0$. ∎

Combining Proposition 5.6 and Proposition 5.8, we obtain the following theorem.

THEOREM 5.9. *Let $T \in \mathcal{L}(E)$ be a compactoid operator. Then for each $r \in D_T$, we have $M_r = \sup_{|\lambda| \leq r} \|(I - \lambda T)^{-1}\| < \infty$.*

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