EXTRACTA MATHEMATICAE doi:10.17398/2605-5686.39.1.97
Vol. **39**, Num. 1 (2024), 97 – 118 Available online May 7, 2024

On lifts of symplectic vector bundles and connections to Weil bundles

E. Hinamari Mang-massou [1], A. Ntyam [2]

[1] *Department of Mathematics and Computer Science, Faculty of Science University of Ngaoundéré, PO.BOX 454, Ngaoundéré, Cameroon*

[2] *Department of Mathematics, Higher Teacher Training College University of Yaoundé 1, PO.BOX 47 Yaoundé, Cameroon*

hinamariemmanuel@yahoo.com , achille.ntyam@univ-yaounde1.cm

Received December 30, 2023 Presented by M. de León
Accepted March 11, 2024

Abstract: In this paper, we generalize to Frobenius-Weil bundles some lifts of symplectic manifolds and symplectic vector bundles.

Key words: Symplectic vector bundle, differential form, connection, Weil functor, lift.

MSC (2020): Primary 53D05; 58A10; Secondary 53B05; 58A32.

1. Introduction

The concept of symplectic geometry emerged in the early nineteenth century in the study of classical mechanical systems, such as planetary orbits. Many important geometric problems can be naturally formulated in the context of symplectic geometry, thus it is also a widely useful language in mathematical physics, representation theory etc. Over time, it became an important and independent mathematical subject which is an extension of complex geometry. A *symplectic manifold* is a smooth manifold M endowed with a 2-form ω on M which is closed and nondegenerate. The precised definition and properties may be seen in [14]. A *linear symplectic manifold* (or a special symplectic manifold in [6]) is a symplectic manifold E, where E is the total space of a vector bundle $E \to M$ and ω is a linear 2-form on E (see Section 4). A *symplectic vector bundle* over a manifold M is a pair (E, ω) consisting of a real vector bundle $q : E \to M$ and a smooth section ω of the vector bundle $\bigwedge^2 E^* \to M$ such that (E_x, ω_x) is a symplectic vector space for all $x \in M$. Each linear symplectic manifold induces a symplectic vector bundle (TE, ω) over E. Kurek and Mikulski described all natural symplectic structures from a smooth manifold M to its tangent bundles TM (see [11]) and they studied

ISSN: 0213-8743 (print), 2605-5686 (online)

the complete lifts of symplectic structures to tangent bundles of higher order $T^r M$ (see [12]).

Okassa studied the lifts of symplectic structures to bundles of infinitely near points (see [16]). Lifts of symplectic structures to Frobenius-Weil bundles $T^A M$ were studied by several authors namely [2, 3, 4, 9] where the authors deduced almost symplectic forms on $T^A M$ from prolongations of almost symplectic structures on a maniflod M.

In this paper, we study the lifting of symplectic vector bundles, linear symplectic manifolds and the Poisson manifold associated to a linear symplectic manifold using a Frobenius-Weil functor. We begin by giving an intrinsic description of the structure of linear k-forms developped in [10]. We then show that lifts of k-forms, symplectic manifolds and symplectic vector bundles with respect to tangent functors of high order may be generalized to Frobenius-Weil functors. Finally, we prove that the complete lift of a symplectic or a semi-Riemannian connection is also a symplectic or a semi-Riemannian connection. These results are the continuation of those developped over last 25 years by many authors, some of whom have been cited above. In particular, symplectic structures are involved in the Hamilton equation of motion. For this reason the results of this paper are also interesting from the point of view of theorical mechanics. This article is divided into two main parts: the preliminaries and the main results.

2. Preliminaries

Weil algebra [8]: A *Weil algebra* is a finite-dimensional quotient of the algebra of germs $\mathcal{E}_p = C_0^\infty(\mathbb{R}^p, \mathbb{R})$ $(p \in \mathbb{N}^*)$. Let us denote by $\mathcal{M}_p \subseteq \mathcal{E}_p$ the ideal of germs vanishing at 0; hence $(\mathcal{E}_p, \mathcal{M}_p)$ is a local algebra. For a Weil algebra $A = \mathcal{E}_p/I$, there exists a non negative integer k such the ideal I contains the power $\mathcal{M}_p^k$ of the maximal ideal $\mathcal{M}_p$. We denote r the width of A, i.e., the smallest integer such that $I \supseteq \mathcal{M}_p^{r+1}$; hence $A = \mathbb{R}{\cdot}1_A \oplus N$ where $N = \left(\left\{e_\alpha,\, \alpha \in \mathbb{N}^k,\, 1 \leq |\alpha| \leq r\right\}\right)$ with $e_\alpha := X^\alpha + I$ is the vector subspace $\left\langle\left\{e_\alpha,\, \alpha \in \mathbb{N}^k,\, 1 \leq |\alpha| \leq r\right\}\right\rangle$ spanned by vectors e_α, $|\alpha| \leq r$; N is in fact the nilpotent ideal of A and (A, N) is a local algebra. Conversely, Given a real commutative, associative, unital algebra A such that $\dim_{\mathbb{R}}(A) < +\infty$ and $A = \mathbb{R}{\cdot}1_A \oplus N$ with N a nilpotent ideal of A, if $(X_1, \ldots, X_p)$ is a basis of N and r a non negative integer such that $N^{r+1} = 0$, the surjective algebras homomorphism $\mathcal{E}_p \to A$, $[f]_0 \mapsto \sum\limits_{\alpha \in \mathbb{N}^p} \frac{1}{\alpha!} D_\alpha f(0) (X_1)^{\alpha_1} \cdots (X_p)^{\alpha_p}$ induces an algebra isomorphism $\mathcal{E}_p/I \to A$ with I its kernel.

EXAMPLE 2.1. $\mathbb{R} = \mathcal{E}_p/\mathcal{M}_p$ and $\mathbb{D} = \mathcal{E}_1/\mathcal{M}_1$ are Weil algebras; more generally, $\mathbb{D}_p^r := \mathcal{E}_p/\mathcal{M}_p^{r+1}$ is Weil algebra isomorphic to the algebra $J_0^r(\mathbb{R}^p, \mathbb{R})$ of jets of smooth functions on $\mathbb{R}^p$ vanishing at 0.

FROBENIUS-WEIL ALGEBRA: A Weil algebra $A = \mathbb{R}\cdot 1_A \oplus N$ is called a *Frobenius-Weil algebra* if there is a nondegenerate bilinear form $\sigma : A \times A \to \mathbb{R}$ such that $\sigma(ab, c) = \sigma(a, bc)$, for all a, b, c in A. Equivalently, A is a Frobenius-Weil algebra if there exists a linear map $\lambda_0 : A \to \mathbb{R}$ such that $\ker \lambda_0$ contains no nonzero ideal of A. More precisely, when σ is given, $\lambda_0(a) = \sigma(a, 1_A) = \sigma(1_A, a)$ and when λ_0 is given, $\sigma(a, b) = \lambda_0(ab)$. Let $\mathfrak{I}(A)$ be the set of non trivial ideal of an algebra A. A minimal element of $(\mathfrak{I}(A), \subseteq)$ is called a *minimal ideal*, i.e., a non zero ideal I of A which contains no other non zero ideal. The *socle* of a Weil algebra $A = \mathbb{R}\cdot 1_A \oplus N$ is the set $\mathrm{Soc}(A)$ of a in A such that $au = 0_A$, for all u in N; $\mathrm{Soc}(A)$ is an ideal and hence a vector subspace of A. Each minimal ideal I of A is contained into $\mathrm{Soc}(A)$ [3, Proposition 2], since $1_A - u$ is invertible for all nilpotent element u. The correct wording of [3, Proposition 3] is: "Minimal ideals of A are 1-dimensional vector subspaces of $\mathrm{Soc}(A)$." Indeed, for a non zero element x of $\mathrm{Soc}(A)$, $I = \mathbb{R}\cdot x$ is clearly a minimal ideal. Conversely, given a non zero element x of a minimal ideal I, the relation $\{0_A\} \neq (x) \subseteq I$ implies $I = (x) = Ax = \{tx : t \in \mathbb{R}\} = \mathbb{R}x$, since $x \in \mathrm{Soc}(A)$. By [3, Proposition 4], A is a Frobenius-Weil algebra if and only if A has a unique minimal ideal.

EXAMPLE 2.2. When $A = \mathbb{D}_p^r$, $\mathrm{Soc}(A)$ is the vector subspace spanned by e_α, $|\alpha| = r$ hence $\dim_{\mathbb{R}} \mathrm{Soc}(A) = \binom{p+r-1}{r}$. Thus $\mathbb{D}_p^r$ is a Frobenius-Weil algebra if and only if $p + r - 1 = r$, i.e., $p = 1$.

COVARIANT DESCRIPTION OF A WEIL FUNCTOR $T^A : \mathcal{M}f \to \mathcal{FM}$: Let us denote by $\mathcal{M}f$ the category of finite dimensional differential manifolds and mappings of class C^∞, $\mathcal{FM}$ the category of fibered manifolds and fibered manifolds morphisms and $\mathcal{VB} \subseteq \mathcal{FM}$ the subcategory of vector bundles and morphisms of vector bundles. Let $A = \mathcal{E}_p/I$ be a Weil algebra and consider a manifold M. In the set of smooth maps $\varphi \in C^\infty(\mathbb{R}^p, M)$ such that $\varphi(0) = x$, one defines an equivalence relation $\underset{x}{\sim}$ by: $\varphi \underset{x}{\sim} \psi$ if and only if $[h]_x \circ [\psi]_0 - [h]_x \circ [\varphi]_0 \in I$, for all germs $[h]_x \in C_x^\infty(M, \mathbb{R})$. The equivalence class of φ is denoted by $j^A\varphi$ and is called the $A-$*velocity* of φ at 0; the class $j^A\varphi$ depends only on the germ of φ at 0. The quotient set is denoted by $(T^AM)_x$ and the disjoint union of $(T^AM)_x$, $x \in M$ by T^AM. The mapping $\pi_{A,M} : T^AM \to M$, $j^A\varphi \mapsto$

$\varphi(0)$, defines a bundle structure on T^AM and for any differentiable mapping $f : M \to N$, one can associate a bundle morphism $T^Af : T^AM \to T^AN$ over f by: $T^Af\left(j^A\varphi\right) = j^A(f \circ \varphi)$. The correspondence $T^A : \mathcal{M}f \to \mathcal{FM}$ is a product-preserving bundle functor ([8]).

Example 2.3. When $A = J_0^r(\mathbb{R}^p, \mathbb{R})$, then T^A is equivalent to the functor T_p^r of (p, r)-velocities and when $A = \mathcal{E}_1/\mathcal{M}_1^2$, then $T^A = T$ is the tangent functor.

Remarks 2.4. (1) Weil functors preserve immersions, embeddings, submersions, surjective submersions, transversal maps, . . .

(2) Let $T^A, T^B : \mathcal{M}f \to \mathcal{FM}$ be Weil functors. Hence $T^A \circ T^B$ is also a Weil functor; its corresponding Weil algebra is canonically isomorphic to the tensor product $B \otimes_{\mathbb{R}} A$ of A and B. Moreover there is a bijective correspondence between the set of natural transformations $T^A \to T^B$ and the set of all algebra homomorphisms $A \to B$.

For a Weil algebra $A = \mathbb{R}{\cdot}1_A \oplus N$, we fix a subset $\Lambda \subseteq \left\{\alpha \in \mathbb{N}^p : 1 \leq |\alpha| \leq r\right\}$ such that $e_\alpha := j^A(z^\alpha)$, $\alpha \in \Lambda$ constitute a basis N; hence $(e_\alpha)_{\alpha\in\{0\}\cup\Lambda}$ is a basis of $A = T^A\mathbb{R}$.

Local coordinate system: For a local coordinate system $\left(u^i\right)_{1\leq i\leq m}$ on U of a differential manifold M, one can associate an adapted local coordinate system $\left(u^i, \overline{u}^i_\beta\right)$ defined on $\pi^{-1}_{A,M}(U)$ by

$$\begin{cases} u^i(j_A\varphi) = u^i(\varphi(0)), \\ \overline{u}^i_\beta(j_A\varphi) = \frac{1}{\beta!}D_\beta(u^i \circ \varphi)(0) + \sum\limits_{\substack{|\alpha|\leq r \\ \alpha\notin\{0\}\cup\Lambda}} \frac{1}{\alpha!}D_\alpha(u^i \circ \varphi)(0)\lambda^\beta_\alpha\,, \end{cases} \tag{2.1}$$

for $1 \leq i \leq m$, $\beta \in \Lambda$, where $e_\alpha = \sum\limits_{\beta\in\Lambda} \lambda^\beta_\alpha e_\beta$, for all $\alpha \in \mathbb{N}^p\backslash\Lambda$ and $1 \leq |\alpha| \leq r$.

The flow operator of T^A: For a smooth vector field X on a differential manifold M, let us denote $Fl^X : \Omega \to M$ its maximal flow. One can define a smooth vector field on T^AM by:

$$X^c(u) = \frac{d}{dt}T^A(Fl^X_t)(u)\,|_{t=0}\,.$$

This vector field is called the *complete lift* of X related to T^A. One defines in this way a natural operator (see [8]), $\mathcal{F}^A : T \rightsquigarrow TF$, given for all manifold M by:

$$\left(\mathcal{F}^A\right)_M : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(T^A M), \quad X \longmapsto X^c, \tag{2.2}$$

called the *flow operator* of T^A.

Remark 2.5. X^c is a projectable vector field since the following diagram

$$\begin{array}{ccc} T^A M & \xrightarrow{X^c} & TT^A M \\ {\scriptstyle \pi^A_M}\big\downarrow & & \big\downarrow{\scriptstyle T(\pi^A_M)} \\ M & \underset{X}{\xrightarrow{\pi_M}} & TM \end{array}$$

commutes. In particular, (2.2) is a Lie algebra homomorphism.

The canonical flow natural equivalence $\kappa : T^A \circ T \to T \circ T^A$ [8]: Let $A = \mathcal{E}_p/I$ be a Weil algebra. A natural transformation $i : T^A \circ T \to T \circ T^A$ is called a *flow natural transformation* if the following diagram

$$\begin{array}{ccc} T^A M & \xrightarrow{\mathcal{F}^A X} & TT^A M \\ {\scriptstyle T^A X}\big\downarrow & {\scriptstyle i_M}\nearrow & \big\downarrow{\scriptstyle \pi_{T^A M}} \\ T^A T M & \underset{T^A \pi_M}{\longrightarrow} & T^A M \end{array}$$

commutes for all manifold M and all vector field X on M.

Now, let M be a manifold. For any $\zeta = j^A \varphi \in T^A TM$, there is a differentiable mapping $\Phi : \mathbb{R}^p \times \mathbb{R} \to M$ such that $\varphi(z) = \frac{d}{dt}\Phi_z(t) \mid_{t=0}$, in a neighbourhood of $0 \in \mathbb{R}^p$ (see [8]). By this result, one can define a natural equivalence

$$\kappa : T^A \circ T \longrightarrow T \circ T^A \tag{2.3}$$

as follows:

$$\kappa_M(\zeta) = \frac{d}{dt}\eta(t) \mid_{t=0},$$

where $\eta : \mathbb{R} \to TM$, $t \mapsto j^A \Phi^t$, in a neighbourhood of $0 \in \mathbb{R}$. (2.3) is called the *canonical flow natural equivalence* associated to the bundle functor T^A.

Natural transformations $s_f : T^A \circ T^* \to T^* \circ T^A$ [2]: Let us consider a linear map function $f : A \to \mathbb{R}$; there is a natural transformation

$$s_f : T^A \circ T^* \longrightarrow T^* \circ T^A \tag{2.4}$$

defined for all manifold M as follows:

$$\left[(s_f)_M \left(j^A\varphi\right)\right] \left(\kappa_M \left(j^A\eta\right)\right) := f\left(j^A \langle\varphi, \eta\rangle_{TM}\right),$$

for all $j^A\varphi \in T^AT^*M$, $j^A\eta \in T^ATM$ such that $T^A\pi_M^* \left(j^A\varphi\right) = T^A\pi_M \left(j^A\eta\right)$ with $\langle,\rangle_{TM} : TM \oplus T^*M \to \mathbb{R}$ the usual pairing.

Frobenius-Weil functors: A *Frobenius-Weil functor* is a Weil functor T^A with A a Frobenius-Weil algebra. Given two Frobenius-Weil functors T^A and T^B, the fiber product $T^A \oplus T^B$ defined by

$$T^A \oplus T^B (M) = T^AM \times_M T^BM,$$
$$T^A \oplus T^B (f) = T^Af \times_f T^Bf,$$

is Frobenius-Weil functor; the composition $T^A \circ T^B$ is also a Frobenius-Weil functor.

The internalization map of a vector bundle: Let T^A be a Frobenius-Weil functor with $\lambda_0 : A \to \mathbb{R}$ as the associated linear function.

For a vector bundle (E, M, q), let us consider the vector bundles $\left(T^AE, T^AM, T^Aq\right), \left(T^AE^*, T^AM, T^Aq^*\right)$ and the non-degenerate bilinear form $\langle\langle,\rangle\rangle_E : T^AE \oplus_{T^AM} T^AE^* \to \mathbb{R}$ given by $\langle\langle,\rangle\rangle_E := \lambda_0 \circ T^A \langle,\rangle_E$. The induced vector bundles isomorphism

$$I_E^A : T^AE^* \longrightarrow \left(T^AE\right)^* \tag{2.5}$$

over T^AM is called the *internalization map of* (E, M, q) associated to T^A.

• When $T^A = T$, $I_E = I_E^{\mathbb{D}} : TE^* \to T^\bullet E$ is an isomorphism of *double vector bundles* over E^* and TM from $(TE^*; E^*, TM; M)$ to $(T^\bullet E; E^*, TM; M)$ the horizontal dual of $(TE; E, TM; M)$ (see [13]).

• When $E = TM$ is the tangent bundle of M, it is clear that $(s_{\lambda_0})_M = \left(\kappa_M^{-1}\right)^* \circ I_{TM}$, where $(\kappa_M)^*$ denotes the transpose over T^AM of $\kappa_M^{-1} : TT^AM \to T^ATM$; the natural equivalence s_{λ_0} is often denoted

$$\varepsilon^A : T^A \circ T^* \longrightarrow T^* \circ T^A \tag{2.6}$$

and called the *Tulczyjew natural isomorphism* associated to T^A. Moreover, by [3, Proposition 6], (2.4) is a natural equivalence if and only if A is a Frobenius-Weil algebra (with the associated linear form f).

3. Prolongation of some tensor fields

In all the section, T^A is a Weil functor.

Natural transformations $\chi_\alpha : T^A \to T^A$: Given a vector bundle (E, M, q), the fibered multiplication $m^E : \mathbb{R} \times E \to E$ is a vector bundle morphism over the projection $\mathbb{R} \times M \to M$; the induced map $T^A\left(m^E\right) : A \times T^A E \to T^A E$ determines for each a in A a natural transformation

$$\overline{Q}(a) : T^A \longrightarrow T^A \tag{3.1}$$

by $\overline{Q}(a)_E := T^A m^E(a, \cdot)$.

When $e_\alpha = j^A\left(z^\alpha\right)$, $\alpha \in \mathbb{N}^p$, the natural transformation $\overline{Q}(e_\alpha)$ is denoted $\chi_\alpha : T^A \to T^A$. It is clear that

$$\left(\chi_\alpha\right)_E\left(j^A\varphi\right) = j^A\left(z^\alpha\varphi\right), \tag{3.2}$$

for all smooth map $\varphi : \mathbb{R}^p \to E$.

Prolongation of functions: Let us recall these tools of [4].

Let $f : M \to \mathbb{R}$ be a smooth function. The λ-*lift* of f is $f^{(\lambda)} := \lambda \circ T^A f$, for $\lambda : A \to \mathbb{R}$ a linear map. It is easy to check that $(f \circ h)^{(\lambda)} = f^{(\lambda)} \circ T^A h$, for $h : N \to M$ a smooth map and $(f_1 + f_2)^{(\lambda)} = f_1^{(\lambda)} + f_2^{(\lambda)}$, for all smooth functions f_1, f_2 on M. One denotes

$$f^{(\alpha)} := e_\alpha^* \circ T^A f \tag{3.3}$$

the lift of $f \in C^\infty(M, \mathbb{R})$ associated to the linear form e_α^*, $\alpha \in \{0\} \cup \Lambda$, with the convention $f^{(\alpha)} = 0$, for all α in $\mathbb{Z}^p \backslash \{0\} \cup \Lambda$. $f^c := f^{(0)} = f \circ p_M^A$ is called the *complete lift* of f. In particular when $\left(u^i, \overline{u}_\beta^i\right)$ is the adapted local coordinate system (2.1) of $T^A M$ associated to $\left(u^i\right)$, we have

$$\begin{cases} \left(u^i\right)^{(0)} = u^i, \\ \left(u^i\right)^{(\alpha)} = \overline{u}_\alpha^i \ \text{ for } \alpha \text{ in } \Lambda\,. \end{cases}$$

This implies that functions $f^{(\alpha)}$, $\alpha \in \{0\} \cup \Lambda$ generates the algebra $C^\infty\left(T^A M\right)$ of smooth functions on $T^A M$.

In particular, when $f : E \to \mathbb{R}$ is constant or linear on fibres of a vector bundle $q : E \to M$, the λ-lift of f is also constant or linear on fibres.

Prolongation of vector fields: For a vector bundle (E, M, q), a smooth section $\underline{\sigma} \in \Gamma(E)$ and an element a of A, one can define a smooth section

$$\underline{\sigma}^{(a)} := \overline{Q}(a)_E \circ T^A \underline{\sigma} \tag{3.4}$$

of the vector bundle $\left(T^A E, T^A M, T^A q\right)$. In particular, given a smooth vector field X on M, one can associate a vector field on $T^A M$,

$$X^{(a)} = \kappa_M \circ \overline{Q}(a)_{TM} \circ FX = Q(a)_M \circ \mathcal{F}_M X, \tag{3.5}$$

where $Q(a) : TT^A \to TT^A$ is the natural affinor defined by $Q(a)_M = \kappa_M \circ Q(a)_M \circ \kappa_M^{-1}$.

Let $\lambda : A \to \mathbb{R}$ a linear map and $\lambda_a : A \to \mathbb{R}$ the linear map given by $\lambda_a(x) = \lambda(ax)$, for $a \in A$. The following equalities hold (see [4]):

$$X^{(a)}\left(f^{(\lambda)}\right) = (X(f))^{(\lambda_a)} \tag{3.6}$$

and

$$\left[X^{(a)}, Y^{(b)}\right] = [X, Y]^{(ab)}, \tag{3.7}$$

for all smooth function f, vector fields X, Y on M and a, b in A.

Similarly, one denotes

$$X^{(\alpha)} := Q(e_\alpha)_M \circ \mathcal{F}_M X \tag{3.8}$$

the lift of $X \in \mathfrak{X}(M)$ associated to the vector e_α, $\alpha \in \mathbb{N}^p$; its is clear that $X^{(\alpha)} = 0$, for $|\alpha| > r$. We have

$$\begin{cases} X^{(0)}\left(f^{(\beta)}\right) = [X(f)]^{(\beta)} \quad \text{if } \beta \in \{0\} \cup \Lambda, \\ X^{(\alpha)}\left(f^{(0)}\right) = 0 \quad \text{if } 0 \neq \alpha \in \mathbb{N}^p, \\ X^{(\alpha)}\left(f^{(\beta)}\right) = [X(f)]^{(0)} + \sum\limits_{\gamma \in \Lambda,\ \alpha+\gamma \in \Lambda} \delta^\beta_{\alpha+\gamma} [X(f)]^{(\gamma)} \\ \qquad + \sum\limits_{\gamma \in \Lambda,\ \alpha+\gamma \notin \Lambda} \delta^\beta_{\alpha+\gamma} [X(f)]^{(\gamma)} \quad \text{if } \alpha, \beta \in \Lambda. \end{cases}$$

In particular, we have $X^c(f^c) = (X(f))^c$.

Local expression: Let $X \in \mathfrak{X}(M)$ with $X|_U = \sum_{i=1}^{m} X^i \frac{\partial}{\partial u^i}$; we have

$$\begin{aligned} X^c\left(u^i\right) &= \left(X\left(u^i\right)\right)^{(0)} = X^i \circ p_M^A\,, \\ X^c\left(\overline{u}^i_\beta\right) &= \left(X^i\right)^{(\beta)}\,, \qquad \beta \in \Lambda\,, \\ X^{(\alpha)}\left(u^i\right) &= 0 \qquad \text{if } \alpha \in \mathbb{N}^p,\ \alpha \neq 0\,, \\ X^{(\alpha)}\left(\overline{u}^i_\beta\right) &= X^i \circ p_M^A + \sum_{\gamma\in\Lambda,\ \alpha+\gamma\in\Lambda} \delta^\beta_{\alpha+\gamma}\left(X^i\right)^{(\gamma)} \\ &\quad + \sum_{\gamma\in\Lambda,\ \alpha+\gamma\notin\Lambda} \lambda^\beta_{\alpha+\gamma}\left[X(f)\right]^{(\gamma)} \qquad \text{if } \alpha,\beta \in \Lambda\,; \end{aligned}$$

hence

$$X^c|_{T^AU} = X^i \circ p_M^A \frac{\partial}{\partial u^i} + \sum_{\beta\in\Lambda} \left(X^i\right)^{(\beta)} \frac{\partial}{\partial \overline{u}^i_\beta} \tag{3.9}$$

and

$$X^{(\alpha)}|_{T^AU} = \sum_{\beta\in\Lambda}\left[X^i \circ p_M^A + \sum_{\substack{\gamma\in\Lambda \\ \alpha+\gamma\in\Lambda}} \delta^\beta_{\alpha+\gamma}\left(X^i\right)^{(\gamma)} + \sum_{\substack{\gamma\in\Lambda \\ \alpha+\gamma\notin\Lambda}} \lambda^\beta_{\alpha+\gamma}\left(X^i\right)^{(\gamma)}\right]\frac{\partial}{\partial \overline{u}^i_\beta}\,,$$

for $\alpha \neq 0$ in $\mathbb{N}^p$.

One may also deduce that

$$\left(\frac{\partial}{\partial u^i}\right)^c = \frac{\partial}{\partial u^i} \quad \text{and} \quad \left(\frac{\partial}{\partial u^i}\right)^{(\alpha)} = \frac{\partial}{\partial \overline{u}^i_\alpha}\,, \quad 1 \leq i \leq m \ \text{ and } \ \alpha \in \Lambda\,.$$

Prolongations of k-forms: Each k-form ω on a manifold M may be viewed as a skew symmetric k-linear function $\widetilde{\omega} : \bigoplus^k TM \to \mathbb{R}$. Since $\kappa_M : T^ATM \to TT^AM$ is an isomorphism of vector bundles over T^AM, one defines in [4] a k-form $\omega^{(\lambda)}$ on T^AM by:

$$\widetilde{\omega^{(\lambda)}} = \lambda \circ T^A\left(\widetilde{\omega}\right) \circ \bigoplus^k \kappa_M^{-1}\,, \tag{3.10}$$

for a linear fonction $\lambda : A \to \mathbb{R}$. The following properties are satisfied by $\omega^{(\lambda)}$:

PROPOSITION 3.1. ([4]) *For all $a_1, \dots, a_k \in A$, all $X_1, \dots, X_k \in \mathfrak{X}(M)$ and all smooth function f on M, we have:*

$$\begin{cases} \omega^{(\lambda)}\left(X_1^{(a_1)}, \dots, X_k^{(a_k)}\right) = \left(\omega\left(X_1, \dots, X_k\right)\right)^{\left(\lambda_{a_1 \cdots a_k}\right)}, \\ \left(T^A f\right)^* \left(\omega^{(\lambda)}\right) = \left(f^* \omega\right)^{(\lambda)}, \\ d\omega^{(\lambda)} = (d\omega)^{(\lambda)}. \end{cases} \tag{3.11}$$

In particular, if ω is closed, then $\omega^{(\lambda)}$ also closed.

Remark 3.2. Since ω may also be viewed as a skew symmetric $(k-1)$-linear morphism $\omega^\flat : \bigoplus^{k-1} TM \to T^*M$, $\omega^{(\lambda)}$ is also given by

$$\left[\omega^{(\lambda)}\right]^\flat = \left(s_\lambda\right)_M \circ T^A\left(\omega^\flat\right) \circ \bigoplus^{k-1} \kappa_M^{-1}, \tag{3.12}$$

where $s_\lambda : T^A T^* \to T^* T^A$ is the natural transformation (2.4). In particular, when (M, ω) is a symplectic manifold, T^A a Frobenius-Weil functor and λ_0 the associated linear function, hence

$$\left[\omega^{(\lambda_0)}\right]^\flat := \varepsilon_M^A \circ T^A\left(\omega^\flat\right) \circ \kappa_M^{-1}$$

is a vector bundle isomorphism over $id_{T^A M}$, so $\omega^{(\lambda_0)}$ is a closed nondegenerate 2-form on $T^A M$, i.e., $\left(T^A M, \omega^{(\lambda_0)}\right)$ is a symplectic manifold. $\omega^{(\lambda_0)}$ is denoted ω^c and called the complete lift of ω to $T^A M$.

4. SOME LINEAR TENSOR FIELDS ON VECTOR BUNDLES

DOUBLE VECTOR BUNDLE:

DEFINITION 4.1. (See [13] or [6]) A *double vector bundle* is a system $(D; A, B; M)$ of four vector bundle structures

$$\begin{array}{ccc} D & \xrightarrow{q_B^D} & B \\ {\scriptstyle q_A^D}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow{q_A} & M \end{array} \tag{4.1}$$

where D is a vector bundle on bases A and B, which are themselves vector bundles on M, such that each of the four structure maps of each vector bundle structure on D (projection, addition, scalar multiplication and the zero section) is a vector bundle morphism with respect to other structure.

Remark 4.2. The double tangent vector bundle of a vector bundle (E, M, q)

$$\begin{array}{ccc} TE & \xrightarrow{Tq} & TM \\ \pi_E \downarrow & & \downarrow \pi_M \\ E & \xrightarrow{q} & M \end{array} \tag{4.2}$$

allows the concept of *linear vector fields*, i.e., sections of $TE \to E$ that are morphisms of vector bundles with respect to the vector bundle structure $TE \to TM$. This may be generalize to an arbitrary double vector bundle.

THE VERTICAL DUAL OF THE TANGENT DOUBLE VECTOR BUNDLE: Given a vector bundle (E, M, q), the vertical dual

$$\begin{array}{ccc} T^*E & \xrightarrow{r_E} & E^* \\ \pi_E^* \downarrow & & \downarrow q^* \\ E & \xrightarrow{q} & M \end{array} \tag{4.3}$$

of the tangent double vector bundle (4.2) is defined as follows: $T^*E \to E$ is the dual of the tangent bundle $TE \to E$; if $\tau_E : E \times_M E \to VE \subset TE$, $(e, e') \mapsto \frac{d}{dt}(e + te')\,|_{t=0}$ is the *vertical lift* ([8]) of E, $r_E = p_2 \circ \tau_E^*$ where $p_2 : q^*(E^*) \to E^*$ is the canonical projection. The fiber over $\theta \in E_x^*$ is the set of all linear functions $\Phi : T_eE \to \mathbb{R}$ $(e \in E_x)$ such that $\Phi \circ \tau_E(e, \cdot) = \theta$. Moreover given a local coordinate system $\left(x^i, y^j\right)$ of E deduced from a fibered chart,

$$\begin{cases} r_E\left(dx^i\right) = 0^{E^*} \circ q\,, \\ r_E\left(dy^j\right) = \varepsilon^j \circ q\,, \end{cases} \tag{4.4}$$

where $0^{E^*} : M \to E^*$ is the zero section and ε^j the local section corresponding the linear function y^j. Finally, the addition and the multiplication of $T^*E \to E^*$ are defined on fibres by

$$\begin{cases} \left(\Phi \underset{E^*}{+} \Phi'\right)(\xi'') = \Phi(\xi) + \Phi'(\xi')\,, \\ \left(s \underset{E^*}{\cdot} \Phi\right)\left(s \underset{TM}{\cdot} \xi\right) = s\Phi(\xi)\,, \end{cases} \tag{4.5}$$

where $\Phi \in T_e^*E$, $\Phi' \in T_{e'}^*E$, $\xi'' \in T_{e+e'}E$, $\xi'' = \xi \underset{TM}{+} \xi'$ with $\xi \in T_eE$ and $\xi' \in T_{e'}E$.

LINEAR k-FORMS: Let (E, M, q) be a vector bundle. A smooth k-form $\omega : E \to \bigwedge^k T^*E$ $(k \geq 1)$ is said *linear* if the associated morphism of vector bundles $\omega^\flat : \bigoplus^{k-1} TE \to T^*E$ over E is also a morphism of vector bundles

$$\begin{array}{ccc} \bigoplus^{k-1} TE & \xrightarrow{\omega} & T^*E \\ {\scriptstyle \bigoplus^{k-1} Tq}\downarrow & & \downarrow{\scriptstyle r_E} \\ \bigoplus^{k-1} TM & \xrightarrow{\underline{\omega}} & E^* \end{array}$$

over a smooth multilinear map $\underline{\omega} : \bigoplus^{k-1} TM \to E^*$. Equivalently, if

$$\begin{array}{cccc} \widetilde{\omega} : & \bigoplus^k TE & \longrightarrow & \mathbb{R} \\ & \bigoplus^k T_eE \ni (\xi_1, \ldots, \xi_k) & \longmapsto & \omega(e) \cdot (\xi_1, \ldots, \xi_k) \end{array}$$

denotes the corresponding multilinear function, hence ω is linear if and only if $\widetilde{\omega}$ is a morphism of vector bundles

$$\begin{array}{ccc} \bigoplus^k TE & \xrightarrow{\widetilde{\omega}} & \mathbb{R} \\ {\scriptstyle \bigoplus^k Tq}\downarrow & & \downarrow \\ \bigoplus^k TM & \longrightarrow & \{pt\} \end{array}$$

over a constant map (see [10]). In local coordinate (x^i, y^j) of E, each linear k-form $(\omega, \underline{\omega})$ can be written

$$\begin{aligned} \omega \mid_{q^{-1}(U)} = & \frac{1}{(k-1)!} \underline{\omega}_{i_1 \ldots i_{k-1} j} dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} \wedge dy^j \\ & + \frac{1}{k!} \omega_{i_1 \ldots i_k j} y^j dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \end{aligned} \tag{4.6}$$

where $\underline{\omega}\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_{k-1}}}\right) = \underline{\omega}_{i_1 \ldots i_{k-1} j} \varepsilon^j$ with (ε^j) the local frame of E^* associated to linear functions $y^j : q^{-1}(U) \to \mathbb{R}$.

THE STRUCTURE OF LINEAR k-FORMS [10]: We give there an intrinsic description of the structure of a linear k-form on E. Let us denote $\Omega^h(M; G)$ the module of G-valued h-forms on M, i.e., the module of smooth sections of the vector bundle $\bigwedge^h T^*M \otimes G$ over M. If $\ell_E : \Gamma(E^*) \to C^\infty_\ell(E)$ is the canonical isomorphism of modules over $C^\infty(M)$, we have $\ell_{q^*(E)}(q^*(\sigma))_e = \ell_E(\sigma)_{q(e)}$ and there exists a morphism of modules over $C^\infty(M)$,

$$\Omega^h(E; q^*(E^*)) \longrightarrow \Omega^h(E), \quad \varphi \longmapsto \widetilde{\varphi},$$

given by $\widetilde{\varphi}(X_1,\ldots,X_{k-1}) = \ell_{q^*(E)}(\varphi(X_1,\ldots,X_{k-1}))$.

Let $(\omega,\underline{\omega})$ be a linear k-form on E. $\underline{\omega} : \bigoplus^{k-1} TM \to E^*$ is a E^*-valued $(k-1)$-form on M, i.e., $\underline{\omega} \in \Omega^{k-1}(E;E^*)$; its pull-back by the projection $q : E \to M$ is $q^*(\underline{\omega}) \in \Omega^{k-1}(E;q^*(E^*))$, hence $\widetilde{\underline{\omega}} := \widetilde{q^*(\underline{\omega})}$ is a $(k-1)$-form on E. If locally $\underline{\omega}\mid_U = \frac{1}{(k-1)!}\underline{\omega} i_1 \ldots i_{k-1} j\, dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}}\otimes\varepsilon^j$, it is clear by (4.6) that $\widetilde{\underline{\omega}}\mid_{q^{-1}(U)} = \frac{1}{(k-1)!}\,\underline{\omega}_{i_1\ldots i_{k-1}j}\circ q y^j dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}}$; let us consider $\mu := (-1)^{k-1}\,\widetilde{\underline{\omega}} \in \Omega^{k-1}(E)$.

Proposition 4.3. *We have*

$$\omega = d\mu + \nu\,,$$

where $\nu \in \Omega^{k-1}(E;T^*E)$ is a linear k-form over the zero map. Moreover, in the case of closed k-forms, $\underline{\omega}$ determines ω.

Proof. Indeed $\nu := \omega - d\mu$ is clearly a linear k-form on E and since $d\mu = \frac{1}{(k-1)!}\partial_{i_k}\underline{\omega}_{i_1\ldots i_{k-1}j}y^j dx^{i_1}\wedge\cdots\wedge dx^{i_k} + \frac{1}{(k-1)!}\underline{\omega}_{i_1\ldots i_{k-1}j}dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}}\wedge dy^j$,

$$\nu\mid_{q^{-1}(U)} := \left(\frac{1}{k!}\omega_{i_1\ldots i_k j} - \frac{1}{(k-1)!}\partial_{i_k}\underline{\omega}_{i_1\ldots i_{k-1}j}\right) y^j dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

is a linear k-form on $E\mid_U$ over the zero multilinear map by (4.4). Moreover, $d\omega = 0$ iff $d\nu = 0$, i.e., $\nu = 0$, hence $\omega = d\mu$ is entirely determined by $\underline{\omega}$. ∎

Remark 4.4. For each morphism of vector bundles $\rho : E \to T^*M$, the pull-back of the Liouville 1-form $\lambda_M \in \Omega^1(T^*M)$ by ρ is equal to $\widetilde{\rho^*}$, i.e., $\rho^*(\lambda_M) = \widetilde{\rho^*}$. Indeed if $\lambda_M\mid_{\pi_M^{*-1}(U)} = p_i dx^i$ and $\rho^*(\partial_i) = \underline{\omega}_{ij}\varepsilon^j$, we have $\rho^*(\lambda_M)\mid_{q^{-1}(U)} = p_i\circ\rho d\left(x^i\circ\rho\right) = p_i\circ\rho dx^i$; but

$$\begin{aligned} p_i\circ\rho(e) &= \rho(e)\left((\partial_i)_{q(e)}\right) = y^j(e)\,\rho(\varepsilon_j)\left((\partial_i)_{q(e)}\right)\\ &= y^j(e)\,\underline{\omega}_{kj}(q(e))\,dx^k\left((\partial_i)_{q(e)}\right) = \left(\underline{\omega}_{ij}\circ q y^j\right)(e), \end{aligned}$$

hence $\rho^*(\lambda_M)\mid_{q^{-1}(U)} = \underline{\omega}_{ij}\circ q y^j dx^i$.

Symplectic forms: Now, let $(\omega,\underline{\omega})$ be a linear 2-form on E; hence ω is a morphism of double vector bundles over E and $\underline{\omega}$; let us denote $\rho : E \to T^*M$ its *core morphism*. Since the transpose ω^* of ω is a morphism of double vector bundles over E and ρ with $\underline{\omega}^*$ as core morphism ([13, Proposition 9.2.1]), the equality $\omega^* = -\omega$ implies $\rho^* = -\underline{\omega}$. The following result follows:

Proposition 4.5. ([6]) *ω is closed if and only if ω is the pull-back of the canonical symplectic form ω_M on T^*M by ρ, i.e., $\rho^*\omega_M = \omega$.*

Proof. ω is closed if and only if $\omega = d\left(-\widetilde{\underline{\omega}}\right)$ by Proposition 4.3; moreover $\rho^* = -\underline{\omega}$ hence $\omega = d\left(\widetilde{\rho^*}\right) = d\left(\rho^*\left(\lambda_M\right)\right) = \rho^*\left(d\lambda_M\right) = \rho^*\left(\omega_M\right)$. ∎

Euler vector field of a vector bundle: Let (E, M, q) be a vector bundle.

The group of homotheties induces a 1-parameter group

$$h : \mathbb{R} \times E \longrightarrow E\,, \quad (t, u) \longmapsto e^t \cdot u\,;$$

the associated vector field $\xi_E \in \mathfrak{X}(E)$ is given by:

$$\xi_E\left(u\right) = \frac{d}{dt} e^t \cdot u \mid_{t=0}\ .$$

Moreover ξ_E is a linear vertical vector field since

$$\begin{array}{ccc} E & \xrightarrow{\xi_E} & TE \\ q\big\downarrow & & \big\downarrow T(q) \\ M & \xrightarrow{0^{TM}} & TM \end{array}$$

is a morphism of vector bundles. If $\left(x^i, y^j\right)$ is local coordinate system of E deduced from a fibered chart $\left(q^{-1}\left(U\right), \varphi\right)$ and $\xi_E \mid_{q^{-1}(U)} = \xi^j \frac{\partial}{\partial y^j}$, we have

$$\xi^j\left(u\right) = \dot{y}^j \circ \xi_E\left(u\right) = \frac{d}{dt} y^j\left(e^t \cdot u\right) \mid_{t=0} = \frac{d}{dt} e^t y^j\left(u\right) \mid_{t=0} = y^j\left(u\right),$$

for all $u \in q^{-1}\left(U\right)$, hence

$$\xi_E \mid_{q^{-1}(U)} = y^j \frac{\partial}{\partial y^j}\ .$$

ξ_E is called the *Euler-Liouville vector field* associated to E ([5]). ξ_E is clearly complete and for all vector bundle morphism $f : E \to F$, ξ_E and ξ_F are f-related.

Remarks 4.6. (1) Let us denote $q^*C^\infty\left(M\right) = \left\{h \circ q\, :\, h \in C^\infty\left(M\right)\right\}$ the module of smooth functions $E \to \mathbb{R}$ constant on fibres and $C^\infty_\ell\left(E\right)$ that of functions linear on fibres. Since each linear vector field is determined by its values on $q^*C^\infty(M)$ and $C^\infty_\ell(E)$ (see [13]), it is also clear that ξ_E is the only linear vertical vector field on E such that

$$\mathcal{L}_{\xi_E} f = \xi_E(f) = f, \tag{4.7}$$

for all $f \in C^\infty_\ell(E)$.

(2) More generally, let $\overline{\varphi} : E \to \bigwedge^k T^*E$ be a linear k-form, i.e., a k-form such that $\overline{\varphi} : \bigoplus^k TE \to \mathbb{R}$ is a linear function when TE is endowed with its vector bundle structure on TM. Hence

$$\mathcal{L}_{\xi_E}\overline{\varphi} = \overline{\varphi}\,.$$

Indeed for all u in E,

$$\begin{aligned}
\mathcal{L}_{\xi_E}\overline{\varphi}(u) &= \frac{d}{dt}\left(Fl_t^{\xi_E}\right)^* \overline{\varphi}(u)\ |_{t=0} \\
&= \frac{d}{dt}\overline{\varphi}_{e^t u} \circ \overset{k}{\oplus}\left[e^t \underset{TM}{\cdot} id_{TE}\right] |_{t=0} \\
&= \frac{d}{dt} e^t \overline{\varphi}_u\ |_{t=0} \qquad (\text{since } \overline{\varphi} \text{ is linear}) \\
&= \overline{\varphi}_u\,.
\end{aligned}$$

5. Main results

In this section, T^A is a Frobenius-Weil functor with λ_0 as the associated linear form

Prolongations of Euler vector fields: Let E be a vector bundle. According to (3.8), one can define some lifts

$$\xi_E^{(\alpha)} := Q(e_\alpha)_E \circ \mathcal{F}_E \xi_E\,, \tag{5.1}$$

of the Euler-Liouville vector field ξ_E of E, associated to e_α, $\alpha \in \mathbb{N}^p$.

Proposition 5.1. *$\xi_E^{(0)} = (\xi_E)^c$ is the Euler-Liouville vector field $\xi_{T^A E}$ of the vector bundle $\left(T^A E, T^A M, T^A q\right)$.*

Proof. In a fibered chart $(q^{-1}(U), x^i, y^j)$ of E, $\xi_E|_{q^{-1}(U)} = y^j \frac{\partial}{\partial y^j}$ then in the local coordinate $(x^i, \overline{x}^i_\alpha, y^j, \overline{y}^j_\alpha)$ of $T^A E$, we have

$$(\xi_E)^c \mid_{T^A q^{-1}(U)} = y^j \circ p^A_E \frac{\partial}{\partial y^j} + \sum_{\beta \in \Lambda} \overline{y}^j_\beta \frac{\partial}{\partial \overline{y}^j_\beta},$$

according to (3.9). ∎

COROLLARY 5.2. (i) *$\xi_{T^A E}$ is the only linear vertical vector field on $T^A E$ such that $\xi_{T^A E}(f^{(\alpha)}) = f^{(\alpha)}$, for all f in $C^\infty_\ell(E)$ and $\alpha \in \mathbb{N}^p$.*

(ii) *Moreover for any linear k-form on E, we have:*

$$\mathcal{L}_{\xi_{T^A E}} \overline{\varphi}^{(\alpha)} = \overline{\varphi}^{(\alpha)}, \qquad \alpha \in \{0\} \cap \Lambda .$$

Proof. (i) By Remark 4.6 (1), $\xi_{T^A E}$ is the only linear vertical vector field on $T^A E$ such that $\xi_{T^A E}(\widetilde{f}) = \widetilde{f}$, for all $\widetilde{f}$ in $C^\infty_\ell\left(T^A E\right)$ and since this module is generated by lifts $f^{(\alpha)}$, $\alpha \in \mathbb{N}^p$ of f in $C^\infty_\ell(E)$, $\xi_{T^A E}$ is the only linear vertical vector field on $T^A E$ such that $\xi_{T^A E}(f^{(\alpha)}) = f^{(\alpha)}$, for all f in $C^\infty_\ell(E)$ and $\alpha \in \mathbb{N}^p$.

(ii) By Remark 4.6 (2) since $\overline{\varphi}^{(\alpha)}$ is a linear k-form. ∎

PROPOSITION 5.3. *For any vector bundle morphism $f : E \to F$, Euler vector fields $\xi_{T^A E}$ and $\xi_{T^A F}$ are $T^A f$-related.*

Proof. Indeed

$$\begin{aligned} TT^A f \circ \xi_{T^A E} &= TT^A f \circ \kappa_E \circ T^A(\xi_E) \\ &= \kappa_F \circ T^A T f \circ T^A(\xi_E) \quad \text{(since } \kappa \text{ is a natural transformation)} \\ &= \kappa_F \circ T^A(Tf \circ \xi_E) \\ &= \kappa_F \circ T^A(\xi_F \circ f) \quad \text{(since } \xi_F \text{ are } f\text{-related)} \\ &= \xi_{T^A F} \circ T^A f, \end{aligned}$$

hence $TT^A f \circ \xi_{T^A E} = \xi_{T^A F} \circ T^A f$. ∎

PROLONGATIONS OF LINEAR k-FORMS: For a linear k-form $(\omega, \underline{\omega})$ on E, let us consider its complete lift ω^c on $T^A E$ given in Remark 3.2 by:

$$[\omega^c]^\flat = \varepsilon^A_E \circ T^A \omega^\flat \circ \bigoplus^{k-1} \kappa_E^{-1}.$$

THEOREM 5.4. *Hence* $(\omega^c, \underline{\omega}^c)$ *is a linear k-form with* $\underline{\omega}^c = I_E^A \circ T^A \underline{\omega} \circ \bigoplus^{k-1} \kappa_M^{-1}$. *In particular, if* $(\omega, \underline{\omega})$ *is a linear symplectic form,* $(\omega^c, \underline{\omega}^c)$ *is also a linear symplectic form with the core morphism* $\rho^c := \varepsilon_M^A \circ T^A \rho$.

Proof. ω^c is a k-form on $T^A E$ by Remark 3.2 and since $r_{T^A E} = I_E^A \circ T^A r_E \circ \left(\varepsilon_E^A\right)^{-1} : T^* T^A E \to \left(T^A E\right)^*$, the second part of the proof is clear. ∎

Let $(\omega, \underline{\omega})$ be a linear 2-form on E and $\rho : E \to T^* M$ a morphism of vector bundles over M.

COROLLARY 5.5. *Hence* ω^c *is closed if and only if* $(\rho^c)^* \omega_M^c = \omega^c$, *where* ω_M^c *denotes the complete lift of the canonical symplectic form* ω_M *on* $T^* T^A M$.

PROLONGATIONS OF SYMPLECTIC VECTOR BUNDLES: Let (E, M, q) be a vector bundle of rank $2n$.

A *symplectic form* on (E, M, q) is a fibrewise smooth bilinear function $\omega : E \oplus E \to \mathbb{R}$ endowed with a symplectic structure on each fiber. A pair (E, ω) is called a *symplectic vector bundle* if ω is a symplectic form on (E, M, q). Given two symplectic vector bundles (E, ω) and (E', ω'), a vector bundle isomorphism $f : E \to E'$ is called a *symplectomorphism* if $f^*(\omega') = \omega$, i.e., $f_x^* \left(\omega'_{\underline{f}(x)}\right) = \omega_x$, for all x in M. It is clear that each symplectic manifold (M, ω) induces a symplectic vector bundle (TM, ω).

Let $\omega^\flat : E \to E^*$ be the vector bundle isomorphism associated to a symplectic form ω on (E, M, q); there is a well-defined symplectic form ω^A on the vector bundle $(T^A E, T^A M, T^A q)$ induced by the vector bundle isomorphism $I_E^A \circ T^A \omega^\flat : T^A E \to \left(T^A E\right)^*$. We have

$$\omega^A = \lambda_0 \circ T^A \omega : T^A E \oplus T^A E \longrightarrow \mathbb{R}\,. \tag{5.2}$$

PROPOSITION 5.6. *Hence* $\left(T^A E, \omega^A\right)$ *is a symplectic vector bundle.*

DEFINITION 5.7. ω^A is called the *complete lift* of ω to $T^A E \to T^A M$. The symplectic vector bundle $\left(T^A E, \omega^A\right)$ is called the *complete lift* of (E, ω) to $T^A E \to T^A M$.

PROPOSITION 5.8. *Let* $(E, \omega), (F, \mu)$ *be two symplectic vector bundles and* $f : E \to F$ *a symplectomorphism. Then* $T^A f : T^A E \to T^A F$ *is also a symplectic isomorphism.*

Proof. Indeed

$$(T^A f)^* \mu^A = (f^* \mu)^A = \omega^A.$$

So $T^A f$ is a symplectomorphism. ∎

Prolongations of symplectic connections: Let (E, ω) be a symplectic vector bundle. A linear connection on (E, M, q) given by its covariant derivative $\nabla : (X, \sigma) \mapsto \nabla_X \sigma$ is said *symplectic* if its covariant derivative $\nabla_X \omega$ along each smooth vector field X on M vanishes, i.e.,

$$\nabla_X \omega (\sigma_1, \sigma_2) := X \cdot \omega (\sigma_1, \sigma_2) - \omega (\nabla_X \sigma_1, \sigma_2) - \omega (\sigma_1, \nabla_X \sigma_2) = 0 .$$

In [17] the author defined the complete lift $\mathcal{T}^A \Gamma$ of an arbitrary connection on a fibered manifold. In the particular case of linear connections on a vector bundle (E, M, q), the following results are generalizations of some results of [1]:

Proposition 5.9. ([15]) *Let Γ be a linear connection on (E, M, q), ∇ its covariant derivative, $\mathcal{T}^A \Gamma$ the complete lift of Γ to $(T_A E, T_A M, T_A \pi)$ and ∇^A the covariant derivative associated to $\mathcal{T}^A \Gamma$. Then $\mathcal{T}^A \Gamma$ is the unique linear connection on $(T_A E, T_A M, T_A \pi)$ such that*

$$\nabla^A_{X^{(\alpha)}} \sigma^{(\beta)} = (\nabla_X \sigma)^{(\alpha+\beta)}, \qquad \alpha, \beta \in \mathbb{N}^p, \tag{5.3}$$

for all smooth sections $\sigma : M \to E$ and $X \in \mathfrak{X}(M)$.

Corollary 5.10. ([15]) *Let Γ be a linear connection on M, ∇ its covariant derivative, Γ^c the image of $\mathcal{T}^A \Gamma$ by the vector bundles isomorphism $\kappa_M : T^A TM \to TT^A M$. Then Γ^c is the unique linear connection on $T_A M$ such that*

$$\nabla^c_{X^{(\alpha)}} Y^{(\beta)} = (\nabla_X Y)^{(\alpha+\beta)}, \qquad \alpha, \beta \in \mathbb{N}^p, \tag{5.4}$$

for all vector fields X, $Y \in \mathfrak{X}(M)$.

Now, let Γ be a linear connection on (E, M, q).

Theorem 5.11. *If Γ is a symplectic connection on (E, ω) then $\mathcal{T}^A \Gamma$ is also a symplectic connection on $\left(T^A E, \omega^A\right)$. In particular, the complete lift Γ^c of a symplectic connection Γ on TM is a symplectic connection on $TT^A M$.*

Proof. Indeed, for all smooth sections σ_1, $\sigma_2 : M \to E$ and $X \in \mathfrak{X}(M)$,

$$\begin{aligned}
\nabla^A_{X^{(\alpha)}}\omega^A\left(\sigma_1^{(\beta)},\sigma_2^{(\gamma)}\right) &= X^{(\alpha)}\cdot\omega^A\left(\sigma_1^{(\beta)},\sigma_2^{(\gamma)}\right) - \omega^A\left(\nabla^A_{X^{(\alpha)}}\sigma_1^{(\beta)},\sigma_2^{(\gamma)}\right)\\
&\quad - \omega^A\left(\sigma_1^{(\beta)},\nabla^A_{X^{(\alpha)}}\sigma_2^{(\gamma)}\right)\\
&= X^{(\alpha)}\cdot\left(\omega\left(\sigma_1,\sigma_2\right)\right)^{\left((\lambda_0)_{e_{\beta+\gamma}}\right)} - \omega^A\left(\left(\nabla_X\sigma_1\right)^{(\alpha+\beta)},\sigma_2^{(\gamma)}\right)\\
&\quad - \omega^A\left(\sigma_1^{(\beta)},\left(\nabla_X\sigma_2\right)^{(\alpha+\gamma)}\right)\\
&= \left[X\cdot\omega\left(\sigma_1,\sigma_2\right)\right]^{\left((\lambda_0)_{e_{\alpha+\beta+\gamma}}\right)} - \left[\omega\left(\nabla_X\sigma_1,\sigma_2\right)\right]^{\left((\lambda_0)_{e_{\alpha+\beta+\gamma}}\right)}\\
&\quad - \left[\omega\left(\sigma_1,\nabla_X\sigma_2\right)\right]^{\left((\lambda_0)_{e_{\alpha+\beta+\gamma}}\right)}\\
&= \left[X\cdot\omega\left(\sigma_1,\sigma_2\right) - \omega\left(\nabla_X\sigma_1,\sigma_2\right) - \omega\left(\sigma_1,\nabla_X\sigma_2\right)\right]^{\left((\lambda_0)_{e_{\alpha+\beta+\gamma}}\right)}\\
&= \left[\nabla_X\omega\left(\sigma_1,\sigma_2\right)\right]^{\left((\lambda_0)_{e_{\alpha+\beta+\gamma}}\right)},
\end{aligned}$$

by the definitions, (3.11), (3.6) and (5.3). Since the set of all sections $\sigma^{(\alpha)}$, $\sigma : M \to E$ smooth section of E and $\alpha \in \mathbb{N}^p$, spans the module of smooth sections of the vector bundle $(T_AE, T_AM, T_A\pi)$, the result follows. ∎

Remark 5.12. Replacing (E,ω) with a semi-Riemannian vector bundle (E,g), a linear connexion ∇ on (E,M,q) is called a metric connection if the covariant derivative $\nabla_X g$ of g along each smooth vector field X on M vanishes. The tangent bundle TM of a semi-Riemaniann manifold (M,g) is a semi-Riemannian vector bundle (TM,g).

Now, let Γ be a linear connection on (E,M,q).

Corollary 5.13. *If Γ is a semi-Riemaniann connection on (E,g) then $\mathcal{T}^A\Gamma$ is also a semi-Riemaniann connection on $\left(T^AE, g^A\right)$. In particular, the complete lift Γ^c of a semi-Riemaniann connection Γ on TM is a semi-Riemaniann connection on TT^AM.*

Applications in Hamiltonian mechanics: Let (E,ω) a linear symplectic manifold and $\omega^\flat : TE \to T^*E$ its associated morphism of double vector bundles. The Poisson morphism of the induced linear Poisson manifold (E,π) is $\pi^\sharp = (\omega^\flat)^{-1} : T^\star E \to TE$. Hence, $\{G,H\} = \omega\left(X_G, X_H\right)$, where $X_G := \pi^\sharp(dG)$, $X_H := \pi^\sharp(dH)$ are the Hamiltonian vector fields associated to functions $G, H \in C^\infty(E)$. In particular if $H \in C^\infty_\ell(E)$ is linear on fibers, then X_H is a linear vector field. H is called a *Hamiltonian function* and X_H the *Hamiltonian vector field* associated to H.

Let $H^c = H^{(0)} = H \circ p_E^A$, the complete lift of H to $T^A E$; it is clear that:

1. $(H \circ h)^c = H^c \circ T^A h$, for any morphism of vector bundles $h : F \to E$.
2. $(H_1 + H_2)^c = H_1^c + H_2^c$ and $dH^c = (dH)^c$, for all $H_1, H_2, H \in C_\ell^\infty(E)$.

Remark 5.14. Let $\left(T^A E, \pi^c\right)$ be the Poisson manifold associated to the linear symplectic manifold $\left(T^A E, \omega^c\right)$; we have

$$\begin{cases} (\pi^c)^\sharp = \kappa_E \circ T^A \pi^\sharp \circ (\varepsilon_E^A)^{-1} : T^* T^A E \to T T^A E\,, \\ X_{H^c} = (X_H)^c\,, \end{cases} \tag{5.5}$$

for all $H \in C_l^\infty(E)$. Indeed $(\pi^c)^\sharp = \left[(\omega^c)^\flat\right]^{-1} = \kappa_E \circ T^A \pi^\sharp \circ (\varepsilon_E^A)^{-1}$ and

$$\begin{aligned} X_{H^c} = (\pi^c)^\sharp (dH^c) &= \kappa_E \circ T^A \pi^\sharp \circ (\varepsilon_E^A)^{-1} \left((dH)^c\right) \\ &= \kappa_E \circ T^A \pi^\sharp \circ (\varepsilon_E^A)^{-1} \left[\varepsilon_E^A \circ T^A (dH)\right] \\ &= \kappa_E \circ T^A X_H = (X_H)^c\,, \end{aligned}$$

hence X_{H^c} is the complete lift of X_H to $T^A E$.

PROPOSITION 5.15. *$\{G^c, H^c\} = \{G, H\}^c$, for all $G, H \in C_l^\infty(E)$.*

Proof. Indeed

$$\begin{aligned} \{G^c, H^c\} &= X_{G^c}(H^c) = (X_G)^c (H^c) \\ &= (X_G(H))^c = \{G, H\}^c\,, \end{aligned}$$

hence $\{G^c, H^c\} = \{G, H\}^c$, for all $G, H \in C_l^\infty(E)$. ∎

DEFINITION 5.16. Let (E, ω) a linear symplectic manifold and $H \in C_\ell^\infty(E)$ a Hamiltonian function.

(1) The triple (E, ω, H) is called a Hamiltonian mechanical system.

(2) An integral of motion for (E, ω, H) is a function f with $\{H, f\} = X_H(f) = 0$, i.e., f is constant on any trajectory generated by X_H. Note that H itself is an integral of motion for (E, ω, H) (conservation of energy). The integrals of motion for (E, ω, H) form a sub-Poisson algebra of $C^\infty(E)$.

Remarks 5.17. (a) Let φ_t be the flow of X_H. Then $\varphi_t^*\omega = \omega$ for all $t \in \mathbb{R}$, i.e., φ_t is symplectic. Hence φ_t^c is symplectic.

(b) Let (E, ω, H) be a Hamiltonian mechanical system and $(T^A E, \omega^c, H^c)$ its complete lift. Hence, if f is an integral of motion for (E, ω, H) then f^c is also an integral of motion for $(T^A E, \omega^c, H^c)$.

References

[1] L.A. Cordero, C.T.J. Dodson, M. de León, "Differential geometry of frame bundles", Kluwer Academic Publishers, Dordrecht, 1989.

[2] M. Doupovec, Natural transformations of the composition of Weil and cotangent functors, *Ann. Polon. Math.* **77** (2) (2001), 105 – 117.

[3] M. Doupovec, M. Kureš, Some geometric constructions on Frobenius Weil bundles, *Differential Geom. Appl.* **35** (2014), 143 – 149.

[4] J. Gancarzewicz, W.M. Mikulski, Z. Pogoda, Lifts of some tensor fields and connections to product preserving functors, *Nagoya Math. J.* **135** (1994), 1 – 14.

[5] K. Grabowska, J. Grabowski, Z. Ravanpak, VB-structures and generalizations, *Ann. Global Anal. Geom.* **Volume 62** (2022), 235 – 284.

[6] K. Konieczna, P. Urbański, Double vector bundles and duality, *Arch. Math. (Brno)* **35** (1999), 59 – 95.

[7] I. Kolář, On the natural operators on vector fields, *Ann. Global Anal. Geom.* **6** (1988), 109 – 117.

[8] I. Kolář, P.W. Michor, J. Slovák, "Natural operations in differential geometry", Springer-Verlag, Berlin, 1993.

[9] P.M. Kouotchop Wamba, G.F. Wankap Nono, A. Ntyam, Prolongations of G-structures related to Weil bundles and some applications, *Extracta Math.* **37** (1) (2022), 111 – 138.

[10] Y. Kosmann-Schwarzbach, Multiplicativity, from Lie groups to generalized geometry, in "Geometry of jets and fields", Banach Center Publ., 110, Polish Academy of Sciences, Institute of Mathematics, Warsaw, 2016, 131 – 166.

[11] J. Kurek, W.M. Mikulski, Symplectic structures on the tangent bundles of symplectic and cosymplectic manifolds, *Ann. Polon. Math.* **82** (3) (2003), 273 – 285.

[12] J. Kurek, W.M. Mikulski, Canonical symplectic structures on the rth order tangent bundle of a symplectic manifold, *Extracta Math.* **21** (2) (2006), 159 – 166.

[13] K.C.H. Mackenzie, "General theory of Lie groupoids and Lie algebroids", London Math. Soc. Lecture Note Ser., 213, Cambridge University Press, Cambridge, 2005.

[14] D. McDuff, D. Salamon, "Introduction to symplectic topology", 2nd edition, Oxford Math. Monogr., Oxford University Press, New York, 1998.

[15] A. Ntyam, J.W. Kamga, New versions of curvature and torsion formulas for the complete lifting of a linear connection to Weil bundles, *Ann. Polon. Math.* **82** (3) (2003), 233 – 240.

[16] E. Okassa, Relèvements des structures symplectiques et pseudo-Riemanniennes à des variétés de points proches, *Nagoya Math.* **115** (1989), 63 – 71.

[17] J. Slovák, Prolongations of connections and sprays with respect to Weil functors, *Rend. Circ. Mat. Palermo (2) Suppl.* (14) (1987), 143 – 155.