EXTRACTA MATHEMATICAE
Vol. **39**, Num. 1 (2024), 57 – 95

doi:10.17398/2605-5686.39.1.57
Available online February 19, 2024

# The current state of play in the Landsberg-Berwald problem of Finsler geometry

M. Crampin

*Orchard Rising, Herrings Lane, Burnham Market, Norfolk, UK*

*m.crampin@btinternet.com*

Received October 17, 2023
Accepted January 17, 2024

Presented by T. Arias-Marco

*Abstract*: A progress report on the (still unresolved) Landsberg-Berwald problem of Finsler geometry: whether there can be non-Berwaldian regular Landsberg spaces.

*Key words*: Finsler spaces; Landsberg spaces; Berwald spaces.

MSC (2020): 53B40.

## 1. Introduction

A long standing, intriguing — one could without exaggeration say famous — problem of Finsler geometry is the following:

> Is every regular (y-global) Landsberg space, over a manifold of dimension at least 3, a Berwald space?

Nobody knows for sure. There is no known example of such a Landsberg space which is not Berwaldian, but no proof that there cannot be any. There are, however, numerous known results of the form " A regular Landsberg space which has such-and-such an additional property — property P say — is a Berwald space ". This paper is an attempt to collect them all together in one place; and along the way it adds a few new ones.

The body of the paper therefore consists of a succession of theorems of the form:

Theorem. *A regular Landsberg space which has property* P *is a Berwald space*

for a number of different properties P. If a year is quoted, it is the year of first publication of the result: if none is given the result is likely to be new. Once

ISSN: 0213-8743 (print), 2605-5686 (online)

or twice the result turns out in fact to be: a regular Landsberg space which has property P is a Riemannian space — but of course that counts.

My main new result on the Landsberg-Berwald problem is (roughly) this:

Theorem. *A complete regular Landsberg space whose Berwald scalar curvature is geodesically invariant is a Berwald space. In particular, a Landsberg space over a compact base whose Berwald scalar curvature is geodesically invariant is a Berwald space.*

(Theorem 4 below: there is in fact an additional boundedness condition, which I have omitted here for the sake of simplicity, but which will be explained in context.) This is an improvement, in the sense that it involves a weakening of the hypothesis, on the previous strongest result in this general area, which is

Theorem. *A Landsberg space whose Berwald scalar curvature vanishes is a Berwald space.*

Apart from the proof of the new result there is little in the way of proof in the main section of the paper. For published work I have simply relied on references. Most of the new detailed argument is given in appendices: so it is in these appendices that most of the original material in the paper is to be found. New results, relevant to the Landsberg-Berwald problem but not sufficiently directly to warrant inclusion in the main part of the paper, include:

Proposition. *A function $\kappa$ on $\mathbb{R}^n_\circ$ which is positively-homogeneous of degree 0 satisfies*

$$g^{ij}\frac{\partial^2\kappa}{\partial y^i\partial y^j} = 0$$

*if and only if it is constant.*

(Proposition 2 in Appendix B.) This is similar in appearance to a result I proved in [12], but it applies to functions positively-homogeneous degree 0 while the previous one was for functions homogeneous of degree 1.

I used the latter in [12] to prove that a weakly-Berwald Landsberg space is a Berwald space; and I use it here to prove the following related result concerning isometries between Minkowski spaces. (A Minkowski space induces a Riemannian metric on $\mathbb{R}^n_\circ$: by 'an isometry' here I mean a diffeomorphism $\mathbb{R}^n_\circ \to \mathbb{R}^n_\circ$ which preserves the corresponding metrics, and is positively-homogeneous of degree 1.)

PROPOSITION. *An isometry between two Minkowski spaces of the same dimension is linear if and only if its Jacobian determinant is constant.*

(Proposition 5(2) in Appendix E; the Jacobian is to be calculated with respect to linear coordinates.) This result seems rather important, given that from one point of view the key issue is whether parallel translation, which is an isometry between tangent Minkowski spaces in a Landsberg space, is linear, which is what is required for the space to be a Berwald space. It is in effect a new version of M. Li's equivalence theorem for Minkowski spaces from [27].

1.1. NON-EXISTENCE OF UNICORNS Many Finsler geometers nowadays refer to non-Berwaldian Landsberg metrics as unicorns, a usage introduced by Bao in [6]. I have decided against following this trend, as will be clear from my title, and I feel I should explain why.

The problem is that in calling a non-Berwaldian Landsberg metric a unicorn no consideration is given as to whether or not the space is regular: a unicorn may very well fail to be y-global, and indeed Bao himself refers in [6] to Asanov's metrics as unicorns. Consider for example Tayebi's 2021 paper 'A survey on unicorns in Finsler geometry' [46]. This turns out to be closely focussed on results about non-regular Landsberg spaces and contains, I think it is fair to say, very little of interest about the regular case. I wish to make it completely clear that by contrast the present paper deals only with regular Landsberg spaces. It is for this reason that I abjure the use of the term unicorn, and instead refer to the conjecture that all regular Landsberg spaces are Berwald spaces as the Landsberg-Berwald conjecture, and to the problem of resolving whether this conjecture is true or false as the Landsberg-Berwald problem: I continue to use the terminology that I have adopted in the past, in other words. You will find no unicorns here.

It seems to me in fact that Bao's coinage, while witty, is not terribly apt. Surely nobody in the 21st century thinks unicorns exist: Bao himself describes them as mythical. So to call non-Berwaldian regular Landsberg spaces unicorns implies in effect that there are none, when this is tantalisingly still an open question; whereas to call Asanov's metrics unicorns denies their evident existence and is simply wrong. Marco Polo, having observed on his travels what he claimed were unicorns, described them in terms which makes it clear that what he actually saw were rhinoceroses. I suggest that non-regular non-Berwaldian Landsberg metrics would be more appropriately called rhinos. Whether it is correct to call regular non-Berwaldian Landsberg metrics unicorns, on the other hand, still waits for the resolution of the Landsberg-

Berwald problem.

At the risk of seeming over-repetitive I state again my general insistence on the qualification 'regular'. I hope that I have now used the word enough to make it unnecessary to continually repeat it in future. You will find no rhinos here either.

1.2. Background and notation I take $M$ to be a differentiable manifold whose dimension $n$ is at least 3. I denote by $\pi : T^\circ M \to M$ the slit tangent bundle of $M$. I shall mostly work in terms of coordinates $x^i$ on $M$ and corresponding fibre coordinates $y^i$. Let $F$ be a regular Finsler function on $T^\circ M$: positive, smooth, positively-homogeneous of degree 1, and strongly convex. From another perspective, $F$ is an assignment to each $x \in M$ of a Minkowski norm on $T^\circ_x M$, depending smoothly on $x$.

I denote by $\mathcal{I} = \{(x,y) \in T^\circ M : F(x,y) = 1\}$ the indicatrix bundle of $F$, by $L = \frac{1}{2}F^2$ its energy Lagrangian, by

$$g_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j}$$

its fundamental tensor, by

$$h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j}$$

its angular metric, by

$$C_{ijk} = \frac{\partial g_{ij}}{\partial y^k}$$

its Cartan tensor, and by $C_i = g^{jk}C_{ijk}$ its mean Cartan tensor.

I should perhaps pause here and elucidate what I mean by 'tensor' in this context. I denote by $V(T^\circ M)$ the vertical sub-bundle of $T(T^\circ M)$, whose fibre at $(x,y) \in T^\circ M$ is $V_{(x,y)}T^\circ M$, the vertical subspace of $T_{(x,y)}T^\circ M$. It is of course a vector bundle over $T^\circ M$. By 'tensor', or more properly $\mathfrak{V}$-tensor, I mean a section of a tensor bundle constructed from $V(T^\circ M)$.

Note that a $\mathfrak{V}$-tensor $K$, when restricted to any fibre $T^\circ_x M$ of $\pi : T^\circ M \to M$, defines there a linear-tensor field $K_x$ of the same type, that is, a field that transforms tensorially in the manner specified by its type under linear transformations of the canonical fibre coordinates. Conversely, an assignment to each fibre $T^\circ_x M$ of a linear-tensor field $K_x$ of a given type, depending smoothly on $x$ in an obvious sense, defines a $\mathfrak{V}$-tensor $K$ on $T^\circ M$ of that type.

The fundamental tensor, the angular metric, and the Cartan tensor are all examples of $\mathfrak{V}$-tensors, of type $(0,2)$, $(0,2)$, and $(0,3)$ respectively, all completely symmetric.

Any tensor field on $M$ induces in a natural way a $\mathfrak{V}$-tensor on $T^\circ M$, which may be called its natural lift. Such a $\mathfrak{V}$-tensor is distinguished by the fact that its components with respect to canonical fibre coordinates $y^i$ are independent of $y$ — they are the same as the components of the original tensor field with respect to the base coordinates $x^i$. In other words, such $\mathfrak{V}$-tensors are constant on fibres.

Type $(1,0)$ $\mathfrak{V}$-tensors are simply vertical vector fields. In particular, in accordance with the remarks in the prevous paragraph, a vector field $X$ on $M$ gives rise to a type $(1,0)$ $\mathfrak{V}$-tensor or vertical vector field $X^{\mathsf{v}}$ on $T^\circ M$, its vertical lift, where

$$X^{\mathsf{v}} = X^i \frac{\partial}{\partial y^i} \qquad \text{if } \; X = X^i \frac{\partial}{\partial x^i}$$

relative to coordinates $x^i$ on $M$ with corresponding fibre coordinates $y^i$.

This discussion of $\mathfrak{V}$-tensors, which is a précis of a more extensive one in [16], is included for completeness' sake. In practice I shall work almost always in terms of components. And from that perspective a $\mathfrak{V}$-tensor looks like, and transforms as if it were, just an ordinary tensor of the same type: its components however are, in general, local functions on $T^\circ M$. Not only shall I use components in tensorial calculations, I shall generally refer to a $\mathfrak{V}$-tensor by its component representation, with indices in place, reprehensible though this may seem to purists: it does, however, have the advantage of convenience, especially since it distinguishes between covariant and contravariant forms. I shall, however, from time to time refer for example to the fundamental tensor simply as $g$ where I judge no confusion can arise.

Though I shall mostly use component representations of $\mathfrak{V}$-tensors, the vertical lift construction does give me the opportunity to express some formulae in index-free form. So for example the fundamental tensor is determined by the fact that $g(X^{\mathsf{v}}, Y^{\mathsf{v}}) = X^{\mathsf{v}}(Y^{\mathsf{v}}(L))$ for all vector fields $X$ and $Y$ on $M$; likewise the angular metric satisfies $h(X^{\mathsf{v}}, Y^{\mathsf{v}}) = FX^{\mathsf{v}}(Y^{\mathsf{v}}(F))$, while for the Cartan tensor $C(X^{\mathsf{v}}, Y^{\mathsf{v}}, Z^{\mathsf{v}}) = X^{\mathsf{v}}(Y^{\mathsf{v}}(Z^{\mathsf{v}}(L)))$.

These formulae, and some others below, give just a few tastes of the index-free alternative to the component notation adopted in this paper. For a fully comprehensive index-free approach to these matters I recommend the text of Szilasi et al. [44].

I denote by $\Gamma$ the geodesic spray of $F$, in other words the Euler-Lagrange field of $L$: it is determined by the Euler-Lagrange equations $\Gamma(X^{\mathsf{v}}(L)) - X^{\mathsf{c}}(L) = 0$, holding for all vector fields $X$ on $M$, where $X^{\mathsf{c}}$ is the complete lift of $X$ to $T^\circ M$. In coordinates

$$\Gamma = y^i \frac{\partial}{\partial x^i} - 2\Gamma^i(x,y)\frac{\partial}{\partial y^i},$$

where

$$2\Gamma^i = \gamma^i_{jk} y^j y^k, \qquad \gamma^i_{jk} = \tfrac{1}{2} g^{il}\left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right).$$

So $\gamma^i_{jk}$ looks the same as a Christoffel symbol for the Levi-Civita connection of a Riemannian metric: however, here $g_{ij}$ will in general depend on $y$. (In fact $\gamma^i_{jk}$ is called a formal Christoffel symbol by Bao et al. in [7].)

The associated horizontal distribution on $T^\circ M$ is spanned by the local vector fields

$$\mathsf{H}_i = \frac{\partial}{\partial x^i} - \Gamma^j_i \frac{\partial}{\partial y^j}, \qquad \Gamma^j_i = \frac{\partial \Gamma^j}{\partial y^i}.$$

A vector field $X$ on $M$ determines a horizontal vector field $X^{\mathsf{h}}$ on $T^\circ M$, by

$$X^{\mathsf{h}} = \tfrac{1}{2}([X^{\mathsf{v}}, \Gamma] + X^{\mathsf{c}}).$$

The $\mathsf{H}_i$ are the horizontal lifts of the coordinate vector fields on $M$, and $X^{\mathsf{h}} = X^i \mathsf{H}_i$.

It will sometimes be useful to denote the vertical lifts of the coordinate vector fields on $M$ by $\mathsf{V}_i$:

$$\mathsf{V}_i = \frac{\partial}{\partial y^i},$$

and $X^{\mathsf{v}} = X^i \mathsf{V}_i$.

I use the Berwald connection throughout. I point this out because this choice is by no means universal: for example Bao et al., in [7], use the Chern, or Chern-Rund, connection rather than the Berwald connection.

The covariant derivative operator of the Berwald connection is essentially determined by the rules

$$\nabla_{X^{\mathsf{h}}} V = [X^{\mathsf{h}}, V], \qquad \nabla_{X^{\mathsf{v}}} V = [X^{\mathsf{v}}, V],$$

where $X$ is a vector field on $M$, and $V$, $[X^{\mathsf{h}}, V]$, and $[X^{\mathsf{v}}, V]$, all of which are vertical vector fields on $T^\circ M$, are to be interpreted as type $(1,0)$ $\mathfrak{V}$-tensors.

The connection coefficients of the horizontal part of the Berwald connection are $\Gamma^i_{jk}$ where

$$\Gamma^i_{jk} = \frac{\partial \Gamma^i_j}{\partial y^k} = \frac{\partial^2 \Gamma^i}{\partial y^j \partial y^k} = \frac{\partial \Gamma^i_k}{\partial y^j} = \Gamma^i_{kj}.$$

The Riemann curvature $\mathfrak{V}$-tensor is most directly defined in terms of the bracket of horizontal vector fields:

$$[\mathsf{H}_i, \mathsf{H}_j] = -R^k_{ij}\mathsf{V}_k.$$

Note that $R^k_{ij}$ is positively-homogeneous of degree 1. For each fixed $i$ and $j$, $R^k_{ij}\mathsf{V}_k$ is a vertical vector field which I call the $(i, j)$ Riemann curvature vector. Alternatively one could define a Riemann curvature vector for each pair of vector fields $X$, $Y$ on $M$ as

$$[X, Y]^{\mathsf{h}} - [X^{\mathsf{h}}, Y^{\mathsf{h}}] = R^k_{ij}X^iY^j\mathsf{V}_k.$$

The Riemann curvature is also a component part of the curvature of the Berwald connection. In that context I must explain my notation for covariant derivatives. I use a semi-colon for the horizontal Berwald covariant derivative, a comma for the vertical one. Thus for example for a vector field $\xi^i$ (type $(1, 0)$ $\mathfrak{V}$-tensor)

$$\xi^i_{;j} = \mathsf{H}_j(\xi^i) + \Gamma^i_{jk}\xi^k, \qquad \xi^i_{,j} = \mathsf{V}_j(\xi^i).$$

It is an obvious fact that vertical covariant derivatives commute; and for this reason I shall simplify the notation by writing (for example) $\xi^i_{,jk}$ instead of $\xi^i_{,j,k}$.

Of course $\Gamma$ is itself horizontal: $\Gamma = y^i\mathsf{H}_i$. The operator of horizontal Berwald covariant differentiation along $\Gamma$ is called the dynamical covariant derivative and denoted simply by $\nabla$: thus for example

$$\nabla(\xi^i) = y^j\xi^i_{;j} = \Gamma(\xi^i) + \Gamma^i_j\xi^j.$$

On the other hand, $\Delta = y^i\mathsf{V}_i$ is the Liouville vector field. It serves also as a vertical covariant derivative. It is frequently invoked when dealing with quantities which are positively homogeneous: for example, the fact that $F$ is positively-homogeneous of degree 1 can be expressed as $\Delta(F) = F$.

The Riemann curvature appears in other forms as follows:

$$R^i_{ljk} = R^i_{jk,l}\,, \qquad R^i_j = R^i_{jk}y^k = R^i_{ljk}y^ky^l,$$
$$R^i_{jk} = R^i_{ljk}y^l = \tfrac{1}{3}(R^i_{j,k} - R^i_{k,j}).$$

The type $(1,1)$ $\mathfrak{V}$-tensor $R^i_j$ is often called the Jacobi endomorphism, because of its role in the Jacobi equation.

The Berwald curvature, which is the other part of the curvature of the Berwald connection, is the type $(1,3)$ $\mathfrak{V}$-tensor whose components $B^i_{jkl}$ are given by

$$B^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial y^l} = \frac{\partial^2 \Gamma^i_j}{\partial y^k \partial y^l} = \frac{\partial^3 \Gamma^i}{\partial y^j \partial y^k \partial y^l}.$$

It can also be thought of as defining, for each choice of the indices $i$, $j$ and $k$, a vertical vector field, the Berwald curvature vector $B^l_{ijk}\mathsf{V}_l$.

The mean Berwald curvature, or E-curvature, is the trace of $B^l_{ijk}$:

$$E_{ij} = B^k_{kij} = \frac{\partial \Gamma^k_{ki}}{\partial y^j} = \frac{\partial \Gamma^k_{kj}}{\partial y^i} = \frac{\partial^2 \Gamma^k_k}{\partial y^i \partial y^j}.$$

I call $\varepsilon = g^{ij}E_{ij}$ the Berwald scalar curvature.

The Berwald connection is not in general $g$-compatible: it is however always the case that $\nabla(g_{ij}) = 0$.

A Landsberg space is a Finsler space for which $g_{ij;k} = 0$, that is, for which the fundamental tensor is constant for the horizontal Berwald covariant derivative. In a Landsberg space, therefore, the Berwald connection is one degree more $g$-compatible than is generally the case. The tensor $g_{ij;k}$ is often called the Landsberg tensor: so a Landsberg space is a Finsler space whose Landsberg tensor vanishes.

In a Landsberg space

$$\mathsf{H}_k(g_{ij}) - \Gamma^l_{ik}g_{lj} - \Gamma^l_{jk}g_{il} = 0,$$

which may be solved for the connection coefficient in the usual way to give

$$\Gamma^i_{jk} = \tfrac{1}{2}g^{il}\big(\mathsf{H}_j(g_{lk}) + \mathsf{H}_k(g_{jl}) - \mathsf{H}_l(g_{jk})\big).$$

In general, of course, $\Gamma^i_{jk}$ will depend on $y$: differentiating with respect to $y^l$ is irresistible, and leads to

$$2B_{ijkl} = 2g_{im}B^m_{jkl} = 2g_{im}\frac{\partial \Gamma^m_{jk}}{\partial y^l} = C_{ikl;j} + C_{ijl;k} - C_{jkl;i}.$$

Now $B_{ijkl}$ is always symmetric in its last three indices, $C_{ijk;l}$ in its first three indices; it follows from the relation above that in a Landsberg space both $B_{ijkl}$ and $C_{ijk;l}$ must be symmetric in all indices.

This is in fact one of several alternative equivalent ways of stating necessary and sufficient tensorial conditions for a Finsler space to be Landsberg, which are established in Appendix A:

(1) $y^i B_{ijkl} = 0$;

(2) $\nabla(C_{ijk}) = 0$;

(3) $B_{ijkl}$ is symmetric;

(4) $C_{ijk;l}$ is symmetric.

I note also that the following useful relations hold in a Landsberg space:

(5) $2B_{ijkl} = C_{ijk;l}$;

(6) $2E_{kl} = C_{k;l}$;

(7) $R_{ijkl} + R_{jikl} + R^m_{kl} C_{ijm} = 0$.

(I remark in passing that the last of these says that a Landsberg space has vanishing stretch tensor — see for example [31].) I have published these results previously in [10], and discuss them here at greater length in Appendix A.

It so happens that

$$\tfrac{1}{2} g^{il} \big(\mathsf{H}_j(g_{lk}) + \mathsf{H}_k(g_{jl}) - \mathsf{H}_l(g_{jk})\big)$$

is the expression for the connection coefficient for the Chern-Rund connection (see [7]). That is to say, in a Landsberg space the Chern-Rund connection and the Berwald connection coincide.

A Berwald space is a Finsler space whose Berwald connection is linear. A necessary and sufficient condition for a Finsler space to be a Berwald space is that $B^i_{jkl} = 0$. Evidently every Berwald space is a Landsberg space.

The Landsberg condition is most naturally thought of as a condition on the derivatives of the metric, in other words on the Cartan tensor, the Berwald condition as a condition on the Berwald curvature: but in fact in this context the Cartan and Berwald tensors are effectively interchangeable, since $B_{ijkl} = \frac{1}{2} C_{ijk;l}$ in a Landsberg space. Indeed, $C_{ijk;l} = 0$ is necessary and sufficient for a Landsberg space to be a Berwald space.

Here are two versions of the Landsberg-Berwald conjecture: the first is entirely in terms of the Berwald tensor, the second is in terms of the Cartan tensor:

- In any Finsler space, if $B_{ijkl}$ is symmetric then it vanishes.
- In any Finsler space, if $C_{ijk;l}$ is symmetric then it vanishes.

These formulations of the Landsberg-Berwald conjecture seem to me to capture the difficulty of the Landsberg-Berwald problem: it is hard to see why the mere symmetry of a certain type-$(0,4)$ $\mathfrak{V}$-tensor can be sufficient to ensure it vanishes.

## 2. Results

If one is to prove a theorem of the form

Theorem. *A regular Landsberg space which has property* P *is a Berwald space.*

then P had better be a possible property of Berwald spaces, and the obvious place to start is with properties P that Berwald spaces obviously have.

### 2.1. Projectively-Berwald Landsberg spaces

Theorem 1. (1993; [4]) *A Landsberg space which is projectively equivalent to a Berwald space is a Berwald space.*

A Finsler space is projectively equivalent to a Berwald space if and only if its Douglas tensor vanishes. The Douglas tensor $D$ is given by

$$D^i_{jkl} = B^i_{jkl} - \frac{1}{n+1}(E_{jk,l}y^i + E_{kl}\delta^i_j + E_{jl}\delta^i_k + E_{jk}\delta^i_l).$$

So Theorem 1 says that a Landsberg space whose Douglas tensor vanishes is a Berwald space. The first published proof of this result is to be found in [4]; for later ones see [5, 10, 39, 48].

### 2.2. Weakly-Berwald Landsberg spaces

Recall that a Finsler space is said to be weakly Berwald if its mean Berwald curvature $E_{ij} = B^k_{kij}$ vanishes.

Theorem 2. (2018; [12, 27]) *A weakly-Berwald Landsberg space is a Berwald space.*

As was pointed out in [27], this answered a challenge set by Shen in [41]. See also the interesting discussion by Bácsó and Yoshikawa in [5]. These authors take, in a way, an opposite point of view to the present one: they regard Landsberg, Douglas, and weakly-Berwald spaces as alternative generalisations of Berwald spaces, and investigate the relations between them. But they were not able to resolve the question of whether a Finsler space which is both Landsberg and weakly Berwald is a Berwald space.

At the time Theorem 2 seemed like a breakthrough: but stronger versions, described below, have since come to light.

In fact $E_{ij} = 0$ if and only if the Berwald scalar curvature $\varepsilon = g^{ij}E_{ij} = 0$ [12, 17, 28], so (making even less of an initial concession in the Berwald direction):

THEOREM 3. (2018) *A Landsberg space whose Berwald scalar curvature vanishes is a Berwald space.*

We may equally well express these results in terms of the Cartan tensor. In a Landsberg space $E_{ij} = \frac{1}{2}C_{i;j}$, so a Landsberg space for which $C_{i;j} = 0$ is a Berwald space by Theorem 2, and a Landsberg space for which $g^{ij}C_{i;j} = 0 = C^i_{;i}$ is a Berwald space by Theorem 3.

The function

$$\mathfrak{c} = g^{ij}\frac{\partial^2(\log\det g)}{\partial y^i \partial y^j} = g^{ij}C_{i,j}$$

on $T^\circ M$ plays an important role in the next, new, result. If the restriction of $\mathfrak{c}$ to the indicatrix bundle $\mathcal{I}$ is bounded I shall say that the Finsler space is a $\mathfrak{c}$-bounded space. If the space is geodesically complete, forward or backward, I shall simply say that it is complete.

I say that a function $f$ on $T^\circ M$ is geodesically invariant if $\Gamma(f) = 0$. A geodesically invariant function $f$ is constant along geodesics, or to be more precise, if $t \mapsto \gamma(t)$ is a geodesic (a curve in $M$ such that $t \mapsto (\gamma(t), \dot\gamma(t))$ is an integral curve of $\Gamma$), then $f(\gamma(t), \dot\gamma(t))$ is constant.

THEOREM 4. *A $\mathfrak{c}$-bounded complete Landsberg space whose Berwald scalar curvature is geodesically invariant is a Berwald space. In particular, a Landsberg space over a compact base whose Berwald scalar curvature is geodesically invariant is a Berwald space.*

*Proof.* In a Landsberg space $\nabla(C_i) = 0$. So by Proposition 1 of Appendix A we have

$$0 = (\nabla C_i)_{,j} = \nabla(C_{i,j}) + C_{i;j} = \nabla(C_{i,j}) + 2E_{ij},$$

and therefore

$$\varepsilon = -\tfrac{1}{2}g^{ij}\nabla(C_{i,j}) = -\tfrac{1}{2}\nabla(g^{ij}C_{i,j}) = -\tfrac{1}{2}\Gamma(\mathfrak{c}).$$

Consider a geodesic $\gamma(t)$, parametrized by arc length so that $(\gamma(t),\dot\gamma(t))\in\mathcal{I}$, with $(\gamma(0),\dot\gamma(0))=(x,y)\in\mathcal{I}$. Set

$$c(t)=g^{ij}(\gamma(t),\dot\gamma(t))C_{i,j}(\gamma(t),\dot\gamma(t))=\mathfrak{c}(\gamma(t),\dot\gamma(t)).$$

Now $\varepsilon(\gamma(t),\dot\gamma(t))$ is constant, say $\varepsilon(\gamma(t),\dot\gamma(t))=\varepsilon_0=\varepsilon(x,y)$. But

$$\dot c(t)=-2\varepsilon(\gamma(t),\dot\gamma(t))=-2\varepsilon_0,$$

so $c(t)=c(0)-2t\varepsilon_0$. By assumption $c(t)$ is bounded. On the other hand, $\gamma(t)$ is defined for all positive (negative) $t$, so $c(t)$ can be bounded only if $\varepsilon_0=0=\varepsilon(x,y)$. This holds for all $(x,y)\in\mathcal{I}$, and then by homogeneity $\varepsilon=0$ on $T^\circ M$. So the space is a Berwald space by Theorem 3.

The indicatrix bundle of a Finsler space over a compact base is compact, so such a space is $\mathfrak{c}$-bounded; moreover, a Finsler space over a compact base is complete. ∎

2.3. R-quadratic Landsberg spaces A Finsler space whose Riemann curvature tensor is independent of $y$, that is, for which $R^i_{jkl,m}=0$, is said to be R-quadratic — a mildly confusing terminology to my mind, but one has to remember that it is actually the Jacobi endomorphism that is quadratic (in $y$). Evidently Berwald spaces are R-quadratic. On the other hand, in an R-quadratic Landsberg space $\nabla(B_{ijkl})=0$, as I shall show below, and so certainly $\Gamma(\varepsilon)=0$. So one consequence of Theorem 4 is this:

Theorem 5. *An R-quadratic $\mathfrak{c}$-bounded complete Landsberg space is a Berwald space. In particular, an R-quadratic Landsberg space over a compact base is a Berwald space.*

This is a small improvement on a result I proved in [10]. In fact a forward-complete R-quadratic Finsler space for which (in an appropriate sense) $C_{ijk}$ is bounded is a Landsberg space [40], and a forward-complete R-quadratic Finsler space for which both $C_{ijk}$ and $C_{ijk,l}$ are bounded is a Berwald space. In particular, an R-quadratic Finsler space over a compact base is a Berwald space — there is no need to invoke the Landsberg condition.

The proof of Theorem 4 is based on the proofs of the results above in [10], which in turn were extensions of arguments in Shen's book [39] and paper [40].

It is clear that there must be a stronger result than Theorem 5 coming from Theorem 4. To obtain it I start from the fact that

$$\nabla(B^m_{ijk})=-y^lR^m_{jkl,i}=-R^m_{k,ij}+R^m_{ikj}+R^m_{jki},$$

which follows from the Bianchi identity $B^m_{ijk;l} - B^m_{ijl;k} = -R^m_{jkl,i}$. Thus

$$\nabla(E_{ij}) = -R^k_{k,ij} + R^k_{ikj} + R^k_{jki}$$

and

$$\Gamma(\varepsilon) = g^{ij}\nabla(B^k_{ijk}) = -g^{ij}(R^k_{k,ij} - 2R^k_{ikj}).$$

The terms on the right-hand side are usually named after Ricci by Finsler geometers: see for example the discussion by B. Li and Shen in [26]. I find Finsler Ricci terminology confused and confusing, so I shall avoid using it: but let me list the variations. First of all, the scalar $R^k_k$ —the trace of the Jacobi endomorphism— is often called the Ricci curvature. There are at least three alternative suggestions for a Ricci curvature tensor:

- $\frac{1}{2}R^k_{k,ij}$, proposed by Akbar-Zadeh in [3];
- $R^k_{ikj}$, directly following the model of Riemannian geometry;
- $\frac{1}{2}(R^k_{ikj} + R^k_{jki})$, introduced by B. Li and Shen in [26], on the grounds that $R^k_{ikj}$ need not be symmetric in Finsler geometry.

Each of these has an associated Ricci curvature scalar, obtained by contracting it with $g^{ij}$. The Ricci curvature scalars corresponding to the second and third variants evidently coincide, but the first is in general different: in fact $g^{ij}(R^k_{k,ij} - 2R^k_{ikj}) = 0$ is just the condition for all three to be equal. This holds in a Berwald space since $R^i_{jkl}$ is independent of $y$, so that

$$R^k_{k,ij} = (R^k_{pkq}y^p y^q)_{,ij} = R^k_{ikj} + R^k_{jki},$$

whence clearly $g^{ij}(R^k_{k,ij} - 2R^k_{ikj}) = 0$.

THEOREM 6. *A $\mathfrak{c}$-bounded complete Landsberg space for which $g^{ij}(R^k_{k,ij} - 2R^k_{ikj}) = 0$ is a Berwald space. In particular, a Landsberg space over a compact base for which $g^{ij}(R^k_{k,ij} - 2R^k_{ikj}) = 0$ is a Berwald space.*

### 2.4. Landsberg spaces of scalar curvature

THEOREM 7. (1975; [38]) *A Landsberg space of non-zero scalar curvature is a Berwald space.*

In fact such a space is a Riemannian space of constant curvature, as Numata showed in [38]. Shibata [43] showed that the result still holds if the Landsberg condition is relaxed to vanishing stretch tensor. Matsumoto discussed both results in [31].

In the light of Theorem 4 it is of interest to observe that the flag curvature of a Finsler space of scalar flag curvature is constant if and only if the Berwald scalar curvature of the space is geodesically invariant. This result was originally due to Akbar-Zadeh [3]. I discuss it from a more modern perspective in Appendix C. So the special case of Theorem 7 in which the scalar curvature is constant actually follows from Theorem 4, albeit with extra provisos about $\mathfrak{c}$-boundedness and completeness.

2.5. The S-function of a Landsberg space I remind the reader that the S-function of a Finsler space is defined as $S = \Gamma(\tau)$, where

$$\tau = \log\left(\frac{\sqrt{\det g}}{\omega}\right)$$

is the distortion relative to the Busemann form $\omega(dx)^n$, a volume form on $M$.

Theorem 8. (2019) *A Landsberg space whose S-function vanishes is a Berwald space.*

This is part of the main result of [27]. Alternatively, it is known that $E_{ij} = 0$ if and only if $\varepsilon = 0$ if and only if $S = 0$ [17], so the result follows from Theorem 2.

The condition $S = 0$ can be relaxed, at the expense of introducing the $\chi$-vector (see Appendix D).

Theorem 9. *A $\mathfrak{c}$-bounded complete Landsberg space for which $g^{ij}\chi_{i,j} = 0$ is a Berwald space. In particular a $\mathfrak{c}$-bounded complete Landsberg space whose $\chi$-vector vanishes is a Berwald space. A Landsberg space over a compact base, for which $g^{ij}\chi_{i,j} = 0$, is a Berwald space.*

*Proof.* We have $g^{ij}\chi_{i,j} = \Gamma(\varepsilon)$ (Appendix D), and so $\Gamma(\varepsilon) = 0$. The result follows from Theorem 4. ∎

In fact this result is just Theorem 6 in disguise (see Appendix D).

2.6. Volume forms and Landsberg spaces The next result concerns volume forms on $T^\circ M$. Such a form is said to be vertically invariant if its Lie derivative by the vertical lift of every vector field on $M$ vanishes, and geodesically invariant if its Lie derivative by the geodesic spray vanishes. It is quite hard for a volume form to be vertically invariant and geodesically invariant at the same time.

Theorem 10. (2022; [14]) *A Landsberg space which admits a geodesically-invariant vertically-invariant volume form is a Berwald space.*

In fact if a Landsberg space admits a geodesically-invariant vertically-invariant volume form then that volume form must be a constant multiple of the Busemann volume form. It must also be a scalar multiple of the metric volume form, by a scalar factor which is geodesically invariant. The scalar factor is closely related to the distortion, which consequently must be geodesically invariant, which is to say that the S-function vanishes.

2.7. The averaged metric of a Landsberg space One test of whether a Finsler space is a Berwald space is this: a Finsler space is a Berwald space if and only if there is a Riemannian metric $a$ on $M$ such that $a_{ij;k} = 0$ (where the $a_{ij}$ are to be thought of as the components of the natural lift of $a$ to $T^\circ M$, and of course are independent of $y$, and the $a_{ij;k}$ are the components of its horizontal Berwald covariant derivative as defined by the Finsler structure). This result, in slightly different form, is due to Aikou [1]; see also [16, 45].

There is a by now familiar method of obtaining a Riemannian metric on $M$ from a Finsler structure, namely by averaging: see for example [2, 8, 10, 11, 33, 47, 49]. There are in fact several different ways of carrying out such a construction, described in [11]. I shall here use what it is natural to call the averaged metric, specified below.

Set

$$\bar{g}_{ij}(x) = \frac{1}{\nu_x} \int_{B_x} g_{ij}(x, y)\omega_x,$$

where $g$ is the fundamental tensor; $\omega_x$ is the metric volume form on $T^\circ_x M$ induced by $g$:

$$\omega_x = \sqrt{\det g(x, y)}dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n = \sqrt{\det g(x, y)}(dy)^n;$$

$B_x = \{y \in T^\circ_x M : F(x, y) \leq 1\}$ is the unit ball in $T^\circ_x M$ determined by $F$; and $\nu_x = \int_{B_x} \omega_x$. Then (see for example [10]) for each $x \in M$ $(\bar{g}_{ij}(x))$ is a positive-definite quadratic form, that is, an Euclidean metric, on $T_xM$, and $\bar{g}$ is a Riemannian metric on $M$, the averaged metric of the Finsler space.

It is shown in [10] that in a Landsberg space

$$\frac{1}{\nu}\int_B \mathsf{H}_k(g_{ij})\omega = \frac{\partial \bar{g}_{ij}}{\partial x^k}.$$

But also, in a Landsberg space, since $g_{ij;k} = 0$

$$\mathsf{H}_k(g_{ij}) = \Gamma^l_{ik} g_{lj} + \Gamma^l_{jk} g_{il}.$$

Thus

$$\frac{\partial \bar{g}_{ij}}{\partial x^k} = \frac{1}{\nu}\int_B (\Gamma^l_{ik} g_{lj} + \Gamma^l_{jk} g_{il})\omega.$$

It follows that if $\Gamma^k_{ij}$ is independent of $y$, so that the space is a Berwald space, then

$$\frac{\partial \bar{g}_{ij}}{\partial x^k} = \Gamma^l_{ik}\bar{g}_{lj} + \Gamma^l_{jk}\bar{g}_{il}.$$

This shows, first, that $\bar{g}_{ij;k} = 0$, so we can take $\bar{g}$ for $a$ in Aikou's result; and furthermore that the $\Gamma^k_{ij}$ are just the Christoffel symbols of the Levi-Civita connection of $\bar{g}$.

On the other hand, if $\bar{g}_{ij;k} = 0$ then

$$\frac{\partial \bar{g}_{ij}}{\partial x^k} = \Gamma^l_{ik}\bar{g}_{lj} + \Gamma^l_{jk}\bar{g}_{il},$$

whence

$$\Gamma^i_{jk} = \tfrac{1}{2}\bar{g}^{il}\left(\frac{\partial \bar{g}_{lk}}{\partial x^j} + \frac{\partial \bar{g}_{jl}}{\partial x^k} - \frac{\partial \bar{g}_{ij}}{\partial x^l}\right)$$

by the usual method, so $\Gamma^i_{jk} = \bar{\Gamma}^i_{jk}$ and the space is a Berwald space. So we have the following variant of Aikou's result [1].

THEOREM 11. *A Landsberg space whose averaged metric satisfies* $\bar{g}_{ij;k} = 0$ *is a Berwald space.*

There is a formula for the Christoffel symbols $\bar{\Gamma}^i_{jk}$ of the averaged metric in terms of averages of the connection coefficients of the Berwald connection, for a Landsberg space which is not necessarily Berwald, which is worth mentioning. Set $\Gamma_{ijk} = g_{il}\Gamma^l_{jk}$, and denote by $\overline{\Gamma_{ijk}}$ the average of $\Gamma_{ijk}$:

$$\overline{\Gamma_{ijk}} = \frac{1}{\nu}\int_B \Gamma_{ijk}\,\omega.$$

Then

$$\frac{\partial \bar{g}_{ij}}{\partial x^k} = \overline{\Gamma_{ijk}} + \overline{\Gamma_{jik}},$$

from which it follows that

$$\bar{\Gamma}^i_{jk} = \bar{g}^{il}\overline{\Gamma_{ljk}}.$$

(I gave a slightly different version of this formula in [10].) When the space actually is a Berwald space we have

$$\bar{\Gamma}^i_{jk} = \bar{g}^{il}\overline{\Gamma_{ljk}} = \bar{g}^{il}\bar{g}_{lm}\Gamma^m_{jk} = \Gamma^i_{jk}$$

as before.

2.8. Fibre isometries in a Landsberg space The fundamental tensor $g$ of a Finsler space defines on each punctured tangent space $T^\circ_x M$ a Riemannian metric $g_x$, which I call the fibre metric at $x$. By a fibre isometry I mean a diffeomorphism between two fibres $T^\circ_{x_1}M$ and $T^\circ_{x_2}M$, which is positively-homogeneous of degree 1, and is an isometry of the fibre metrics $g_{x_1}$ and $g_{x_2}$. The same terminology applies, mutatis mutandis, to any Minkowski space. We may of course take $x_1 = x_2$: the set of (self) fibre isometries of a single Minkowski space forms a group.

In [51], Xu and Matveev prove the following striking and significant results.

Theorem. (Xu and Matveev) *Suppose $F$ is a Minkowski norm on $\mathbb{R}^n$ with $n \geq 3$, which is invariant with respect to the standard block diagonal action of the group $SO(k) \times SO(n-k)$ with $1 \leq k \leq n-1$. Let $G_0$ be the connected isometry group for the Hessian metric $g = \frac{1}{2}d^2F^2$ on $\mathbb{R}^n\backslash\{0\}$. Then, every element $\Phi \in G_0$ is linear. Moreover, if $F$ is not Euclidean, then $G_0$ together with its action coincides with $SO(k) \times SO(n-k)$.*

The "Hessian metric $g = \frac{1}{2}d^2F^2$" is of course the fibre metric, whose components are

$$g_{ij} = \tfrac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j};$$

and $G_0$ consists of the fibre isometries of the (single) punctured vector space $\mathbb{R}^n_\circ$ with that fibre metric.

Corollary. (Xu and Matveev) *Let $(M, F)$ be a Finsler manifold of dimension $n \geq 3$. Assume that for every point $p \in M$, there exist linear coordinates in $T_pM$ such that the restriction $F|_{T_pM}$ is invariant with respect* *to the standard block diagonal action of the group $SO(k) \times SO(n-k)$ with $1 \leq k \leq n-1$. Then, if the Landsberg curvature vanishes, $F$ is Berwald.*

So the conclusion is:

THEOREM 12. (2022) *A Landsberg space whose fibre metric on each fibre is invariant under the linear action of the group $SO(k) \times SO(n-k)$ in the fibre, as described in the theorem of Xu and Matveev, is a Berwald space.*

The reason that this result is so significant is that, as Xu and Matveev point out, it covers the standard test cases of Finsler geometry: $(\alpha, \beta)$ spaces and general $(\alpha, \beta)$ spaces are examples with $k = 1$, $(\alpha_1, \alpha_2)$ spaces examples with more general $k$.

On the other hand, we have no reason to suppose that the fibre metrics of Landsberg spaces in general have the required pattern of isometries, indeed no reason to suppose that the fibre metrics of a given Landsberg space have any non-trivial fibre isometries at all. At the other end of the spectrum we have

THEOREM 13. *A $\mathfrak{c}$-bounded complete Landsberg space whose fibre metric at each point admits no non-trivial fibre isometries is a Berwald space.*

The key point here is that the relationship $R_{ijkl} + R_{jikl} + R^m_{kl}C_{ijm} = 0$, the vanishing of the stretch curvature, which holds in a Landsberg space, says that the Riemann curvature vector field $R^k_{ij}\mathsf{V}_k$ is an infinitesimal isometry of the fibre metric in every fibre, that is to say, it satisfies Killing's equation, which for a vector field $\xi = \xi^k\mathsf{V}_k$ is

$$\xi^k \frac{\partial g_{ij}}{\partial y^k} + \frac{\partial \xi^k}{\partial y^i} g_{kj} + \frac{\partial \xi^k}{\partial y^j} g_{ik} = 0 = \xi^k C_{ijk} + \frac{\partial \xi^k}{\partial y^i} g_{kj} + \frac{\partial \xi^k}{\partial y^j} g_{ik}.$$

So if there are no infinitesimal isometries we must have $R^k_{ij} = 0$, and the space is a Berwald space by Theorem 5. This again is a slight improvement on a result I gave in [10].

If the fibre metric of a Landsberg space admits a large group of (linear) isometries, as specified in the theorem of Xu and Matveev, then the space is a Berwald space; while if the fibre metric of a Landsberg space admits no isometries, of any kind, then the space is a Berwald space.

2.9. Isometries of Landsberg spaces We now move on to isometries of Landsberg spaces themselves, as distinct from their fibre metrics. Recall that an isometry of a Finsler space is a diffeomorphism $\psi$ of $M$ onto itself which preserves the Finsler function, in the sense that $F(\psi(x), \psi_{*x}y) = F(x, y)$ [19]. The isometries of a Finsler space form a Lie group $\mathfrak{I}$. A Finsler space whose isometries act transitively on $M$ is said to be homogeneous.

It is conjectured by Xu and Deng in [50] that any homogeneous Landsberg space must be a Berwald space. The paper is actually mainly concerned with showing that any Landsberg $(\alpha_1, \alpha_2)$ space is a Berwald space. From the point of view of the theorem of Xu and Matveev, the significant feature of $(\alpha_1, \alpha_2)$ spaces is the symmetry of the fibre metric. This is indeed what leads Xu and Deng to make their conjecture, as they report in [50]. But the key issue, it seems to me, is not whether the Landsberg space is homogeneous, but (supposing it to have isometries) the nature of the isotropy groups. As I show in Appendix F, the isotropy group $\mathfrak{I}_x$ at each $x \in M$ acts by linear transformations of $T_xM$, and is a subgroup of $SO(n)$.

Theorem 14. *A Landsberg space on which a group $\mathfrak{I}$ of isometries acts, such that the isotropy group $\mathfrak{I}_x$ at each $x \in M$ contains one of the groups identified in the theorem of Xu and Mateveev, is a Berwald space.*

This does not require the isometry action to be transitive on $M$, of course. However, it will in general be the case that if the dimension of the orbit of $x$ is $m$ then the dimension of $\mathfrak{I}_x$ is at most $\frac{1}{2}m(m-1)$. To achieve the case $k = 1$ of the theorem of Xu and Matveev, for example, we must have $m = n - 1$ if the action is not transitive. This is indeed possible: spherically symmetric Finsler spaces (see [54]) provide an obvious example.

2.10. Parallel transport in Landsberg spaces One way, to my mind indeed the most geometrically appealing way, of defining a Landsberg space is that it is a Finsler space for which parallel transport is an isometry between the Riemannian metrics on tangent spaces. This well-known result was due originally to Ichijyō [24].

We also have the following familiar result, which could be regarded as the geometrical crux of the Landsberg-Berwald problem:

Theorem 15. (1976; [23]) *A Landsberg space with the property that its fibre isometries are all linear is a Berwald space.*

M. Li, in [27] for instance, calls two Minkowski spaces which are linked by a linear isometry equivalent. So we may rephrase this result as follows: a Landsberg space with the property that its fibres are all pairwise equivalent as Minkowski spaces is a Berwald space.

I have included a discussion of isometries and equivalences of Minkowski spaces at Appendix E. I discussed at length the correspondence between results about parallel transport and their tensorial equivalents in [16].

Geometrically appealing Ichijyō's theorem may be — but unfortunately it does not appear to help very much when it comes to investigating the Landsberg-Berwald problem.

One possible further line of attack might be to consider holonomy, that is, parallel transport around closed curves. In the case of a Landsberg space Ichijyō's theorem shows that holonomy transformations are fibre isometries. There is some pretty heavyweight discussion of holonomy of Finsler spaces in the literature, such as [18, 25, 36]. Of more immediate interest is the interpretation of the vanishing of the stretch curvature as showing that the Riemann curvature vector field is an infinitesimal isometry of the fibre metric (Subsection 2.8): the curvature vector field is a generator of holonomy transformations after all.

2.11. Landsberg metrizability In [35] Muzsnay discusses the Landsberg metrizability problem: under what conditions can a given spray be the geodesic spray of a Finsler metric of Landsberg type. He notes that in a Landsberg space the Finsler function $F$ is invariant under horizontal vector fields: $\mathsf{H}_i(F) = 0$ (this is of course true for any Finsler space). Moreover, the Landsberg condition $B_{lijk}y^l = 0$ amounts to $B^l_{ijk}\mathsf{V}_l(F) = 0$. Thus in order to find a Landsberg Finsler function whose geodesic spray is the given spray one must find a positive regular solution to these conditions, considered as a system of first-order partial differential equations for the function $F$, which also satisfies the requirements of homogeneity and strong convexity.

From our perspective the point of interest is this. Suppose given a Finsler space with Finsler function $F$. Denote by $\mathfrak{L}$ the smallest integrable distribution on $T^\circ M$ containing the $\mathsf{H}_i$ and the vertical fields $B^l_{ijk}\mathsf{V}_l$, the Berwald curvature vector fields. Then the space is a Landsberg space if and only if the integral manifolds of $\mathfrak{L}$ lie in the level sets of $F$.

Muzsnay devotes a section of his paper to the Landsberg-Berwald problem, in which he proposes essentially that investigating the properties of the distribution $\mathfrak{L}$ would be a 'promising strategy' for tackling the problem. Unfortunately this proposal has not yet led to any significant progress so far as I am aware.

However, the fact that a necessary and sufficient condition for a Finsler space to be a Landsberg space is that the Berwald curvature vector fields must be tangent to the indicatrix is worth recording.

2.12. Special classes of Landsberg spaces I shall now focus on familiar special classes of Landsberg spaces.

2.12.1. Semi-C-reducible spaces A Finsler space is said to be semi-C-reducible if

$$C_{ijk} = K(C_i h_{jk} + C_j h_{ik} + C_k h_{ij}) + L C_i C_j C_k$$

for some functions $K$ and $L$. (In fact some care must be taken over the domains of definition of $K$ and $L$, a matter which is discussed in detail in [13].) A semi-C-reducible space for which $L = 0$ is said to be C-reducible.

Theorem 16. (1979; [32]) *A Landsberg semi-C-reducible space is a Berwald space.*

For more recent discussions of this result see [13, 22].

This theorem of course subsumes the corresponding result for C-reducible spaces, proved by Matsumoto in 1972 [29].

2.12.2. $(\alpha, \beta)$ spaces

Theorem 17. (2009) *A Landsberg $(\alpha, \beta)$ space is a Berwald space.*

There are by now several different proofs of this result:

- by direct calculation [42]: the necessary and sufficient condition for an $(\alpha, \beta)$ space to be Landsberg is that the 1-form $b = b_i dx^i$ is parallel with respect to the Riemannian metric $a = a_{ij} dx^i \otimes dx^j$, and this is also the condition for the space to be Berwald;
- if the 1-form $b$ of an $(\alpha, \beta)$ space is parallel then the S-function vanishes [9, 15], so Theorem 8 applies;
- $(\alpha, \beta)$ spaces have the $SO(n-1)$ fibre-symmetry property of the theorem of Xu and Matveev, as was pointed out in [53] (see also the discussion in the Postscript to my paper [13]), so Theorem 12 applies;
- $(\alpha, \beta)$ spaces are semi-C-reducible [13, 22, 30], so Theorem 16 applies.

2.12.3. General $(\alpha, \beta)$ spaces

Theorem 18. (2019) *A Landsberg general $(\alpha, \beta)$ space is a Berwald space.*

In fact there are two kinds of general $(\alpha, \beta)$ spaces: fairly general and quite general. This result for fairly general $(\alpha, \beta)$ spaces is proved by direct calculation by Zhou, Wang and B. Li in [55]. The result for quite general $(\alpha, \beta)$ spaces is proved by Feng, Han and M. Li in [21], using the methods established by M. Li in [27]. Both versions follow from Theorem 12 (the theorem of Xu and Matveev).

2.12.4. $(\alpha_1, \alpha_2)$ spaces

Theorem 19. (2014) *A Landsberg $(\alpha_1, \alpha_2)$ space is a Berwald space.*

Xu and Deng gave the first proof, by calculation, in [50] (first posted on the arXiv in 2014). In fact they prove two theorems:

1. The S-curvature of any Landsberg $(\alpha_1, \alpha_2)$ metric vanishes identically.
2. Any Landsberg $(\alpha_1, \alpha_2)$ space is a Berwald space.

They go on to say: 'We remark here that logically Theorem (1) is a corollary of Theorem (2), since any Berwald space must have vanising S-curvature. However, our proof of the second theorem relies heavily on the techniques developed in the proof of the first one.' From the present perspective we see that in fact the main result follows from the vanishing of the S-function by Theorem 8.

Theorem 19 also follows directly from Theorem 12.

2.12.5. Spherically-symmetric Finsler spaces

Theorem 20. (2014) *A Landsberg spherically-symmetric Finsler space is a Berwald space.*

This is proved by direct calculation by Mo and Zhou in [34]. It also follows from Theorem 14.

According to Elgendi [20] such a space is in fact Riemannian.

## 3. Conclusion

I have to confess that I approached my task in writing this paper in the belief that the Landsberg-Berwald conjecture is true. In the course of writing it my confidence has been undermined somewhat. The problem, of course, is that the positive evidence, while quite extensive, is not cumulative. I am no clearer about how one could prove the pure Landsberg-Berwald conjecture at the end of the process than I was at the beginning.

## A. More background

In this appendix I give further tensorial results and discuss the various ways of reformulating the Landsberg conditions. I reiterate that I always assume regularity, and that $n \geq 3$.

The relationships between second covariant derivatives are governed by the Ricci identities: for example

$$\begin{array}{ll} \text{(vertical)} & \xi^i_{,j,k} - \xi^i_{,k,j} = 0\,, \\ \text{(mixed)} & \xi^i_{;j,k} - \xi^i_{,k;j} = B^i_{jkl}\xi^l\,, \\ \text{(horizontal)} & \xi^i_{;j;k} - \xi^i_{;k;j} = -R^i_{ljk}\xi^l + R^l_{jk}\xi^i_{,l}, \end{array}$$

the final term coming from the bracket of horizontal vector fields. Incidentally, the mixed Ricci identity for covariant derivatives applied to a function $f$ on $T^\circ M$ is just $f_{;i,j} - f_{,j;i} = 0$, while the horizontal Ricci identity in such circumstances gives back the original definition of $R^k_{ij}$.

The following tensor result is often useful.

Proposition 1. *For a covariant tensor* $T_{i_1 i_2 \ldots i_p}$,

$$(\nabla(T_{i_1 i_2 \ldots i_p}))_{,j} = \nabla(T_{i_1 i_2 \ldots i_p,j}) + T_{i_1 i_2 \ldots i_p;j}.$$

*Proof.* I give the proof for a covector field $\pi_i$: the proof of the general case is similar, and only notationally more complicated.

I have to prove that

$$(\nabla(\pi_i))_{,j} = \nabla(\pi_{i,j}) + \pi_{i;j}.$$

I use the mixed Ricci identity for Berwald covariant differentiation, which for a covector $\pi_k$ is

$$\pi_{i,j;k} - \pi_{i;k,j} = B^l_{ijk}\pi_l.$$

Since $y^k B^l_{ijk} = 0$, on contracting with $y^k$ we get

$$\nabla(\pi_{i,j}) = y^k \pi_{i;k,j} = (y^k \pi_{i;k})_{,j} - \pi_{i;j} = (\nabla(\pi_i))_{,j} - \pi_{i;j}$$

as claimed. ∎

It even holds that for a function $\phi$,

$$(\Gamma(\phi))_{,i} = \nabla(\phi_{,i}) + \phi_{;i},$$

though the proof is rather different in this case.

One should bear it in mind that $y^i_{;j} = 0$ and that in general $\nabla(g_{ij}) = 0$.

I now turn to the Landsberg conditions.

First of all, recall that it is a property of the horizontal distribution in any Finsler space that $L$ is horizontally constant: $\mathsf{H}_i(L) = 0$. Taking two successive vertical derivatives of this gives (with $B_{lijk} = g_{lm} B^m_{jkl}$, as before)

$$g_{jk;i} = y^l B_{lijk}.$$

Using Proposition 1, first with $\pi_i = y_i = g_{ij} y^j$, then with $T_{ij} = g_{ij}$, we obtain, in any Finsler space,

(1) $y_{i;j} = 0$;

(2) $g_{ij;k} = -\nabla(g_{ij,k}) = -\nabla(C_{ijk})$.

Moreover, from the mixed Ricci identity, in any Finsler space

$$C_{ijk;l} = g_{ij,k;l} = g_{ij;l,k} + B_{ijkl} + B_{jikl}.$$

Notice that

$$\begin{aligned} B_{ijkl} + B_{jikl} &= C_{ijk;l} - g_{ij;l,k}, \\ B_{jkil} + B_{kjil} &= C_{ijk;l} - g_{jk;l,i}, \\ B_{kijl} + B_{ikjl} &= C_{ijk;l} - g_{ki;l,j}. \end{aligned}$$

Adding the first and last and subtracting the second, and taking account of symmetries, we find that

$$2B_{ijkl} = C_{ijk;l} - g_{ij;l,k} + g_{jk;l,i} - g_{ki;l,j}.$$

Finally, from the Ricci identity for horizontal derivatives

$$g_{ij;k;l} - g_{ij;l;k} = R_{ijkl} + R_{jikl} + R^m_{kl} g_{ij,m}.$$

The assertions about a Landsberg space made at the start of the paper follow essentially by setting $g_{ij;k} = 0$. Firstly, in a Landsberg space

(1) $y^l B_{lijk} = 0$;

(2) $\nabla(C_{ijk}) = 0$;

(3) $2B_{ijkl} = C_{ijk;l}$;

(4) $B_{ijkl}$ is symmetric;

(5) $C_{ijk;l}$ is symmetric;

(6) $R_{ijkl} + R_{jikl} + R^m_{kl} C_{ijm} = 0$.

Conversely, if $y^l B_{lijk} = 0$ or $\nabla(C_{ijk}) = 0$ holds in a Finsler space then $g_{jk;i} = 0$ and the space is a Landsberg space. If $B_{ijkl}$ is symmetric then $y^l B_{lijk} = y^l B_{ijkl} = 0$ and the space is a Landsberg space. If $C_{ijk;l}$ is symmetric then $\nabla(C_{ijk}) = y^l C_{ijk;l} = (y^l C_{ijl})_{;k} = 0$ and the space is a Landsberg space. So conditions (1), (2), (4) and (5) above are necessary and sufficient for a Finsler space to be a Landsberg space.

## B. A result about homogeneous functions

This appendix is concerned mainly with Minkowski spaces. A Minkowski space is a punctured vector space, which I take to be $\mathbb{R}^n_\circ$ with its standard coordinates, equipped with a Minkowski norm $F$ and corresponding fundamental tensor $g_{ij}$.

Proposition 2. *A function $\kappa$ on $\mathbb{R}^n_\circ$ which is positively-homogeneous of degree 0 satisfies*

$$g^{ij} \frac{\partial^2 \kappa}{\partial y^i \partial y^j} = 0$$

*if and only if it is constant.*

*Proof.* Let $V$ be the vector field on $\mathbb{R}^n_\circ$ given by

$$V = \left( (\det g) g^{ij} \kappa \frac{\partial \kappa}{\partial y^j} \right) \frac{\partial}{\partial y^i}.$$

Then $V$ is tangent to the level hypersurfaces of $F$:

$$V^i \frac{\partial F}{\partial y^i} = (\det g) \kappa g^{ij} \frac{\partial \kappa}{\partial y^j} \frac{\partial F}{\partial y^i} = \left( \frac{(\det g)\kappa}{F} \right) y^i \frac{\partial \kappa}{\partial y^i} = 0$$

since $\kappa$ is homogeneous of degree 0. So for any volume form $\Omega$ on $\mathbb{R}^n_\circ$, for any $a$, $b$ with $0 < a < b$,

$$\int_{B^b_a} (\mathrm{div}_\Omega V)\Omega = \int_{B^b_a} d(i_V \Omega) = 0, \qquad \text{where } B^b_a = \{y : a \le F(y) \le b\},$$

by the Stokes-Cartan Theorem, since the $(n-1)$-form $i_V\Omega$ vanishes when pulled back to any level hypersurface of $F$ (or as one might say, the flux of $V$ over the boundary $\{F(y)=a\}\cup\{F(y)=b\}$ of $B_a^b$ vanishes). Now with $\Omega=(dy)^n$,

$$\begin{aligned}\mathrm{div}_\Omega V=\frac{\partial V^i}{\partial y^i}&=\frac{\partial}{\partial y^i}((\det g)g^{ij})\kappa\frac{\partial\kappa}{\partial y^j}+(\det g)g^{ij}\left(\frac{\partial\kappa}{\partial y^i}\frac{\partial\kappa}{\partial y^j}+\kappa\frac{\partial^2\kappa}{\partial y^i\partial y^j}\right)\\&=(\det g)(C_ig^{ij}-C^j)\kappa\frac{\partial\kappa}{\partial y^j}+(\det g)g^{ij}\frac{\partial\kappa}{\partial y^i}\frac{\partial\kappa}{\partial y^j}\\&=(\det g)g^{ij}\frac{\partial\kappa}{\partial y^i}\frac{\partial\kappa}{\partial y^j}.\end{aligned}$$

Thus

$$\int_{B_a^b}\left((\det g)g^{ij}\frac{\partial\kappa}{\partial y^i}\frac{\partial\kappa}{\partial y^j}\right)(dy)^n=0.$$

Evidently

$$(\det g)g^{ij}\frac{\partial\kappa}{\partial y^i}\frac{\partial\kappa}{\partial y^j}\geq 0.$$

If there were any point $y\in\mathbb{R}^n_\circ$ at which this function took a positive value then by choosing $a$ and $b$ so that $y\in B_a^b$ we would ensure that

$$\int_{B_a^b}\left((\det g)g^{ij}\frac{\partial\kappa}{\partial y^i}\frac{\partial\kappa}{\partial y^j}\right)(dy)^n>0$$

since the argument is never negative and would be positive on some open set contained in $B_a^b$. This would contradict the fact that every such integral must vanish. It follows that

$$(\det g)g^{ij}\frac{\partial\kappa}{\partial y^i}\frac{\partial\kappa}{\partial y^j}=0,$$

and therefore (since $\det g$ cannot vanish)

$$\frac{\partial\kappa}{\partial y^i}=0.$$

∎

Consider, for example, the function

$$\mathfrak{c}=g^{ij}\frac{\partial^2(\log\det g)}{\partial y^i\partial y^j}$$

on $T^\circ M$, which plays an important role in Theorem 4. Clearly $\log\det g$ is positively-homogeneous of degree 0. So if $\mathfrak{c}$ vanishes on any fibre $T_x^\circ M$ then

$\log\det g_x$ is constant (that is, independent of $y$), and so $g_x$ is constant by Deicke's Theorem. It follows that if $\mathfrak{c} = 0$ on $T^\circ M$ then the space is Riemannian. It is hard to resist the temptation to think of $\mathfrak{c}$ as measuring by how much the Finsler space diverges from being a Riemannian space.

In [12] I proved, by similar methods, a result analogous to the proposition above for functions which are positively-homogeneous of degree 1:

PROPOSITION 3. *A function $\kappa$ on $\mathbb{R}^n_\circ$ which is positively-homogeneous of degree 1 satisfies*

$$g^{ij}\frac{\partial^2\kappa}{\partial y^i \partial y^j} = 0$$

*if and only if it is linear.*

## C. Scalar flag curvature

A Finsler space is of scalar flag curvature $\kappa$ if

$$R^i_j = \kappa(F^2\delta^i_j - y^i y_j), \qquad y_j = g_{jk}y^k = F\frac{\partial F}{\partial y^j},$$

equivalently if $R_{ij} = \kappa F^2 h_{ij}$ where $R_{ij} = g_{ik}R^k_j$, and $h_{ij}$ is the angular metric. Here in general $\kappa$ is a function on $T^\circ M$ (a function of both $x$ and $y$), which is positively-homogeneous of degree 0. In a Finsler space $R_{ij}$ is symmetric (see for example [39]).

The Finslerian version of Schur's Lemma states that for $n \geq 3$, if $\kappa$ is constant on the fibres of $T^\circ M \to M$, that is, independent of $y$, then it must be constant, so that the space is a space of constant curvature.

The key formula in Numata's proof, in [38], that a Landsberg space of non-zero scalar curvature is a Berwald space (indeed, a Riemannian space of constant curvature) is

$$\begin{aligned} &3\nabla(B_{ijkl}) + h_{ij}K_{kl} + h_{ik}K_{jl} + h_{il}K_{jk} \\ &\qquad\qquad - 2y_i(h_{kl}\kappa_{,j} + h_{jl}\kappa_{,k} + h_{jk}\kappa_{,l} + 3\kappa C_{jkl}) = 0, \end{aligned}$$

which holds for any space of scalar flag curvature $\kappa$, where

$$K_{jk} = F^2\kappa_{,jk} + y_j\kappa_{,k} + y_k\kappa_{,j}.$$

This formula is derived from one of the Bianchi identities for the curvatures of the Berwald connection, namely $B^m_{ijk;l} - B^m_{ijl;k} = -R^m_{jkl,i}$. In a Landsberg space $y^i B_{ijkl} = y_i B^i_{jkl} = 0$, and therefore

$$h_{kl}\kappa_{,j} + h_{jl}\kappa_{,k} + h_{jk}\kappa_{,l} + 3\kappa C_{jkl} = 0.$$

Then provided $\kappa \neq 0$ the space is C-reducible:

$$C_{jkl} = \frac{1}{n+1}(h_{kl}C_j + h_{jl}C_k + h_{jk}C_l),$$

and is therefore a Berwald space (Theorem 16, [29]). It then follows from the original formula that

$$h_{ij}K_{kl} + h_{ik}K_{jl} + h_{il}K_{jk} = 0,$$

whence

$$g^{kl}K_{kl} = 0 = g^{kl}\frac{\partial^2\kappa}{\partial y^k \partial y^l},$$

since $\kappa$ is homogeneous of degree 0. So $\kappa$ is fibrewise constant by Proposition 2, and constant by the Schur Lemma for Finsler spaces. Moreover $C_{jkl} = 0$ and the space is Riemannian.

It follows from the general formula for $\nabla(B_{ijkl})$ for a space of scalar curvature that

$$\Gamma(\varepsilon) = g^{ij}g^{kl}\nabla(B_{ijkl}) = -\tfrac{1}{3}(n+1)g^{kl}K_{kl}.$$

The following result was originally due to Akbar-Zadeh [3]. A more modern treatment may be found in [37].

Proposition 4. *A necessary and sufficient condition for a Finsler space of scalar flag curvature to be of constant curvature is that its Berwald scalar curvature is geodesically invariant.*

*Proof.* Evidently if $\kappa$ is constant then $\Gamma(\varepsilon) = 0$. Conversely, if $\Gamma(\varepsilon) = 0$ then $g^{kl}K_{kl} = 0$, whence $\kappa$ is constant, as before. ∎

Proposition 2 here replaces a somewhat more complicated argument in [37] which appeals to the Hopf maximum principle.

## D. $\chi$-vector

The $\chi$-(co)vector is defined (see [26]) by

$$\chi_i = \Gamma\left(\frac{\partial S}{\partial y^i}\right) - \frac{\partial S}{\partial x^i} = \nabla(S_{,i}) - S_{;i}$$

where $S$ is the S-function. Then by Proposition 1

$$\chi_{i,j} = (\nabla(S_{,i}) - S_{;i})_{,j} = \nabla(S_{,ij}) + S_{,i;j} - S_{;i,j}.$$

Now for any function $f$ on $T^\circ M$, $f_{,i;j} = f_{;j,i}$ (the mixed Ricci identity for functions), and so

$$\begin{aligned}\chi_{i,j} + \chi_{j,i} &= \nabla(S_{,ij}) + S_{,i;j} - S_{;i,j} + \nabla(S_{,ji}) + S_{,j;i} - S_{;j,i} \\ &= 2\nabla(S_{,ij}) = 2\nabla(E_{ij}).\end{aligned}$$

Thus ([26, Lemma 3.2])

$$g^{ij}\chi_{i,j} = \Gamma(\varepsilon).$$

It is also shown in [26, Lemma 3.2] that

$$\chi_{i,j} + \chi_{j,i} = R^k_{ikj} + R^k_{jki} - R^k_{k,ij},$$

which prompts the remark in Subsection 2.5 that Theorem 9 " is just Theorem 6 in disguise".

## E. Isometries of Minkowski spaces

This appendix again is concerned entirely with Minkowski spaces.

Suppose we have two Minkowski norms $F$ and $\hat{F}$ on $\mathbb{R}^n_\circ$. By an isometry of the corresponding Minkowski spaces I mean a diffeomorphism $\phi : \mathbb{R}^n_\circ \to \mathbb{R}^n_\circ$ which is an isometry of the Riemannian metrics $g$ and $\hat{g}$ derived from the Minkowski norms $F$ and $\hat{F}$. I shall also require, as part of the definition, that $\phi$ is positively-homogeneous of degree 1, and orientation preserving. I do not assume ab initio that $\phi$ is linear: indeed, of particular interest are conditions for an isometry to be linear. Recall that a function $f$ on $\mathbb{R}^n_\circ$ which is positively-homogeneous of degree 1 is linear if and only if

$$g^{ij}\frac{\partial^2 f}{\partial y^i \partial y^j} = 0,$$

by Proposition 3.

The condition for $\phi$ to be an isometry is

$$(\hat{g}_{kl} \circ \phi)\frac{\partial \phi^k}{\partial y^i}\frac{\partial \phi^l}{\partial y^j} = g_{ij}.$$

It follows, by differentiation, that

$$\frac{\partial g_{ij}}{\partial y^k} - \left(\frac{\partial \hat{g}_{pq}}{\partial y^r}\circ\phi\right)\frac{\partial \phi^p}{\partial y^i}\frac{\partial \phi^q}{\partial y^j}\frac{\partial \phi^r}{\partial y^k} = (\hat{g}_{pq}\circ\phi)\left(\frac{\partial^2 \phi^p}{\partial y^i \partial y^k}\frac{\partial \phi^q}{\partial y^j} + \frac{\partial \phi^p}{\partial y^i}\frac{\partial^2 \phi^q}{\partial y^j \partial y^k}\right).$$

The left-hand side is completely symmetric in $i$, $j$ and $k$, so the right-hand side must be symmetric also. So we must have

$$(\hat{g}_{pq}\circ\phi)\frac{\partial^2 \phi^p}{\partial y^i \partial y^k}\frac{\partial \phi^q}{\partial y^j} = (\hat{g}_{pq}\circ\phi)\frac{\partial^2 \phi^p}{\partial y^i \partial y^j}\frac{\partial \phi^q}{\partial y^k}.$$

By homogeneity it follows that

$$(\hat{g}_{pq}\circ\phi)\frac{\partial^2 \phi^p}{\partial y^i \partial y^j}\phi^q = 0;$$

indeed the two conditions are easily seen to be equivalent. This is an additional property that a homogeneous isometry of metrics arising from Minkowski norms must have, as a consequence of the special nature of the metrics.

So

$$\frac{\partial g_{ij}}{\partial y^k} = \left(\frac{\partial \hat{g}_{pq}}{\partial y^r}\circ\phi\right)\frac{\partial \phi^p}{\partial y^i}\frac{\partial \phi^q}{\partial y^j}\frac{\partial \phi^r}{\partial y^k} + 2(\hat{g}_{pq}\circ\phi)\frac{\partial^2 \phi^p}{\partial y^i \partial y^j}\frac{\partial \phi^q}{\partial y^k}.$$

I write

$$\frac{\partial g_{ij}}{\partial y^k} = C_{ijk}, \qquad g^{ij}\frac{\partial g_{ij}}{\partial y^k} = g^{ij}C_{ijk} = C_k = \frac{\partial \log\det g}{\partial y^k}$$

in the usual way. Thus

$$d(\log\det g) = g^{ij}\frac{\partial g_{ij}}{\partial y^k}dy^k = C_k dy^k,$$

where $d$ here denotes the exterior derivative on $\mathbb{R}^n_\circ$, that is, with respect to the $y^i$ alone, so that

$$d(\log\det g) = \phi^* d(\log\det\hat{g}) + 2g^{ij}\frac{\partial^2 \phi^p}{\partial y^i \partial y^j}\phi^*(\hat{g}_{pq}dy^q).$$

It follows that an isometry $\phi$ is linear if and only if $d(\log\det g) = \phi^* d(\log\det\hat{g})$. Clearly if $\phi$ is linear then $d(\log\det g) = \phi^* d(\log\det\hat{g})$. Conversely, if $d(\log\det g) = \phi^* d(\log\det\hat{g})$ then

$$g^{ij}\frac{\partial^2 \phi^p}{\partial y^i \partial y^j}\phi^*(\hat{g}_{pq}dy^q) = 0.$$

But the 1-forms $g_{ij}dy^j$, $i = 1, 2, \dots, n$, are linearly independent, so the 1-forms $\phi^*(\hat{g}_{pg}dy^q)$, $p = 1, 2, \dots, n$, are linearly independent, so

$$g^{ij}\frac{\partial^2\phi^p}{\partial y^i\partial y^j} = 0,$$

and $\phi$ is linear.

Proposition 5. *An isometry between two Minkowski spaces of the same dimension is linear if and only if*

(1) $(\hat{C}_l \circ \phi)\dfrac{\partial\phi^l}{\partial y^k} = C_k$;

*equivalently,*

(2) *its Jacobian determinant is constant.*

*Proof.* We know that $\phi$ is linear if and only if $d(\log\det g) = \phi^* d(\log\det\hat{g})$, that is,

$$(\hat{C}_l \circ \phi)\frac{\partial\phi^l}{\partial y^k} = C_k.$$

But from the definition of an isometry, taking determinants

$$\det g = (\phi^*\det\hat{g})J^2,$$

where $J = \det\left(\dfrac{\partial\phi^i}{\partial y^j}\right)$ is the Jacobian determinant of $\phi$. Thus

$$d(\log\det g) = \phi^* d(\log\det\hat{g}) + 2d(\log J),$$

and so $d(\log\det g) = \phi^* d(\log\det\hat{g})$ if and only if $dJ = 0$, that is, if and only if $J$ is constant. ∎

I emphasise that I work throughout with linear coordinates on $\mathbb{R}^n_\circ$: if $J$ is constant with respect to one linear coordinate system, it is constant with respect to all.

Proposition 5 (1) is M. Li's equivalence theorem of Minkowski spaces [27]. The argument above is based on the discussion of Li's result at the end of my paper [12].

The consequence of Proposition 5 (2) for parallel translation in Landsberg spaces is Theorem 2: the condition for the Jacobian determinant of parallel translation between fibres to be independent of $y$ is just $\Gamma^k_{ki,j} = 0 = E_{ij}$.

An interesting application of Proposition 5 (2) is afforded by the Legendre transformation ([7, Section 14.8], [51]). A Minkowski norm $F$ on $\mathbb{R}^n_\circ$ induces another, its dual $F^*$, on $\mathbb{R}^{n*}_\circ$, such that $F^*(y_i) = F(y^i)$. The Legendre transformation is the map $\phi : \mathbb{R}^n_\circ \to \mathbb{R}^{n*}_\circ$ by $\phi_i(y) = y_i = g_{ij}(y)y^j$. It is clearly an isometry between $F$ and $F^*$. We have

$$\frac{\partial \phi_i}{\partial y^j} = g_{ij},$$

from which it is clear that $\phi$ is linear if and only if $g_{ij}$ is constant. But the proposition says that in this case if $\det g$ is constant then $\phi$ is linear, which indeed is so by Deicke's Theorem.

Notice that an isometry (linear or not) preserves norms. The condition for $\phi$ to be an isometry is

$$(\hat{g}_{kl} \circ \phi)\frac{\partial \phi^k}{\partial y^i}\frac{\partial \phi^l}{\partial y^j} = g_{ij}.$$

Moreover by the homogeneity assumption

$$y^j \frac{\partial \phi^i}{\partial y^j} = \phi^i.$$

Then

$$\begin{aligned}(\hat{g}_{kl} \circ \phi)\frac{\partial \phi^k}{\partial y^i}\frac{\partial \phi^l}{\partial y^j}y^i y^j &= g_{ij}y^i y^j = 2F^2 \\ &= (\hat{g}_{kl} \circ \phi)\phi^k\phi^l = 2(\hat{F} \circ \phi)^2.\end{aligned}$$

Proposition 5 (2) specializes of course to (self) isometries of a single Minkowski space: one might call this Deicke's Theorem for isometries.

There is an analogous result for infinitesimal isometries, or Killing fields, of a single Minkowski space, which is actually somewhat simpler to prove. The vector field

$$\xi = \xi^i \frac{\partial}{\partial y^i}$$

on $\mathbb{R}^n_\circ$ is an infinitesimal isometry if and only if it satisfies Killing's equation

$$\xi^k \frac{\partial g_{ij}}{\partial y^k} + \frac{\partial \xi^k}{\partial y^i} g_{kj} + \frac{\partial \xi^k}{\partial y^j} g_{ik} = 0;$$

I assume that $\xi$ is positively homogeneous of degree 0, that is, $\xi^i$ is positively homogeneous of degree 1. Then $\xi$ is linear, so that $\xi^i = A^i_j y^j$ for constants

$A^i_j$, if and only if

$$\frac{\partial \xi^k}{\partial y^k} = \operatorname{tr}\left(\frac{\partial \xi^i}{\partial y^j}\right) \text{ is constant.}$$

The proof goes as follows. A similar analysis of the isometry condition to that carried out in the preliminary steps to Proposition 5 leads to the result that $\xi^i$ must satisfy

$$g_{jl}\frac{\partial^2 \xi^l}{\partial y^i \partial y^k} = g_{kl}\frac{\partial^2 \xi^l}{\partial y^i \partial y^j}.$$

Thus

$$g^{ij}g_{jl}\frac{\partial^2 \xi^l}{\partial y^i \partial y^k} = \frac{\partial}{\partial y^k}\left(\frac{\partial \xi^l}{\partial y^l}\right) = g^{ij}\frac{\partial^2 \xi^l}{\partial y^i \partial y^j}g_{kl}.$$

Clearly if $\xi$ is linear then the left-hand side vanishes. On the other hand, if the left-hand side vanishes then

$$g^{ij}\frac{\partial^2 \xi^l}{\partial y^i \partial y^j} = 0;$$

but then (since it is positively homogeneous of degree 1) $\xi^l$ must be linear.

I now show that a linear isometry of a Minkowski space can be represented as an orthogonal transformation relative to certain linear coordinates. For this purpose I introduce an auxiliary Euclidean metric on $\mathbb{R}^n_\circ$, effectively the averaged metric from Subsection 2.7. I denote by $\omega$ the metric volume form on $\mathbb{R}^n_\circ$ induced by the metric $g_{ij}$:

$$\omega = \sqrt{\det g(y)}dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n = \sqrt{\det g(y)}(dy)^n.$$

Set

$$\bar{g}_{ij} = \frac{1}{\nu}\int_B g_{ij}(y)\omega$$

where $B = \{y \in \mathbb{R}^n_\circ : F(y) \leq 1\}$ is the unit ball determined by $F$, and $\nu = \int_B \omega$ is its volume. Then $(\bar{g}_{ij})$ is a positive-definite quadratic form, that is, an Euclidean metric on $\mathbb{R}^n_\circ$. It is just the averaged metric of the Minkowski space.

For any linear isometry $\phi : \mathbb{R}^n \to \mathbb{R}^n$, say $\phi^i(y) = A^i_j y^j$, we have $\phi(B) = B$, and therefore

$$\begin{aligned}\int_{\phi(B)} g_{ij}\,\omega &= \int_B g_{ij}\,\omega = \int_B \phi^*(g_{ij}\,\omega)\\ &= \int_B A^k_i A^l_j g_{kl}\,\omega = A^k_i A^l_j \int_B g_{kl}\,\omega\end{aligned}$$

and

$$\int_{\phi(B)} \omega = \int_B \phi^*(\omega) = \int_B \omega,$$

so

$$A_i^k A_j^l \bar{g}_{kl} = \bar{g}_{ij};$$

that is, the linear map $A$ is orthogonal with respect to $\bar{g}$. In particular, we may choose linear coordinates on $\mathbb{R}^n$ such that $\bar{g}$ is the standard Euclidean metric, and with respect to such coordinates $\phi$ is represented by an orthogonal matrix, that is, an element of $SO(n)$. The value of the (constant) Jacobian determinant of a linear self isometry of a Minkowski space is 1.

## F. Isometries of Finsler spaces

Any differentiable map $\psi : M \to M$ of a manifold $M$ to itself lifts naturally to a differentiable map $\tilde{\psi} : TM \to TM$ of the tangent bundle to itself, its complete lift, where

$$\tilde{\psi}(x, y) = (\psi(x), \psi_{*x} y), \qquad \text{with } (\psi_{*x} y)^i = \frac{\partial \psi^i}{\partial x^j}(x) y^j;$$

note that $\tilde{\psi}$ acts linearly in the fibres.

According to Deng and Hou, [19], an isometry of a Finsler space is a diffeomorphism $\psi$ of $M$ onto itself for which $\tilde{\psi}$ preserves the Finsler function:

$$\tilde{\psi}^* F = F, \qquad \text{or} \qquad F(\psi(x), \psi_{*x} y) = F(x, y).$$

For any Finsler space the group of isometries $\mathfrak{I}$ is a Lie transformation group of $M$, and for each $x \in M$ the isotropy subgroup $\mathfrak{I}_x$ is compact.

I shall discuss isometries of Finsler spaces, not necessarily Landsberg spaces, in this final section, as applications of some of the ideas introduced earlier; my main purpose is to establish a result that was needed for Theorem 14. I shall assume that $M$ is orientable, and deal only with orientation-preserving isometries $\psi$. I use again the averaged metric to give a Riemannian metric $\bar{g}$ on $M$:

$$\bar{g}_{ij}(x) = \frac{1}{\nu_x} \int_{B_x} g_{ij}(x, y) \omega_x$$

where $B_x = \{y \in T_x^\circ M : F(x, y) \leq 1\}$ and $\nu_x = \int_{B_x} \omega_x$.

For each $x \in M$, $\psi_{*x} : T_x M \to T_{\psi(x)} M$ is an isomorphism of vector spaces, and its restriction to $T_x^\circ M$ is an isometric map between the fibre metrics $g_x$

and $g_{\psi(x)}$. In fact

$$g_{kl}(\psi(x), \psi_{*x}y)\frac{\partial\psi^k}{\partial x^i}(x)\frac{\partial\psi^l}{\partial x^j}(x) = g_{ij}(x, y).$$

In particular, if $\psi \in \mathfrak{I}_x$ then the restriction of $\psi_{*x}$ to $T_x^\circ M$ is an isometry of $g_x$.

A Finslerian isometry $\psi$ maps unit balls to unit balls: $B_{\psi(x)} = \psi_{*x}(B_x)$, which is to say, $z \in B_{\psi(x)} \subset T_{\psi(x)}M$ if and only $z = \psi_{*x}y$ where $y \in B_x \subset T_xM$.

I now show that if $\psi$ is an isometry of the Finsler space then it is an isometry of the averaged metric $\bar{g}$: that is to say

$$(\bar{g}_{kl} \circ \psi)\frac{\partial\psi^k}{\partial x^l}\frac{\partial\psi^l}{\partial x^j} = \bar{g}_{ij}.$$

(A similar result, obtained however using the so-called Binet-Legendre metric on $M$ rather than the averaged metric, is to be found in [33].) Firstly, since $\psi_{*x}$ is a fibre isometry we have

$$\nu_{\psi(x)} = \int_{B_{\psi(x)}} \omega_{\psi(x)} = \int_{\psi_{*x}(B_x)} \omega_{\psi(x)} = \int_{B_x} (\psi_{*x})^*(\omega_{\psi(x)}) = \int_{B_x} \omega_x = \nu_x.$$

Furthermore

$$\begin{aligned}
\bar{g}_{kl}(\psi(x))\frac{\partial\psi^k}{\partial x^l}(x)\frac{\partial\psi^l}{\partial x^j}(x) &= \left(\int_{B_{\psi(x)}} g_{kl}(\psi(x), z)\,\omega_{\psi(x)}\right)\frac{\partial\psi^k}{\partial x^l}(x)\frac{\partial\psi^l}{\partial x^j}(x) \\
&= \int_{B_{\psi(x)}} g_{kl}(\psi(x), z)\frac{\partial\psi^k}{\partial x^l}(x)\frac{\partial\psi^l}{\partial x^j}(x)\,\omega_{\psi(x)} \\
&= \int_{\psi_{*x}(B_x)} g_{kl}(\psi(x), z)\frac{\partial\psi^k}{\partial x^l}(x)\frac{\partial\psi^l}{\partial x^j}(x)\,\omega_{\psi(x)} \\
&= \int_{B_x} g_{ij}(x, y)\,\omega_x = \bar{g}_{ij}(x),
\end{aligned}$$

using the fact that $z \in B_{\psi(x)}$ if and only $z = \psi_{*x}y$ where $y \in B_x$.

The group $\mathfrak{I}$ of orientation-preserving isometries of a Finsler space with Finsler function $F$ is therefore contained in the Lie group of isometries of the Riemannian space with metric $\bar{g}$.

Consider the isotropy group $\mathfrak{I}_x$ of a point $x \in M$, that is, the subgroup of $\mathfrak{I}$ consisting of those $\psi \in \mathfrak{I}$ such that $\psi(x) = x$. An element of it defines a linear isomorphism $\Psi_x$ of $T^\circ_x M$:

$$\Psi_x(y)^i = \frac{\partial \psi^i}{\partial x^j}(x) y^j,$$

which is an isometry of $g_x$.

Now $\Psi_x$ is also an isometry of $\bar{g}_x$. We may choose coordinates $(x^i)$ on $M$ which are orthonormal with respect to $\bar{g}_x$, that is, such that $\bar{g}_{kl}(x) = \delta_{kl}$. With respect to such coordinates, and the corresponding natural fibre coordinates, the matrix $(\partial\psi^i/\partial x^j(x))$ is orthogonal: in other words, $\mathfrak{I}_x$ is a subgroup of $SO(n)$. This result is assumed in the statement of Theorem 14.

The elements of $\mathfrak{I}_x$ leave the Minkowski norm $F_x$ invariant, and they map the Minkowskian unit sphere to itself, as well as mapping the Euclidean unit sphere to itself. Moreover, $\det g(x)$ is invariant under the action of $\mathfrak{I}_x$. If $\mathfrak{I}_x$ acts transitively, on either unit sphere and therefore on both of them, then $\det g(x)$ is constant over $T^\circ_x M$, and so by Deicke's Theorem $g_{ij}(x)$ itself is constant over $T^\circ_x M$. Thus if $\mathfrak{I}_x$ acts transitively on the unit spheres then the Minkowski space is actually Euclidean, and $g(x)$ and $\bar{g}(x)$ coincide. If $\mathfrak{I}_x$ acts transitively on unit spheres for all $x \in M$ then $g = \bar{g}$, and the Finsler space is actually Riemannian. This extends a result of Wang of 1947, which is discussed in Chapter VIII §4 of Yano's classic text on the Lie derivative [52].

## References


[1] T. Aikou, Some remarks on Berwald manifolds and Landsberg manifolds, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* **26** (2010), 139 – 148.

[2] T. Aikou, Averaged Riemannian metrics and connections with application to locally conformal Berwald manifolds, *Publ. Math. Debrecen* **81**-(1-2) (2012), 179 – 198.

[3] H. Akbar-Zadeh, Sur les espaces de Finsler à courbures sectionnelles constantes, *Acad. Roy. Belg. Bull. Cl. Sci. (5)* **74** (1988), 271 – 322.

[4] S. Bácsó, F. Ilosvay, B. Kis, Landsberg spaces with common geodesics, *Publ. Math. Debrecen* **42** (1-2) (1993), 139 – 144.

[5] S. Bácsó, R. Yoshikawa, Weakly-Berwald spaces, *Publ. Math. Debrecen* **61** (1-2) (2002), 219 – 231.

[6] D. Bao, On two curvature-driven problems in Riemann-Finsler geometry, in " Finsler geometry, Sapporo 2005 – in memory of Makoto Matsumoto ", Adv. Stud. Pure Math., 48, Mathematical Society of Japan, Tokyo, 2007, 19 – 71.

[7] D. Bao, S.-S. Chern, Z. Shen, " An introduction to Riemann-Finsler geometry ", Grad. Texts in Math., 200, Springer-Verlag, New York, 2000.

[8] P. Centore, Volume forms in Finsler spaces, *Houston J. Math.* **25** (1999), 625 – 640.

[9] X. Cheng, The $(\alpha, \beta)$-metrics of scalar flag curvature, *Differential Geom. Appl.* **35** (2014), 361 – 369.

[10] M. Crampin, On Landsberg spaces and the Landsberg-Berwald problem, *Houston J. Math.* **37** (4) (2011), 1103 – 1124.

[11] M. Crampin, On the construction of Riemannian metrics for Berwald spaces by averaging, *Houston J. Math.* **40** (3) (2014), 737 – 750.

[12] M. Crampin, A condition for a Landsberg space to be Berwaldian, *Publ. Math. Debrecen* **93** (1-2) (2018), 143 – 155.

[13] M. Crampin, Finsler spaces of $(\alpha, \beta)$ type and semi-C-reducibility, *Publ. Math. Debrecen* **98** (3-4) (2021), 419 – 454.

[14] M. Crampin, Invariant volumes, weakly-Berwald Finsler spaces, and the Landsberg-Berwald problem, *Publ. Math. Debrecen* **100** (1-2) (2022), 101 – 118.

[15] M. Crampin, Isometries and geodesic invariants of Finsler spaces of $(\alpha, \beta)$ type, 2022, DOI:10.13140/RG.2.2.10839.55203.

[16] M. Crampin, On parallel transport in Finsler spaces, *Publ. Math. Debrecen* **92** (3-4) (2023), 343 – 370.

[17] M. Crampin, S-curvature, E-curvature, and Berwald scalar curvature of Finsler spaces, *Differential Geom. Appl.* **92** (2024), paper no. 102080, 12 pp.

[18] M. Crampin, D.J. Saunders, Holonomy of a class of bundles with fibre metrics, *Publ. Math. Debrecen* **81** (1-2) (2012), 199 – 234.

[19] S. Deng, Z. Hou, The group of isometries of a Finsler space, *Pacific J. Math.* **207** (1) (2002), 149 – 155.

[20] S.G. Elgendi, On the classification of Landsberg spherically symmetric Finsler metrics, *Int. J. Geom. Methods Mod. Phys.* **18** (14) (2021), paper no. 2150232, 26 pp.

[21] H. Feng, Y. Han, M. Li, An equivalence theorem of a class of Minkowski norms and its applications, *Sci. China Math.* **64** (2021), 1429 – 1446.

[22] H. Feng, M. Li, An equivalence theorem for a class of Minkowski spaces, 2018, arXiv:1812.11938v2.

[23] Y. Ichijyō, Finsler manifolds modeled on a Minkowski space, *J. Math. Kyoto Univ.* **16** (1976), 639 – 652.

[24] Y. Ichijyō, On holonomy mappings associated with a nonlinear connection, *J. Math. Tokushima Univ.* **17** (1983), 1 – 9.

[25] L. Kozma, On holonomy groups of Landsberg manifolds, *Tensor (N.S.)* **62** (1) (2000), 87 – 90.

[26] B. Li, Z. Shen, Ricci curvature tensor and non-Riemannian quantities, *Canad. Math. Bull.* **58** (3) (2015), 530 – 537.

[27] M. Li, Equivalence theorems of Minkowski spaces and an application in Finsler geometry, *Acta Math. Sinica (Chin. Ser.)* **62** (2) (2019), 177 – 190 (in Chinese: for the English version see arXiv:1504.04475v2 (2018)).

[28] M. Li, L. Zhang, Properties of Berwald scalar curvature, *Front. Math. China* **15** (6) (2020), 1143 – 1153.

[29] M. Matsumoto, On C-reducible Finsler spaces, *Tensor (N.S.)* **24** (1972), 29 – 37.

[30] M. Matsumoto, Theory of Finsler spaces with $(\alpha, \beta)$-metric, *Rep. Math. Phys.* **31** (1992), 43 – 83.

[31] M. Matsumoto, An improved proof of Numata and Shibata's theorems on Finsler spaces of scalar curvature, *Publ. Math. Debrecen* **64** (3-4) (2004), 489 – 500.

[32] M. Matsumoto, C. Shibata, On semi-$C$-reducibility, $T$-tensor= 0, and $S4$-likeness of Finsler spaces, *J. Math. Kyoto Univ.* **19** (1979), 301–314.

[33] V.S. Matveev, M. Troyanov, The Binet-Legendre metric in Finsler geometry, *Geom. Topol.* **16** (2012), 2135 – 2170.

[34] X. Mo, L. Zhou, The curvatures of spherically symmetric Finsler metrics in $\mathbb{R}^n$, 2014, arXiv:1202.4543.

[35] Z. Muzsnay, The Euler-Lagrange PDE and Finsler metrizability, *Houston J. Math.* **32** (1) (2006), 79 – 98.

[36] Z. Muzsnay, P.T. Nagy, Finsler manifolds with non-Riemannian holonomy, *Houston J. Math.* **38** (1) (2012), 77 – 92.

[37] B. Najafi, Z. Shen, A. Tayebi, Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties, *Geom. Dedicata* **131** (2008), 87 – 97.

[38] S. Numata, On Landsberg spaces of scalar curvature, *J. Korean Math. Soc.* **12** (2) (1975), 97 – 100.

[39] Z. Shen, " Differential geometry of spray and Finsler spaces ", Kluwer Academic Publishers, Dordrecht, 2001.

[40] Z. Shen, On $R$-quadratic Finsler spaces, *Publ. Math. Debrecen* **58** (1-2) (2001), 263 – 274.

[41] Z. Shen, Landsberg curvature, $S$-curvature and Riemann curvature, in " A sampler of Riemann-Finsler geometry " (edited by David Bao, Robert L. Bryant, Shiing-Shen Chern and Zhongmin Shen), Math. Sci. Res. Inst. Publ., 50, Cambridge University Press, Cambridge, 2004, 303 – 355.

[42] Z. Shen, On a class of Landsberg metrics in Finsler geometry, *Canad. J. Math.* **61** (2009), 1357 – 1374.

[43] C. Shibata, On the curvature tensor $R_{hijk}$ of Finsler spaces of scalar curvature, *Tensor (N.S.)* **32** (1978), 311 – 317.

[44] J. Szilasi, R. Lovas, D. Kertész, " Connections, sprays and Finsler structures ", World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014.

[45] J. Szilasi, R.L. Lovas, D.Cs. Kertész, Several ways to a Berwald manifold — and some steps beyond, *Extracta Math.* **26** (2011), 89 – 130.

[46] A. Tayebi, A survey on unicorns in Finsler geometry, *J. Math. Com.* **2** (2) (2021), 239 – 250.

[47] R.G. Torrome, F. Etayo, On a rigidity condition for Berwald spaces, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.* **104** (1) (2010), 69 – 80.

[48] Sz. Vattamány, Projection onto the indicatrix bundle of a Finsler manifold, *Publ. Math. Debrecen* **58** (1-2) (2001), 193 – 221.

[49] Cs. Vince, A new proof of Szabó's theorem on the Riemann-metrizability of Berwald manifolds, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* **21** (2005), 199 – 204.

[50] M. Xu, S. Deng, The Landsberg equation of a Finsler space, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **22** (1) (2021), 31 – 51.

[51] M. Xu, V.S. Matveev, Proof of Laugwitz conjecture and Landsberg unicorn conjecture for Minkowski norms with $SO(k) \times SO(n-k)$-symmetry, *Canad. J. Math.* **74** (5) (2022), 1486 – 1516.

[52] K. Yano, "The theory of Lie derivatives and its applications", Bibliotheca Mathematica 3, North-Holland, Amsterdam, 1957.

[53] C. Yu, H. Zhu, On a new class of Finsler metrics, *Differential Geom. Appl.* **29** (2) (2011), 244 – 254.

[54] L. Zhou, Spherically symmetric Finsler metrics in $\mathbb{R}^n$, *Publ. Math. Debrecen* **80** (1-2) (2012), 67 – 77.

[55] S. Zhou, J. Wang, B. Li, On a class of almost regular Landsberg metrics, *Sci. China Math.* **62** (2019), 935 – 960.