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A topological characterization of an almost Boolean algebra

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Abstract: For any Boolean space X and a discrete almost distributive lattice D, it is proved that the set $\mathcal{C}(X, D)$ of all continuous mappings of X into D, when D is equipped with the discrete topology, is an almost Boolean algebra under pointwise operations. Conversely, it is proved that any almost Boolean algebra is a homomorphic image of $\mathcal{C}(X, D)$ for a suitable Boolean space X and a discrete almost distributive lattice D.

Key words: almost distributive lattice (ADL); almost Boolean algebra (ABA); maximal element; discrete ADL; discrete topology; Boolean space.

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1. Introduction

After the notion of Boolean algebra came to light, several generalizations have come up in which the lattice theoretic generalizations like distributive lattices, implicative lattices, post algebras, pseudo-complemented distributive lattices, stone lattices, relatively complemented lattices, etc. The notion of an almost distributive lattice (ADL) was introduced by Swamy and Rao [5]. An ADL $(A, \wedge, \vee, 0)$ is an algebra of type $(2, 2, 0)$ which satisfies all the axioms of a distributive lattice with 0, except the commutativity of the operations $\wedge$, $\vee$ and the right distributivity of $\vee$ over $\wedge$. In fact, these three conditions are equivalent to each other in any ADL. The concept of an almost Boolean algebra (ABA) is introduced by Swamy and Rao [5] which is an ADL $(A, \wedge, \vee, 0)$ with a maximal element satisfying the condition that, for any $x \in A$, there exists $y \in A$ such that $x \wedge y = 0$ and $x \vee y$ is maximal.

It is well known that, every Boolean algebra is isomorphic to an algebra

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of the form $\mathcal{C}(X, 2)$. In this paper, we prove that an ADL A with a maximal element is an ABA if and only if it is homomorphic image of $\mathcal{C}(X, D)$, the ADL of continuous mappings of a Boolean space X into a discrete ADL D, where D is equipped with the discrete topology.

2. Preliminaries

In this section, we collect certain definitions and properties of ADLs from [1, 2, 3, 4, 5] that are required in the main text of this paper.

Definition 2.1. An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an almost distributive lattice (abbreviated as ADL) if it satisfies the following identities:

(1) $0 \wedge a = 0$;

(2) $a \vee 0 = a$;

(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;

(4) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$;

(5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$;

(6) $(a \vee b) \wedge b = b$.

Example 2.2. Every non-empty set A can be regarded as an ADL as follows. Let $a_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$a \vee b = \begin{cases} a & \text{if } a \neq a_0, \\ b & \text{if } a = a_0; \end{cases} \qquad a \wedge b = \begin{cases} b & \text{if } a \neq a_0, \\ a_0 & \text{if } a = a_0. \end{cases}$$

Then $(A, \vee, \wedge, a_0)$ is an ADL (where a_0 is the zero element).

Definition 2.3. Let $(A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, define

$$a \leq b \qquad \text{if } a = a \wedge b \text{ (equivalently } a \vee b = b).$$

Then $\leq$ is a partial order on A.

Theorem 2.4. *If $(A, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in A$, we have the following:*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$;

(2) $a \vee b = b \Leftrightarrow a \wedge b = a$;
(3) $\wedge$ *is associative in* A;
(4) $a \wedge b \wedge c = b \wedge a \wedge c$;
(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$;
(6) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$;
(7) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$;
(8) $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$;
(9) $a \leq a \vee b$ *and* $a \wedge b \leq b$;
(10) $a \wedge a = a$ *and* $a \vee a = a$;
(11) $0 \vee a = a$ *and* $a \wedge 0 = 0$;
(12) *If* $a \leq c$, $b \leq c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$;
(13) $a \vee b = b \vee a$ *whenever* $a \wedge b = 0$;
(14) $a \vee b = (a \vee b) \vee a$.

Definition 2.5. A homomorphism between ADL $(A, \vee, \wedge, 0)$ into an ADL A', we mean, a mapping $f : A \to A'$ satisfying the following:

(1) $f(a \vee b) = f(a) \vee f(b)$;
(2) $f(a \wedge b) = f(a) \wedge f(b)$;
(3) $f(0) = 0$.

A nonempty subset I of an ADL A is called an ideal of A if $x \vee y \in I$ and $x \wedge a \in I$ whenever $x, y \in I$ and $a \in A$. For any $X \subseteq A$, the ideal generated by X is

$$(X] = \left\{ \Big(\bigvee_{i=1}^{n} a_i \Big) \wedge x \,:\, a_i \in X,\ x \in A,\ n \in \mathbb{Z}^+ \right\}.$$

If $X = \{x\}$, then we write $(x]$ for $(X]$ and this is called a principal ideal generated by x. The set of all principal ideals of A is a distributive lattice. A proper ideal P of A is called prime if for any $x, y \in A, x \wedge y \in P$ then $x \in P$ or $y \in P$. For any $x, y \in A$ with $x \leq y, [x, y] = \{t \in A : x \leq t \leq y\}$ is a bounded distributive lattice with respect to the operations induced from those on A. An element m is maximal in $(A, \leq)$ if and only if $m \wedge x = x$ for all $x \in A$. An ADL A is said to be discrete if every nonzero element is maximal. The ADL given in the example 2.2 is a discrete ADL. For any $X \subseteq A$, $X^* = \{a \in A : x \wedge a = 0, \forall x \in X\}$ is an ideal of A and X^* is called the annihilator of X.

Lemma 2.6. *Let A be an ADL and I is an ideal of A. Then, for any $a, b \in A$, we have the following:*

(1) $(a] = \{a \wedge x : x \in A\}$;

(2) $a \in (b] \Leftrightarrow b \wedge a = a$;

(3) $a \wedge b \in I \Leftrightarrow b \wedge a \in I$;

(4) $(a] \cap (b] = (a \wedge b] = (b \wedge a]$;

(5) $(a] \vee (b] = (a \vee b] = (b \vee a]$;

(6) $(a] = A \Longleftrightarrow a$ *is maximal.*

Lemma 2.7. *Let A be an ADL and $x, y \in A$. Then the following statements hold:*

(1) $\{x \vee y\}^* = \{x\}^* \cap \{y\}^*$;

(2) $\{x \wedge y\}^* = \{y \wedge x\}^*$;

(3) $\{x\}^{***} = \{x\}^*$;

(4) $x \leq y \Rightarrow \{y\}^* \subseteq \{x\}^*$;

(5) $\{x \wedge y\}^{**} = \{x\}^{**} \cap \{y\}^{**}$.

Definition 2.8. An ADL $(A, \wedge, \vee, 0)$ is said to be relatively complemented if every interval in A is a Boolean algebra.

Theorem 2.9. *Let A be an ADL. Then the following are equivalent to each other:*

(1) *for any $a, b \in A$ there exists $x \in A$ such that $a \wedge x = 0$ and $a \vee x = a \vee b$;*

(2) *for any $a \leq b$ in A, $[a, b]$ is a complemented lattice;*

(3) *for any $a \in A$, $[0, a]$ is complemented lattice.*

Definition 2.10. A nontrivial ADL A is called an almost Boolean algebra (ABA) if it has a maximal element and satisfies one, and hence all of the equivalent conditions given in Theorem 2.9.

Theorem 2.11. *Let A be an ADL with a maximal element. Then the following are equivalent to each other:*

(1) *A is an almost Boolean algebra;*

(2) *for any* $a \in A$*, there exists* $b \in A$ *such that* $a \wedge b = 0$ *and* $a \vee b$ *is maximal;*

(3) $[0, m]$ *is a Boolean algebra for all maximal elements* m*;*

(4) *there exists a maximal element* m *such that* $[0, m]$ *is a Boolean algebra.*

THEOREM 2.12. *Let* A *be an ADL and* m *and* n *be maximal elements in* A*. Then the lattices* $[0, m]$ *and* $[0, n]$ *are isomorphic to each other. Moreover, the Boolean algebras* $[0, m]$ *and* $[0, n]$ *are isomorphic when* A *is almost Boolean algebra.*

THEOREM 2.13. *Let* $(A, \wedge, \vee, 0)$ *be an ABA. Then for any* a *and* b *in* A *there exists a unique* $x \in A$ *such that* $a \wedge x = 0$ *and* $a \vee x = a \vee b$*.*

DEFINITION 2.14. A nontrivial ADL A is called dense if $a \wedge b \neq 0$ for all $a \neq 0$ and $b \neq 0$ (equivalently, $\{a\}^* = \{0\}$, for any $0 \neq a \in A$).

3. $\mathcal{C}(X, D)$

The set of all continuous mappings of a topological space X into a topological space Y is denoted by $\mathcal{C}(X, Y)$. It can be easily proved that, the set $\mathcal{C}(X, D)$ is an ADL under the point-wise operations, where D is an ADL equipped with the discrete topology. Further, if m is a maximal element in D, then the constant map $\overline{m}$ is a maximal element in the ADL $\mathcal{C}(X, D)$, and conversely, if f is a maximal element in $\mathcal{C}(X, D)$ then for any $x \in X$, $f(x)$ is maximal element in D. In the following we consider a special case when X is a Boolean space (that is; compact, Hausdorff and totally disconnected space) and D is a non-trivial discrete ADL, and prove that $\mathcal{C}(X, D)$ and its homomorphic image are ABA's. First we start with the following.

THEOREM 3.1. *Let* A *be an ADL with maximal element and*

$$D = \big\{x \in A : \{x\}^* = \{0\}\big\} \cup \big\{0\big\}.$$

Then D *is a dense sub-ADL of* A *containing all maximal elements of* A*. Moreover, if* A *is an ABA, then* D *is discrete.*

Proof. By (1) of Lemma 2.7, $x \vee y \in D$, for any x and $y \in D$. Also, $x \wedge y \in D$; for, $z \in \{x \wedge y\}^* \Rightarrow x \wedge y \wedge z = 0 \Rightarrow y \wedge z \in \{x\}^* = \{0\} \Rightarrow y \wedge z = 0 \Rightarrow z \in \{y\}^* = \{0\} \Rightarrow z = 0$. Thus D is a sub-ADL of A. And, if m is a maximal element in A and $m \wedge x = 0$ implies $x = 0$. Therefore $m \in D$.

Further, let $0 \neq a \in D$. Then $\{a\}^* = \{0\}$. Since A is an ABA, there exists $b \in A$ such that $a \wedge b = 0$ and $a \vee b$ is maximal. It follows that $b = 0$. Now, $a \vee b = a \vee 0 = a$ which is maximal. Thus D is discrete. ∎

THEOREM 3.2. *Let A and B be are ADL's and B is a homomorphic image of A. If A is an ABA, then so is B.*

Proof. Let $f\colon A \to B$ be a epimorphism. Then, it can be easily verified that, for any maximal element m in A, $f(m)$ is a maximal element in B. Suppose that A is an ABA. Let $y \in B$. Then $f(x) = y$ for some $x \in A$. Since A is an ABA, there exists $x' \in A$ such that $x \wedge x' = 0$ and $x \vee x'$ is maximal in A. Now,

$$\begin{aligned} f(x) \wedge f(x') &= f(x \wedge x') = f(0) = 0, \\ f(x) \vee f(x') &= f(x \vee x') \ \text{ which is maximal.} \end{aligned}$$

Thus B is also an ABA. ∎

THEOREM 3.3. *Let X be a Boolean space and D be a discrete ADL equipped with discrete topology. Then the ADL $\mathcal{C}(X, D)$ of all continuous mappings of X into D is an ABA under point-wise operations. And, hence any homomorphic image of $\mathcal{C}(X, D)$ is an ABA.*

Proof. Let $f \in \mathcal{C}(X, D)$ and fix a maximal element m in D.
Define $g\colon X \to D$ by

$$g(x) = \begin{cases} m & \text{if } f(x) = 0, \\ 0 & \text{if } f(x) \neq 0. \end{cases}$$

Since D is a discrete space and f is continuous, it follows that $f^{-1}(D - \{0\})$ is a clopen (closed and open) set in X and it implies that g is continuous and hence $g \in \mathcal{C}(X, D)$. It is clear that $f \wedge g = \overline{0}$, the zero element in $\mathcal{C}(X, D)$. Further, $(f \vee g)(x) = f(x) \vee g(x)$.

$$\begin{aligned} f(x) = 0 \quad &\Rightarrow \quad g(x) = m \\ &\Rightarrow \quad f(x) \vee g(x) = 0 \vee m = m = \overline{m}(x) \\ &\Rightarrow \quad f \vee g = \overline{m}, \ \text{maximal in } \mathcal{C}(X, D), \end{aligned}$$

and

$$\begin{aligned} f(x) \neq 0 \quad &\Rightarrow \quad g(x) = 0 \\ &\Rightarrow \quad f(x) \text{ is maximal (since } D \text{ is discrete)} \\ &\Rightarrow \quad f(x) \vee g(x) = f(x) \\ &\Rightarrow \quad (f \vee g)(x) = f(x) \\ &\Rightarrow \quad f \vee g = f, \text{ maximal in } \mathcal{C}(X, D). \end{aligned}$$

Thus $\mathcal{C}(X, D)$ is an ABA. ∎

Next we shall prove a converse of Theorem 3.3 and it is a characterization of ABA's. Before going to the main theorem, let us recall from [5] that, for any almost Boolean algebra (ABA) A, $\mathrm{Spec}(A)$ denotes the space of all prime ideals of A together with the hull-kernal topology for which $\{X_a : a \in A\}$ is a base, where $X_a = \{P \in \mathrm{Spec}(A) : a \notin P\}$ and that $\mathrm{Spec}(A)$ is a Boolean space.

THEOREM 3.4. *Any ABA is a homomorphic image of $\mathcal{C}(X, D)$ for a suitable Boolean space X and a discrete ADL D.*

Proof. Let A be an ABA and X the Boolean space $\mathrm{Spec}(A)$. Let D be the set of all dense elements of A together with 0; that is,

$$D = \big\{x \in A : \{x\}^* = \{0\}\big\} \cup \{0\}.$$

Then, by Theorem 3.1, D is a discrete ADL. Now, define $\alpha \colon \mathcal{C}(X, D) \to A$ as follows. Let $f \in \mathcal{C}(X, D)$. As X is compact and f is continuous, $f(X)$ is a compact subset of the discrete space D. So that $f(X)$ is finite, say $\{d_1, d_2, \ldots, d_n\}$. Also, for each $1 \leq i \leq n$, $f^{-1}(\{d_i\})$ is a clopen subset of X and hence $f^{-1}(\{d_i\}) = X_{a_i}$ for some $a_i \in A$. Now, it can be easily seen that

$$\bigcup_{i=1}^{n} X_{a_i} = X_{\bigvee_{i=1}^{n} a_i} = X$$

and

$$X_{a_i} \cap X_{a_j} = X_{a_i \wedge a_j} = \emptyset \quad \text{for } i \neq j.$$

So $a_i \wedge a_j = 0$ for $i \neq j$. Now define

$$\alpha(f) = \bigvee_{i=1}^{n} (a_i \wedge d_i).$$

Since $(a_i \wedge d_i) \wedge (a_j \wedge d_j) = (a_i \wedge a_j) \wedge (d_i \wedge d_j) = 0$ and by (13) of Theorem 2.4, we get $\bigvee_{i=1}^{n} (a_i \wedge d_i)$ is l.u.b.$\{a_i \wedge d_i : 1 \leq i \leq n\}$. So that α is well-defined. We shall prove that α is an epimorphism. Let $f, g \in \mathcal{C}(X, D)$ and let $f(X) = \{d_1, d_2, \ldots, d_n\}$, $g(X) = \{e_1, e_2, \ldots, e_m\}$, $f^{-1}(\{d_i\}) = X_{a_i}$ and $g^{-1}(\{e_j\}) = X_{b_j}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Then $\alpha(f) = \bigvee_{i=1}^{n} (a_i \wedge d_i)$ and $\alpha(g) = \bigvee_{j=1}^{m} (b_j \wedge e_j)$. Now we can easily verified that

$$\begin{aligned}
(f \wedge g)(X) &= \{d_i \wedge e_j : 1 \leq i \leq n,\ 1 \leq j \leq m\},\\
(f \vee g)(X) &= \{d_i \vee e_j : 1 \leq i \leq n,\ 1 \leq j \leq m\},\\
(f \wedge g)^{-1}\big(\{d_i \wedge e_j\}\big) &= X_{a_i \wedge b_j},\\
(f \vee g)^{-1}\big(\{d_i \vee e_j\}\big) &= X_{a_i \vee b_j}.
\end{aligned}$$

Implies, α is a homomorphism of ABA's. Finally, to prove α is onto, let $x \in A$. Then there exists $y \in A$ such that $x \wedge y = 0$ and $x \vee y$ is maximal, say m. Define $g \colon X \to D$ by

$$g(P) = \begin{cases} m & \text{if } P \in X_x,\\ 0 & \text{if } P \in X - X_x = X_y. \end{cases}$$

Since X_x is clopen, g is continuous so that $g \in \mathcal{C}(X, D)$. Further,

$$g(X) = \{m, 0\}, \quad g^{-1}(\{m\}) = X_x \quad \text{and} \quad g^{-1}(\{0\}) = X_y.$$

Now $\alpha(g) = (x \wedge m) \vee (y \wedge 0) = (x \wedge m) \vee 0 = x \wedge m = x \wedge (x \vee y) = x$. Therefore α is onto. Thus α is an epimorphism of $\mathcal{C}(X, D)$ onto A. ∎

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