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Unbounded generalized B-Fredholm operators

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Abstract: In this paper, we investigate a new class of unbounded linear operators, that is, the unbounded generalized B-Fredholm operators in Banach space. More explicitly, we provide a characterization of this class of operators and some of its basic properties on a Banach space. Moreover, we study the generalized B-Fredholm spectrum and we prove a perturbation result of an unbounded generalized B-Fredholm operator under a commuting power finite-rank operator.

Key words: Unbounded generalized B-Fredholm operators, operator of Saphar type, generalized B-Fredholm spectrum, quasi-Fredholm operator.

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1. Introduction

Let $\mathcal{C}(X)$ denotes the set of all closed linear operators defined from a Banach space X to X. For $A \in \mathcal{C}(X)$ and for each integer $n \in \mathbb{N}$, the domain $\mathcal{D}(A^n)$, the kernel $\mathcal{N}(A^n)$ and the range $\mathcal{R}(A^n)$ of the power operator A^n are defined, respectively, by

$$\mathcal{D}(A^n) = \big\{x \in X \,:\, x, Ax, \ldots, A^{n-1}x \in \mathcal{D}(A)\big\},$$
$$\mathcal{N}(A^n) = \big\{x \in \mathcal{D}(A^n) \,:\, A^n x = 0\big\}$$

and

$$\mathcal{R}(A^n) = \big\{y \in X \,:\, A^n x = y \text{ for } x \in \mathcal{D}(A^n)\big\}.$$

If $n = 0$, one has

$$A^0 = I, \quad D(A^0) = X, \quad \mathcal{N}(A^0) = 0, \quad \mathcal{R}(A^0) = X,$$

where I is the identity operator defined from X to X. For all $n \geq 1$, we have $A^n(x) = AA^{n-1}(x)$, where $x \in \mathcal{D}(A^n)$. We clearly have:

$$\mathcal{D}(A^{n+1}) \subseteq \mathcal{D}(A^n),$$

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for all $n \in \mathbb{N}$. Let $\sigma(A)$ (resp. $\rho(A)$) denote the usual spectrum (resp. the resolvent set) of A.

For $A \in \mathcal{C}(X)$ and $n \in \mathbb{N}$, let $A_n : \mathcal{R}(A^n) \to \mathcal{R}(A^n)$ be the restriction of the operator A to $\mathcal{R}(A^n)$ into $\mathcal{R}(A^n)$. The domain $\mathcal{D}(A_n)$, the kernel $\mathcal{N}(A_n)$ and the range $\mathcal{R}(A_n)$ of A_n are defined respectively by

$$\mathcal{D}(A_n) = \mathcal{D}(A) \cap \mathcal{R}(A^n),$$
$$\mathcal{N}(A_n) = \mathcal{N}(A) \cap \mathcal{R}(A^n),$$
$$\mathcal{R}(A_n) = \mathcal{R}(A^{n+1}).$$

The class of B-Fredholm operators was first introduced by M. Berkani in [2] in the case of bounded operators acting on a Banach space. This notion of operators was generalized by Berkani and Castro-González in [4] to unbounded operators in Hilbert space. Recently, this class of operators was extended and studied by O. García et al. in [12] to generalized bounded B-Fredholm operators acting on a Banach space. Here, in this paper, we will consider unbounded generalized B-Fredholm operators defined on a Banach space. Our work, in this paper, extend some results obtained in [12] to the case of unbounded operators. After an introductory section, we recall in section 2 a list of well known definitions which are must be required, in this paper. We know that, J.P. Labrousse proved in [6] two decomposition theorems of closed quasi-Fredholm operators on a Hilbert space. As mentioned in [6, p. 206], these theorems are still true in the case of Banach spaces if, the subspaces $\mathcal{N}(A) \cap \mathcal{R}(A^d)$ and $\mathcal{R}(A) + \mathcal{N}(A^d)$, where $d = \mathrm{dis}(A)$, are closed and complemented. Using this result, we characterize in Theorem 3.1 a closed generalized B-Fredholm operator as a direct sum of a closed operator of Saphar type and a nilpotent one. Besides, we show in Theorem 3.2 an important result says that, if A is a closed generalized B-Fredholm operator with a non empty resolvent set, then there exists an integer $n \in \mathbb{N}$ such that $\mathcal{R}(A^n)$ is closed and such that the restriction operator A_n is of Saphar type. In Proposition 3.1, we show that if A is a closed generalized B-Fredholm operator densely defined on a Banach space, then its adjoint operator is also a generalized B-Fredholm operator. Based on Theorem 3.1, we prove in Proposition 4.1 that the generalized B-Fredholm spectrum of a closed operator A defined from X to X is a closed subset of the complex plane $\mathbb{C}$. Next, we characterize in Proposition 4.2 the generalized B-Fredholm spectrum of a closed linear operator A in terms of the corresponding spectrum of its bounded inverse. The end of section 4 contains a perturbation result of an unbounded generalized B-Fredholm operator under a commuting power finite-rank operator.

2. Preliminaries

In this section, we collect a list of well known definitions which are relevant to the development of this paper.

First, we give the following algebraic result which must be required:

Lemma 2.1. ([1]) *Let A be a linear operator defined on a vector space. Then, the following conditions are equivalent:*

(1) *for all* $s \in \mathbb{N}$, $\mathcal{N}(A) \subseteq \mathcal{R}(A^s)$,

(2) *for all* $n \in \mathbb{N}$, $\mathcal{N}(A^n) \subseteq \mathcal{R}(A)$,

(3) *for all* $s, n \in \mathbb{N}$, $\mathcal{N}(A^n) \subseteq \mathcal{R}(A^s)$,

(4) *for all* $s, n \in \mathbb{N}$, $\mathcal{N}(A^n) = A^s(\mathcal{N}(A^{n+s}))$.

In the following, we define the classes of semi-regular operators and the operators of Saphar type, which are the key tool for the study of unbounded generalized B-Fredholm operators. It is well known that, these classes of operators were been studied by several authors, we can see for instance the works of [9, 13, 14] and elsewhere.

Definition 2.1. An operator $A \in \mathcal{C}(X)$ is said to be semi-regular if, $\mathcal{R}(A)$ is closed and it verifies one of the equivalent conditions of Lemma 2.1.

Definition 2.2. An operator $A \in \mathcal{C}(X)$ is said to be of Saphar type if it is semi-regular, and $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are complemented subspaces of X.

Remark 2.1. We see that a Fredholm operator $A \in \mathcal{C}(X)$, that is, $\dim(\mathcal{N}(A)) < \infty$ and codim $(\mathcal{R}(A)) < \infty$ is an operator of Saphar type.

Next, we introduce an important class of linear operators which have a close link to unbounded generalized B-Fredholm operators, that is, the quasi-Fredholm operators. It is well known that, this notion of operators was first discovered by J.P. Labrousse in the famous paper [6] as a generalization of Fredholm operators on a Hilbert space, and it was also studied by [1, 2, 10, 11] and others.

Definition 2.3. ([6]) The degree of stable iteration, dis(A), of the operator A is defined as

$$\text{dis}(A) = \inf(\triangle(A)),$$

where, $\triangle(A) := \big\{n \in \mathbb{N} : \forall m \in \mathbb{N}, m \geq n \Rightarrow \big(\text{ran}(A^n) \cap \mathcal{N}(A)\big) \subset \big(\text{ran}(A^m) \cap \mathcal{N}(A)\big)\big\}$. If $\triangle(A) = \emptyset$, then dis$(A) = \infty$.

Definition 2.4. An operator $A \in \mathcal{C}(X)$ is said to be a quasi-Fredholm of degree $d \in \mathbb{N}$ if, the following three conditions are fulfilled:

(i) $\mathrm{dis}(A) = d$,

(ii) $R(A^n)$ is a closed subspace of X for each $n \geq d$,

(iii) $\mathcal{N}(A^d) + \mathcal{R}(A)$ is a closed subspace of X.

In the sequel, the set of quasi-Fredholm operators of degree d is denoted by $QF(d)$.

Remark 2.2. Note that Definition 2.4 is equivalent to the definitions given in the case of bounded operators in [10, 12]. In the case of Hilbert space, it is equivalent to the definition given in [6].

The following definition is due to J.T. Marti in [8].

Definition 2.5. ([8]) Let X be a Banach space, $A : D(A) \subset X \to X$ and $T : D(T) \subset X \to X$ two linear operators. We say that A commutes with T and we denote $AT = TA$, if

(i) $D(A) \subset D(T)$.

(ii) $Tx \in D(A)$ whenever $x \in D(A)$.

(iii) $AT = TA$ on $\{x \in D(A) \,:\, Ax \in D(T)\}$.

Lemma 2.2. ([3]) *Let A and T be two closed linear operators on a Banach space X such that $AT = TA$. Then,*

$$(\lambda I - A)^{-1}(\lambda I - T)^{-1} = (\lambda I - T)^{-1}(\lambda I - A)^{-1}$$

for each $\lambda \in \rho(A) \cap \rho(T)$.

3. Properties of unbounded generalized B-Fredholm operators

It is well known that, the class of B-Fredholm operators was first introduced by M. Berkani in [2] in the bounded case in Banach space, and it was generalized by Berkani and Castro-González in [4] to unbounded operators in Hilbert space. Newly, this notion of operators was extended by García et al. in [12] to generalized B-Fredholm operators, in the case of bounded linear operators defined on a Banach space. In this section, we shall study this theory in the case of unbounded operators defined from X to X. In order to give our main results in this section, we recall the following definition inspired from [5].

DEFINITION 3.1. ([5]) Let $A \in \mathcal{C}(X)$. The operator A is called B-Fredholm if there exists an integer $d \in \mathbb{N}$ such that $A \in QF(d)$, and such that $\dim(\mathcal{N}(A) \cap \mathcal{R}(A^d)) < \infty$ and $\operatorname{codim}[\mathcal{N}(A^d) + \mathcal{R}(A)] < \infty$.

DEFINITION 3.2. Let $A \in \mathcal{C}(X)$. The operator A is called generalized B-Fredholm if there exists an integer $d \in \mathbb{N}$ such that $A \in QF(d)$, $\mathcal{N}(A) \cap \mathcal{R}(A^d)$ and $\mathcal{N}(A^d) + \mathcal{R}(A)$ are complemented subspaces of X.

The set of generalized B-Fredholm operators defined from X to X is denoted by $\Phi_B^g(X)$.

Remark 3.1. (i) As a finite dimension or codimension subspace on a Banach space is complemented, it follows from Definition 3.1 that each B-Fredholm operator is a generalized B-Fredholm one.

(ii) Since a nilpotent operator is a B-Fredholm one, then it is a generalized B-Fredholm operator.

In [6, Theorem 3.2.1], Labrousse proved a decomposition theorem for closed quasi-Fredholm operators in Hilbert space. This theorem remains true in the case of Banach space if the subspaces $\mathcal{N}(A) \cap \mathcal{R}(A^d)$ and $\mathcal{N}(A^d) + \mathcal{R}(A)$, where $d = \operatorname{dis}(A)$, are closed and complemented in this space, as shown in [6, p. 206]. Based on this decomposition theorem, we establish the following characterization result of unbounded generalized B-Fredholm operators defined from X to X.

THEOREM 3.1. *Let $A \in \mathcal{C}(X)$ be such that $\rho(A) \neq \emptyset$. Then, A is a generalized B-Fredholm operator with $d = \operatorname{dis}(A)$ if and only if there exist two closed invariant subspaces V and W of X such that:*

(i) $X = V \oplus W$, $A(D(A) \cap V) \subseteq V$, $A(W) \subseteq W$, $W \subseteq \mathcal{N}(A^d)$ *and* $W \nsubseteq \mathcal{N}(A^{d-1})$;

(ii) $A_0 = A_{/V}$ *is a closed operator of Saphar type defined on* V *to* V;

(iii) $A_1 = A_{/W}$ *is a nilpotent operator of degree* d.

Proof. Suppose that A is a generalized B-Fredholm operator with $d = \operatorname{dis}(A)$, that is, $A \in QF(d)$, $\mathcal{N}(A) \cap \mathcal{R}(A)^d$ and $\mathcal{R}(A) + \mathcal{N}(A^d)$ are complemented subspaces of X. From [6, Theorem 3.2.1] there exist two closed subspaces V and W such that the conditions (i) and (iii) of the present theorem are satisfied and the operator $A_0 = A_{/V}$ is a closed semi-regular operator defined on V to V. Let us prove that the operator A_0 is of Saphar type. Using equations (3.2.22) and (3.2.23) of the proof of [6, Theorem 3.2.1], we get

$$\mathcal{R}(A) + \mathcal{N}(A^d) = \mathcal{R}(A_0) \oplus W \tag{3.1}$$

$$, \mathcal{N}(A_0) = \mathcal{N}(A) \cap V = \mathcal{N}(A) \cap \mathcal{R}(A)^d. \tag{3.2}$$

Since the subspace $\mathcal{N}(A) \cap \mathcal{R}(A)^d$ is complemented in X, then there exists a closed subspace M of X such that $X = (\mathcal{N}(A) \cap \mathcal{R}(A)^d) \oplus M$. Hence from equality (3.2) we obtain that

$$V = \mathcal{N}(A_0) \oplus (M \cap V).$$

On the other hand, since the subspace $\mathcal{R}(A) + \mathcal{N}(A^d)$ is complemented in X, then there exists a closed subspace S of X such that $X = [\mathcal{R}(A) + \mathcal{N}(A^d)] \oplus S$. Consider the linear projection $P_V : X \to V$ onto V along W. Then, using equality (3.1), we get $P_V(X) = V = \mathcal{R}(A_0) \oplus P_V(S)$, which shows that $\mathcal{R}(A_0)$ is complemented in V. Consequently, the operator A_0 is being of Saphar type.

Conversely, assume that there exist two closed subspaces V and W satisfying conditions (i), (ii) and (iii) of the present theorem.

We have $A^d(V \cap D(A^d)) \subseteq A(V \cap D(A)) \subseteq V$ and $A^d(W \cap D(A^d) \subseteq A(W \cap D(A)) \subseteq W$. Let $n \geq d$, then $\mathcal{R}(A^n) = \mathcal{R}(A^n_{/V})$. Since $\rho(A) \neq \emptyset$, then $\rho(A_{/V}) \neq \emptyset$ and the fact that $A_{/V}$ is a semi-regular operator, this show from [9, Proposition 3.5] that $A^n_{/V}$ is a semi-regular operator and so it has a closed range, for all $n \geq d$. Thus,

$$\begin{aligned}\mathcal{N}(A) \cap \mathcal{R}(A^d) &= \mathcal{N}(A) \cap \mathcal{R}(A^d_{/V}) \\ &= \mathcal{N}(A) \cap \mathcal{R}(A^d) \cap V = \mathcal{N}(A_{/V}) \cap \mathcal{R}(A^d).\end{aligned}$$

Or the operator $A_{/V}$ is semi-regular, because it is of Saphar type, which gives from Lemma 2.1 that $\mathcal{N}(A_{/V}) \subseteq \mathcal{R}(A^n_{/V})$, for every $n \in \mathbb{N}$. So, we get $\mathcal{N}(A_{/V}) \cap \mathcal{R}(A^d_{/V}) = \mathcal{N}(A_{/V})$ and therefore $\mathcal{N}(A_{/V}) = \mathcal{N}(A) \cap \mathcal{R}(A^d)$. We have

$$\begin{aligned}\mathcal{R}(A) + \mathcal{N}(A^d) &= \mathcal{R}(A_{/V}) + \mathcal{R}(A_{/W}) + \mathcal{N}(A^d_{/V}) + \mathcal{N}(A^d_{/W}) \\ &= \mathcal{R}(A_{/V}) + \mathcal{N}(A^d_{/V}) + \mathcal{R}(A_{/W}) \oplus W.\end{aligned}$$

Since the operator $A^d_{/V}$ is semi-regular, then from Lemma 2.1 we get $\mathcal{N}(A^d_{/V}) \subseteq \mathcal{R}(A_{/V})$, which entails that $\mathcal{R}(A_{/V}) + \mathcal{N}(A^d_{/V}) = \mathcal{R}(A_{/V})$. Therefore, $\mathcal{R}(A)+$

$\mathcal{N}(A^d) = \mathcal{R}(A_{/V}) \oplus W$. Since the operator $A_0 = A_{/V}$ is of Saphar type, then there exist two closed subspaces L and M such that

$$\mathcal{N}(A_0) \oplus L = V, \tag{3.3}$$

$$\mathcal{R}(A_0) \oplus M = V. \tag{3.4}$$

Using equality (3.4), we get $X = W \oplus V = W \oplus \mathcal{R}(A_0) \oplus M = (\mathcal{R}(A) + \mathcal{N}(A^d)) \oplus M$. From the Neubauer Lemma [6, Proposition 2.1.1], this means that $\mathcal{R}(A) + \mathcal{N}(A^d)$ is a closed subspace of X and so $A \in QF(d)$. From equality (3.3), we obtain that $X = W \oplus V = W \oplus \mathcal{N}(A_0) \oplus L = \mathcal{N}(A_0) \oplus W \oplus L$ and therefore we get $(\mathcal{N}(A) \cap \mathcal{R}(A^d))$ and $(\mathcal{R}(A) + \mathcal{N}(A^d))$ are complemented subspaces of X. ∎

THEOREM 3.2. *Let $A \in \mathcal{C}(X)$ be such that $\rho(A) \neq \emptyset$. If A is a generalized B-Fredholm operator, then there exists an integer $n \in \mathbb{N}$ such that $\mathcal{R}(A^n)$ is closed and such that the operator A_n is of Saphar type.*

Proof. Assume that A is a generalized B-Fredholm operator, and let $d = \mathrm{dis}(A)$. Then $\mathcal{R}(A^d)$ is closed. Consider the operator $A_d : \mathcal{R}(A^d) \to \mathcal{R}(A^d)$. If A is a generalized B-Fredholm operator, then $\mathcal{N}(A) \cap \mathcal{R}(A^d)$ and $\mathcal{R}(A) + \mathcal{N}(A^d)$ are complemented subspaces in X and therefore there exist two closed subspaces L and M such that

$$X = [\mathcal{N}(A) \cap \mathcal{R}(A^d)] \oplus L, \tag{3.5}$$

$$X = [\mathcal{R}(A) + \mathcal{N}(A^d)] \oplus M. \tag{3.6}$$

Hence, using equality (3.5), we get $\mathcal{R}(A^d) = \mathcal{N}(A_d) \oplus (L \cap \mathcal{R}(A^d))$, which means that $\mathcal{N}(A_d)$ is a complemented subspace. We have $\mathcal{R}(A_d) = \mathcal{R}(A^{d+1})$ is a closed subspace, because A is a quasi-Fredholm of degree d. Now, it remains to show that $\mathcal{R}(A_d)$ is complemented. Since $\rho(A) \neq \emptyset$, from [7, Lemma 1.1], we have $X = \mathcal{D}(A^d) + \mathcal{R}(A)$. Then, we get

$$\begin{aligned} A^d(X) &= A^d(\mathcal{D}(A^d) + \mathcal{R}(A)) \\ &\subseteq A^d(\mathcal{D}(A^d)) + A^d(\mathcal{R}(A)) \subseteq A^d(\mathcal{D}(A^d)) = \mathcal{R}(A^d). \end{aligned}$$

Thus, we get $\mathcal{R}(A^d) = A^d(X)$ and from equality (3.6) we have

$$\mathcal{R}(A^d) = A^d(X) = A^d(\mathcal{R}(A) + \mathcal{N}(A^d)) \oplus A^d(M) = \mathcal{R}(A^{d+1}) \oplus A^d(M).$$

From the Neubauer Lemma [6, Proposition 2.1.1] we obtain that $A^d(M)$ is a closed subspace. Therefore, we obtain that $\mathcal{R}(A_d)$ is complemented. Accordingly, the operator A_d is of Saphar type. ∎

We notice that if $A \in \mathcal{C}(X)$ is a densely defined linear operator, then the adjoint operator A^* exists, belongs to $\mathcal{C}(X^*)$ and is a densely defined linear operator, where X^* is the dual space of X. If M is a subspace of X, then by $M^\perp$ we denote the annihilator of M as a subspace of X^*. Clearly, $M^\perp$ is a closed subspace of X^*.

Proposition 3.1. *Let $A \in \mathcal{C}(X)$ be densely defined. If A is a generalized B-Fredholm operator, then A^* is a generalized B-Fredholm operator.*

Proof. If A is a generalized B-Fredholm operator, then it is a quasi-Fredholm of degree $d \in \mathbb{N}$. Then, it follows from [6, Proposition 3.3.5] that $A^* \in QF(d)$. Hence, we get

$$\mathcal{R}(A) + \mathcal{N}(A^d) = \left[\mathcal{N}(A^*) \cap \mathcal{R}(A^*)^d\right]^\perp,$$
$$\mathcal{R}(A^*) + \mathcal{N}(A^*)^d = \left[\mathcal{N}(A) \cap \mathcal{R}(A)^d\right]^\perp.$$

Since $\mathcal{N}(A) \cap \mathcal{R}(A^d)$ and $\mathcal{N}(A^d) + \mathcal{R}(A)$ are complemented in X, then we get $\mathcal{N}(A^*) \cap \mathcal{R}(A^*)^d$ and $\mathcal{R}(A^*) + \mathcal{N}(A^*)^d$ are complemented in X^*. This prove that A^* is a generalized B-Fredholm operator. ∎

4. Generalized B-Fredholm spectrum

In this section, we define and we study an essential spectrum related to the class of unbounded generalized B-Fredholm operators named the generalized B-Fredholm spectrum, which is defined as follows:

Definition 4.1. Let $A \in \mathcal{C}(X)$. The generalized B-Fredholm spectrum of A is defined by:

$$\sigma^g_{bf}(A) := \left\{\lambda \in \mathbb{C} \,:\, A - \lambda I \notin \Phi^g_B(X)\right\}$$

and the generalized B-Fredholm set of A is defined by

$$\rho^g_{bf}(A) = \mathbb{C} \backslash \sigma^g_{bf}(A).$$

Proposition 4.1. *Let $A \in \mathcal{C}(X)$ be such that $\rho(A) \neq \emptyset$. Then, the generalized B-Fredholm spectrum $\sigma^g_{bf}(A)$ of A is a closed subset of $\mathbb{C}$ contained in the usual spectrum $\sigma(A)$ of A.*

Proof. If $\lambda \notin \sigma(A)$ then $A-\lambda I$ is invertible and therefore $A-\lambda I$ is a B-Fredholm operator. Hence from Remark 3.1 we get $\lambda \notin \sigma^g_{bf}(A)$. If $\alpha \notin \sigma^g_{bf}(A)$, then $A-\alpha I$ is a generalized B-Fredholm operator. Set $S = A-\alpha I$. By Theorem 3.1, there exist two closed subspaces M and N invariant under A of X such that $X = M \oplus N$ and $S = S_{/M} \oplus S_{/N}$, where $S_{/M}$ is of type Saphar and $S_{/N}$ is a nilpotent operator. Since $S_{/M}$ is of Saphar type, then from [14, Theorem 2] there exists an open disc $D(0,\varepsilon)$ centered at 0 such that $S_{/M}-\lambda I$ is of Saphar type, for all $\lambda \in D(0,\varepsilon)\backslash\{0\}$. As $S_{/N}$ is a nilpotent operator, then we get $S_{/N}-\lambda I$ is invertible, for all $\lambda \neq 0$. Then, using Theorem 3.1, we get $S-\lambda I$ is a generalized B-Fredholm operator, for all $\lambda \in D(\alpha,\varepsilon)\backslash\{\alpha\}$. So $\rho^g_{bf}(A)$ is open in $\mathbb{C}$ or equivalently $\sigma^g_{bf}(A)$ is a closed subset of $\mathbb{C}$. ∎

Remark 4.1. Note that, the generalized B-Fredholm spectrum can be empty. For example, if A is a nilpotent operator, then $\sigma_{bf}(A) = \emptyset$ and since $\sigma^g_{bf}(A) \subseteq \sigma_{bf}(A)$, then we get $\sigma^g_{bf}(A) = \emptyset$, where $\sigma_{bf}(A) = \{\lambda \in \mathbb{C}, A-\lambda I$ is not B-Fredholm$\}$, is the B-Fredholm spectrum.

PROPOSITION 4.2. *Let $A \in \mathcal{C}(X)$ be a closed invertible operator with a dense domain. Then,*

$$\sigma^g_{bf}(A) = \left\{\lambda^{-1} : \lambda \in \sigma^g_{bf}(A^{-1})\backslash\{0\}\right\}.$$

Proof. Using the relations proved in [5, Proposition 3.3 and Proposition 3.4], we obtain that $A-\lambda I$ is a generalized B-Fredholm operator if and only if $A^{-1}-\lambda^{-1}I$ is also, for all $\lambda \neq 0$. ∎

COROLLARY 4.1. *Let $A, T \in \mathcal{C}(X)$ be two closed invertible operators with a dense domain and such that $AT = TA$. If the bounded operator $A^{-1}-T^{-1}$ is of power finite-rank, then*

$A-\lambda I$ is generalized B-Fredholm $\Rightarrow$ $T-\lambda I$ is quasi-Fredholm, for all $\lambda \neq 0$.

Proof. Let $\lambda \neq 0$. If the operator $A-\lambda I$ is generalized B-Fredholm, then from Proposition 4.2 we have the same also for the bounded operator $A^{-1}-\lambda^{-1}I$. Since $A^{-1}-T^{-1}$ is of power finite-rank operator, then it follows from [12, Theorem 3.1] and Lemma 2.2 that, $T^{-1}-\lambda^{-1}I = T^{-1}-\lambda^{-1}I-A^{-1}+A^{-1}$ is a bounded quasi-Fredholm operator, and therefore from [5, Theorem 3.6] the operator $T-\lambda I$ is also quasi-Fredholm. ∎

Corollary 4.2. *Let $A, T \in \mathcal{C}(X)$ be two closed invertible operators with a dense domain and such that $AT = TA$. If the bounded operator $A^{-1} - T^{-1}$ is nilpotent, then*

$$A - \lambda I \text{ is generalized } B\text{-Fredholm} \quad \Rightarrow \quad T - \lambda I \text{ is quasi-Fredholm, for all } \lambda \neq 0.$$

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