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c-Continuous polynomials on ℓ_1

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Abstract: In this article we study the n-homogeneous polynomials P that are c-continuous on bounded subsets of ℓ_1. We show that P can be decomposed in the form $R+Q$, where Q and R are n-homogeneous polynomials, with R weakly star continuous and $Q(x)=0$ for all $x \in \ker u$ for $u=(1,1,\dots,1,\dots)$.We conclude that $P=\sum_{j=0}^{n} u^{n-j} \otimes R_j$, where R_j is a weakly star continuous j-homogeneous polynomial for $j=0,1,\dots,n$.

Key words: Polynomials, Banach, holomorphic, weak.

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1. Introduction

Let E and F be Banach spaces and Φ be an arbitrary subset of E'. A function $f : E \to F$ is said to be Φ-continuous on bounded subsets of E, if for each bounded set $\Omega \subset E$, $a \in \Omega$ and $\varepsilon > 0$, there are $\phi_1,\dots,\phi_p$ in Φ and $\delta > 0$, such that if $x \in \Omega$, $|\phi_j(x-a)| < \delta$, for $j = 1,2,\dots,p$, then $\|f(x) - f(a)\| < \varepsilon$. In a similar way we define uniform Φ-continuity on bounded subsets of E.

In [1] is showed that in every Banach space E, every m-homogeneus polynomial $P : E \to F$ which is weakly continuous on bounded sets of E is weakly uniformly continuous on bounded sets. The corresponding problem for holomorphic functions is still open.

Problem 1. If $f : E \to \mathbb{C}$ is a holomorphic function which is weakly continuous on bounded sets, is f weakly uniformly continuous?

This problem was raised in 1982 by Aron et al. in [1] and cited in many works, such as [1, 2, 3, 5, 8]. It is obvious that the problem has an affirmative

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answer if E is reflexive. However, Dineen in [6] showed that this problem has an affirmative answer if $E = c_0$ and more generally in [4], it is shown that this problem also has an affirmative answer in every Banach space space with the U property and without a copy of ℓ_1. In particular, this is true for every Banach space that is an M-ideal in its bidual, such as Banach spaces with a shrinking and unconditional Schauder basis.

The Problem 1 is also so-called "*the* ℓ_1*-problem*", since Aron et al., showed in [1, Example 3.5], that if Problem 1 has an affirmative answer for the space ℓ_1, then it has an affirmative answer for all Banach spaces E.

Every entire function $f : \ell_1 \to \mathbb{C}$, which is c_0-continuous on bounded sets of ℓ_1, is c_0-uniformly continuous on bounded sets, since every bounded set is relatively $\sigma(\ell_1, c_0)$-compact. However, it changes if we consider the space c of the convergent sequences and the topology $\sigma(\ell_1, c)$ in ℓ_1, since the bounded subsets of ℓ_1 are not relatively $\sigma(\ell_1, c)$-compact. In fact, the sequence of vectors (e_n) of the canonical basis of ℓ_1 does not converge in this topology. Thus we raise the next problem apparently weaker than ℓ_1-problem.

Problem 2. Is every c-continuous holomorphic function on bounded subsets of ℓ_1, c-uniformly continuous?

This paper is motivated by the question mentioned above. We focus our attention on polynomials and entire functions on ℓ_1 that are c-continuous on bounded sets.

2. Notations

If E is a complex Banach space, $B(E)$ and E' will denote the closed unit ball and the topological dual of E, respectively. For each positive integer m, $\mathcal{L}(^m E)$ is the space of continuous m-linear mappings from $E \times \cdots \times E$ to $\mathbb{C}$ and $\mathcal{P}(^m E)$ is the space of continuous m-homogeneous polynomials from E to $\mathbb{C}$. For each polynomial $P \in \mathcal{P}(^m E)$, there exists a unique symmetric mapping $\overset{\vee}{P} \in \mathcal{L}(^m E)$ such that $P(x) = \overset{\vee}{P}(x, \ldots, x) = \overset{\vee}{P}(x^m)$. When $m = 1$, we have that $\mathcal{L}\left(^1 E\right) = \mathcal{P}\left(^1 E\right) = E'$ and for $m = 0$, $\mathcal{P}\left(^0 E\right)$ and $\mathcal{L}\left(^0 E\right)$ are associated to $\mathbb{C}$.

The space $\mathcal{L}(^m E)$ is a Banach space, under the norm

$$A \in \mathcal{L}(^m E) \longmapsto \|A\| = \sup\{|A(x_1, x_2, \ldots, x_m)| : x_j \in E,\ \|x_j\| \leq 1\},$$

and therefore for every $x, y \in E$ and every integer positive j, with $0 \leq j \leq m$, we have that

$$\left|A\left(x^{m-j},y^{j}\right)\right|\leq\|A\|\left\|x^{m-j}\right\|\left\|y^{j}\right\|.$$

Also, $\mathcal{P}\left({}^{m}E\right)$ is a Banach space with respect to the norm

$$\|P\|=\sup_{x\in B(E)}|P\left(x\right)|$$

and we have that

$$\|P\|\leq\left\|\overset{\vee}{P}\right\|\leq\frac{m^{m}}{m!}\|P\|.$$

We refer to [9] or [5] for the general theory of polynomials and holomorphic mappings on Banach spaces.

Let $\Phi\subset E'$ be an arbitrary family. We say that a bounded sequence $(x_n)\subset E$, is Φ-Cauchy if for all $\phi\in\Phi$, the numerical sequence $\phi\left(x_n\right)$ converges. We say that $(x_n)\subset E$, is Φ-convergent if there exists $x\in E$ such that $\lim_n\phi\left(x_n\right)=\phi\left(x\right)$, for every $\phi\in\Phi$. In this case we write $\Phi-\lim_n x_n=x$. For example, in the space ℓ_1 space, the sequence of canonical basis vectors (e_n) is c-Cauchy, but (e_n) is not c-convergent. We denote by $\mathcal{P}_\Phi\left({}^{m}E\right)$ the space of all Φ-sequentially continuous polynomials on bounded subsets of E. $\mathcal{P}_\Phi\left({}^{m}E\right)$ is a norm-closed subspace of $\mathcal{P}\left({}^{m}E\right)$.

The following result is an immediate consequence of [1, Lemma 2.4, Lemma 2.6, Proposition 2.8].

THEOREM 1. *Let E be a complex Banach space and Φ be any separable subspace of E'.*

(i) *If $P\in\mathcal{P}_\Phi({}^{m}E)$, then for every bounded Φ-Cauchy sequence (x_n), the sequence of $(m-1)$-homogeneous polynomials $T_n\left(x\right)=\overset{\vee}{P}\left(x_n,x^{m-1}\right)$ converges in norm. In particular, if (x_n) is Φ-convergent to 0 then (T_n) converges in norm to the null polynomial.*

(ii) *If $P\in\mathcal{P}_\Phi\left({}^{m}E\right)$ then the m-linear mapping $\overset{\vee}{P}:E\times\cdots\times E\to\mathbb{C}$ is Φ-continuous. Besides, for each $a\in E$ and every integer j with $0\leq j\leq m$, the mapping $T_j\left(x\right)=\overset{\vee}{P}\left(a^{j},x^{m-j}\right)$ is Φ-continuous on bounded subsets of E.*

3. c-CONTINUOUS POLYNOMIALS

The canonical basis (e_j) of ℓ_1 is c-Cauchy and therefore by Theorem 1, given a polynomial $P\in\mathcal{P}_c\left({}^{m}\ell_1\right)$ the sequence of polynomials $T_k\left(x\right)=$

$\overset{\vee}{P}\left(e_k, x^{m-1}\right)$ converges in the norm. If $P \in \mathcal{P}_{c_0}\left({}^m\ell_1\right)$, then T_k converges to 0 in norm, since $c_0 - \lim_k e_k = 0$.

If $\phi \in \mathcal{P}\left({}^1\ell_1\right) = \ell_\infty$ is c-continuous on bounded subsets of ℓ_1 then $\phi \in c$. In fact, suppose that $\phi = (\phi_1, \phi_2, \dots)$. Since the sequence (e_k) is c-Cauchy, then by Theorem 1, the sequence $(\phi_k) = (\phi(e_k))$ converges, that is $(\phi_k) \in c$. In the same way, we show that if $\phi \in \mathcal{P}\left({}^1\ell_1\right)$ is c_0-continuous on bounded subsets of ℓ_1, then $\phi \in c_0$. However, this last result is a particular case of [7, Theorem V.5.6].

We denote by (e_n^*) the associated sequence of coefficient functionals for the basis (e_n) of ℓ_1.

Proposition 1. *Let (f_n) be a sequence of complex-valued functions defined on ℓ_1. If (f_n) is pointwise bounded, then for all $x, y \in \ell_1$ the series $\sum_{j=1}^\infty e_j^*(x) f_j(y)$ converges. Moreover, we have that:*

(i) *If $(R_n) \subset \mathcal{P}\left({}^m\ell_1\right)$ converges to 0 pointwise and*

$$P(x) = \sum_{j=1}^\infty e_j^*(x) R_j(x),$$

then $P \in \mathcal{P}\left({}^{m+1}\ell_1\right)$.

(ii) *If $\Phi = c$ or $\Phi = c_0$ and $(R_n) \subset P_\Phi\left({}^m\ell_1\right)$ converges to 0 in norm and*

$$P(x) = \sum_{j=1}^\infty e_j^*(x) R_j(x),$$

then $P \in \mathcal{P}_\Phi\left({}^m\ell_1\right)$.

Proof. Let $\left(e_j^*\right)$ be the coordinate functionals associated with the canonical basis (e_j) of ℓ_1. For each $y \in \ell_1$ we have $(f_i(y)) \in \ell_\infty$ and therefore $\sum_{j=1}^\infty e_j^*(x) f_j(y)$ converges.

(i) Since (R_n) converges to 0 pointwise, then (R_n) is uniformly bounded on $B(\ell_1)$ by [9, Theorem 2.6], that is, $\sup_{j\geq 1} \|R_j\| < \infty$. Thus $|R_j(x)| \leq \|R_j\| \|x\|^m$, for all $x \in B(\ell_1)$ and $j \geq 1$. Obviously $R(x) = \sum_{j=1}^\infty e_j^*(x) R_j(x)$ is an $(m+1)$-homogeneous polynomial and

$$|P(x)| = \left|\sum_{j=1}^\infty e_j^*(x) R_j(x)\right| \leq \sup_{j\geq 1} |R_j(x)| \sum_{j=1}^\infty \left|e_j^*(x)\right| \leq \sup_{j\geq 1} \|R_j\| \|x\|^{m+1},$$

hence

$$\|P\| = \sup_{x\in B(\ell_1)} |P(x)| \leq \sup_{j\geq 1} \|R_j\|,$$

and therefore it is continuous.

(ii) For each $k \in \mathbb{N}$ define $T_k(x) := \sum_{j=1}^{k} e_j^*(x) R_j(x)$. Since $\left(e_j^*\right) \subset \Phi$ and $(R_j) \subset P_\Phi({}^m\ell_1)$, then $(T_k) \subset P_\Phi\left({}^{m+1}\ell_1\right)$. Now, for all $x \in B(\ell_1)$ and $m, n \in \mathbb{N}$ with $n > m$, we have

$$\begin{aligned} |T_m(x) - T_n(x)| &\leq \left| \sum_{j=m+1}^{n} e_j^*(x) R_j(x) \right| \\ &\leq \sup_{j=m+1,\ldots,n} |R_j(x)| \sum_{j=m+1}^{n} \left|e_j^*(x)\right| \\ &\leq \sup_{j=m+1,\ldots,n} \|R_j\| \|x\|^{m+1} \leq \sup_{j\geq m+1} \|R_j\| \|x\|^{m+1}, \end{aligned}$$

and therefore $\|T_m - T_n\| \leq \sup_{j\geq m+1} \|R_j\|$. Since $\lim \|R_j\| = 0$, it follows that (T_m) is a Cauchy sequence in the space $P_\Phi({}^m\ell_1)$ and therefore convergent in norm. Since $P(x) = \lim_k T_k(x)$ for all $x \in \ell_1$, it follows that $P \in P_\Phi({}^m\ell_1)$. ∎

Our interest in the ℓ_1 space is due to the following result.

PROPOSITION 2. *Let E be a Banach space with a bounded unconditional Schauder basis (b_n), $m \in \mathbb{N}$ and let $(P_j) \subset \mathcal{P}({}^mE)$ be a sequence such that for all $x \neq 0$ we have $\lim_j P_j(x) \neq 0$. If for all $x = \sum_{j=1}^{\infty} x_j b_j \in E$ the function $Q(x) := \sum_{j=1}^{\infty} x_j P_j(x)$ is defined and continuous on E, then E is isomorphic to ℓ_1.*

Proof. In fact, let be $x = \sum_{j=1}^{\infty} x_j b_j \neq 0$ and $(\theta_j) \subset \mathbb{C}$ with $|\theta_j| = 1$ for all $j = 1, 2, \ldots$ such that $\theta_j x_j P_j(x) = |x_j P_j(x)|$, then $\bar{x} = \sum_{j=1}^{\infty} x_j \theta_j b_j \in E$ and therefore

$$Q(\bar{x}) = \sum_{j=1}^{\infty} x_j \theta_j P_j(x) = \sum_{j=1}^{\infty} |x_j| |P_j(x)|.$$

Since $\lim_j P_j(x) \neq 0$, then there exists an positive integer j_0 and $\delta > 0$ such that $|P_j(x)| > \delta$, for $j \geq j_0$. Hence we have that

$$Q(\bar{x}) \geq \sum_{j=1}^{j_0} |x_j| |P_j(x)| + \delta \sum_{j=j_0+1}^{\infty} |x_j|.$$

Thus $(x_j) \in \ell_1$. This proves that $(b_j) \succ (e_j)$. Since (b_j) is bounded then $\sum_{j=1}^{\infty} |x_j| < \infty$ implies that $\sum_{j=1}^{\infty} x_j b_j \in E$. Thus, $(e_j) \succ (b_j)$ and therefore E is isomorphic to ℓ_1. ∎

The conclusion of Proposition 1 (ii) is not true if the sequence (P_j) converges to 0. In fact, if $E = \ell_2$ and $P_j(x_1, x_2, \dots) = 1/j$, then $Q(x_1, x_2, \dots) = \sum_{j=1}^{\infty} x_j P_j(x) \in \mathcal{P}\left({}^2\ell_2\right)$.

Corollary 1. *Let $(R_j) \subset \mathcal{P}_c({}^m\ell_1)$ be a sequence of polynomials convergent in norm. If $P(x) = \sum_{j=1}^{\infty} x_j R_j(x)$ then $P \in \mathcal{P}_c(\ell_1)$.*

Proof. Since $\mathcal{P}_c({}^m\ell_1)$ is a closed subspace of $\mathcal{P}({}^m\ell_1)$, then $R = \lim R_j \in \mathcal{P}_c({}^m\ell_1)$. Now, if $u = (1, 1, \dots) \in c$ then

$$\begin{aligned} P(x) &= \sum_{j=1}^{\infty} e_j^*(x)\,(R_j(x) - R(x)) + \sum_{j=1}^{\infty} e_j^*(x)\,R(x) \\ &= \sum_{j=1}^{\infty} e_j^*(x)\,(R_j(x) - R(x)) + u(x)\,R(x)\,. \end{aligned}$$

Since $\lim_j \|R_j - R\| = 0$, then by Proposition 1(2) the polynomial

$$Q(x) = \sum_{j=1}^{\infty} e_j^*(x)\,(R_j(x) - R(x))\,,$$

is c-continuous on bounded sets. Obviously $S(x) := u(x)\,R(x)$ is also c-continuous on bounded subsets of ℓ_1. ∎

Lemma 1. *Let E be a Banach space. If $\phi \in E'$, $R \in P\left({}^{m-1}E\right)$ and $Q(x) := \phi(x)\,R(x)$, then for all $x, y \in E$ we have*

$$\overset{\vee}{Q}\left(x, y^{m-1}\right) = \frac{1}{m}\phi(x)\,R(y) + \left(1 - \frac{1}{m}\right)\phi(y)\,\overset{\vee}{R}\left(x, y^{m-2}\right).$$

Proof. Let $T : E \times \cdots \times E \to \mathbb{C}$ be the m-linear map defined by

$$T(z_1, z_2, \dots, z_m) = \phi(z_1)\,\overset{\vee}{R}(z_2, z_3, \dots, z_m)\,.$$

Then $Q(x) = T(x, x, \ldots, x)$, and by [9, Proposition 1.6] we have

$$\overset{\vee}{Q}(z_1, z_2, \ldots, z_n) = \frac{1}{m!} \sum_{\sigma \in S_m} T\left(z_{\sigma(1)}, z_{\sigma(2)}, \ldots, z_{\sigma(m)}\right)$$
$$= \frac{1}{m!} \sum_{\sigma \in S_m} \phi\left(z_{\sigma(1)}\right) \overset{\vee}{R}\left(z_{\sigma(2)}, z_{\sigma(3)}, \ldots, z_{\sigma(m)}\right).$$

If $z_2 = z_3 = \cdots = z_m = z$, then we obtain

$$\phi\left(z_{\sigma(1)}\right) \overset{\vee}{R}\left(z_{\sigma(2)}, z_{\sigma(3)}, \ldots, z_{\sigma(n)}\right) = \begin{cases} \phi(z_1) \overset{\vee}{R}(z, z \ldots, z) & \text{if } \sigma(1) = 1, \\ \phi(z) \overset{\vee}{R}(z_1, z, \ldots, z) & \text{if } \sigma(1) \neq 1. \end{cases}$$

Therefore, if $K = \{\sigma \in S_m : \sigma(1) = 1\}$, then $\#K = (m-1)!$ and

$$\overset{\vee}{Q}\left(z_1, z^{m-1}\right) = \frac{1}{m!}\left(\sum_{\sigma \in K} \phi(z_1) \overset{\vee}{R}(z, z \ldots, z) + \sum_{\sigma \in S_m - K} \phi(z) \overset{\vee}{R}(z_1, z, \ldots, z)\right)$$
$$= \frac{1}{m!}\Big((m-1)!\phi(z_1) R(z) + (m! - (m-1)!)\,\phi(z) \overset{\vee}{R}\left(z_1, z^{m-2}\right)\Big)$$
$$= \frac{1}{m}\phi(z_1) R(z) + \left(1 - \frac{1}{m}\right) \phi(z) \overset{\vee}{R}\left(z_1, z^{m-2}\right).$$

∎

LEMMA 2. *For $m \geq 1$, let $(R_j) \subset \mathcal{P}\left({}^{m-1}\ell_1\right)$ be a pointwise convergent sequence to zero and $P(x) = \sum_{j=1}^{\infty} e_j^*(x) R_j(x)$. Then for all $x, y \in \ell_1$ we have*

$$\overset{\vee}{P}\left(x, y^{m-1}\right) = \frac{1}{m} \sum_{j=1}^{\infty} e_j^*(x) R_j(y) + \left(1 - \frac{1}{m}\right) \sum_{j=1}^{\infty} e_j^*(y) \overset{\vee}{R}_j\left(x, y^{m-2}\right).$$

Proof. Let $Q_j(x) = e_j^*(x) R_j(x)$. Lemma 1 implies that for all $x, y \in \ell_1$ we have

$$\overset{\vee}{Q}_j\left(x, y^{m-1}\right) = \frac{1}{m} e_j^*(x) R_j(y) + \left(1 - \frac{1}{m}\right) e_j^*(y) \overset{\vee}{R}_j\left(x, y^{m-2}\right).$$

Since (R_j) converges pointwise to zero, then by [9, Theorem 2.6], (R_j) is bounded in norm. Hence, by Proposition 1, the series $\sum_{j=1}^{\infty} e_j^*(x) R_j(y)$ converges. Let (S_j) be a sequence of $(m-1)$-homogeneous polynomials defined by $S_j(y) = \overset{\vee}{R}_j\left(x, y^{m-2}\right)$. Then the sequence (S_j) converges pointwise to zero by the polarization formula [9, Theorem 1.10]. Therefore, by Proposition 1 the series $\sum_{j=1}^{\infty} e_j^*(y) \overset{\vee}{R}_j\left(x, y^{m-2}\right)$ converges and since

$$P(x) = \sum_{j=1}^{\infty} e_j^*(x) R_j(x) = \sum_{j=1}^{\infty} Q_j(x),$$

it follows by linearity that $\overset{\vee}{P}\left(x, y^{m-1}\right) = \sum_{j=1}^{\infty} \overset{\vee}{Q}_j\left(x, y^{m-1}\right)$. So

$$\overset{\vee}{P}\left(x, y^{m-1}\right) = \sum_{j=1}^{\infty} \frac{1}{m} e_j^*(x) R_j(y) + \sum_{j=1}^{\infty} \left(1 - \frac{1}{m}\right) e_j^*(y) \overset{\vee}{R}_j\left(x, y^{m-2}\right).$$

∎

It follows from Lemma 2 that if $P(x) = \sum_{j=1}^{\infty} e_j^*(x) R_j(x)$ and $y = (y_1, y_2, \dots) \in \ell_1$, then

$$\overset{\vee}{P}\left(e_k, y^{m-1}\right) = \frac{1}{m} R_k(y) + \left(1 - \frac{1}{m}\right) \sum_{j=1}^{\infty} e_j^*(y) \overset{\vee}{R}_j\left(e_k, y^{m-2}\right).$$

We do not know if the converse of Proposition 1(2) is true for all $m \in \mathbb{N}$. However, the following proposition shows that if $\Phi = c_0$, the pointwise convergence of (R_n) is necessary.

Proposition 3. *Let $(R_n) \subset \mathcal{P}\left({}^m\ell_1\right)$, be a sequence of c_0-continuous polynomials and for each $x \in \ell_1$ define*

$$P(x) = \sum_{n=1}^{\infty} e_n^*(x) R_n(x).$$

If P is c_0-continuous in the bounded subsets of ℓ_1, then (R_n) converges pointwise to zero.

Proof. We prove the assertion by induction on m. Recall that if (e_k) is the canonical basis of ℓ_1 and $(\phi_j) \subset c_0$ is a bounded sequence such that $\lim_{n\to\infty} \phi_n(e_k) = 0$ for every k, then $\lim_j \phi_j(a) = 0$ for all $a \in \ell_1$.

Consider the bounded sequence $(\phi_n) \subset c_0 = \mathcal{P}_{c_0}\left({}^1\ell_1\right)$, and the polynomial $P(x) = \sum_{n=1}^{\infty} e_n^*(x) \phi_n(x)$. Assume that the polynomial P is c_0-continuous on bounded subsets of ℓ_1. Then, by Lemma 2, we have

$$\overset{\vee}{P}(e_k, y) = \frac{1}{2} \phi_k(y) + \frac{1}{2} \sum_{j=1}^{\infty} e_j^*(y) \phi_j(e_k),$$

thus

$$\overset{\vee}{P}(e_k, e_l) = \frac{1}{2}(\phi_k(e_l) + \phi_l(e_k)). \tag{3.1}$$

As $P \in \mathcal{P}_{c_0}(\ell_1)$, then for each l we have $\lim_k \overset{\vee}{P}(e_k, e_l) = 0$, also for each l we have $\lim_k \phi_l(e_k) = 0$ because $\phi_l \in c_0$. Thus, Equation 3.1 implies that for each l we have $\lim_k \phi_k(e_l) = 0$ and therefore for all $a \in \ell_1$, we have $\lim \phi_n(a) = 0$. This shows the assertion for $m = 1$.

We assume the assertion true for m. Let $(R_n) \in P_{c_0}\left({}^{m+1}\ell_1\right)$ be a bounded sequence and $P(x) = \sum_{n=1}^{\infty} e_n^*(x) R_n(x)$. Assume that $P \in \mathcal{P}_{c_0}\left({}^{m+2}\ell_1\right)$. By Lemma 2, we have

$$\overset{\vee}{P}\left(e_k, y^{m+1}\right) = \frac{1}{m+2} R_k(y) + \left(1 - \frac{1}{m+2}\right) \sum_{j=1}^{\infty} e_j^*(y) \overset{\vee}{R}_j(e_k, y^m). \tag{3.2}$$

As $P \in P_{c_0}\left({}^{m+2}\ell_1\right)$, then by Theorem 1, the polynomial $T_k(y) = \overset{\vee}{P}(e_k, y^m)$ is c_0-continuous on bounded subsets of ℓ_1. Also by hypothesis $R_k \in P_{c_0}({}^m\ell_1)$, thus the identity 3.2 implies that for each k, the polynomial

$$S_k(y) := \sum_{j=1}^{\infty} y_j \overset{\vee}{R}_j\left(e_k, y^{m-1}\right),$$

is c_0-continuous on bounded subsets of ℓ_1. By Theorem 1, for each k, j, the polynomial $U_j(y) = \overset{\vee}{R}_j\left(e_k, y^{m-1}\right)$ is c_0- continuous on bounded subsets of ℓ_1. Also, by [9, Theorem 2.2] we have

$$\sup_j \|U_j\| \le \sup_j \left\|\overset{\vee}{R}_j\right\| < \frac{m^m}{m!} \sup_j \|R_j\| < \infty.$$

Thus, $(U_j) \subset \mathcal{P}_{c_0}\left({}^{m-1}\ell_1\right)$ is a bounded sequence and by induction hypothesis, given k and $y \in \ell_1$, we have

$$\lim_j \overset{\vee}{R}_j\left(e_k, y^{m-1}\right) = 0. \tag{3.3}$$

For each j and $x \in \ell_1$ define $\psi_j(x) = \overset{\vee}{R}_j\left(x, y^{m-1}\right)$. Since $R_j \in \mathcal{P}_{c_0}\left({}^{m+1}\ell_1\right)$ then we have that $(\psi_j) \subset c_0$ by Theorem 1, and

$$\|\psi_j\| \le \frac{m^m}{m!} \sup_j \|R_j\| \|y\|^{m-1} < \infty \qquad \text{for all } j \ge 1.$$

Equation 3.3 implies that for each k we have that $\lim_j \psi_j(e_k) = 0$ and therefore for all $x \in \ell_1$ we have $\lim_j \psi_j(x) = 0$. In particular $\lim_j \psi_j(y) = 0$, that is $\lim R_j(y) = 0$. This proves our assertion for $m+1$, and the proof is complete. ∎

We recall that if $\psi \in c \subset \ell_\infty$ then $\psi = \lambda u + \phi$, where $\phi \in c_0$, $u = (1, 1, \dots, 1, \dots) \in c$ and $\lambda \in \mathbb{C}$. Now, for each j we have $\|\psi\| \geq |\psi(e_j)| = |\lambda u(e_j) + \phi(e_j)| = |\lambda_j + \phi_j(e_k)|$ and letting $j \to \infty$ we have $\|\psi\| \geq |\lambda|$. Therefore, if $(\lambda_j) \subset \mathbb{C}$, $(\phi_j) \subset c_0$ and $\lim_j \|\lambda_j u + \phi_j\| = 0$ then $\lim_j \lambda_j = 0$ and $\lim_j \|\phi_j\| = 0$. This result can be generalized to polynomials in the space $\mathcal{P}_c({}^m\ell_1)$.

THEOREM 2. *Let $(Q_j) \subset \mathcal{P}_c({}^m\ell_1)$ and $(R_j) \subset \mathcal{P}_{c_0}({}^m\ell_1)$ be sequences of polynomials such that $Q_j(x) = 0$ for all $x \in \ker u$. If $\lim_j \|Q_j + R_j\| = 0$, then $\lim_j \|R_j\| = 0$ and $\lim_j \|Q_j\| = 0$.*

Proof. Let $P_j = Q_j + R_j$ then $P_j \in P_c({}^m\ell_1)$ for every j. Now, if $z \in \ker u$ then

$$|P_j(z)| = |R_j(z)|.$$

Thus

$$\sup_{x \in B(\ell_1) \cap \ker u} |R_j(x)| = \sup_{x \in B(\ell_1) \cap \ker u} |P_j(x)| \leq \sup_{x \in B(\ell_1)} \|P_j(x)\| = \|P_j\|.$$

Therefore

$$\lim_{j \to \infty} \sup_{x \in B(\ell_1) \cap \ker u} |R_j(x)| = 0. \tag{3.4}$$

If $x = \sum_{j=1}^\infty \alpha_j e_j \in B(\ell_1)$, then for every n we have

$$x = \sum_{j=1}^\infty \alpha_j (e_j - e_{j+n}) + \sum_{j=1}^\infty \alpha_j e_{j+n}.$$

Note that

$$y_n := \sum_{j=1}^\infty \alpha_j (e_j - e_{j+n}) \in 2B(\ell_1) \cap \ker u, \quad \text{and} \quad z_n := \sum_{j=1}^\infty \alpha_j e_{n+j} \in B(\ell_1).$$

By Leibniz's formula [9, Theorem 1.8], we have

$$R_j(x) = R_j(y_n + z_n) = R_j(y_n) + \sum_{k=0}^{m-1} \binom{n}{k} \overset{\vee}{R}_j\left(y_n^k, z_n^{m-k}\right).$$

Since $R_j \in P_{c_0}(\ell_1)$, $\|z_n\| \leq 2$ for every n, and $c_0 - \lim_n z_n = 0$, then by Theorem 1, for each $k = 0, 1, \ldots, m-1$, we have

$$\lim_n \left(\sup_{y \in B(\ell_1)} \left| \overset{\vee}{R}_j \left(y^k, z_n^{m-k} \right) \right| \right) = 0.$$

Thus, for each $k = 0, 1, \ldots, m-1$, and $\varepsilon > 0$, there exists $n_0, n_1, \ldots, n_{m-1}$ such that

$$\sup_{x \in B(\ell_1)} \left| \overset{\vee}{R}_j \left(x^k, z_n^{m-k} \right) \right| < \frac{\varepsilon}{2^{m+1}}, \quad \text{for all } n \geq n_k, \ k = 0, 1, \ldots, m-1,$$

and therefore

$$\sup_{x \in B(\ell_1)} \left| \overset{\vee}{R}_j \left(x^k, z_n^{m-k} \right) \right| < \frac{\varepsilon}{2^{m+1}}, \text{ for all } n \geq \max\{n_0, n_1, \ldots, n_{m-1}\}.$$

Thus, for all $n \geq \max\{n_0, n_1, \ldots, n_{m-1}\}$, we obtain

$$\begin{aligned} |R_j(x)| &= \left| R_j(y_n) + \sum_{k=0}^{m-1} \binom{m}{k} \overset{\vee}{R}_j \left(y_n^k, z_n^{m-k} \right) \right| \\ &= |R_j(y_n)| + \sum_{k=0}^{m-1} \binom{m}{k} \left| \overset{\vee}{R}_j \left(y_n^k, z_n^{m-k} \right) \right| \\ &\leq \sup_{y \in 2B(\ell_1) \cap \ker u} |R_j(y)| + \sum_{k=0}^{m-1} \binom{m}{k} \sup_{x \in B(\ell_1)} \left| \overset{\vee}{R}_j \left(x^k, z_n^{m-k} \right) \right| \qquad (3.5) \\ &\leq \sup_{y \in 2B(\ell_1) \cap \ker u} |R_j(y)| + \sum_{k=0}^{m-1} \binom{m}{k} \frac{\varepsilon}{2^{m+1}} \\ &= \sup_{y \in 2B(\ell_1) \cap \ker u} |R_j(y)| + \frac{\varepsilon}{2}. \end{aligned}$$

By 3.4 we have $\lim_j \sup_{y \in B(\ell_1) \cap \ker u} |R_j(y)| = 0$, hence there exists j_0 such that for $j \geq j_0$ we have

$$\sup_{y \in 2B(\ell_1) \cap \ker u} |R_j(y)| < \frac{\varepsilon}{2},$$

By relation 3.5 we obtain

$$|R_j(x)| < \varepsilon, \text{ for all } x \in B(\ell_1) \text{ and } j \geq j_0.$$

Thus for all $j \geq j_0$, $\|R_j\|_{B(\ell_1)} < \varepsilon$. This shows that $\lim_j \|R_j\| = 0$. Now $Q_j = P_j - R_j$, implies that $\lim_j \|Q_j\| \leq \lim_j \|P_j\| + \lim_j \|R_j\| = 0$. ∎

Theorem 3. *Every polynomial $P \in \mathcal{P}_c\left({}^m\ell_1\right)$ can be decomposed in the form $P = Q + R$, where $Q \in P_c\left({}^{m-1}\ell_1\right)$ with $Q(x) = 0$ for all $x \in \ker u$ and $R \in P_{c_0}\left({}^{m-1}\ell_1\right)$.*

Proof. For $m = 1$ the statement is obvious. Suppose it is true for $m-1$. Let $P \in \mathcal{P}_c\left({}^m\ell_1\right)$ and for each j consider the polynomials $T_j(x) = \overset{\vee}{P}\left(e_j, x^{m-1}\right)$. Then by Theorem 1 we have that $T_j \in P_c\left({}^{m-1}\ell_1\right)$ and by induction hypothesis we have

$$\overset{\vee}{P}\left(e_j, x^{m-1}\right) = T_j(x) = Q_j(x) + R_j(x),$$

where $Q_j \in P_c\left({}^{m-1}\ell_1\right)$, $R_j \in P_{c_0}\left({}^{m-1}\ell_1\right)$ and $Q_j(x) = 0$ for all $x \in \ker u$ for all j. Since (e_j) is c-Cauchy, then (T_j) converges in norm to a polynomial $\overline{P} \in P_c\left({}^{m-1}\ell_1\right)$ and by induction hypothesis we have $\overline{P} = \overline{Q} + \overline{R}$, with $\overline{Q} \in P_c(\ell_1)$, $\bar{R} \in P_{c_0}(\ell_1)$ and $\overline{Q}(x) = 0$ for all $x \in \ker u$. By Lemma 2 $\lim_j R_j = \overline{R}$ and $\lim_j Q_j = \overline{Q}$ in norm. So

$$P(x) = \sum_{j=1}^{\infty} e_j^*(x)\, \overset{\vee}{P}\left(e_j, x^{m-1}\right) = \sum_{j=1}^{\infty} e_j^*(x)\left(Q_j(x) + R_j(x)\right)$$

And therefore

$$\begin{aligned}
P(x) &= \sum_{j=1}^{\infty} e_j^*(x)\left(Q_j(x) - \bar{Q}(x)\right) + \sum_{j=1}^{\infty} e_j^*(x)\,\bar{Q}(x) \\
&\quad + \sum_{j=1}^{\infty} e_j^*(x)\left(R_j(x) - \bar{R}(x)\right) + \sum_{j=1}^{\infty} e_j^*(x)\,\bar{R}(x) \\
&= \sum_{j=1}^{\infty} e_j^*(x)\left(Q_j(x) - \bar{Q}(x)\right) + \bar{Q}(x)\,u(x) \\
&\quad + \sum_{j=1}^{\infty} e_j^*(x)\left(R_j(x) - \bar{R}(x)\right) + \bar{R}(x)\,u(x) \\
&= \sum_{j=1}^{\infty} e_j^*(x)\left(Q_j(x) - \bar{Q}(x)\right) + \left(\bar{Q}(x) + \bar{R}(x)\right)u(x) \\
&\quad + \sum_{j=1}^{\infty} e_j^*(x)\left(R_j(x) - \bar{R}(x)\right).
\end{aligned}$$

Since $\lim_j \left\| Q_j(x) - \bar{Q}(x) \right\| = 0$, the polynomial

$$x \longmapsto \sum_{j=1}^{\infty} e_j^*(x) \left(Q_j(x) - \bar{Q}(x) \right)$$

is c-continuous on bounded subsets of ℓ_1 by Proposition 1 and vanishes on $\ker u$. Also the polynomial $x \mapsto \left(\bar{Q}(x) + \bar{R}(x) \right) u(x)$ vanishes on $\ker u$. Since $\lim_j \left\| R_j(x) - \bar{R}(x) \right\| = 0$, and $R_j, \bar{R} \in \mathcal{P}_{c_0}\left({}^{m-1}\ell_1 \right)$, Proposition 1 implies that $x \mapsto \sum_{j=1}^{\infty} e_j^*(x) \left(R_j(x) - \bar{R}(x) \right)$ is a c_0-continuous polynomial on bounded subsets of ℓ_1.

We define

$$Q(x) = \left(\bar{Q}(x) + \bar{R}(x) \right) u(x) + \sum_{j=1}^{\infty} e_j^*(x) \left(Q_j(x) - \bar{Q}(x) \right),$$
$$R(x) = \sum_{j=1}^{\infty} e_j^*(x) \left(R_j(x) - \bar{R}(x) \right).$$

∎

LEMMA 3. *Let E be a Banach space, $\phi \in E'$ and $Q \in \mathcal{P}\left({}^m E \right)$ be a polynomial such that $Q(x) = 0$ for all $x \in \ker \phi$. Then there exists a polynomial $R \in P\left({}^{m-1} E \right)$ such that $Q = \phi R$.*

Proof. Pick $a \in E$ such that $\phi(a) = 1$ and define the map $T : E \to E$ by $T(x) = \phi(x) a - x$. Then T is a continuous linear operator and $T(x) \in \ker \phi$ for all $x \in E$. By Leibniz's formula, we have

$$\begin{aligned} Q(x) &= Q(\phi(x) a - T(x)) \\ &= \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} \overset{\vee}{Q} \left((\phi(x) a)^j, (T(x))^{m-j} \right) \\ &= \sum_{j=1}^{m} \binom{m}{j} (-1)^{m-j} \overset{\vee}{Q} \left((\phi(x) a)^j, (T(x))^{m-j} \right) + Q(T(x)) \\ &= \sum_{j=1}^{m} \binom{m}{j} (-1)^{m-j} \phi^j(x) \overset{\vee}{Q} \left(a^j, (T(x))^{m-j} \right) \\ &= \phi(x) \sum_{j=1}^{m} \binom{m}{j} (-1)^{m-j} \phi^{j-1}(x) \overset{\vee}{Q} \left(a^j, (T(x))^{m-j} \right). \end{aligned}$$

Note that for each j the map $x \mapsto \phi^{j-1}(x) \overset{\vee}{Q}\left(a^j, (T(x))^{m-j}\right)$ is an $(m-1)$-homogeneous polynomial. So

$$R(x) := \sum_{j=0}^{m-1} \binom{m}{j} (-1)^{m-j} \phi^{j-1}(x) \overset{\vee}{Q}\left(a^j, (T(x))^{m-j}\right),$$

is a continuous $(m-1)$-homogeneous polynomial and

$$Q(x) = \phi(x) R(x).$$

∎

Corollary 2. *Let $Q \in \mathcal{P}_c(^m\ell_1)$ such that $Q(x) = 0$ for all $x \in \ker u$, then there exists $R \in \mathcal{P}_c\left(^{m-1}\ell_1\right)$ such that $Q(x) = u(x) R(x)$ for all $x \in \ell_1$.*

Proof. We define the map $T : \ell_1 \to \ell_1$ by $T(x) = u(x) e_1 - x$, then T is obviously a c-continuous linear operator and $T(x) \in \ker u$ for all $x \in \ell_1$. By Lemma 3 we have

$$Q(x) = Q(u(x) e_1 - T(x)) = u(x) \sum_{j=1}^{m} \binom{m}{j} u^{j-1}(x) \overset{\vee}{Q}\left(e_1^j, (T(x))^{m-j}\right).$$

Since $Q \in \mathcal{P}_c(^m\ell_1)$ then for each $j = 1, 2, \ldots, m$, the polynomial $S_j : \ell_1 \to \mathbb{C}$ given by $S_j(z) = \overset{\vee}{Q}\left(e_1^j, z^{m-j}\right)$, is c-continuous on bounded subsets of ℓ_1. Therefore $S_j \circ T \in \mathcal{P}_c(\ell_1)$ for $j = 1, 2, \ldots, m$ and we have that

$$\begin{aligned} R(x) &= \sum_{j=1}^{m-1} \binom{m}{j} u^{j-1}(x) (S_j \circ T)(x) \\ &= \sum_{j=1}^{m-1} \binom{m}{j} u^{j-1}(x) \overset{\vee}{Q}\left(e_1^j, (T(x))^{m-j}\right), \end{aligned}$$

is a c-continuous polynomial on bounded sets, and $Q = uR$. ∎

Theorem 4. *If $P \in \mathcal{P}_c(^m\ell_1)$, then for $j = 0, 1, 2, \ldots, m$ there are polynomials $R_j \in P_{c_0}\left(^j\ell_1\right)$, such that*

$$P(x) = R_0(x) u^m(x) + u^{m-1}(x) R_1(x) + \cdots + u(x) R_{m-1}(x) + R_m(x).$$

Proof. By Theorem 3 we have $P = Q_m + R_m$, where $Q_m \in \mathcal{P}_c\left({}^m\ell_1\right)$, $R_m \in \mathcal{P}_{c_0}\left({}^m\ell_1\right)$ and $Q(x) = 0$ for all $x \in \ker u$. By Lemma 3, $Q_m = uS_{m-1}$ with $S_{m-1} \in \mathcal{P}_c\left({}^m\ell_1\right)$. Thus, we have

$$P = uS_{m-1} + R_m.$$

Since $S_{m-1} \in \mathcal{P}_c\left({}^{m-1}\ell_1\right)$, then by Theorem 3 we have $S_{m-1} = Q_{m-1} + R_{m-1}$, where $Q_{m-1} \in P_c\left({}^{m-1}\ell_1\right)$, $Q_{m-1}(x) = 0$ for all $x \in \ker u$ and $R_{m-1} \in P_{c_0}\left({}^{m-1}\ell_1\right)$ and therefore

$$\begin{aligned} P &= u\left(Q_{m-1} + R_{m-1}\right) + R_m(x) \\ &= uQ_{m-1} + R_{m-1}u + R_m(x). \end{aligned}$$

By Lemma 3 we have that $Q_{m-1} = uS_{m-2}$, with $S_{m-2} \in \mathcal{P}_c\left({}^m\ell_1\right)$. Therefore we have

$$P(x) = u(x)^2 S_{m-2} + R_{m-1}u + R_m(x).$$

Proceeding in this way we find for each $j = 0, 1, 2, \ldots, m$, the polynomials $R_j \in \mathcal{P}_{c_0}\left({}^j\ell_1\right)$, and $S_j \in \mathcal{P}_c\left({}^j\ell_1\right)$, such that

$$P(x) = u^m R_0 + R_1 u^{m-1} + \cdots + R_{m-1}u + R_m(x),$$

where $R_0 := S_0$. ∎

4. c-Continuous entire functions

Let Ω be an open subset of complex Banach space E. A mapping $f : \Omega \subset E \to \mathbb{C}$ is said to be holomorphic, if for each $a \in \Omega$ there exists a ball $B(a, r) \subset \Omega$ and a sequence of polynomials (P_m) with $P_m \in \mathcal{P}\left({}^m\ell_1\right)$, $m = 0, 1, 2 \ldots$, such that $f(x) = \sum_{m=0}^{\infty} P_m(x)$ uniformly for $x \in B(a, r)$. We denote by $\mathcal{H}(\Omega)$ the vector space of all holomorphic mappings from Ω into $\mathbb{C}$. A holomorphic function $f \in \mathcal{H}(E)$ is said to be of bounded type if it maps bounded sets into bounded sets. We denote by $\mathcal{H}_b(E)$ the space of the holomorphic functions on E of bounded type.

Let $\Phi \subset E'$, we denote by $\mathcal{H}_\Phi(E)$ the space of all functions $f \in \mathcal{H}(E)$ that are Φ-continuous on bounded subsets of E, and by $\mathcal{H}_{\Phi u}(E)$ the space of all functions $f \in \mathcal{H}(E)$ that are uniformly Φ-continuous on bounded subsets of E.

In 1982 Aron et al. in [1] have shown that the ℓ_1 problem has a positive answer if $\mathcal{H}_{\ell_\infty}(\ell_1) \subset \mathcal{H}_b(\ell_1)$. On the other hand, it is obviously that

$\mathcal{H}_{c_0}(\ell_1) \subset \mathcal{H}_b(\ell_1)$ because every bounded set of ℓ_1 is relatively $\sigma(\ell_1, c_0)$-compact, but bounded subsets of ℓ_1 are not necessarily relatively $\sigma(\ell_1, c)$-compact. These considerations have motivated us to raise the following question.

Problem 3. If $f : \ell_1 \to \mathbb{C}$ is a holomorphic function which is c-continuous on bounded sets, is f of bounded type?

An affirmative answer to this problem would answer affirmatively Problem 2.

We denote by $\mathcal{P}_{c_0}^{(m)}(\ell_1)$ the space of all polynomials of the form $Q = \sum_{j=0}^{m} Q_j$, with $Q_j \in \mathcal{P}_{c_0}\left({}^j\ell_1\right)$ for all $j = 0, 1, 2, \ldots, m$. If $U_m(x) := \sum_{j=0}^{m} u^{m-j}(x)$, for all $x \in \ell_1$, we define the m-homogeneous polynomial $U \otimes Q \in \mathcal{P}_c({}^m\ell_1)$ by

$$(U \otimes Q)(x) = \sum_{j=0}^{m} u^{m-j}(x) Q_j(x).$$

We denote by $\mathcal{P}_{f^*}({}^m\ell_1)$ the space of continuous polynomials of finite type that are c_0-continuous on bounded subsets of ℓ_1.

Lemma 4. *If $R(x) \in \mathcal{P}_{c_0}^{(m)}(\ell_1)$, then given $\varepsilon > 0$ there exists a polynomial $Q = \sum_{j=0}^{m} Q_j$ with $Q_j \in \mathcal{P}_{f^*}\left({}^j\ell_1\right)$ such that $\|U_m \otimes (R - Q)\| < \varepsilon$.*

Proof. If $x \in \ell_1$, we denote by

$$q^n(x) = \sum_{j=1}^{n} e_j^*(x)\, e_j \qquad \text{and} \qquad q_n(x) = \sum_{j=n+1}^{\infty} e_j^*(x)\, e_j.$$

Then $x = q^n(x) + q_{n+1}(x)$. Now, if $\phi = (\phi_j)_{j \in \mathbb{N}} \in c_0$, then $\lim_n \max_{i \geq n} |\phi_i| = 0$. As $\max_{i \geq n} |\phi_i| = \sup_{x \in B(\ell_1)} \phi(q_n(x))$ we have $\lim_n \sup_{x \in B(\ell_1)} \phi(q_n(x)) = 0$. That is

$$\lim_n \sup_{x \in B(\ell_1)} \phi(x - q^n(x)) = 0. \tag{4.1}$$

Let $R = \sum_{j=0}^{m} R_j$, with $R_j \in \mathcal{P}_{c_0}\left({}^j\ell_1\right)$. Then by [1], for each $j = 0, 1, 2, \ldots, m$, the polynomial R_j is c_0-uniformly continuous on bounded sets. By 4.1, this implies that given $\varepsilon > 0$, there exists an n_0 such that $|R_j(x) - R_j(q^n(x))| <$

$\varepsilon / (m+2)$, for all $n \geq n_0$, $x \in B(\ell_1)$ and $j = 0, 1, 2 \ldots, m$. Thus $\|R_j - R_j q^n\| \leq \varepsilon / (m+2)$ for $n \geq n_0$ and $j = 0, 1, \ldots, m$. Therefore we have

$$\begin{aligned} \left\|u^{m-j} \otimes (R_j - R_j q^n)\right\| &= \sup_{x \in B(\ell_1)} \left|u^{m-j}(x)(R_j(x) - R_j q^n(x))\right| \\ &\leq \sup_{x \in B(\ell_1)} |R_j(x) - R_j q^n(x)| \\ &= \|R_j - R_j q^n\| \leq \frac{\varepsilon}{m+2}. \end{aligned}$$

Thus, for $n \geq n_0$ we have

$$\|R - Rq^n\| = \left\| \sum_{j=0}^{m} U_m \otimes (R - Rq^n) \right\| \leq \sum_{j=0}^{m} \|R - Rq^n\| < \varepsilon.$$

Since $R \in \mathcal{P}_{c_0}({}^m)(\ell_1)$ and $q^n : \ell_1 \to \ell_1$ is a finite range operator, we have that $Rq^n \in \mathcal{P}_{f^*}({}^m \ell_1)$. ∎

If $f = \sum_{n=0}^{\infty} P_n \subset H_b(\ell_1)$ is a holomorphic function of bounded type with $P_n \in P_c({}^n \ell_1)$ for all $n \in \mathbb{N}$, then using the same arguments as in [1], it is not difficult to show that $f \in H_c(\ell_1)$.

PROPOSITION 4. *The following statements are equivalent.*

(i) *Every holomorphic function $f \in H_c(\ell_1)$ of the form $f = \sum_{m=0}^{\infty} U_m \otimes Q_m$ with $Q_m \in \mathcal{P}_{c_0}^{(m)}(\ell_1)$ is of bounded type.*

(ii) *Every holomorphic function $f \in H_c(\ell_1)$ of the form $f = \sum_{m=0}^{\infty} U_m \otimes Q_m \in H_c(\ell_1)$ with $Q_m \in \mathcal{P}_{f^*}^{(m)}(\ell_1)$, is of bounded type.*

Proof. The implication (i) ⇒ (ii) is obvious since $\mathcal{P}_{f^*}({}^m \ell_1) \subset \mathcal{P}_{c_0}({}^m \ell_1)$. Let us prove (ii) ⇒ (i). Let $f = \sum_{m=0}^{\infty} U_m \otimes Q_m \in H_b(\ell_1)$ with $Q_m \in \mathcal{P}_{c_0}^{(m)}(\ell_1)$ for every m. Since $Q_m \in \mathcal{P}_{c_0}({}^m \ell_1)$, by Lemma 4 there exists a $R_m \in \mathcal{P}_{f^*}^{(m)}(\ell_1)$, such that $\|U_m \otimes (Q_m - R_m)\|^{1/m} < \frac{1}{m^m}$. Thus $\lim \|U_m \otimes (Q_m - R_m)\|^{1/m} = 0$ and by [6, p. 206], the holomorphic function $g = \sum U_m \otimes (Q_m - R_m)$ is of bounded type and therefore $g \in H_c(\ell_1)$. Then $f - g = \sum U_m \otimes R_m \in H_c(\ell_1)$. By hypothesis $h = f - g \in H_b(\ell_1)$ and therefore $f = g + h \in H_b(\ell_1)$. ∎

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