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Classes of homothetic convex sets

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Abstract: This is a survey of known results and still open problems on characteristic properties of classes of homothetic convex sets in the n-dimensional Euclidean space. These properties are formulated in terms of orthogonal projections, plane sections, homothety classes, Choquet simplices, and homothetic tilings and partitions.

Key words: convex, homothetic, symmetric, section, projection, partition, tiling.

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Contents

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1. Introduction

Characteristic properties of special classes of convex sets became an established topic of convex geometry on the turn of 20th century. Bonnesen and Fenchel [13, §14–16] gave an overview of known results in this field, published prior to 1934. The canonical list of such special classes traditionally includes regular polytopes, balls, ellipsoids, centrally symmetric convex bodies, and bodies of constant width. Numerous properties of these classes were extensively studied in the literature, with books and surveys entirely devoted to them (see, for instance, Coxeter [22], Heil and Martini [59], Martini, Montejano, and Oliveros [70], McMullen [75], Petty [85], and Soltan [108]).

Nowadays, a wider interpretation of the concept of special convex set also includes various classes of polytopes, regular and strictly convex sets, and families of convex bodies defined by means of certain groups of transformations (like homotheties, symmetries, congruences, similarities, affine transformations, etc).

This survey describes known results and still open problems on characteristic properties of classes of pairwise homothetic convex sets in the n-dimensional Euclidean space $\mathbb{R}^n$. Despite a wide interest towards homothety classes of convex sets, no separate collection of results or surveys on this topic are known in the literature. The present paper aims to fill in this gap (at least partly) and to give a uniform presentation of existing results and still open problems. It is divided into various sections, as given by the above table of contents.

Besides the intuitive geometric appeal and simplicity of their description, the homothety classes have multiple connections with various branches of convex geometry. Homothetic convex sets are studied in Brunn-Minkowski theory (see Schneider [95]), geometric inequalities, and geometric tomography, designed to cover the area of mathematics dealing with retrieval of information about a geometric object from data about its sections and projections (see, e.g., Gardner [35]).

Crystallography and the geometry of numbers often deal with tilings of space by translates of a given polytope. Such polytopes are called parallelohedra, and there is a large variety of research on this topic (see, e.g., Gruber [55]).

Furthermore, combinatorial and discrete geometry of convex sets deals with numerous problems on packing and covering that involve various families of translates and homothety classes of convex bodies (see, for instance, Boltyanski, Martini, Soltan [10], Böröczky [14], and Brass, Moser, Pach [15]).

To achieve the uniformity of presentation, we assume throughout the text that all convex sets in question are n-dimensional. Such a restriction does not affect the generality of the argument, since we always can consider the sets in their affine spans, where they become full-dimensional.

We conclude this section with necessary definitions and terminology (see, e.g., [109] for a detailed account). Nonempty sets X_1 and X_2 in the n-dimensional Euclidean space $\mathbb{R}^n$ are called *homothetic* if $X_1 = v + \lambda X_2$ for a suitable vector (point) $v \in \mathbb{R}^n$ and a nonzero scalar λ. Furthermore, X_1 and X_2 are *directly* (*inversely*) homothetic if $\lambda > 0$ (respectively, $\lambda < 0$). In particular, X_1 and X_2 are *translates of each other* if $\lambda = 1$, and are *symmetric to each other* if $\lambda = -1$. If $\lambda \neq 1$, then $X_1 = s + \lambda(X_2 - s)$, where the point $s = v/(1 - \lambda)$ is called the *center* of homothety. A nonempty set $X \subset \mathbb{R}^n$ is *centrally symmetric* provided there is a point $v \in \mathbb{R}^n$ such that $X = v - X$ (in this case $s = v/2$ is the center of symmetry of X). The origin (zero vector) of $\mathbb{R}^n$ is denoted o.

A *plane* $L \subset \mathbb{R}^n$ of dimension m, $0 \le m \le n$, is a translate of an m-dimensional subspace $S \subset \mathbb{R}^n$: $L = c + S$ for a suitable vector $c \in \mathbb{R}^n$. Planes L_1 and L_2 are *parallel* provided they are translates of each other. A parallel (orthogonal) projection f of $\mathbb{R}^n$ onto the plane $L = c + S$ is a mapping of the form $f(x) = c + g(x - c)$, where g is a linear (orthogonal) projection of $\mathbb{R}^n$ onto the subspace S.

A *hyperplane* is a plane of dimension $n - 1$; it can be described as

$$H = \{x \in \mathbb{R}^n : x \cdot e = \gamma\}, \qquad e \neq o, \ \gamma \in \mathbb{R}, \tag{1}$$

where $x \cdot e$ means the dot product of x and e. Every hyperplane of the form (1) determines the opposite *closed halfspaces*

$$V_1 = \{x \in \mathbb{R}^n : x \cdot e \le \gamma\} \quad \text{and} \quad V_2 = \{x \in \mathbb{R}^n : x \cdot e \ge \gamma\}. \tag{2}$$

Positive multiples λe of the vector e in (2) are called *outward* (*inward*) normal vectors of V_1 (of V_2).

In a standard way, a hyperplane $H \subset \mathbb{R}^n$ *supports* a nonempty set $X \subset \mathbb{R}^n$ if H meets X such that X is contained in one of the closed halfspaces determined by H. A closed halfspace $V \subset \mathbb{R}^n$ *supports* X if $X \subset V$ and the boundary hyperplane of V supports X.

By a *convex solid* in $\mathbb{R}^n$ we will mean an n-dimensional closed convex set in $\mathbb{R}^n$, possibly unbounded. A *convex body* is a bounded convex solid. A convex set is called *line-free* if it contains no line. A nonempty intersection of finitely many closed halfspaces of $\mathbb{R}^n$ is called a *polyhedron*, and a bounded polyhedron is a *polytope*.

A nonempty set C in $\mathbb{R}^n$ is called a *cone* with apex $p \in \mathbb{R}^n$ if $p+\lambda(x-p) \in C$ whenever $\lambda \geq 0$ and $x \in C$. (Obviously, this definition implies that $p \in C$, although a stronger condition $\lambda > 0$ can be beneficial; see, e.g., [65].) The cone C is called convex if it is a convex set.

For a nonempty set $X \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$, the *generated cone* $C_p(X)$ with apex p is defined by

$$C_p(X) = \{p + \lambda(x - p) \,:\, x \in X,\ \lambda \geq 0\}.$$

The cone $C_p(X)$ is convex if X convex. Furthermore, for any point $v \in \mathbb{R}^n$ and a scalar $\lambda > 0$, one has

$$C_{v+\lambda p}(v + \lambda X) = v + \lambda C_p(X) = v + (\lambda - 1)p + C_p(X). \tag{3}$$

The *recession cone* of a convex set $K \subset \mathbb{R}^n$ is defined by

$$\operatorname{rec} K = \{e \in \mathbb{R}^n \,:\, x + \lambda e \in K \text{ whenever } x \in K \text{ and } \lambda \geq 0\}.$$

If K is closed, then $\operatorname{rec} K$ is a closed convex cone with apex o, and $\operatorname{rec} K \neq \{o\}$ if and only if K is unbounded. The *lineality space* of K is the subspace given by $\operatorname{lin} K = \operatorname{rec} K \cap (-\operatorname{rec} K)$. If K is closed, then $\operatorname{lin} K = \{o\}$ if and only if K contains no lines.

2. Homothety conditions

This section contains a brief account of geometric results in which the concept of homothety appear as a requirement or a necessary tool. We start with the following theorem that plays an important role in classical convex geometry.

Theorem 2.1. (Brunn-Minkowski) *For convex bodies $K, L \subset \mathbb{R}^n$ and a scalar $0 < \lambda < 1$, the volumes of K, L and $(1-\lambda)K + \lambda L$ satisfy the inequality*

$$V((1-\lambda)K + \lambda L)^{1/n} \geq (1-\lambda)V(K)^{1/n} + \lambda V(L)^{1/n}. \tag{4}$$

Furthermore, equality in (4) *holds if and only if K and L are directly homothetic.*

Theorem 2.1 was discovered by Brunn (see [16, Chapter III] and [17, Chapter III]) for dimensions $n \leq 3$. Its importance was emphasized by Minkowski, who gave an analytic proof for the n-dimensional case (see [78, § 56 and § 57]) and characterized the equality case. Numerous variations and generalizations of this result, called nowadays the Brunn-Minkowski Theory, form one of the central fields of modern convex geometry (see, e.g., the survey of Gardner [34] and the monograph of Schneider [95] for historical references and a variety of related facts).

The next result is related to the study of *intersection bodies*, which has an essential role in the dual Brunn-Minkowski theory and in geometric tomography. We recall that a star body $K \subset \mathbb{R}^n$ is a nonempty set whose radial function $\rho_K(e)$, defined on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$, is positive and continuous. Accordingly,

$$K = \{te : 0 \leq t \leq \rho_K(e),\ e \in \mathbb{S}^{n-1}\}.$$

The intersection body IK of a star-body K was introduced and studied by Lutwak [67], who defined IK as the star-body given by the positive radial function $\rho_{IK}(e)$:

$$\rho_{IK}(e) = \mathrm{Vol}_{n-1}(K \cap e^\perp), \qquad e \in \mathbb{S}^{n-1},$$

where Vol_{n-1} stands for the $(n-1)$-dimensional volume and $e^\perp$ denotes the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to e.

In 1956 Busemann and Petty posed the problem (see [20], Problem 5) whether solid ellipsoids are the only convex bodies for which the family of special inscribed cones have the same volume. Reformulating this problem in affine terms, Lutwak [68] asked whether solid ellipsoids centered at the origin o are the only star-bodies $K \subset \mathbb{R}^n$ for which the bodies $I(IK)$ and K are directly homothetic; equivalently, that $I(IK) = cK$, where $c = c(K)$ is a suitable positive scalar. The recent preprint of Milman, Shabelman, and Yehudayoff [76] affirmatively answers Lutwak's question, as given below.

THEOREM 2.2. *Let K be a star-body in $\mathbb{R}^n$, $n \geq 3$. Then $I(IK) = cK$ for a suitable scalar $c > 0$ if and only if K is a solid ellipsoid centered at o.*

Another instance is the homothety problem for *floating bodies*. We recall that, given a convex body $K \subset \mathbb{R}^n$ and a scalar $\delta > 0$, the floating body K_δ is defined as the intersection (possibly empty) of all closed halfspaces whose bounding hyperplanes cut off a set of volume δ from K. Sharpening the assertions from [97] and [115], Werner and Ye [122] proved the following result.

Theorem 2.3. *Given a convex body $K \subset \mathbb{R}^n$, there exists a scalar $\delta(K) > 0$ such that the conditions below are equivalent:*

(a) *the floating body K_δ is directly homothetic to K for some $0 < \delta < \delta(K)$,*

(b) *K is a solid ellipsoid.*

One more example gives the *Choquet representation theory* in ordered vector spaces (see Choquet [21] and Phelps [87]). Namely, if $C = \{x \in E \mid x \geq o\}$ is the positive cone of an ordered vector space E, and if S is a convex base of C, then E is a vector lattice if and only if S is a Choquet simplex; which means that every nonempty intersection of directly homothetic copies of S is again a directly homothetic copy of S or a singleton:

$$(u + \lambda S) \cap (v + \mu S) = w + \nu S, \qquad u, v, w \in E, \ \ \lambda, \mu > 0, \ \ \nu \geq 0.$$

In finite dimension, line-free closed Choquet simplices are precisely the usual simplices and simplicial cones (see the survey [99]).

3. Homothetic projections

It is easy to see that parallel projections of homothetic sets X_1 and X_2 on a plane $L \subset \mathbb{R}^n$ are homothetic to each other (see Figure 1). Indeed, let $X_1 = v + \lambda X_2$, with $v \in \mathbb{R}^n$ and $\lambda \neq 0$. If $L = c + S$ for a suitable vector

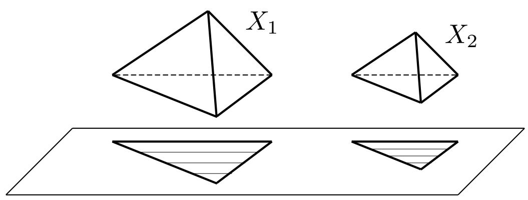

Figure 1: Parallel projections of homothetic sets.

$c \in L$ and a subspace S of $\mathbb{R}^n$, then a parallel projection f of $\mathbb{R}^n$ on L can be given as $f(x) = c + g(x - c)$, where g is a linear projection of $\mathbb{R}^n$ on S. With $u = (1 - \lambda)(c - g(c)) + g(v)$, we have

$$
\begin{aligned}
f(X_1) &= f(v+\lambda X_2) = c + g(v+\lambda X_2 - c) \\
&= (1-\lambda)(c-g(c)) + g(v) + \lambda(c + g(X_2 - c)) = u + \lambda f(X_2).
\end{aligned}
$$

A natural question here is whether the sets X_1 and X_2 in $\mathbb{R}^n$ are homothetic themselves provided their parallel (or even only orthogonal) projections on every proper plane of $\mathbb{R}^n$ are homothetic. Generally, this is not true (see examples below), but the question has an affirmative answer for the case of convex bodies, as described below.

In 1926, Bonnesen [11] (see also [12, p. 128]) proved that two convex bodies in $\mathbb{R}^3$ are directly homothetic if and only if the orthogonal projections of these bodies on every plane are directly homothetic, where the similarity ratio of the projections is the same for all projection planes. Three years later, Süss [116] announced a similar statement for a more general case, which allows a similarity ratio of the projections be depend on the projection plane. However, as mentioned by Bonnesen and Fenchel [13, p. 34], the proof there is not formulated correctly, and an accurate and simplified presentation was given later by Süss [117]. Following [116], other analytical proofs of this fact were given by Kubota [64] and Nakajima [82, 83].

Since the argument of Süss [117] can be routinely extended to the case $n \geq 3$, Bonnesen and Fenchel [13, p. 13] formulated in 1934 the following result as a known fact.

THEOREM 3.1. ([117]) *Convex bodies K_1 and K_2 in $\mathbb{R}^n$, $n \geq 3$, are directly homothetic if and only if the orthogonal projections of these bodies on every hyperplane are directly homothetic, where the homothety ratio may depend on the projection hyperplane.*

The following example shows that the assumption on K_1 and K_2 in Theorem 3.1 cannot be relaxed by assuming that they are only solid and bounded (possibly nonclosed).

EXAMPLE 3.2. It is easy to see that the following distinct convex sets K_1 and K_2 in $\mathbb{R}^2$ have identical parallel projections on every line in $\mathbb{R}^2$:

$$
\begin{aligned}
K_1 &= \{(x,y) : 0 < x \leq 1, 0 \leq y \leq 1\} \cup \{(0,0)\}, \\
K_2 &= \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\} \setminus \{(0,1)\}.
\end{aligned}
$$

An important particular case of Theorem 3.1 states that convex bodies K_1 and K_2 in $\mathbb{R}^n$ are translates of each other if and only if the orthogonal projections of these bodies on every hyperplane of $\mathbb{R}^n$ are translates of each other. This case was considered by Alexandrov [5, § 5], Leichtweiß[66, pp. 241–243], Ryabogin [92], Schneider [95, p. 351], and, probably, some others.

The assertion of Theorem 3.1 can be refined by considering certain families of projection hyperplanes. Let $F \subset \mathbb{R}^n$ be a set of unit vectors, symmetric about the origin of $\mathbb{R}^n$, and let $\mathcal{H}(F)$ denote the family of $(n-1)$-dimensional subspaces in $\mathbb{R}^n$ whose unit normals belong to F. Then F is called a *homothety-set*, provided any two convex bodies in $\mathbb{R}^n$ are directly homothetic if and only if their orthogonal projections on every subspace from $\mathcal{H}(F)$ are directly homothetic. Székely [118] showed that F is a homothety-set if and only if it contains three non-collinear vectors and the closure of F meets every big $(n-2)$-dimensional sphere of the unit sphere in $\mathbb{R}^n$. For the case of translates of compact convex sets in $\mathbb{R}^n$ this fact was lately obtained by Golubyatnikov [40].

Given a pair of nonempty sets X and Y in $\mathbb{R}^n$ and an integer m, $2 \le m \le n-1$, one may consider the following property $P_m(X,Y)$: the orthogonal projections of X and Y on every m-dimensional plane in $\mathbb{R}^n$ are homothetic. As shown above, $P_m(X,Y)$ holds if X and Y are homothetic. Furthermore, $P_k(X,Y) \Rightarrow P_m(X,Y)$ provided $k > m$. Indeed, if an m-dimensional plane $M \subset \mathbb{R}^n$ is contained in a k-dimensional plane $L \subset \mathbb{R}^n$, then the orthogonal projection f of $\mathbb{R}^n$ on M can be expressed as the composition $f = g \circ h$, where h is the orthogonal projection of $\mathbb{R}^n$ on L, and g is the orthogonal projection of L on M.

In view of this argument, Theorem 3.1 can be sharpened by considering orthogonal projections on planes of dimension smaller than $n-1$. Groemer [47] proved that convex bodies K_1 and K_2 in $\mathbb{R}^n$, one of them being centrally symmetric, are directly homothetic if and only if there is an integer m, $2 \le m \le n-1$, such that the orthogonal projections of K_1 and K_2 on every m-dimensional plane in $\mathbb{R}^n$ are directly homothetic. Hadwiger [58] observed that Groemer's result holds without any symmetry requirements and gave a backward induction proof. Finally, Rogers [89] provided a simple proof of this assertion for the case $m = 2$ and observed that it holds for any integer m, $2 \le m \le n-1$.

THEOREM 3.3. ([58, 89]) *Convex bodies K_1 and K_2 in $\mathbb{R}^n$, $n \ge 3$, are directly homothetic if and only if there is an integer m, $2 \le m \le n-1$, such that the orthogonal projections of these bodies on every m-dimensional plane* *of* $\mathbb{R}^n$ *are directly homothetic, where the homothety ratio may depend on the projection plane.*

In view of the above results, we pose the following problem.

PROBLEM 3.4. Let $K_1, \dots, K_t$ be convex bodies in $\mathbb{R}^n$, $n \geq 3$, and m, $2 \leq m \leq n-1$, be an integer. Suppose that a convex body $K \subset \mathbb{R}^n$ satisfies the following condition: for any m-dimensional plane $L \subset \mathbb{R}^n$, the orthogonal projection $f(K)$ of K on L is a translate (directly homothetic copy) of one of the respective orthogonal projections $f(K_1), \dots, f(K_t)$. Is it true that K is a translate (directly homothetic copy) of one of the sets $K_1, \dots, K_t$?

Theorem 3.3 can be further sharpened, as shown below, by choosing a reduced family of m-dimensional planes (see Hadwiger [58] for $m = n-1 \geq 3$ and Sallee [93] for $m = 2$ and $r = 1$).

THEOREM 3.5. ([105]) *Convex bodies* K_1 *and* K_2 *in* $\mathbb{R}^n$, $n \geq 3$, *are directly homothetic if and only if there are integers* r *and* m, *with* $1 \leq r \leq m-1 \leq n-2$, *and an* r*-dimensional plane* $L \subset \mathbb{R}^n$ *such that the orthogonal projections of* K_1 *and* K_2 *on every* m*-dimensional plane containing* L *are directly homothetic, where the homothety ratio may depend on the projection plane.*

The following example shows that the plane L in Theorem 3.5 cannot be chosen in advance (see Groemer [53] for similar examples).

EXAMPLE 3.6. Let K_1 and K_2 be triangular prisms in $\mathbb{R}^3$, given by

$$K_1 = \{(x, y, z) : x \geq 0,\ y \geq 0,\ x + y \leq 1,\ 0 \leq z \leq 1\},$$
$$K_2 = \{(x, y, z) : x \leq 0,\ y \leq 0,\ x + y \geq -1,\ 0 \leq z \leq 1\},$$

and let L be the z-axis of $\mathbb{R}^3$. For any 2-dimensional subspace $H \subset \mathbb{R}^3$ containing L, the orthogonal projection of K_1 on H is a translate of the respective orthogonal projection of K_2, while K_1 and K_2 are not directly homothetic.

There are several stability results concerning orthogonal projections of translates of convex bodies K_1 and K_2 on hyperplanes, formulated in terms of the *translative Hausdorff distance*

$$\tilde{\delta}(K_1, K_2) = \inf\{\delta(K_1, x + K_2) : x \in \mathbb{R}^n\},$$

where $\delta(K_1, K_2)$ denotes the standard Hausdorff distance on the family of compact sets in $\mathbb{R}^n$. Correcting an assertion of Golubyatnikov [38], Groemer [52] proved that if the orthogonal projections K_1' and K_2' of convex bodies K_1 and K_2 in $\mathbb{R}^n$ on every $(n-1)$-dimensional subspace satisfy the condition $\tilde{\delta}(K_1', K_2') \le \varepsilon$, then $\tilde{\delta}(K_1, K_2) \le (1 + 2\sqrt{2})\varepsilon$; furthermore, $\tilde{\delta}(K_1, K_2) \le \varepsilon$ provided both K_1 and K_2 are centrally symmetric.

Considering a special distance function on the family of compact sets in $\mathbb{R}^n$, which is invariant under direct homotheties, Groemer [52] established similar estimates concerning orthogonal projections of directly homothetic copies of K_1 and K_2.

The remaining part of this section is devoted to various extensions of the above results that involve arbitrary homotheties and, possibly, unbounded solids. We start with the following result from [105].

Theorem 3.7. ([105]) *For compact (respectively, closed) convex sets K_1 and K_2 in $\mathbb{R}^n$, $n \ge 3$, and an integer m, $2 \le m \le n-1$ (respectively, $3 \le m \le n-1$), the following conditions are equivalent.*

(a) *K_1 and K_2 are homothetic.*

(b) *The orthogonal projections of K_1 and K_2 on every m-dimensional plane $L \subset \mathbb{R}^n$ are homothetic, where the homothety ratio and its sign may depend on the projection plane.*

We observe that Theorem 3.7 cannot be routinely reduced to Theorem 3.3 by using compactness or continuity arguments. The main obstacle along this way represents the case when the orthogonal projections $f(K_1)$ and $f(K_2)$ on L are centrally symmetric. Namely, if these centrally symmetric projections are related as $f(K_1) = u + \lambda f(K_2)$, then u and λ (but not the absolute value of λ) are not uniquely determined. For instance, the homothetic squares K_1 and K_2 in Figure 2 have two distinct centers of homothety.

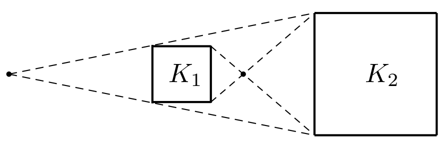

Figure 2: Squares with two centers of homothety.

The following example shows that the inequality $m \geq 3$ in Theorem 3.7 is sharp for the case of unbounded convex solids.

Example 3.8. Let K_1 and K_2 be solid paraboloids in $\mathbb{R}^3$, given, respectively, by

$$K_1 = \big\{(x, y, z) \,:\, x^2 + y^2 \leq z\big\} \quad \text{and} \quad K_2 = \big\{(x, y, z) \,:\, 2x^2 + y^2 \leq z\big\}.$$

Obviously, K_1 and K_2 are not homothetic. On the other hand, their parallel projections $f(K_1)$ and $f(K_2)$ on every plane $L \subset \mathbb{R}^3$ are directly homothetic. Indeed, this is obvious if $\dim L = 1$. If $\dim L = 2$, then either $f(K_1) = f(K_2) = L$, or $f(K_1)$ and $f(K_2)$ are convex solids in L bounded by parabolas with parallel axes of symmetry, and thus are directly homothetic.

Based on Example 3.8, one might pose the following problem.

Problem 3.9. What is the relation between convex solids K_1 and K_2 in $\mathbb{R}^n$, $n \geq 3$, such that their orthogonal projections on every 2-dimensional plane of $\mathbb{R}^n$ are homothetic?

The next two results describe some particular extensions of Theorem 3.7 to the case of 2-dimensional projections.

Theorem 3.10. ([101]) *Given a convex solid $K_1 \subset \mathbb{R}^n$, $n \geq 3$, there is a line $l \subset \mathbb{R}^n$ (depending on K_1) with the following property: a convex solid $K_2 \subset \mathbb{R}^n$ is a translate of K_1 if and only if the orthogonal projection of K_2 on every 2-dimensional plane L containing l is a translate of the respective orthogonal projection of K_1 on L.*

Theorem 3.11. ([113]) *Polyhedra P_1 and P_2 in $\mathbb{R}^n$, $n \geq 3$, are homothetic if and only if their orthogonal projections on every 2-dimensional plane of $\mathbb{R}^n$ are homothetic.*

The assertion of Theorem 3.11 also holds for the case of closed convex sets which are sums of polytopes and closed convex cones, while there are examples of non-homothetic M-decomposable sets (that is, sums of compact convex sets and closed convex sets, as defined in [37, 110, 111]) whose orthogonal projections on every 2-dimensional plane of $\mathbb{R}^n$ are directly homothetic (see [113]).

4. Generated cones

Geometrically, a parallel projection of a nonempty set $X \subset \mathbb{R}^n$ on a hyperplane $H \subset \mathbb{R}^n$ can be viewed as the intersection of H with the both-way infinite cylinder $X + l$, where l is a suitable 1-dimensional subspace. Clearly, the parallel projections of nonempty sets X_1 and X_2 on H are homothetic if and only if the cylinders $X_1 + l$ and $X_2 + l$ are homothetic. This argument suggests an interpretation of central projections in terms of generated cones. Namely, the projection with center $p \in \mathbb{R}^n$ of a nonempty set $X \subset \mathbb{R}^n$ on a hyperplane $H \subset \mathbb{R}^n$ may be viewed as the intersection of the generated cone $C_p(X)$ and H (see Figure 3).

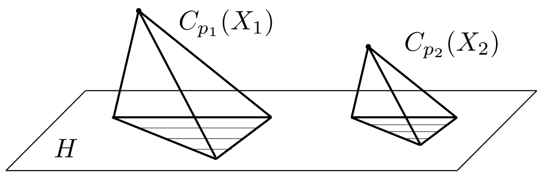

Figure 3: Generated cones of homothetic sets.

Suppose that nonempty sets X_1 and X_2 are directly homothetic: $X_1 = v + \lambda X_2$ for suitable $v \in \mathbb{R}^n$ and $\lambda > 0$. For any point $p_1 \in \mathbb{R}^n \setminus X_1$, let $p_2 = \lambda^{-1}(p_1 - v)$. Then

$$p_1 = v + \lambda p_2, \qquad p_2 \in \mathbb{R}^n \setminus \lambda^{-1}(X_1 - v) = \mathbb{R}^n \setminus X_2$$

and, by (3),

$$\begin{aligned} C_{p_1}(X_1) &= C_{v+\lambda p_2}(v + \lambda X_2) = v + (\lambda - 1)p_2 + C_{p_2}(X_2) \\ &= p_1 - p_2 + C_{p_2}(X_2). \end{aligned}$$

Based on this argument, we can formulate the assertion below.

Theorem 4.1. ([113]) *Let K_1 and K_2 be proper closed convex sets in $\mathbb{R}^n$, $n \geq 3$, with $\dim(\operatorname{lin} K_1) \leq n - 3$. The following conditions are equivalent.*

(a) *K_1 and K_2 are directly homothetic.*

(b) *For any point $p_1 \in \mathbb{R}^n \setminus K_1$ there is a point $p_2 \in \mathbb{R}^n \setminus K_2$ such that $C_{p_1}(K_1)$ is a translate of $C_{p_2}(K_2)$.*

Remark 4.2. It is known that for a closed convex set $K \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n \setminus K$, the generated cone $C_p(K)$ has a unique apex. Consequently, condition (b) in Theorem 4.1 can be equivalently reformulated as follows:

(b') For any point $p_1 \in \mathbb{R}^n \setminus K_1$ there is a point $p_2 \in \mathbb{R}^n \setminus K_2$ such that

$$C_{p_1}(K_1) = p_1 - p_2 + C_{p_2}(K_2). \tag{5}$$

We also observe that the point p_2 in condition (b') may be not uniquely determined by p_1. For instance, if $K_1 = K_2 = \{(x, y) : x, y \geq 0\}$, then the equality (5) holds for any choice of points p_1 and p_2 in the open domain $\{(x, y) : x, y < 0\}$.

The restriction $\dim(\operatorname{lin} K_1) \leq n - 3$ in Theorem 4.1 is essential. Indeed, if the convex sets K_1 and K_2 in $\mathbb{R}^3$ are given by

$$K_1 = \big\{(x, y, z) : x^2 + y^2 \leq 1\big\} \quad \text{and} \quad K_2 = \{(x, y, z) : 0 \leq x, y \leq 1\}.$$

Then $\operatorname{lin} K_1 = \operatorname{lin} K_2$, which is the z-axis. It is easy to see that for any point $p_1 \in \mathbb{R}^3 \setminus K_1$ there is a point $p_2 \in \mathbb{R}^3 \setminus K_2$ such that $C_p(K_1)$ is a translate of $C_p(K_2)$, while K_1 and K_2 are not directly homothetic.

Remark 4.3. Theorem 4.1 allows a reduction to Theorem 3.1 in the case when both sets K_1 and K_2 are compact. We provide here a sketch of the argument. Given a hyperplane $H \subset \mathbb{R}^n$, translate it such that both K_1 and K_2 are contained in the same open halfspace W determined by H. Denote by l the 1-dimensional subspace of $\mathbb{R}^n$ orthogonal to H, and let $h = l \cap V$, where $V = \mathbb{R}^n \setminus W$. For any point $p_1 \in h$, the respective point p_2 satisfying condition (b') belongs to V. If p_1 tends to infinity along h, then the section $C_{p_1}(K_1) \cap H$ tends to the orthogonal projection of K_1 on H. Similarly, the respective section $C_{p_2}(K_2) \cap H$ tends to the orthogonal projection of K_2 on H. Since, by the assumption, the cones $C_{p_1}(K_1)$ and $C_{p_2}(K_2)$ are translates of each other, their sections $C_{p_1}(K_1) \cap H$ and $C_{p_2}(K_2) \cap H$ are directly homothetic. Consequently, the orthogonal projections of K_1 and K_2 on H, as limits of these sections, are directly homothetic. Because the hyperplane H is chosen arbitrarily, the sets K_1 and K_2 are directly homothetic, according to Theorem 3.1.

5. Homothetic sections

Plane sections of convex bodies are often considered as dual operations to their orthogonal projections. It is easy to see that suitable plane sections of homothetic sets X_1 and X_2 in $\mathbb{R}^n$ are also homothetic. For instance, given a point $p_1 \in \mathbb{R}^n$, there is a point $p_2 \in \mathbb{R}^n$ such that for every pair of parallel planes L_1 and L_2 through p_1 and p_2 respectively, the sections $X_1 \cap L_1$ and $X_2 \cap L_2$ are both empty or are homothetic (see Figure 4). Indeed, let $X_1 = v + \lambda X_2$, with $\lambda \neq 0$. If L_1 is a plane through p_1, then it can be expressed

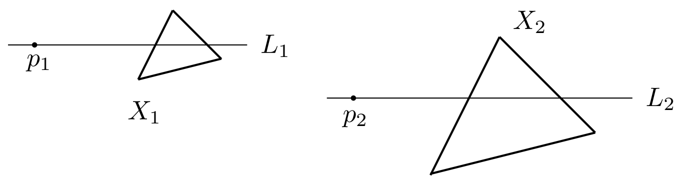

Figure 4: Sections of homothetic sets.

as $L_1 = p_1 + S$ for a suitable subspace S of $\mathbb{R}^n$. With $p_2 = \lambda^{-1}(p_1 - v)$, the plane $L_2 = p_2 + S$ is parallel to L_1 and, due to $\lambda S = S$,

$$\begin{aligned} X_1 \cap L_1 &= (v + \lambda X_2) \cap (p_1 + S) = (v + \lambda X_2) \cap (v + \lambda p_2 + S) \\ &= v + \lambda(X_2 \cap (p_2 + S)) = v + \lambda(X_2 \cap L_2). \end{aligned}$$

The following theorem was proved by Rogers [89] (for the case when the points p_1 and p_2 are interior to the convex bodies K_1 and K_2) and later extended by Burton [18].

Theorem 5.1. ([18, 89]) *Convex bodies K_1 and K_2 in $\mathbb{R}^n$, $n \geq 3$, are directly homothetic if and only if there are points p_1 and p_2 in $\mathbb{R}^n$ such that for every pair of parallel 2-dimensional planes L_1 and L_2 through p_1 and p_2 respectively, the sections $K_1 \cap L_1$ and $K_2 \cap L_2$ are both empty or are directly homothetic.*

Rogers [89] observed that Theorem 5.1 (for the case $p_1 \in \operatorname{int} K_1$ and $p_2 \in \operatorname{int} K_2$) remains true if the parallel planes L_1 and L_2 have some intermediate dimension m, $2 \leq m \leq n-1$. Affirmatively answering Burton's question (see [18]), Burton and Mani [19] proved the following deep result.

Theorem 5.2. ([19]) *Let K_1 and K_2 be convex bodies and p_1 and p_2 be points in $\mathbb{R}^n$, $n \geq 3$. Given an integer m, $2 \leq m \leq n-1$, suppose that for every pair of parallel m-dimensional planes L_1 and L_2 through p_1 and p_2 respectively, the sections $K_1 \cap L_1$ and $K_2 \cap L_2$ are both empty or are homothetic. If there* *is a direct homothety* $h : \mathbb{R}^n \to \mathbb{R}^n$, *such that* $h(K_1) = K_2$ *and* $h(p_1) \neq p_2$, *then* K_1 *and* K_2 *are solid ellipsoids.*

In view of Theorem 3.7, we pose the following problem.

PROBLEM 5.3. Let K_1 and K_2 be convex solids and p_1 and p_2 be points in $\mathbb{R}^n$, $n \geq 3$. Given an integer m, $2 \leq m \leq n-1$, suppose that for every pair of parallel m-dimensional planes L_1 and L_2 through p_1 and p_2 respectively, the sections $K_1 \cap L_1$ and $K_2 \cap L_2$ are both empty or are homothetic, where the homothety ratio and its sign may depend on the choice of planes. Are K_1 and K_2 homothetic?

For the case of unbounded convex solids, Problem 5.3 is partly confirmed by the following results.

THEOREM 5.4. ([101]) *Let* K_1 *and* K_2 *be convex solids in* $\mathbb{R}^n$, $n \geq 3$, *and* $p_1 \in K_1$ *and* $p_2 \in K_2$ *be points such that for every pair of parallel 2-dimensional planes* L_1 *and* L_2 *through* p_1 *and* p_2, *respectively, the section* $K_1 \cap L_1$ *is a translate of* $K_2 \cap L_2$. *Then* K_1 *is a translate of* K_2. *Moreover, if both* K_1 *and* K_2 *are unbounded and line-free, then* $K_1 = u + K_2$ *implies that* $p_1 = u + p_2$.

As follows from Theorem 5.2, a convex body $K \subset \mathbb{R}^n$ is a solid ellipsoid provided there are distinct points p_1 and p_2 in $\mathbb{R}^n$, $n \geq 3$, and an integer m, $2 \leq m < n$, such that for every pair of parallel m-dimensional planes L_1 and L_2 through p_1 and p_2, respectively, the sections $K \cap L_1$ and $K \cap L_2$ are both empty or are homothetic (see [19]). The theorem below partly generalizes this assertion to the case of convex solids.

THEOREM 5.5. ([104]) *Let* $K \subset \mathbb{R}^n$, $n \geq 4$, *be a line-free convex solid and* p_1 *and* p_2 *be distinct points in* $\operatorname{int} K$ *such that every pair of parallel* m-*dimensional planes* L_1 *and* L_2 *through* p_1 *and* p_2 *respectively, the sections* $K \cap L_1$ *and* $K \cap L_2$ *are directly homothetic. Then* K *is either a solid convex quadric or a convex cone whose apex belongs to the line through* p_1 *and* p_2.

Jerónomo-Castro, Montejano, and Morales-Amaya [62] (see also Montejano [79]) proved the following variations of Rogers' result: If K_1 and K_2 are strictly convex bodies in $\mathbb{R}^3$ such that for every 2-dimensional subspace $S \subset \mathbb{R}^3$ one can choose continuously planar sections of K_1 and K_2, parallel to S, which are translated copies (respectively, directly homothetic copies) one of each other, then K_1 is a translate of K_2 (respectively, K_1 and K_2 are directly homothetic).

One more result of similar spirit is due to Olovjanishnikov [84] (see also [108] for the description of related results from this hardly accessible article).

THEOREM 5.6. ([84]) *Let K, K_1, and K_2 be convex bodies in $\mathbb{R}^n$ such that $K_1 \subset \operatorname{int} K_2$, and let p be a point in $\operatorname{int} K$. Suppose that for every hyperplane H through p and for distinct hyperplanes H' and H'', both parallel to H and supporting K_1, the sets $H' \cap K_2$ and $H'' \cap K_2$ are translates of $H \cap K$. If the images of p under these translates always belong to $\operatorname{bd} K_1$, then all three bodies K, K_1, and K_2 are homothetic solid ellipsoids such that K_1 and K_2 have a common center.*

6. Further results on projections and sections

In view of the previous section, it is natural to look for similar assertions that deal with various types of geometric transformations.

A series of results was initiated by Alexandrov [5], who proved that centrally symmetric convex bodies K_1 and K_2 in $\mathbb{R}^n$ are translates of each other if and only if there is an integer m, $2 \le m \le n-1$, such that for every m-dimensional plane $L \subset \mathbb{R}^n$, the orthogonal projections of K_1 and K_2 on L have the same m-dimensional volume. This property, however, does not hold for non-symmetric convex bodies.

A dual form of Alexandrov's theorem states that centrally symmetric convex bodies K_1 and K_2 in $\mathbb{R}^n$, with centers at p_1 and p_2, respectively, are translates of each other if and only there is an integer m, $2 \le m \le n-1$, such that for every pair of parallel m-dimensional planes L_1 and L_2 through p_1 and p_2, respectively, the sections $K_1 \cap L_1$ and $K_2 \cap L_2$ have the same m-dimensional volume (see Gardner [35], Theorem 7.2.6, and p. 291 for historical references). Central symmetry of the bodies is essential here, since there are non-congruent polytopes P_1 and P_2 in $\mathbb{R}^n$ containing the origin o in their interiors and an integer m, $2 \le m \le n-2$, such that for every m-dimensional subspace $S \subset \mathbb{R}^n$, the sections $P_1 \cap S$ and $P_2 \cap S$ have the same m-dimensional volume ([35, Theorem 7.2.13]). These deep results originated a variety of publications on related problems (see, e.g., the surveys of Goodey [45] and Goodey, Schneider, Weil [44]).

Another line of research was originated by Nakajima [82, p. 169] for $n = 3$ and by Petty and McKinney [86] for $n \ge 3$, who studied the following problem: What is the relation between convex bodies K_1 and K_2 in $\mathbb{R}^n$ provided their orthogonal projections on every low-dimensional plane satisfy certain congruence or similarity conditions? Golubyatnikov obtained various results which provide sufficient conditions for K_1 to be either a translate of K_2 or of $-K_2$, or to be a directly homothetic copy of K_2 (see, for instance, [39], [41], and [42]). In the spirit of Groemer's stability result on directly homothetic projections (see Section 3), Golubyatnikov [42, Theorem 2.2.1] established a stability result regarding orthogonal congruent projections on 2-dimensional planes of special types of convex bodies in $\mathbb{R}^n$.

On the other hand, based on a construction of Petty and McKinney [86], Gardner and Volčič [36] gave an example of a pair of centered, coaxial convex bodies of revolution in $\mathbb{R}^n$, $n \geq 3$, whose projections on each two-dimensional plane are similar, but which are not themselves even affinely equivalent.

This example prompted Gardner and Volčič [36] to pose the following question, which concerns a more restricted group to geometric transformations: Suppose that $2 \leq m \leq n-1$ and K_1 and K_2 are convex bodies in $\mathbb{R}^n$ such that their orthogonal projections on every m-dimensional subspaces are congruent. Is K_1 a translate of K_2 or $-K_2$?

Under various additional assumptions, affirmative answers to this question were obtained by Alfonseca, Cordier, Ryabogin [7, 8], Mackey [69], Myroshnychenko and Ryabogin (see [81], [91]). However, Zhang [123] constructed convex bodies K_1 and K_2 in $\mathbb{R}^n$, $n \geq 3$, such that their orthogonal projections on every $(n-1)$-dimensional subspace are congruent, but nevertheless, K_1 and K_2 do not coincide up to a translation or a reflection in the origin.

A dual to the Gardner and Volčič construction reveals the existence of centrally symmetric convex bodies K_1 and K_2 in $\mathbb{R}^n$, with centers at p_1 and p_2, respectively, which are not affinely equivalent to each other, but have the following property: for every pair of parallel 2-dimensional planes L_1 and L_2 through p_1 and p_2, respectively, the sections $K_1 \cap L_1$ and $K_2 \cap L_2$ are similar (see [35, Theorem 7.1.11]).

In view of this example, Gardner [35, p. 289] posed one more question: Suppose that $2 \leq m \leq n-1$ and K_1 and K_2 are star-shaped with respect to o bodies in $\mathbb{R}^n$ such that the sections $K_1 \cap S$ and $K_2 \cap S$ are congruent for every choice of the m-dimensional subspace S of $\mathbb{R}^n$. Is K_1 a translate of K_2 or $-K_2$?

7. Algebra of homothety classes

Given a convex solid $K \subset \mathbb{R}^n$, the family $\mathcal{K}_H$ of all directly homothetic copies of K is called the *homothety class* generated by K. Obviously, the family $\mathcal{K}_H$ is closed with respect to vector addition: if $L = u + \lambda K$ and $M = v + \mu K$, where $\lambda, \mu > 0$, then

$$L + M = (u + v) + (\lambda + \mu)K \in \mathcal{K}_H.$$

Generally, the family $\mathcal{K}_H$ is not closed with respect to n-dimensional intersections (see below the description of homothety classes with this property).

For convex solids K and L in $\mathbb{R}^n$, consider the family

$$\mathcal{K}_H + \mathcal{L}_H = \{K' + L' : K' \in \mathcal{K}_H,\ L' \in \mathcal{L}_H\}.$$

Theorem 7.1. ([102]) *For a pair of line-free convex solids K and L in $\mathbb{R}^n$, the following conditions are equivalent.*

(a) *$\mathcal{K}_H + \mathcal{L}_H$ is contained in a unique homothety class generated by a line-free closed convex solid.*

(b) *$\mathcal{K}_H + \mathcal{L}_H$ is contained in the union of countably many homothety classes generated by line-free closed convex solids.*

(c) *There is a line-free closed convex solid $M \subset \mathbb{R}^n$ such that:*

  (i) $\operatorname{rec} M = \operatorname{rec} K + \operatorname{rec} L$,

  (ii) *each of the sets $K_0 = K + \operatorname{rec} M$ and $L_0 = L + \operatorname{rec} M$ is directly homothetic either to M or to $\operatorname{rec} M$,*

  (iii) *if M is not a cone, then at least one of the sets K_0 and L_0 is not a cone.*

Remark 7.2. The convex solid M from Theorem 7.1 satisfies the inclusion $\mathcal{K}_H + \mathcal{L}_H \subset \mathcal{M}_H$. Furthermore, if K and L are convex bodies in $\mathbb{R}^n$, then $\operatorname{rec} K = \operatorname{rec} L = \{o\}$. Consequently, each of the conditions (a)–(c) from Theorem 7.1 holds if and only if all three bodies K, L, and M are directly homothetic.

Following Hadwiger [57], the *Minkowski difference* $X \sim Y$ of nonempty sets X and Y in $\mathbb{R}^n$ is defined by

$$X \sim Y = \{x \in \mathbb{R}^n : x + Y \subset X\}.$$

If both X and Y are closed convex sets, then the obvious equality

$$X \sim Y = \cap \{X - y : y \in Y\}$$

implies that $X \sim Y$ is also closed and convex (possibly, empty). Given convex solids K and L in $\mathbb{R}^n$, consider the family

$$\mathcal{K}_H \sim \mathcal{L}_H = \big\{K' \sim L' : K' \in \mathcal{K}_H,\ L' \in \mathcal{L}_H,\ \dim(K' \sim L') = n\big\}.$$

An important notion here is that of *tangential set* introduced by Schneider [95, p. 136]: a convex solid $D \subset \mathbb{R}^n$ is a *tangential set* of a convex body K provided $K \subset D$ and through each boundary point of D there is a support hyperplane to D that also supports K.

THEOREM 7.3. ([102]) *For a pair of convex bodies K and L in $\mathbb{R}^n$, the following conditions (a)-(d) are equivalent.*

(a) $\mathcal{K}_H \sim \mathcal{L}_H \subset \mathcal{K}_H$,

(b) $\mathcal{K}_H \sim \mathcal{L}_H$ *is contained in a unique homothety class of a convex body,*

(c) $\mathcal{K}_H \sim \mathcal{L}_H$ *is contained in the union of countably many homothety classes of convex bodies,*

(d) *K is directly homothetic to a tangential set of L.*

The proof of Theorem 7.3 is based on the following assertion (see [95, Lemma 3.1.14] for a particular case).

THEOREM 7.4. ([102]) *Given convex bodes K and L in $\mathbb{R}^n$, the following conditions are equivalent.*

(a) *There is a scalar $\tau > 0$ such that K is a tangential set of τL.*

(b) *There is a scalar $\tau > 0$ such that*

$$K \sim \gamma L = (1 - \gamma/\tau)K, \qquad \forall\, \gamma \in (0, \tau).$$

(c) *There is a scalar $\gamma > 0$ such that $K \sim \gamma L = \lambda K$, where $0 < \lambda < 1$.*

Remark 7.5. Theorem 7.3 cannot be directly generalized to the case of unbounded convex solids. Indeed, if the convex solid K and a convex body L in $\mathbb{R}^2$ are given by

$$K = \{(x, y) : x \geq 0,\ xy \geq 1\}, \qquad L = \{(x, y) : x \geq 0,\ y \geq 0,\ x + y \leq 1\},$$

then $K \sim \gamma L = K$ for all $\gamma > 0$, while K is not directly homothetic to a tangential set of L.

8. Intersections of homothetic copies

We recall that an m-dimensional *simplex* in $\mathbb{R}^n$ is the convex hull of $m+1$ affinely independent points. It is easy to prove (see, e.g., [99]) that a nonempty intersection of two directly homothetic copies of a solid simplex $\Delta \subset \mathbb{R}^n$ is a directly homothetic copy of Δ, possibly degenerated into a point:

$$(x + \lambda\Delta) \cap (y + \mu\Delta) = z + \nu\Delta, \qquad x, y, z \in \mathbb{R}^n, \ \ \lambda, \mu > 0, \ \ \nu \geq 0.$$

Rogers and Shephard [90] proved the following result.

Theorem 8.1. ([90]) *A convex body $K \subset \mathbb{R}^n$ is a solid simplex is and only if every nonempty intersection of K and a translate of K is a directly homothetic copy of K, possibly degenerated into a point:*

$$K \cap (x + K) = z + \lambda K, \qquad x, z \in \mathbb{R}^n, \ \ \lambda \geq 0. \tag{6}$$

Theorem 8.2. ([98]) *For convex bodies K_1 and K_2 in $\mathbb{R}^n$, the following conditions are equivalent.*

(a) *K_1 and K_2 are directly homothetic solid simplices.*
(b) *The solid intersections $(x+\lambda K_1)\cap(z+\mu K_2)$, $x, z \in \mathbb{R}^n$, $\lambda, \mu > 0$, belong to a unique homothety class of convex bodies.*
(c) *The solid intersections $K_1 \cap (x + K_2)$, $x \in \mathbb{R}^n$, belong to a unique homothety class of convex bodies.*
(d) *The solid intersections $K_1 \cap (x+K_2)$, $x \in \mathbb{R}^n$, belong to countably many homothety classes of convex bodies.*

A extension of Theorem 8.1 to the case of line-free unbounded convex solids is due to Bair and Fourneau [9, p. 115].

Theorem 8.3. ([9]) *A line-free unbounded convex solid $K \subset \mathbb{R}^n$ satisfies condition* (6) *is and only if it is a solid simplicial cone.*

Rockafellar [88, p. 154] introduced the notion of *generalized simplex* $\Gamma \subset \mathbb{R}^n$, as a direct sum of an m-dimensional simplex and an $(n-m)$-dimensional simplicial cone, $0 \leq m \leq n$:

$$\Gamma = \operatorname{conv}\{u_0, u_1, \dots, u_m\} + \sum_{i=m+1}^{n} [u_0, u_i\rangle, \qquad 0 \leq m \leq n,$$

where the set $\{u_0, u_1, \dots, u_n\} \subset \mathbb{R}^n$ is affinely independent and $[u_0, u_i\rangle$ denotes the halfline through u_i originated at u_0.

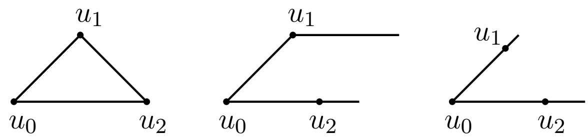

Figure 5: Generalized simplices in the plane.

Clearly, usual solid simplices and solid simplicial cones are particular cases of generalized simplices. Hinrichsen and Krause [60, 61] showed that any line-free convex polyhedron in $\mathbb{R}^n$ can be partitioned into finitely many generalized simplices. Fourneau [33] obtained the following characterization of generalized simplices.

THEOREM 8.4. ([33]) *A convex solid $K \subset \mathbb{R}^n$ is a generalized simplex if and only if all solid intersections $K \cap (x+K)$, $x \in \mathbb{R}^n$, are directly homothetic to K.*

An extension of Theorem 8.4 to the case of translates and homothetic copies of two convex solids is given in [98, 106].

THEOREM 8.5. ([98, 106]) *For line-free convex solids K_1 and K_2 in $\mathbb{R}^n$, conditions (a)–(c) below are equivalent.*

(a) *All solid intersections*

$$(x + \lambda K_1) \cap (z + \mu K_2), \qquad x, z \in \mathbb{R}^n,\ \lambda, \mu > 0,$$

belong to a unique homothety class of convex solids.

(b) *All solid intersections $K_1 \cap (x + K_2)$, $x \in \mathbb{R}^n$, belong to a unique homothety class of convex solids.*

(c) *Both K_1 and K_2 are generalized simplices, and there is a generalized simplex $K \subset \mathbb{R}^n$ such that all solid intersections $K_1 \cap (x + K_2)$, $x \in \mathbb{R}^n$, are homothetic to K. Furthermore, K_1, K_2, and K satisfy either of the following three conditions.*

(1.) *Both K_1 and K_2 are directly homothetic to K.*

(2.) *One of K_1, K_2, say K_1, is directly homothetic to K, and K_2 is a translate of a generated cone $C_u(K)$, where u is a vertex of K.*

(3.) *K_1 and K_2 are translates of generated cones $C_u(K)$ and $C_v(K)$, respectively, where u and v are distinct vertices of K.*

Figure 6 gives the description of all convex solids K_1 and K_2 in the plane that satisfy Theorem 8.5 (the shaded regions are directly homothetic to K).

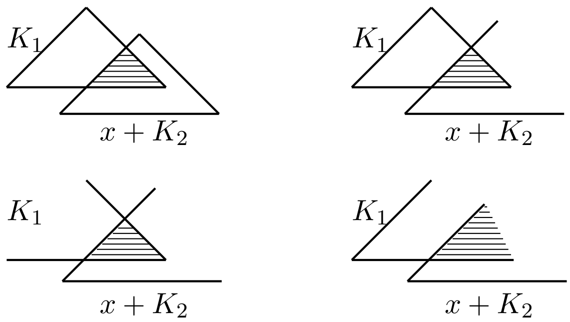

Figure 6: Illustration to Theorem 8.5.

Based on Theorem 8.2, the following problem was posed in [106]

PROBLEM 8.6. Describe all pairs of line-free convex solids K_1 and K_2 in $\mathbb{R}^n$ such that all solid intersections $K_1 \cap (x + K_2)$, $x \in \mathbb{R}^n$, belong to countably many homothety classes of convex solids.

We observe that the solids K_1 and K_2 in Problem 8.6 may be outside the family of generalized simplices even if $K_1 = K_2$. Indeed, for the polyhedral convex cone

$$K = \{(x, y, z) \in \mathbb{R}^3 \, : \, z \geq 0, \; |x| + |y| \leq z\},$$

which is not a generalized simplex, the solid intersections $K \cap (x + K)$, $x \in \mathbb{R}^3$, belong to precisely three homothety classes, generated by the convex solids K, $K \cap ((1, 1, 0) + K)$, and $K \cap ((1, -1, 0) + K)$, respectively. In this regard, the following theorem is proved in [112].

THEOREM 8.7. ([112]) *Let $K \subset \mathbb{R}^n$ be a line-free convex solid such that all solid intersections $K \cap (x + K)$, $x \in \mathbb{R}^n$, belong to finitely many homothety classes of convex solids. Then K can be expressed as the sum of a simplex and a polyhedral cone.*

The description of convex solids satisfying the conditions of Theorem 8.7 are given in Figures 7 and 8 for the cases $n = 2$ and $n = 3$, respectively.

Figure 7: Two-dimensional solids from Theorem 8.7.

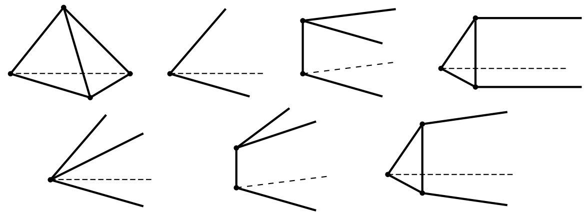

Figure 8: Three-dimensional solids from Theorem 8.7.

Further results and references on intersections of homothetic copies of convex bodies can be found in the survey [99].

Replacing the above condition (6) with a planarity condition on the boundaries of homothetic convex solids leads to a characterization of convex solids with quadratic boundaries. For instance, Gruber [54] proved the following interesting characterizations of solid ellipsoids.

THEOREM 8.8. ([54]) *A convex body $K \subset \mathbb{R}^n$, $n \geq 3$, is a solid ellipsoid provided there is a neighborhood U of the origin o such that for every nonzero point $u \in U$, with $\operatorname{int} K \cap \operatorname{int}(u + K) \neq \emptyset$, there is a hyperplane $H \subset \mathbb{R}^n$ satisfying the inclusion $\operatorname{bd} K \cap \operatorname{bd}(u + K) \subset H$.*

Further development of this topic is due to Goodey [43], who established the following characteristic property of a pair of homothetic ellipsoids.

THEOREM 8.9. ([43]) *Convex bodies K_1 and K_2 in $\mathbb{R}^n$, $n \geq 3$, are homothetic solid ellipsoids if and only if $\operatorname{bd} K_1 \cap \operatorname{bd} K_2'$ is contained in a hyperplane for every translate K_2' of K_2 such that $K_1 \neq K_2'$.*

The theorem below extends Gruber's result to the case of convex solids. We recall that a *line-free convex quadric* in $\mathbb{R}^n$ is a convex hypersurface described in suitable Cartesian coordinates $\xi_1, \dots, \xi_n$ by one of the following conditions (see [108]):

$$
\begin{array}{ll}
a_1\xi_1^2 + \cdots + a_n\xi_n^2 = 1, & \text{(ellipsoid)} \\
a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_n\xi_n^2 = 1,\ \xi_1 \geq 0, & \text{(sheet of elliptic hyperboloid} \\
 & \text{ of two sheets)} \\
a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_n\xi_n^2 = 0,\ \xi_1 \geq 0, & \text{(nappe of elliptic cone)} \\
a_1\xi_1^2 + \cdots + a_{n-1}\xi_{n-1}^2 = \xi_n, & \text{(elliptic paraboloid)}
\end{array}
$$

where all scalars a_i involved are positive.

THEOREM 8.10. ([107]) *Given a line-free convex solid $K \subset \mathbb{R}^n$, $n \geq 3$, and a scalar $\lambda \neq 0$, the following conditions are equivalent.*

(a) *For every homothetic copy $K' = u + \lambda K$ of K, the set $\operatorname{bd} K \cap \operatorname{bd} K'$ lies in a hyperplane.*

(b) $\operatorname{bd} K$ *is a convex quadric.*

In view of Theorem 8.9, one can pose the following problem.

PROBLEM 8.11. Let K_1 and K_2 be line-free convex solids in $\mathbb{R}^n$, $n \geq 3$, such that $\operatorname{bd} K_1 \cap \operatorname{bd} K_2'$ is contained in a hyperplane for every translate K_2' of K_2 satisfying the condition $K_1 \neq K_2'$. Is it true that the solids K_1 and K_2 are homothetic and their boundaries are convex quadrics?

9. Locally homothetic sets

Given a pair of convex solids K_1 and K_2 and a nonzero vector e in $\mathbb{R}^n$, assume the existence of closed halfspaces V_1 and V_2 with outward normal e supporting K_1 and K_2, respectively. The sets

$$F_1(e) = K_1 \cap \operatorname{bd} V_1 \quad \text{and} \quad F_2(e) = K_2 \cap \operatorname{bd} V_2$$

are called *associate* exposed faces of K_1 and K_2, respectively. We will say that K_1 and K_2 are *locally homothetic* provided the sets $F_1(e)$ and $F_2(e)$ are directly homothetic for every choice of e.

Clearly, directly homothetic convex solids K_1 and K_2 are locally homothetic. On the other hand, the converse assertion does not hold. Indeed, the non-homothetic rectangles

$$K_1 = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 2\},$$
$$K_2 = \{(x, y) : 0 \leq x \leq 2,\ 0 \leq y \leq 3\}$$

are locally homothetic. Nevertheless, considering suitable neighborhoods of exposed points and edges, one can obtain the following assertion.

THEOREM 9.1. ([100, 103]) *If K_1 and K_2 are line-free convex solids in $\mathbb{R}^n$, then K_2 is a translate of K_1 if and only if the following conditions are satisfied.*

(a) *Every exposed point a_1 of K_1 has an associate exposed point a_2 of K_2 such that $K_2 \cap (a_2 + U)$ is a translate of $K_1 \cap (a_1 + U)$ for a suitable neighborhood U of o.*

(b) *Every exposed line segment $[a_1, b_1]$ of K_1 has an associate exposed line segment $[a_2, b_2]$ of K_2 which is a translate of $[a_1, b_1]$ and such that for a suitable neighborhood U of o the sets $K_2 \cap (a_2 + U)$ and $K_2 \cap (b_2 + U)$ are translates of $K_1 \cap (a_1 + U)$ and $K_1 \cap (b_1 + U)$, respectively.*

(c) *Every exposed halfline l_1 of K_1 with endpoint a_1 has an associate exposed halfline l_2 of K_2 with endpoint a_2 such that l_2 is a translate of l_1 and $K_2 \cap (a_2 + U)$ is a translate of $K_1 \cap (a_1 + U)$ for a suitable neighborhood U of o.*

Analysis of the proof of Theorem 9.1 implies the following corollary.

COROLLARY 9.2. *Strictly convex bodies K_1 and K_2 in $\mathbb{R}^n$ are directly homothetic if and only if every exposed point a_1 of K_1 has an associate exposed point a_2 of K_2 such that $K_2 \cap (a_2 + U)$ is a homothetic copy of $K_1 \cap (a_1 + U)$ for a suitable neighborhood U of o.*

Kharazishvili [63] proved that a convex body $K \subset \mathbb{R}^n$ is a parallelotope if and only if there is a real number $\lambda \in (0, 1)$ such that all nonempty intersections $K \cap (x + \lambda K)$, $x \in \mathbb{R}^n$, are centrally symmetric. Theorem 9.1 gives a tool to prove the following generalization of this assertion (see also [103]).

THEOREM 9.3. ([100]) *For a pair of convex bodies K_1 and K_2 in $\mathbb{R}^n$, the following three conditions are equivalent.*

(a) *All nonempty intersections $K_1 \cap (x + K_2)$, $x \in \mathbb{R}^n$, are centrally symmetric.*

(b) *K_1 and K_2 are direct sums of the form $K_1 = P_1 \oplus Q_1$ and $K_2 = P_2 \oplus Q_2$ such that: (i) P_1 is a compact convex set of some dimension m, $0 \leq m \leq n$, and $P_2 = z - P_1$ for a suitable vector $z \in \mathbb{R}^n$, (ii) Q_1 and Q_2 are isothetic parallelepipeds, both of dimension $n - m$.*

Theorem 9.1 almost immediately implies that solid polytopes P_1 and P_2 in $\mathbb{R}^n$ are translates of each other provided for any facet $F_1(e)$ of P_1 the associate face $F_2(e)$ of P_2 is a translate of $F_1(e)$. Using induction on k and standard geometric arguments, this fact can be easily generalized as follows.

THEOREM 9.4. *For solid polyhedra P_1 and P_2 in $\mathbb{R}^n$, the following assertions hold.*

(a) *P_1 and P_2 are translates of each other if and only if there is an integer k, $1 \le k \le n-1$, such that for every k-dimensional face $F_1(e)$, $e \ne o$, the associate face $F_2(e)$ of P_2 is a translate of $F_1(e)$.*

(b) *P_1 and P_2 are directly homothetic if and only if there is an integer k, $2 \le k \le n-1$, such that for every k-dimensional face $F_1(e)$, $e \ne o$, the associate face $F_2(e)$ of P_2 is directly homothetic to $F_1(e)$.*

In 1897, Minkowski [77] proved the following result (the original case $n = 3$ is routinely extendable to all $n \ge 3$).

THEOREM 9.5. ([77]) *If P_1 and P_2 are solid polytopes in $\mathbb{R}^n$ such that for any facet $F_1(e)$ of P_1 the associate face $F_2(e)$ of P_2 has the same $(n-1)$-dimensional volume as $F_1(e)$, then P_1 and P_2 are translates of each other.*

The next two theorems are due to Alexandov (see [6, Chapter 6]).

THEOREM 9.6. ([6]) *Let P_1 and P_2 be solid polytopes in $\mathbb{R}^n$ satisfying the following conditions:*

(a) *for every bounded facet $F_1(e)$ of P_1, the associate face $F_2(e)$ of P_2 has the same $(n-1)$-dimensional volume as $F_1(e)$, and vice versa,*

(b) *every unbounded facet $F_1(e)$ of P_1 is a translate of the associate unbounded facet $F_2(e)$ of P_2, and vice versa.*

Then P_1 and P_2 are translates of each other.

THEOREM 9.7. ([4]) *Let P_1 and P_2 be solid polyhedra in $\mathbb{R}^3$ satisfying the following conditions:*

(a) *no bounded facet $F_1(e)$ of P_1 can be translated inside the associate bounded facet $F_2(e)$ of P_2, and vice versa,*

(b) *every unbounded facet $F_1(e)$ of P_1 is a translate of the associate unbounded facet $F_2(e)$ of P_2, and vice versa.*

Then P_1 and P_2 are translates of each other.

Interestingly, Theorem 9.7 cannot be generalized to the case $n > 3$. For instance, in $\mathbb{R}^4$, facets of a cube with edge length 2 cannot be translated inside the associated facets of a parallelotope with edges of length 1, 1, 3, and, 3, while these polytopes are not translates of each other.

10. Homothetic tilings and partitions

In a standard way, a *tiling* of the space $\mathbb{R}^n$ is any collections of convex bodies which cover $\mathbb{R}^n$ without gaps and overlaps. A tiling of $\mathbb{R}^n$ is called *locally finite* if every closed ball in $\mathbb{R}^n$ meets at most finitely many members of the tiling. It is easy to see that any locally finite tiling has at most countably members and each of them is a solid polytope.

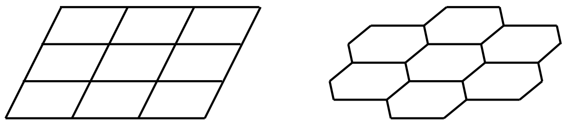

Figure 9: Parallelogons in $\mathbb{R}^2$.

Parallelohedra. There are various types of tilings determined by a given solid polytope $Q \subset \mathbb{R}^n$. Probably, the most simple (and thus most known) are tilings by translates of Q. Traditionally, polytopes which allow such tilings are called *parallelohedra* (*parallelogons* in the plane). Their study was originated by Fedorov [32, Section IV], who described parallelohedra in dimensions two and three. In the plane, there are two types of such parallelogons: parallelograms and centrally symmetric hexagons (see Figure 9). In the 3-space, there are five types of such parallelohedra: parallelepipeds, hexagonal prisms, rhombic dodecahedrons, elongated dodecahedrons, and truncated octahedrons (see Figure 10). Fedorov's arguments were criticized for some omissions, and new simplified and complete methods were given later (see, for instance, Voronoĭ [120, 121], Delaunay [24, 26], Moser [80], and Coxeter [23]).

Description and classification of parallelohedra in higher dimensions is usually realized in terms of combinatorial equivalence of polytopes. (In a standard way, two solid polytopes in $\mathbb{R}^n$ are called *combinatorially equivalent* provided their face latices allow a bijection which preserves face incidence and dimension.) Voronoĭ [120, § 114] described all 52 combinatorial types of parallelohedra in $\mathbb{R}^4$. Delaunay [24], using parallel projections along edges of parallelohedra, described 51 (out of 52) combinatorially distinct parallelohedra in $\mathbb{R}^4$, and the missing type was discovered later by Shtogrin [114]. Similar combinatorial classifications were obtained by Engel [30], Deza and Grishukhin [27]. There are 103769 combinatorial types of parallelohedra in $\mathbb{R}^5$, according to Engel [31].

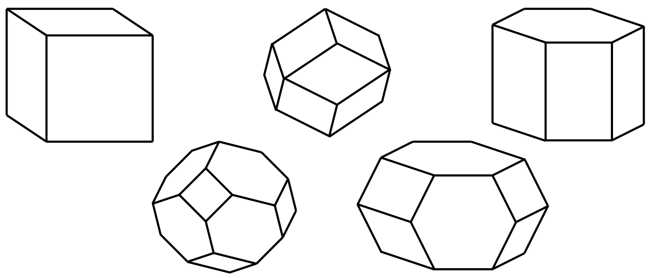

Figure 10: Parallelohedra in $\mathbb{R}^3$.

While describing parallelohedra, Fedorov assumed without proof that every parallelohedron $Q \subset \mathbb{R}^3$ has a center. This fact was established by Minkowski [77] in a slightly more general setting. Namely, he considered a solid polytope $Q \subset \mathbb{R}^3$ satisfying the following condition: there are finitely many vectors $e_1, \ldots, e_k$ such that translates of the form $e + Q$, where

$$e \in \{t_1 e_1 + \cdots + t_k e_k \, : \, t_1, \ldots, t_k \text{ are integers}\},$$

tile $\mathbb{R}^3$. Under this assumption, Minkowski proved that Q is centrally symmetric and then deduced the central symmetry of all faces of Q.

Delaunay [24] and Alexandrov [3] observed that Minkowski's argument can be routinely extended to the case of higher dimensions and formulated the following necessary conditions for a solid polytope $Q \subset \mathbb{R}^n$ to be a parallelohedron:

(P1) Q is centrally symmetric,

(P2) every facet of Q is centrally symmetric,

(P3) every parallel projection of Q on a 2-dimensional plane along an $(n-2)$-dimensional face of Q is either a parallelogram or a centrally symmetric hexagon.

Based on these observations, Venkov [119, Theorem 5] (later also McMullen [73] and Dolbilin [28]) proved the following theorem.

THEOREM 10.1. ([119]) *Any parallelohedron $Q \subset \mathbb{R}^n$ satisfies the above conditions* (P1)–(P3). *Conversely, if a solid polytope $Q \subset \mathbb{R}^n$ satisfies the conditions* (P1)–(P3), *then Q is a parallelohedron.*

As a consequence, Venkov [119] concluded that every parallelohedron in $\mathbb{R}^n$ allows a facet-to-facet tiling of the space. In fact, condition (P1) can be omitted in Theorem 10.1, because (P2) implies (P1) for the case of any solid polytope in $\mathbb{R}^n$. This result was proved by Alexandrov [1] for $n = 3$ and by Shephard [96] for all $n \geq 3$. Furthermore, McMullen [71, 72] (later also Dolbilin and Kozachok [29]) complemented Shephard's result by the following assertion: If $Q \subset \mathbb{R}^n$, $n \geq 4$, is a solid polytope and m is an integer, with $2 \leq m \leq n-2$, such that all m-dimensional faces of Q are centrally symmetric, then Q and all faces of Q are centrally symmetric. This assertion does not hold if $m = n - 1$: McMullen [71] gave an example of a solid non-centrally symmetric polytope $Q \subset \mathbb{R}^n$, $n \geq 4$, all of whose $(n-1)$-dimensional faces are centrally symmetric.

There is a large volume of results and references on combinatorial classification of various types of parallelohedra and their relation to the study of Dirichlet-Voronoĭ cells. This material is beyond the scope of our paper (and deserves a separate survey).

HOMOTHETIC TILES. Delaunay [25] (for $n = 3$), Alexandrov [2, 3] (for $n = 3$ and $n = 4$), and later Groemer [49] (for $n \leq 4$) proved that, if a solid polytope $Q \subset \mathbb{R}^n$ allows a locally finite tiling of the space by directly homothetic copies of Q, then Q is necessarily a parallelohedron (it is assumed in [25] and [49] that the sizes of homothetic tiles of Q are bounded from below and from above). Groemer also proved that if a solid polytope $Q \subset \mathbb{R}^n$, $n \leq 4$, allows a locally finite tiling of the space by directly homothetic copies of Q, not all of them being translates of Q, then Q is a prism based on an $(n-1)$-dimensional parallelohedron. McMullen [73, 74] showed that both assertions hold for all $n \geq 2$.

Expanding the family of direct homotheties to all homotheties in $\mathbb{R}^n$, we can formulate the following problem.

PROBLEM 10.2. Describe all solid polytopes $Q \subset \mathbb{R}^n$ which allow locally finite tilings of $\mathbb{R}^n$ by homothetic (direct or inverse) copies of Q.

For instance, a triangle, a convex quadrilateral, a convex pentagon with a pair of parallel sides, and a convex hexagon with a pair of opposite equal parallel sides, as depicted in Figure 11, allow locally finite homothetic tilings of the plane.

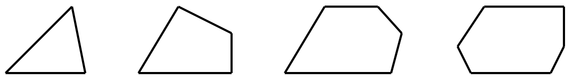

Figure 11: Illustration to Problem 10.2.

A related topic of research concerns tiling of $\mathbb{R}^n$ by congruent copies of a convex body (see, e.g., the survey of Grünbaum and Shephard [56]).

Homothetic partitions. By analogy with homothetic tiling of the space, Groemer [46, 48, 50] considered the problem to describe those convex bodies in $\mathbb{R}^n$ which allow partitions into pairwise homothetic convex pieces. His results can be summarize as follows. Given a convex body $K \subset \mathbb{R}^n$, the assertions below hold:

1. K can be partitioned into convex bodies $K_1, \ldots, K_t$, $t \geq 2$, which are translates of each other, if and only if K is a prism with an $(n-1)$-dimensional base (and all $K_1, \ldots, K_r$ are prisms with an $(n-1)$-dimensional bases).
2. K can be partitioned into convex bodies $K_1, \ldots, K_t$, $t \geq 2$, which are directly homothetic to each other, if and only if K is a prism or a truncated cone with an $(n-1)$-dimensional base (and all $K_1, \ldots, K_r$ are, respectively, prisms or truncated cones).
3. K can be partitioned into convex bodies $K_1, \ldots, K_t$, $t \geq 2$, which are directly homothetic to K, if and only if K is a parallelotope.

Some generalizations of these results, that involve partitions of K into nonconvex pieces, are obtained by Groemer [51] and Sallee [94].

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