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Some applications of Q-points and Lebesgue filters to Banach spaces

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Abstract: We present a new characterization of Q-point ultrafilters and use it to optimize the result of Avilés, Martínez-Cervantes, and Rueda Zoca linking the existence of L-orthogonal sequences and L-orthogonal elements in Banach spaces via ultrafilter limits.

Key words: Q-point, Katětov order, L-sequence, L-element.

MSC (2020): 46B25, 03E35.

1. Introduction

The presence of subspaces isomorphic to ℓ_1 has been a central topic of research in contemporary Banach space theory. The concepts of L-orthogonal sequences and L-orthogonal elements have become relevant, as the existence of a subspace isomorphic to ℓ_1 implies, under a renorming, their existence (see [10, 4]). Avilés, Martínez-Cervantez, and Rueda Zoca [2] inquired into the relation between these concepts. In one direction they present several examples of Banach spaces which have L-orthogonal elements but no L-orthogonal sequences, and in the other, they show that the existence of counter-examples is independent of the usual axioms of set theory. In particular, they show:

(1) For any selective ultrafilter $\mathscr{U}$, and any L-orthogonal sequence $(x_n)_{n\in\mathbb{N}}$ the $\mathscr{U}\text{-}\lim x_n$ in the weak* topology is an L-orthogonal element.

(2) There exists a Banach space X, such that for any ultrafilter $\mathscr{U}$ not a Q-point, there is an L-orthogonal sequence $(x_n)_{n\in\mathbb{N}}$ such that $\mathscr{U}\text{-}\lim x_n$ in the weak* topology is not an L-orthogonal element.

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This suggests the question whether the assumption can be optimized. Our main theorem answers this question in the positive:

Theorem 1.1. (Main theorem) *A free ultrafilter $\mathscr{U}$ on $\mathbb{N}$ is a Q-point if and only if for every Banach space X and every L-orthogonal sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ the $\mathscr{U}$-$\lim x_n \in X^{**}$ with respect to the weak* topology is an L-orthogonal element.*

The theorem will be a consequence of Corollary 4.2 and Corollary 5.6, it also involves a new characterization of Q-points in the spirit of Mathias' characterization of selective ultrafilters (see [12, Theorem 2.12]).

The proof of the theorem requires also knowledge of measure-theoretic properties of filters, in particular, *Lebesgue filters* [9]. We include a short section (Section 3) summarizing known facts about these, in particular, to simplify the proof of the main theorem compared to the one presented in [2].

Finally, we briefly consider a dual theory to the one that involves L-orthogonality, this time involving c_0 [1]. We offer another result involving Q-points that generalizes a result of the same authors.

We outline the contents of the paper. Section 2 presents the relevant definitions as well as basic results. Section 3 is devoted to measure theoretic ultrafilters, the main result being Theorem 3.1 which establishes the equivalence between some of measure theoretic filters present in the literature and which allows, together with Theorem 2.2, to study Lebesgue filters using the Katětov order. Section 4 is devoted to the first half of our main theorem. In contrast with [2], we find a single Banach space and a single L-orthogonal sequence that for any given non-Q ultrafilter $\mathscr{U}$ admits an L-orthogonal subsequence which serves as a counter-example. Section 5 presents the other half of the main theorem and an analogous result concerning c_0.

2. Preliminaries

Let X be a Banach space. We denote by B_X the closed unit ball of X.

(1) A sequence $(x_n)_{n\in\mathbb{N}} \subseteq B_X$ is called an *L-orthogonal sequence* provided that for any $x \in X$, $\lim_{n\to\infty} \|x_n + x\| = 1 + \|x\|$.

(2) A sequence $(x_n)_{n\in\mathbb{N}} \subseteq B_X$ is called an *S-sequence* if for any $x \in X$, $\lim_{n\to\infty} \|x_n + x\| = \max\{\|x\|, 1\}$.

(3) An element $x^{**} \in S_X^{**}$ is called an *L-orthogonal element* if for any $x \in X$, $\|x^{**} + x\| = 1 + \|x\|$.

(4) An element $x^{**} \in S_X^{**}$ is called an *S-element* provided that for any $x \in X$, $\|x^{**} + x\| = \max\{\|x\|, 1\}$.

The "S-" and "L-" notions are dual in that "L-notions" are related to embeddability of ℓ_1 while "S-notions" are related to the embeddability of c_0. The latter notions also come with a different "convex" closure operation: Given a Banach space X, $A \subseteq X$ and $x \in X$ we say that x is a *c_0-convex combination* of elements of A if there are $t_0, \dots, t_n \in \mathbb{R}$ and $x_0, \dots x_n \in A$ such that $\max\{|t_0|, \dots, |t_n|\} = 1$ and $\sum_{i=0}^n t_i x_i = x$. We denote this fact by $x \in \operatorname{conv}_{c_0}(A)$.

Recall that $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is a *filter* if it is closed under taking supersets and finite intersections, and contains all complements of finite sets, while not containing the empty set. Dually, $\mathcal{I}$ is an *ideal* if $\mathcal{I}^* = \{\mathbb{N} \setminus I : I \in \mathcal{I}\}$ is a filter, i.e., I contains all finite subsets of $\mathbb{N}$ and is closed under finite unions and taking subsets, and does not contain $\mathbb{N}$. We denote by $\mathcal{I}^+ = \{J : J \notin \mathcal{I}\}$ the set of all $\mathcal{I}$-positive subsets of $\mathbb{N}$. Finally, the *restriction* of the ideal (filter) to a set X is $\mathcal{I}|_X = \{I \cap X : I \in \mathcal{I}\}$.

Given a and $(a_n)_{n\in\mathbb{N}}$ points in a topological space X and a filter $\mathscr{F} \subseteq \mathcal{P}(\mathbb{N})$ we write $\mathscr{F}\text{-}\lim a_n = a$ if for any U neighborhood of a $\{n \in \mathbb{N} : a_n \in U\} \in \mathscr{F}$. If the sequence $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers we write

(1) $\mathscr{F}\text{-}\limsup a_n = \inf\big\{r : \{n \in \mathbb{N} : a_n \leq r\} \in \mathscr{F}\big\}$, and
(2) $\mathscr{F}\text{-}\liminf a_n = \sup\big\{r : \{n \in \mathbb{N} : a_n \geq r\} \in \mathscr{F}\big\}$.

Similarly, we can also define limit notions with respect to sequences $(A_n)_{n\in\mathbb{N}}$ of sets w.r.t. $\mathscr{F}$ as follows

(3) $\mathscr{F}\text{-}\lim^+ A_n = \big\{x : \{n \in \mathbb{N} : x \in A_n\} \in \mathscr{F}^+\big\}$, and
(4) $\mathscr{F}\text{-}\lim A_n = \big\{x : \{n \in \mathbb{N} : x \in A_n\} \in \mathscr{F}\big\}$.

The following is an easy observation:

Proposition 2.1. *Let $(a_n)_{n\in\mathbb{N}}$ a sequence of reals, $(A_n)_{n\in\mathbb{N}}$ a sequence of sets, and $\mathscr{F} \subseteq \mathcal{P}(\mathbb{N})$ a free filter,*

(1) $\liminf a_n \leq \mathscr{F}\text{-}\liminf a_n \leq \mathscr{F}\text{-}\limsup a_n \leq \limsup a_n$.
(2) $\mathscr{F}\text{-}\lim a_n = a$ *exists if, and only if,* $\mathscr{F}\text{-}\liminf a_n = \mathscr{F}\text{-}\limsup a_n$, *and in this case it is equal to both.*
(3) *For each x, $x \in \mathscr{F}\text{-}\lim^+ A_n$ if, and only if, $\mathscr{F}\text{-}\limsup \chi_{A_n}(x) = 1$; $x \in \mathscr{F}\text{-}\lim A_n$ if, and only if, $\mathscr{F}\text{-}\liminf \chi_{A_n}(x) = 1$.*

(4) $\mathscr{F}\text{-}\limsup a_n = \mathscr{F}\text{-}\liminf -a_n$, *and* $(\mathscr{F}\text{-}\lim^+ A_n)^c = \mathscr{F}\text{-}\lim A_n^c$.

We say an ideal $\mathcal{I} \subseteq \mathscr{P}(\mathbb{N})$ is *tall* if for any infinite $F \subseteq \mathbb{N}$, there is $I \in \mathcal{I}$ such that $|I \cap F| = \mathbb{N}$. We call a family $\mathscr{A} \subseteq [\mathbb{N}]^{\mathbb{N}}$ *countably hitting* if for every collection $\{X_n : n \in \mathbb{N}\}$ of countably many infinite subsets of $\mathbb{N}$ there exists $A \in \mathscr{A}$ such that for every $n \in \mathbb{N}$, $A \cap X_n$ is infinite. Notice that in order to prove that a family $\mathscr{A} \subseteq [\mathbb{N}]^{\mathbb{N}}$ is countably hitting it is enough to prove that for any partition of $\{P_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ into infinite sets there exists $A \in \mathscr{A}$ such that for every $n \in \mathbb{N}$, $A \cap P_n$ is infinite. This is because given an arbitrary family $\{X_n : n \in \mathbb{N}\}$ of countably many infinite subsets of $\mathbb{N}$ we may apply the Disjoint Refinement Lemma to find a pairwise disjoint family $\{P_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, $P_n \subseteq X_n$, P_n is infinite. So, we may find $A \in \mathscr{A}$ such that for every $n \in \mathbb{N}$, $A \cap P_n$ is infinite. It is easy to see this A is the desired witness for the original family. What we are calling a countably hitting is often referred to as ω-hitting.

An important class of filters are the maximal ones, called *ultrafilters*. An ultrafilter $\mathscr{U} \subseteq \mathscr{P}(\mathbb{N})$ is a *Q-point* if for every partition $\{F_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ into finite sets there exists $X \in \mathscr{U}$ such that for every $n \in \mathbb{N}$, $|X \cap F_n| \leq 1$. More generally (and dually), we call an ideal $\mathcal{I} \subseteq \mathscr{P}(\mathbb{N})$ a $Q^+(\mathbb{N})$*-ideal* if for every partition $\{F_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ into finite sets there exists $X \in \mathcal{I}^+$ such that for every $n \in \mathbb{N}$, $|X \cap F_n| \leq 1$.

Ideals are naturally pre-ordered by the *Katětov and Katětov-Blass orders*. Given two ideals $\mathcal{I}, \mathscr{J}$ on $\mathbb{N}$ we write $\mathcal{I} \leq_K \mathcal{J}$ if there is a map $f : \mathbb{N} \to \mathbb{N}$ such that for any $I \in \mathcal{I}$, $f^{-1}[I] \in \mathscr{J}$, if the map is finite-to-one we write $\mathcal{I} \leq_{KB} \mathcal{J}$.

We consider $\mathscr{P}(\mathbb{N})$ equipped with the natural topology inherited from the product topology of $2^{\mathbb{N}}$ via characteristic functions. Whenever we talk about a subset of $\mathscr{P}(\mathbb{N})$ having any topological property: closed, Borel, analytic, etc., we refer to this topology.

We shall mention two tall F_σ ideals on countable sets. The first one is the ideal

$$\mathcal{ED}_{fin} = \big\{A \subseteq \Delta : \exists\, n, m \in \mathbb{N}\ \forall\, k \geq n \big|\{i : (k,i) \in A\}\big| \leq m\big\},$$

on the set $\Delta = \{(n,m) \in \mathbb{N}^2 : m \leq n\}$. The other, the *Solecki's ideal* $\mathcal{S}$ is the ideal on $\mathbb{N} = \big\{A \in \mathrm{Clop}(2^{\mathbb{N}}) : \lambda(A) = 1/2\big\}$ (here λ is the Lebesgue measure on $2^{\mathbb{N}}$) generated by the sets $I_x = \{A \in \mathbb{N} : x \in A\}$, $x \in 2^{\mathbb{N}}$. The ideals are critical for properties considered in the paper:

THEOREM 2.2. (SOLECKI [14]) *Let* $\mathscr{F}$ *be a universally measurable filter.* $\mathscr{F}$ *is Fatou if, and only if, for every* $F \in \mathscr{F}^+$, $\mathcal{S} \not\leq_K \mathscr{F}^* |_F$.

PROPOSITION 2.3. ([5]) *An ultrafilter $\mathscr{F}$ is a Q-point if and only if $\mathcal{ED}_{fin} \not\leq_{KB} \mathscr{F}^*$.*

3. Measure-theoretic filters

The following properties of filters can be found (with slight modifications) in the literature (see e.g. [3, 7, 14]). A filter $\mathscr{F}$ on $\mathbb{N}$ is

(1) *Fubini* [11] if for any finite measure space (Ω, Σ, μ), any sequence $\{X_n \in \Sigma : n \in \mathbb{N}\}$, and any $\epsilon > 0$, $\{n \in \mathbb{N} : \mu(X_n) > \epsilon\} \in \mathscr{F}^+$ implies $\lambda^*(\mathscr{F}\text{-}lim^+ X_n\}) \geq \epsilon$.

(2) *Fatou* [14] if for any σ-finite measure space (Ω, Σ, μ) and any $f_n : \mathbb{N} \to [0, \infty)$ measurable functions [1] $\underline{\int} \mathscr{F}\text{-}\liminf f_n d\mu \leq \mathscr{F}\text{-}\liminf \int f_n d\mu$.

(3) *Lebesgue* [9] if for any (Ω, Σ, μ) σ-finite measure space and any $f_n : \Omega \to [0, \infty)$ measurable functions such that there is $f : \Omega \to [0, \infty)$ integrable such that $|f_n| \leq f$, $\mathscr{F} - \lim f_n = 0$ implies $\mathscr{F}\text{-}\lim \int f_n d\mu = 0$.

THEOREM 3.1. *The following are equivalent:*

(1) *$\mathscr{F}$ is Fatou.*

(2) *$\mathscr{F}$ is Lebesgue.*

(3) *For any finite measure space (Ω, Σ, μ) and any sequence $(X_n)_{n \in \mathbb{N}}$ of elements of Σ, if $\mu(\mathscr{F}\text{-}\lim X_n) = 0$, then $\mathscr{F}\text{-}\lim \mu(X_n) = 0$.*

(4) *$\mathscr{F}$ is Fubini.*

(5) *For any finite measure space (Ω, Σ, μ) and any sequence $(X_n)_{n \in \mathbb{N}}$ of elements of Σ, $\mu_*(\mathscr{F}\text{-}\lim X_n) \leq \mathscr{F}\text{-}\liminf \mu(X_n)$.*

Proof. Let us prove (1) implies (2). It is clear that we can reduce the problem to the case where $(f_n)_n \in \mathbb{N}$ are measurable and non-negative and f is integrable such that $f_n \leq f$, so consider $g_n = f - f_n$. Applying the Fatou property we get

$$\begin{aligned} \int f d\mu = \underline{\int} \mathscr{F}\text{-}\liminf g_n d\mu &\leq \mathscr{F}\text{-}\liminf \int g_n d\mu \\ &= \int f d\mu + \mathscr{F}\text{-}\liminf \int -f_n d\mu. \end{aligned}$$

[1] Where $\underline{\int} f d\mu = \sup\{\int f' d\mu : f \text{ is integrable and } f' \leq f\}$

However, as $\mathscr{F}\text{-}\liminf \int -f_n d\mu = -\mathscr{F}\text{-}\limsup \int f_n d\mu$, we get

$$\mathscr{F}\text{-}\limsup \int f_n d\mu = 0.$$

As all the functions are non-negative we also know that the inferior limit is non-negative, so $\mathscr{F}\text{-}\lim \int f_n d\mu = 0$.

For (2) implies (3), consider $X = \mathscr{F}\text{-}\lim X_n$ and $Y_n = X_n \setminus X$, then $\varnothing = \mathscr{F}\text{-}\lim Y_n$ and for every n, $\mu(Y_n) = \mu(X_n)$, because the space is finite there is a constant that bounds χ_{Y_n} so we get

$$\mathscr{F}\text{-}\lim \mu(X_n) = \mathscr{F}\text{-}\lim \int \chi_{Y_n} d\mu = 0.$$

For (3) implies (4), assume there is an $\epsilon > 0$ such that $\{n \in \mathbb{N} : \mu(X_n) > \epsilon\} \in \mathscr{F}^+$ but $\lambda^*(\mathscr{F}\text{-}\lim^+ X_n) < \epsilon$. Pick $X \in \Sigma$ such that $\mu(X) < \epsilon$ and $\mathscr{F}\text{-}\lim^+ X_n \subseteq X$, define $Y_n = X_n \setminus X$ and $0 < \delta = \epsilon - \mu(X)$, then $\{n \in \mathbb{N} : \mu(Y_n) > \delta\} \in \mathscr{F}^+$ but $\mathscr{F}\text{-}\lim Y_n = \varnothing$. This implies that the sequence of the measures of Y_n converges to 0 in the filter, in particular $\{n \in \mathbb{N} : \mu(Y_n) < \delta\} \in \mathscr{F}$, which is a contradiction.

For (4) implies (5), consider $X_n \in \Sigma$ and assume the property is false, then find $X \in \Sigma$, such that $\mathscr{F}\text{-}\liminf \mu(X_n) < \mu(X)$ and $X \subseteq \mathscr{F}\text{-}\lim X_n$. As always, define $Y_n = X \setminus X_n$, so that if $0 < \delta = \mu(X) - \mathscr{F}\text{-}\liminf \mu(X_n)$ then $\{n \in \mathbb{N} : \mu(Y_n) > \delta\} \in \mathscr{F}$ and $\mathscr{F}\text{-}\lim Y_n = \varnothing$, which is impossible.

For (5) implies (1), it is clear we may restrict, again, our attention to a sequence of non-negative function $(f_n)_{n\in\mathbb{N}}$ bounded by f, all of them defined over a measure space. Define

$$A_f = \{(x,t) \in \Omega \times [0,\infty) : 0 \le t < f(x)\}$$

and for each $n \in \mathbb{N}$,

$$B_n = \{(x,t) \in \Omega \times [0,\infty) : 0 \le t < f_n(x)\}.$$

Define $\nu = \mu \times \lambda$ where λ is the Lebesgue measure. By Fubini's theorem we get $\nu(B_n) = \int f_n d\mu$ and $\underline{\int} \mathscr{F}\text{-}\liminf g_n d\mu = \nu_*(\mathscr{F}\text{-}\lim B_n)$ and that (A_f, ν) is finite and atomless, so we are done. ∎

There are analogs for this properties where we only consider the measure space $\big(2^{\mathbb{N}}, \mathbb{B}(2^{\mathbb{N}}), \lambda\big)$, with λ the Haar measure on the Borel sets of $2^{\mathbb{N}}$. Under an additional hypothesis on the filter these notions are also equivalent. Recall that a filter $\mathscr{F}$ on $\mathbb{N}$ is *universally measurable* if it is measurable by any Borel measure on $2^{\mathbb{N}}$, where we are identifying a set with its characteristic function.

PROPOSITION 3.2. (SOLECKI [14]) *Let $\mathscr{F}$ be an universally measurable filter, assume the filter is Lebesgue with respect to the Haar measure over the Borel subsets of $2^{\mathbb{N}}$, then it is Lebesgue.*

Proof. Consider an arbitrary finite measure space (Ω, Σ, μ) and a sequence $\{X_n \in \Sigma : n \in \mathbb{N}\}$ such that $\mu(\mathscr{F}\text{-}\lim X_n) = 0$. We may assume w.l.o.g. that $\mu(\Omega) = 1$. Call $X = \mathscr{F}\text{-}\lim X_n$, and consider the sets $X' = X \times [0,1]$, $X'_n = X_n \times [0,1]$ and the measure $\mu' = \mu \times \lambda$ on $\Omega' = \Omega \times [0,1]$ where λ is the Lebesgue measure on $[0,1]$ so we get that $\mu'(X') = \mu(X)$ and for any $n \in \omega$, $\mu'(X'_n) = \mu(X_n)$. Notice that μ' is an atomless probability measure. By the Caratheodory and Sikorski theorems [13, Theorem 15.4 and Proposition 15.3] there is a measurable function $f : \Omega' \to 2^{\mathbb{N}}$ such that for any Borel set $B \subseteq 2^{\mathbb{N}}$ $\lambda'(B) = \mu'(f^{-1}(B))$ where λ' is the Haar measure on $2^{\mathbb{N}}$, and $\{B_n \in \mathbb{B}(2^{\mathbb{N}}) : n \in \mathbb{N}\}$ such that for any n, $\mu'(X'_n \triangle f^{-1}(B_n)) = 0$. Then $\lambda'_*(\mathscr{F}\text{-}\lim B_n) = 0$. Let $g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be defined by $g(x)(n) = 1$ if and only if $x \in B_n$. The function g is Borel and $g^{-1}[\mathscr{F}] = \mathscr{F}\text{-}\lim B_n$, so this last set is λ'-measurable. This implies $\lambda'(\mathscr{F}\text{-}\lim B_n) = 0$, so $\mathscr{F}\text{-}\lim \mu(X_n) = \mathscr{F}\text{-}\lim \mu'(X'_n) = \mathscr{F}\text{-}\lim \lambda'(B_n) = 0$ as desired. ∎

By Solecki's theorem and the fact that $\mathcal{S} \not\leq_K \mathcal{ED}_{fin}$ [6] we get the following result that generalizes [2, Proposition 6.2].

COROLLARY 3.3. *Any $\mathscr{F}$ universally measurable filter such that for every $J \in \mathscr{F}^+$, $\mathscr{F}^*|_J \leq_K \mathcal{ED}_{fin}$ is a Lebesgue Filter.*

4. Q-POINT IS NECESSARY

In this section we will present a simple proof of the direct implication of the main theorem. To do so, let

$$J = \big\{A \subseteq \Delta : \forall k \big|\{i : (k,i) \in A\}\big| \leq 1\big\}.$$

Let $K_{\mathcal{ED}_{fin}} = \big\{x \in \{-1,1\}^{\mathbb{N}\times\mathbb{N}} : x^{-1}[\{-1\}] \in J\big\}$, which is clearly compact, and let $X = \mathcal{C}\big(K_{\mathcal{ED}_{fin}}\big)$. Notice that $K_{\mathcal{ED}_{fin}}$ has the following property: Given any open subset U we can find $n_U \in \mathbb{N}$ such that for any $n \geq n_U$, $m \leq n$, and any $\eta \in \{-1,1\}$ there is $y \in U$ such that $y(n,m) = \eta$. This is so because for any open U we can find $s \in \{-1,1\}^{<(\mathbb{N}\times\mathbb{N})}$ such that $\big\{t \in K_{\mathcal{ED}_{fin}} : s \subseteq t\big\} \subseteq U$, then it is enough to take $n_U = \max\{n : \exists\, m (n,m) \in \mathrm{dom}(s)\}$.

As a consequence of this property $K_{\mathcal{ED}_{fin}}$ is perfect, so it is homeomorphic to the Cantor set. We will draw another consequence of this property.

Consider the sequence $\{e_{(i,j)}\}_{(i,j)\in\Delta} \subseteq X$, defined by $e_{(i,j)}(x) = x(n,k)$ if $(i,j) = (n,k)$ and 0 otherwise. We claim it is L-orthogonal (under any enumeration of Δ). Consider $f \in X$, and pick $x \in K_{\mathcal{ED}_{fin}}$ such that $\eta \|f\| = f(x)$, pick $\epsilon > 0$ then for every $n \geq n_\epsilon$ and every $m \leq n$, we can find $y \in K_{\mathcal{ED}_{fin}}$ such that $\eta = y(n,m)$ and $|f(y) - f(x)| < \epsilon$, therefore:

$$\begin{aligned} \|f\| + 1 - \epsilon = |\eta \|f\| + \eta| - \epsilon = |f(x) + y(n,m)| - \epsilon \\ < \left|f(y) + e_{(n,m)}(y)\right| \leq \left\|f + e_{(n,m)}\right\| \leq \|f\| + 1. \end{aligned}$$

Notice that for any $x \in K_{\mathcal{ED}_{fin}}$,

$$\{(i,j) \in \Delta : e_{(i,j)}(x) \neq 1\} = \{(i,j) \in \Delta : x(i,j) = -1\} \in \mathcal{ED}_{fin}.$$

So $(\mathcal{ED}_{fin})^*\text{-}\lim e_{(i,j)}(x) = 1$ for every $x \in K_{\mathcal{ED}_{fin}}$, recalling that $(\mathcal{ED}_{fin})^*$ is a Lebesgue filter we may conclude that $(\mathcal{ED}_{fin})^*\text{-}\lim e_{(i,j)} = 1$ in the weak* topology, by the Riesz representation theorem.

Actually this sequence fulfills a stronger property:

Theorem 4.1. *Let $X = \mathcal{C}(K_{\mathcal{ED}_{fin}})$, $\{e_{(i,j)}\}_{(i,j)\in\Delta} \subseteq X$, and $\mathscr{F}$ such that $\mathcal{ED}_{fin} \leq_{KB} \mathscr{F}^*$, then there is a natural subsequence such that its $\mathscr{F}$-limit in the weak topology is the constant map* 1.

Proof. Assume $\mathcal{ED}_{fin} \leq_{KB} \mathscr{F}^*$ and a witness of $\varphi : \mathbb{N} \to \mathbb{N}^2$. Consider the subsequence $\{e_{\varphi(n)}\}_{n\in\mathbb{N}}$, we already know its L-orthogonal because φ is finite to 1.

But because φ is a witness we get that for any $x \in K_{\mathcal{ED}_{fin}}$, $\mathscr{F}\text{-lim}\, e_{\varphi(n)}(x) = 1$, as the next computation shows

$$\begin{aligned} \{n \in \mathbb{N} : e_{\varphi(n)}(x) \neq 1\} = \{n \in \mathbb{N} : x(\varphi(n)) = -1\} \\ \subseteq \varphi^{-1}\big[\{(i,j) \in \Delta : x(i,j) = -1\}\big] \in \mathscr{F}^*. \end{aligned}$$

Appealing again to the Lebesgue property we get the desired result. ∎

Corollary 4.2. *Given an ultrafilter $\mathscr{U}$ which is not a Q-point, there is a an L-orthogonal sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{C}(K_{\mathcal{ED}_{fin}})$ such that the $\mathscr{U}\text{-}\lim x_n$ in the weak topology is not an L-orthogonal element.*

5. Q-POINT IS SUFFICIENT

In this section we will prove the inverse implication of the main theorem. We will take advantage of the following fact:

THEOREM 5.1. (HRUŠÁK-MEZA-MINAMI, [8]) *Let $\mathcal{I}$ be an analytic ideal, the following are equivalent:*

(1) *$\mathcal{I}$ is a $Q^+(\mathbb{N})$-ideal.*

(2) *$\mathcal{ED}_{fin} \not\leq_{KB} \mathcal{I}$.*

(3) *$\mathcal{I}$ is not countably hitting.*

This fact readily provides a useful characterization of Q-points (compare to the Mathias' characterization of selective ultrafilters [12, Theorem 2.12]).

COROLLARY 5.2. *Let $\mathcal{U}$ be a ultrafilter, the following are equivalent:*

(1) *$\mathscr{U}$ is a Q-point.*

(2) *For every F_σ ideal $\mathcal{I}$ that is countably hitting, $\mathscr{U} \cap \mathcal{I} \neq \varnothing$.*

(3) *For every analytic ideal $\mathcal{I}$ that is countably hitting, $\mathscr{U} \cap \mathcal{I} \neq \varnothing$.*

Proof. Let us start with (1) implies (3). Assume $\mathcal{U}$ is a Q-point. Now pick $\mathcal{I}$ a countably hitting ideal, if $\mathcal{I}$ is analytic, then $\mathcal{I}$ is not a $Q^+(\mathbb{N})$-ideal, so $\mathcal{U} \not\subseteq \mathcal{I}^*$.

(3) implies (2) is trivial.

Now for (2) implies (1). Assume $\mathcal{U}$ is not a Q-point, so take $f : \mathbb{N} \to \mathbb{N}$ witnessing that $\mathscr{U}^* \geq_{KB} \mathcal{ED}_{fin}$, and consider $\mathcal{I} = \{f^{-1}[I] : I \in \mathcal{ED}_{fin}\}$. $\mathcal{J}$ is F_σ and countably hitting because $\mathcal{ED}_{fin}$ is so, but clearly $\mathcal{I} \cap \mathcal{U} = \varnothing$. ∎

We present two applications of this result. To state them we need to introduce two classes of ideals. Let X be a Banach space, $(x_n)_{n\in\mathbb{N}}$ a sequence in B_X, $(\epsilon_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers converging to zero, $Z \subseteq X$ a separable subspace of X, and $(F_n)_{n\in\mathbb{N}}$ an increasing sequence of finite dimensional subspaces of X such that $Z = \overline{\bigcup_{n\in\mathbb{N}} F_n}$. For any $B \subseteq \mathbb{N}$, $n \in \mathbb{N}$ call $B(n)$ the *n-th element* of B, $\overrightarrow{B(n)} = \{A \subseteq B : \min A \geq B(n)\}$, and for $B \subseteq \mathbb{N}$ call $C[B] = \{w \in X : w \in \overline{\mathrm{conv}}\{x_m : m \in B\}\}$ and $C_{c_o}[B] = \{w \in X : w \in \overline{\mathrm{conv}}_{c_0}\{x_m : m \in B\}\}$, then we define the sets

$$\mathcal{L}_{(F_n)_{n\in\mathbb{N}}} = \big\{B \subseteq \mathbb{N} : \forall n \in \mathbb{N},\ \forall A \in \overrightarrow{B(n)},\ \forall w \in C[A],\ \forall y \in F_n,\\ (1-\epsilon_n)(1+\|y\|) \leq \|y+w\| \,\big\}$$

and

$$\mathcal{S}_{(F_n)_{n\in\mathbb{N}}} = \big\{ B \subseteq \mathbb{N} : \forall\, n \in \mathbb{N},\ \forall\, A \in \overrightarrow{B(n)},\ \forall\, w \in C_{c_0}[A],\ \forall\, y \in F_n,$$
$$|(\|y+w\|) - 1| < \epsilon_n \max\{\|y\|, 1\} \big\},$$

and consider $\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ and $\mathcal{J}_{(F_n)_{n\in\mathbb{N}}}$ as the ideals generated respectively by $\mathcal{L}_{(F_n)_{n\in\mathbb{N}}}$ and $\mathcal{S}_{(F_n)_{n\in\mathbb{N}}}$.

It is straightforward to check that $\mathcal{L}_{(F_n)_{n\in\mathbb{N}}}$ and $\mathcal{S}_{(F_n)_{n\in\mathbb{N}}}$ are closed, so $\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ and $\mathcal{J}_{(F_n)_{n\in\mathbb{N}}}$ are F_σ ideals. We shall prove that they are both countably hitting. For the first one we need the following result [2, Lemma 3.1]:

LEMMA 5.3. ([2]) *Let X be a Banach space, $(x_n)_{\in\mathbb{N}}$ an L-orthogonal sequence, $\epsilon > 0$ and $F \subseteq X$ a finite dimensional subspace of X, then, there exists an $m \in \mathbb{N}$ such that for every $n \geq m$, $t \in \mathbb{R}$, and $y \in F$*

$$\|y + tx_n\| \geq (1-\epsilon)(\|y\| + |t|).$$

THEOREM 5.4. *Let X be a Banach space, $(x_n)_{\in\mathbb{N}}$ an L-orthogonal sequence, $(\epsilon_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers, $Z \subseteq X$ a separable subspace of X, and $(F_n)_{n\in\mathbb{N}}$ an increasing sequence of finite dimensional subspaces of X such that $Z = \overline{\bigcup_{n\in\mathbb{N}} F_n}$, $\{P_n : n \in \mathbb{N}\}$ a partition of $\mathbb{N}$ into infinite sets. Then there exists a subsequence $(x_{n_m})_{m\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$, $|\{n_m \in \mathbb{N} : n_m \in P_i\}| = \mathbb{N}$ and for every $k \in \mathbb{N}$, $y \in F_k$, $w \in \overline{\mathrm{conv}}\{x_{n_m} : m \geq k\}$,*

$$\|y + w\| \geq (1-\epsilon_k)(\|y\| + 1).$$

Proof. Consider $f : \mathbb{N} \to \mathbb{N}$ such that $n \in P_{f(n)}$. Pick a sequence $(\delta_n)_{n\in\mathbb{N}}$ of positive real numbers such that for every $n \in \mathbb{N}$, $1 - \epsilon_n < \prod_{i=n+1}^{\infty}(1-\delta_i)$. We will construct the subsequence by recursion, so assume we have built our sequence up to step k, let $E_{k+1} = \langle F_{k+1} \cup \{x_{n_0}, \dots x_{n_k}\} \rangle$ and apply the lemma with δ_{k+1} to get an N such that for any $n \geq N$, $y \in E_{k+1}$, and $\lambda \in \mathbb{R}$,

$$\|y + \lambda x_n\| \geq (1-\delta_{k+1})(\|y\| + |\lambda|)$$

so pick $n_{k+1} \in P_{f(k+1)}$ such that $\max\{N, n_k\} < n_{k+1}$.

Observe that for any $i \in \mathbb{N}$, $\{n_m \in \mathbb{N} : m \in P_i\} \subseteq P_i$, so the first condition is met. To check the second condition consider $k \in \mathbb{N}$, $y \in F_k$, and

$w \in \overline{\text{conv}}\{x_{n_m} : m \geq k\}$. So, we have a sequence $(t_j)_{k \geq j}$ of non-negative reals such that $\sum_{k=j}^{\infty} t_j = 1$ and $w = \sum_{k=j}^{\infty} t_j x_{m_j}$, then for any $l > k$

$$\begin{aligned}
\left\| y - \sum\nolimits_{k \leq j}^{l} t_j x_{m_j} \right\| &\geq (1 - \delta_l)\left(\left\| y - \sum\nolimits_{k \leq j}^{l-1} t_j x_{m_j} \right\| + t_l \right) \\
&\geq (1 - \delta_l)(1 - \delta_{l-1})\left(\left\| y - \sum\nolimits_{k \leq j}^{l-2} t_j x_{m_j} \right\| + t_l + t_{l-1} \right) \\
&\quad \vdots \\
&\geq \prod\nolimits_{k \leq j}^{l} (1 - \delta_j)\left(\|y\| + \sum\nolimits_{k \leq j}^{l} t_j \right) \\
&\geq \prod\nolimits_{k \leq j}^{\infty} (1 - \delta_j)\left(\|y\| + \sum\nolimits_{k \leq j}^{l} t_j \right) \\
&> (1 - \epsilon_k)\left(\|y\| + \sum\nolimits_{k \leq j}^{l} t_j \right).
\end{aligned}$$

It is evident that the sequence $\left(y - \sum_{k \leq j}^{l} t_j x_{m_j}\right)_{l \in \mathbb{N}}$ converges in norm to $y - \sum_{k \leq j}^{\infty} t_j x_{m_j}$, so the previous computation ensures that $\left\| y - \sum_{k \leq j}^{\infty} t_j x_{m_j} \right\| \geq (1 - \epsilon_k)(\|y\| + 1)$. ∎

This result directly implies that the associated ideal $\mathcal{I}_{(F_n)_{n \in \mathbb{N}}}$ is countably hitting. Avilés, Martínez-Cervantes, and Rueda Zoca [2, Lemma 3.3] using Maurey's technique also prove a result which can be stated in the following way.

Lemma 5.5. ([2]) *Let X be a Banach space, $(x_n)_{\in \mathbb{N}}$ an L-orthogonal sequence, $(\epsilon_n)_{n \in \mathbb{N}}$ a sequence of positive real numbers converging to zero, $Z \subseteq X$ a separable subspace of X, and $(F_n)_{n \in \mathbb{N}}$ an increasing sequence of finite dimensional subspaces of X such that $Z = \overline{\bigcup_{n \in \mathbb{N}} F_n}$. If $B \in \mathcal{I}_{(F_n)_{n \in \mathbb{N}}}$ and $u \in \bigcap_{n \in \mathbb{N}} \overline{\text{conv}}^{w^*}\{x_m : m \in B, m \geq n\}$ then for any $y \in Z$, $\|u + y\| = \|u\| + \|y\| = 1 + \|y\|$.*

Corollary 5.6. *Let X be a Banach space, $(x_n)_{\in \mathbb{N}}$ an L-orthogonal sequence, and $\mathcal{U}$ a Q-point, if $x^{**} = \mathcal{U}\text{-}\lim x_n$, then x^{**} is an L-orthogonal element.*

Proof. Consider $z \in X$ and $(\epsilon_n)_{n \in \mathbb{N}}$ a sequence of positive reals converging to zero. Let $\mathcal{I}_{\langle z \rangle}$ be the associated ideal, by Theorem 5.4 we know $\mathcal{I}_{\langle z \rangle}$ is an F_σ, countably hitting ideal of $\mathbb{N}$, so by Theorem 5.1 it is not $Q^+(\mathbb{N})$, which implies that $\mathcal{U} \cap \mathcal{I}_{\langle z \rangle} \neq \varnothing$, so take $I \in \mathcal{U} \cap \mathcal{I}_{\langle z \rangle}$, then there exist $J_0, \ldots, J_m \in \mathcal{L}_{\langle z \rangle}$ such that $I \subset \bigcup_{i \leq m} J_i$. Because $\mathcal{U}$ is an ultrafilter there is $i \leq m$ such that $J_i \in \mathcal{U}$, this implies that $x^{**} \in \mathrm{cl}_{w^*}(\{x_n : n \in J_i\})$ so by the previous lemma $\|u + z\| = 1 + \|z\|$. ∎

The second result follows using similar techniques. We need an analogue to Lemma 5.3 this result is presented as [1, Lemma 2.5].

Lemma 5.7. ([1]) *Let X be a Banach space, $(x_n)_{\in \mathbb{N}}$ an S sequence, $\epsilon > 0$ and $F \subseteq X$ a finite dimensional subspace of X, then, there exists an $m \in \mathbb{N}$ such that for every $n \geq m$, $t \in \mathbb{R}$, and $y \in F$*

$$\big| \|y + tx_n\| - \max\{\|y\|, |t|\} \big| < \epsilon \max\{\|y\|, |t|\}.$$

Theorem 5.8. *Let X be a Banach space, $(x_n)_{\in \mathbb{N}}$ an S sequence, $(\epsilon_n)_{n \in \mathbb{N}}$ a sequence of positive real numbers, $Z \subseteq X$ a separable subspace of X, and $(F_n)_{n \in \mathbb{N}}$ an increasing sequence of finite dimensional subspaces of X such that $Z = \overline{\bigcup_{n \in \mathbb{N}} F_n}$, $\{P_n : n \in \mathbb{N}\}$ a partition of $\mathbb{N}$ into infinite sets. Then there exists a subsequence $(x_{n_m})_{m \in \mathbb{N}}$ such that for every $i \in \mathbb{N}$, $|\{n_m \in \mathbb{N} : n_m \in P_i\}| = \mathbb{N}$ and for every $k \in \mathbb{N}$, $y \in E_k = \langle F_k \cup \{x_{n_i : i < k}\}\rangle$, $w \in \mathrm{conv}_{c_0}\{x_{n_m} : m \geq k\}$,*

$$\big| \|y + w\| - \max\{\|y\|, 1\} \big| < \epsilon_k \max\{\|y\|, 1\}.$$

Proof. Consider, again, $f : \mathbb{N} \to \mathbb{N}$ such that $n \in P_{f(n)}$. Pick a sequence $(\delta_n)_{n \in \mathbb{N}}$ of positive real numbers such that for every $n \in \mathbb{N}$,

$$1 - \epsilon_n < \prod_{i=n+1}^{\infty} (1 - \delta_i) < \prod_{i=n+1}^{\infty} (1 + \delta_i) < 1 + \epsilon_n.$$

We will, once again, construct the subsequence by recursion, so assume we have built our sequence up to step k, consider E_{k+1} and apply the lemma with δ_{k+1} to get an N such that for any $n \geq N$, $y \in E_{k+1}$, and $t \in \mathbb{R}$,

$$\big| \|y + tx_n\| - \max\{\|y\|, |t|\} \big| < \delta_{k+1} \max\{\|y\|, |t|\}.$$

so pick $n_{k+1} \in P_{f(k+1)}$ such that $\max\{N, n_k\} < n_{k+1}$.

Once again, for any $i \in \mathbb{N}$, $\{n_m \in \mathbb{N} : m \in P_i\} \subseteq P_i$. To check the second condition consider $k \in \mathbb{N}$, $y \in E_k$, and $w \in \overline{\mathrm{conv}}_{c_0}\{x_{n_m} : m \geq k\}$. So, we have a sequence $(t_j)_{k=j}^{l}$ of reals such that $\max\{|t_j| : k \le j \le l\} = 1$ and $w = \sum_{k=j}^{l} t_j x_{m_j}$, then

$$\begin{aligned}
\left\| y + \sum\nolimits_{k\le j}^{l} t_j x_{m_j} \right\| &= \left\| \Big(y + \sum\nolimits_{k\le j}^{l-1} t_j x_{m_j} \Big) + t_l x_{m_l} \right\| \\
&< (1+\delta_l)\Big(\max\Big\{ \Big\| y + \sum\nolimits_{k\le j}^{l-1} t_j x_{m_j} \Big\| , |t_l| \Big\}\Big) \\
&= (1+\delta_l)\Big(\max\Big\{ \Big\| \Big(y + \sum\nolimits_{k\le j}^{l-2} t_j x_{m_j} \Big) + t_{l-1} x_{m_{l-1}} \Big\| , |t_l| \Big\}\Big) \\
&< (1+\delta_l)(1+\delta_{l-1})\Big(\max\Big\{ \Big\| \Big(y + \sum\nolimits_{k\le j}^{l-2} t_j x_{m_j} \Big) \Big\| , |t_{l-1}| , |t_l| \Big\}\Big) \\
&\ \ \vdots \\
&< \prod\nolimits_{k\le j}^{l} (1+\delta_j)\Big(\max\big\{ \|y\| , \max\{|t_j| : k \le j \le l\}\big\}\Big) \\
&< (1+\epsilon_n)\max\{\|y\| , 1\}.
\end{aligned}$$

Obviously the opposite inequality is proved in a similar way. ∎

Our final result takes advantage of the following fact [1, Lemma 2.4].

LEMMA 5.9. *Let X be a Banach Space and $\{C_\gamma : \gamma \in \Gamma\}$ a family of bounded convex sets in X^{**} with the finite intersection property and such that for every $\epsilon > 0$ and $x \in X$ there is $\gamma \in \Gamma$ such that for every $x^{**} \in C_\gamma$,*

$$|\|x + x^{**}\| - \max\{\|x\| , 1\}| < \epsilon \max\{\|x\| , 1\}.$$

Then, there is an S-element in $X^{(4)}$.

COROLLARY 5.10. *Let X be a Banach space, $(x_n)_{n\in\mathbb{N}}$ and S-sequence, if there is $\mathscr{U}$ a Q-point, then there is $x \in X^{(4)}$ an S-element.*

Proof. Fix (ϵ_n) converging to zero by applying Theorem 5.8 to the subspace 0 (and relabeling if necessary), we may assume that for any $n \in \mathbb{N}$, any $i \le n$, and any $y \in \mathrm{conv}_{c_0}\{x_j : j > n\}$, $|\|x_i - y\| - 1| < \epsilon_n$. Now, given $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ define $x_{A(n)} = \sum_{i\le A(n)} x_i$. Because of the previous inequality the set $\{x_{A(n)} : n \in \mathbb{N}\}$ is bounded for any A. So we can fix y_A^{**} a w^*-cluster point of it in $B_{X^{**}}$. Define $C_A = \mathrm{conv}\{y_B^{**} : B \subseteq A, B \in \mathscr{U}\}$, it is enough to show that $\{C_A : A \in \mathscr{U}\}$ satisfies the hypothesis of the previous lemma. So, the only missing piece is the inequality, so take $y \in X$ and $\epsilon > 0$. Consider now $\mathcal{J}_{\langle y \rangle}$, it is clear it is F_σ because $\mathcal{S}_{\langle y \rangle}$ is closed, and the previous theorem implies that it is countably hitting, so $\mathscr{U} \cap \mathcal{J}_{\langle y \rangle} \neq \varnothing$, the same argument as before shows there is $A \in \mathscr{U} \cap \mathcal{S}_{\langle y \rangle}$. We claim C_A is the desired witness, pick $z^{**} \in C_A$, and express it as $z^{**} = \sum_{i=0}^{m} t_i y^{**}_{B_i}$, because every $B_i \in \mathscr{U}$ we may choose $k \in \bigcap_{i \leq m} B_i$. Consider

$$D_{A,k} = \operatorname{conv}\{x_{B(n)} : B \subseteq A, B \in \mathscr{U}, k \in B, B(n) \geq k\}.$$

Notice that $x^{**} \in \overline{D_{A,k}}^{w^*}$ and that $D_{A,k} \subseteq \operatorname{conv}_{c_0}\{x_i : i \in A\}$ and even more, if $z \in D_{A,k}$ and we express it as a c_0-convex combination, $z = \sum_{i=0}^{n} t_i x_i$, then $\max\{|t_i| : i \leq n\} = t_k = 1$. Now, the fact that $A \in \mathcal{S}_{\langle y \rangle}$ implies that $|(\|z + y\|) - 1| < \epsilon_n \max\{\|y\|, 1\}$ for any $z \in D_{A,k}$, this easily implies that $|(\|z^{**} + y\|) - 1| < \epsilon_n \max\{\|y\|, 1\}$ for any $z^{**} \in \overline{D_{A,k}}^{w^*}$, in particular for our chosen x^{**}. ∎

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