Unbounded generalized B-Fredholm operators

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Abstract: In this paper, we investigate a new class of unbounded linear operators, that is, the unbounded generalized B-Fredholm operators in Banach space. More explicitly, we provide a characterization of this class of operators and some of its basic properties on a Banach space. Moreover, we study the generalized B-Fredholm spectrum and we prove a perturbation result of an unbounded generalized B-Fredholm operator under a commuting power finite-rank operator.

Key words: Unbounded generalized B-Fredholm operators, operator of Saphar type, generalized B-Fredholm spectrum, quasi-Fredholm operator.

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1. INTRODUCTION

Let \( \mathcal{C}(X) \) denotes the set of all closed linear operators defined from a Banach space \( X \) to \( X \). For \( A \in \mathcal{C}(X) \) and for each integer \( n \in \mathbb{N} \), the domain \( \mathcal{D}(A^n) \), the kernel \( \mathcal{N}(A^n) \) and the range \( \mathcal{R}(A^n) \) of the power operator \( A^n \) are defined, respectively, by

\[
\mathcal{D}(A^n) = \{ x \in X : Ax, \ldots, A^{n-1}x \in \mathcal{D}(A) \},
\]

\[
\mathcal{N}(A^n) = \{ x \in \mathcal{D}(A^n) : A^n x = 0 \}
\]

and

\[
\mathcal{R}(A^n) = \{ y \in X : A^n x = y \text{ for } x \in \mathcal{D}(A^n) \}.
\]

If \( n = 0 \), one has

\[
A^0 = I, \quad \mathcal{D}(A^0) = X, \quad \mathcal{N}(A^0) = 0, \quad \mathcal{R}(A^0) = X,
\]

where \( I \) is the identity operator defined from \( X \) to \( X \). For all \( n \geq 1 \), we have \( A^n(x) = AA^{n-1}(x) \), where \( x \in \mathcal{D}(A^n) \). We clearly have:

\[
\mathcal{D}(A^{n+1}) \subseteq \mathcal{D}(A^n),
\]
for all $n \in \mathbb{N}$. Let $\sigma(A)$ (resp. $\rho(A)$) denote the usual spectrum (resp. the resolvent set) of $A$.

For $A \in \mathcal{C}(X)$ and $n \in \mathbb{N}$, let $A_n : \mathcal{R}(A^n) \to \mathcal{R}(A^n)$ be the restriction of the operator $A$ to $\mathcal{R}(A^n)$ into $\mathcal{R}(A^n)$. The domain $\mathcal{D}(A_n)$, the kernel $\mathcal{N}(A_n)$ and the range $\mathcal{R}(A_n)$ of $A_n$ are defined respectively by

$$\mathcal{D}(A_n) = \mathcal{D}(A) \cap \mathcal{R}(A^n),$$
$$\mathcal{N}(A_n) = \mathcal{N}(A) \cap \mathcal{R}(A^n),$$
$$\mathcal{R}(A_n) = \mathcal{R}(A^{n+1}).$$

The class of B-Fredholm operators was first introduced by M. Berkani in [2] in the case of bounded operators acting on a Banach space. This notion of operators was generalized by Berkani and Castro-González in [3] to unbounded operators in Hilbert space. Recently, this class of operators was extended and studied by O. García et al. in [12] to generalized bounded B-Fredholm operators acting on a Banach space. Here, in this paper, we will consider unbounded generalized B-Fredholm operators defined on a Banach space. Our work, in this paper, extend some results obtained in [12] to the case of unbounded operators. After an introductory section, we recall in section 2 a list of well known definitions which are must be required, in this paper. We know that, J.P. Labrousse proved in [6] two decomposition theorems of closed quasi-Fredholm operators on a Hilbert space. As mentioned in [6, p. 206], these theorems are still true in the case of Banach spaces if, the subspaces $\mathcal{N}(A) \cap \mathcal{R}(A^d)$ and $\mathcal{R}(A) + \mathcal{N}(A^d)$, where $d = \text{dis}(A)$, are closed and complemented. Using this result, we characterize in Theorem 3.1 a closed generalized B-Fredholm operator as a direct sum of a closed operator of Saphar type and a nilpotent one. Besides, we show in Theorem 3.2 an important result says that, if $A$ is a closed generalized B-Fredholm operator with a non empty resolvent set, then there exists an integer $n \in \mathbb{N}$ such that $\mathcal{R}(A^n)$ is closed and such that the restriction operator $A_n$ is of Saphar type. In Proposition 3.1, we show that if $A$ is a closed generalized B-Fredholm operator densely defined on a Banach space, then its adjoint operator is also a generalized B-Fredholm operator. Based on Theorem 3.1, we prove in Proposition 4.1 that the generalized B-Fredholm spectrum of a closed operator $A$ defined from $X$ to $X$ is a closed subset of the complex plane $\mathbb{C}$. Next, we characterize in Proposition 4.2 the generalized B-Fredholm spectrum of a closed linear operator $A$ in terms of the corresponding spectrum of its bounded inverse. The end of section 4 contains a perturbation result of an unbounded generalized B-Fredholm operator under commuting power finite-rank operator.
2. Preliminaries

In this section, we collect a list of well-known definitions which are relevant to the development of this paper.

First, we give the following algebraic result which must be required:

Lemma 2.1. (\[1\]) Let $A$ be a linear operator defined on a vector space. Then, the following conditions are equivalent:

1. for all $s \in \mathbb{N}$, $\mathcal{N}(A) \subseteq \mathcal{R}(A^s)$,
2. for all $n \in \mathbb{N}$, $\mathcal{N}(A^n) \subseteq \mathcal{R}(A)$,
3. for all $s, n \in \mathbb{N}$, $\mathcal{N}(A^n) \subseteq \mathcal{R}(A^s)$,
4. for all $s, n \in \mathbb{N}$, $\mathcal{N}(A^n) = A^s(\mathcal{N}(A^{n+s}))$.

In the following, we define the classes of semi-regular operators and the operators of Saphar type, which are the key tool for the study of unbounded generalized B-Fredholm operators. It is well known that, these classes of operators were been studied by several authors, we can see for instance the works of \[9, 13, 14\] and elsewhere.

Definition 2.1. An operator $A \in \mathcal{C}(X)$ is said to be semi-regular if, $\mathcal{R}(A)$ is closed and it verifies one of the equivalent conditions of Lemma 2.1.

Definition 2.2. An operator $A \in \mathcal{C}(X)$ is said to be of Saphar type if it is semi-regular, and $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are complemented subspaces of $X$.

Remark 2.1. We see that a Fredholm operator $A \in \mathcal{C}(X)$, that is, $\dim(\mathcal{N}(A)) < \infty$ and $\text{codim}(\mathcal{R}(A)) < \infty$ is an operator of Saphar type.

Next, we introduce an important class of linear operators which have a close link to unbounded generalized B-Fredholm operators, that is, the quasi-Fredholm operators. It is well known that, this notion of operators was first discovered by J.P. Labrousse in the famous paper \[6\] as a generalization of Fredholm operators on a Hilbert space, and it was also studied by \[1, 2, 10, 11\] and others.

Definition 2.3. (\[6\]) The degree of stable iteration, $\text{dis}(A)$, of the operator $A$ is defined as

$$\text{dis}(A) = \inf(\triangle(A)),$$

where, $\triangle(A) := \{ n \in \mathbb{N} : \forall m \in \mathbb{N}, m \geq n \Rightarrow (\text{ran}(A^n) \cap \mathcal{N}(A)) \subset (\text{ran}(A^m) \cap \mathcal{N}(A)) \}$. If $\triangle(A) = \emptyset$, then $\text{dis}(A) = \infty$. 

Definition 2.4. An operator \( A \in C(X) \) is said to be a quasi-Fredholm of degree \( d \in \mathbb{N} \) if, the following three conditions are fulfilled:

(i) \( \text{dis}(A) = d \),
(ii) \( R(A^n) \) is a closed subspace of \( X \) for each \( n \geq d \),
(iii) \( \mathcal{N}(A^d) + \mathcal{R}(A) \) is a closed subspace of \( X \).

In the sequel, the set of quasi-Fredholm operators of degree \( d \) is denoted by \( QF(d) \).

Remark 2.2. Note that Definition 2.4 is equivalent to the definitions given in the case of bounded operators in \([10, 12]\). In the case of Hilbert space, it is equivalent to the definition given in \([6]\).

The following definition is due to J.T. Marti in \([8]\).

Definition 2.5. (\([8]\)) Let \( X \) be a Banach space, \( A : D(A) \subset X \to X \) and \( T : D(T) \subset X \to X \) two linear operators. We say that \( A \) commutes with \( T \) and we denote \( AT = TA \), if

(i) \( D(A) \subset D(T) \).
(ii) \( Tx \in D(A) \) whenever \( x \in D(A) \).
(iii) \( AT = TA \) on \( \{ x \in D(A) : Ax \in D(T) \} \).

Lemma 2.2. (\([3]\)) Let \( A \) and \( T \) be two closed linear operators on a Banach space \( X \) such that \( AT = TA \). Then,

\[(\lambda I - A)^{-1}(\lambda I - T)^{-1} = (\lambda I - T)^{-1}(\lambda I - A)^{-1}\]

for each \( \lambda \in \rho(A) \cap \rho(T) \).

3. Properties of unbounded generalized B-Fredholm operators

It is well known that, the class of B-Fredholm operators was first introduced by M. Berkani in \([2]\) in the bounded case in Banach space, and it was generalized by Berkani and Castro-González in \([1]\) to unbounded operators in Hilbert space. Newly, this notion of operators was extended by García et al. in \([12]\) to generalized B-Fredholm operators, in the case of bounded linear operators defined on a Banach space. In this section, we shall study this theory in the case of unbounded operators defined from \( X \) to \( X \). In order to give our main results in this section, we recall the following definition inspired from \([5]\).
Definition 3.1. ([5]) Let $A \in \mathcal{C}(X)$. The operator $A$ is called B-Fredholm if there exists an integer $d \in \mathbb{N}$ such that $A \in \text{QF}(d)$, and such that $\dim(\mathcal{N}(A) \cap \mathcal{R}(A^d)) < \infty$ and $\text{codim}[\mathcal{N}(A^d) + \mathcal{R}(A)] < \infty$.

Definition 3.2. Let $A \in \mathcal{C}(X)$. The operator $A$ is called generalized B-Fredholm if there exists an integer $d \in \mathbb{N}$ such that $A \in \text{QF}(d)$, $\mathcal{N}(A) \cap \mathcal{R}(A^d)$ and $\mathcal{N}(A^d) + \mathcal{R}(A)$ are complemented subspaces of $X$.

The set of generalized B-Fredholm operators defined from $X$ to $X$ is denoted by $\Phi_g^B(X)$.

Remark 3.1. (i) As a finite dimension or codimension subspace on a Banach space is complemented, it follows from Definition 3.1 that each B-Fredholm operator is a generalized B-Fredholm one.

(ii) Since a nilpotent operator is a B-Fredholm one, then it is a generalized B-Fredholm operator.

In [6, Theorem 3.2.1], Labrousse proved a decomposition theorem for closed quasi-Fredholm operators in Hilbert space. This theorem remains true in the case of Banach space if the subspaces $\mathcal{N}(A) \cap \mathcal{R}(A^d)$ and $\mathcal{N}(A^d) + \mathcal{R}(A)$, where $d = \text{dis}(A)$, are closed and complemented in this space, as shown in [6, p. 206]. Based on this decomposition theorem, we establish the following characterization result of unbounded generalized B-Fredholm operators defined from $X$ to $X$.

Theorem 3.1. Let $A \in \mathcal{C}(X)$ be such that $\rho(A) \neq \emptyset$. Then, $A$ is a generalized B-Fredholm operator with $d = \text{dis}(A)$ if and only if there exist two closed invariant subspaces $V$ and $W$ of $X$ such that:

(i) $X = V \oplus W$, $A(D(A) \cap V) \subseteq V$, $A(W) \subseteq W$, $W \subseteq \mathcal{N}(A^d)$ and $W \not\subseteq \mathcal{N}(A^{d-1})$;

(ii) $A_0 = A/V$ is a closed operator of Saphar type defined on $V$ to $V$;

(iii) $A_1 = A/W$ is a nilpotent operator of degree $d$.

Proof. Suppose that $A$ is a generalized B-Fredholm operator with $d = \text{dis}(A)$, that is, $A \in \text{QF}(d)$, $\mathcal{N}(A) \cap \mathcal{R}(A^d)$ and $\mathcal{R}(A) + \mathcal{N}(A^d)$ are complemented subspaces of $X$. From [6, Theorem 3.2.1] there exist two closed subspaces $V$ and $W$ such that the conditions (i) and (iii) of the present theorem are satisfied and the operator $A_0 = A/V$ is a closed semi-regular operator.
defined on $V$ to $V$. Let us prove that the operator $A_0$ is of Saphar type. Using equations (3.2.22) and (3.2.23) of the proof of [6, Theorem 3.2.1], we get

$$\mathcal{R}(A) + \mathcal{N}(A^d) = \mathcal{R}(A_0) \oplus W$$

(3.1)

$$\mathcal{N}(A_0) = \mathcal{N}(A) \cap V = \mathcal{N}(A) \cap \mathcal{R}(A)^d.$$  

(3.2)

Since the subspace $\mathcal{N}(A) \cap \mathcal{R}(A)^d$ is complemented in $X$, then there exists a closed subspace $M$ of $X$ such that $X = (\mathcal{N}(A) \cap \mathcal{R}(A)^d) \oplus M$. Hence from equality (3.2) we obtain that

$$V = \mathcal{N}(A_0) \oplus (M \cap V).$$

On the other hand, since the subspace $\mathcal{R}(A) + \mathcal{N}(A^d)$ is complemented in $X$, then there exists a closed subspace $S$ of $X$ such that $X = [\mathcal{R}(A) + \mathcal{N}(A^d)] \oplus S$. Consider the linear projection $P_V : X \to V$ onto $V$ along $W$. Then, using equality (3.1), we get $P_V(X) = V = \mathcal{R}(A_0) \oplus P_V(S)$, which shows that $\mathcal{R}(A_0)$ is complemented in $V$. Consequently, the operator $A_0$ is being of Saphar type.

Conversely, assume that there exist two closed subspaces $V$ and $W$ satisfying conditions (i), (ii) and (iii) of the present theorem.

We have $A^d(V \cap D(A^d)) \subseteq A(V \cap D(A)) \subseteq V$ and $A^d(W \cap D(A^d)) \subseteq A(W \cap D(A)) \subseteq W$. Let $n \geq d$, then $\mathcal{R}(A^n) = \mathcal{R}(A^n_{|V})$. Since $\rho(A) \neq \emptyset$, then $\rho(A_{|V}) \neq \emptyset$ and the fact that $A_{|V}$ is a semi-regular operator, this show from [9, Proposition 3.5] that $A^n_{|V}$ is a semi-regular operator and so it has a closed range, for all $n \geq d$. Thus, we have

$$\mathcal{N}(A) \cap \mathcal{R}(A^d) = \mathcal{N}(A) \cap \mathcal{R}(A^d_{|V})$$

$$= \mathcal{N}(A) \cap \mathcal{R}(A^d) \cap V = \mathcal{N}(A_{|V}) \cap \mathcal{R}(A^d).$$

Or the operator $A_{|V}$ is semi-regular, because it is of Saphar type, which gives from Lemma 2.1 that $\mathcal{N}(A_{|V}) \subseteq \mathcal{R}(A^n_{|V})$, for every $n \in \mathbb{N}$. So, we get $\mathcal{N}(A_{|V}) \cap \mathcal{R}(A^n_{|V}) = \mathcal{N}(A_{|V})$ and therefore $\mathcal{N}(A_{|V}) = \mathcal{N}(A) \cap \mathcal{R}(A^d)$. We have

$$\mathcal{R}(A) + \mathcal{N}(A^d) = \mathcal{R}(A_{|V}) + \mathcal{R}(A_{|W}) + \mathcal{N}(A^d_{|V}) + \mathcal{N}(A^d_{|W})$$

$$= \mathcal{R}(A_{|V}) + \mathcal{N}(A^d_{|V}) + \mathcal{R}(A_{|W}) \oplus W.$$
\( \mathcal{N}(A^d) = \mathcal{R}(A_{/V}) \oplus W. \) Since the operator \( A_0 = A_{/V} \) is of Saphar type, then there exist two closed subspaces \( L \) and \( M \) such that
\[
\mathcal{N}(A_0) \oplus L = V, \quad (3.3)
\]
\[
\mathcal{R}(A_0) \oplus M = V. \quad (3.4)
\]
Using equality (3.4), we get \( X = W \oplus V = W \oplus \mathcal{R}(A_0) \oplus M = (\mathcal{R}(A) + \mathcal{N}(A^d)) \oplus M. \) From the Neubauer Lemma \[6, Proposition 2.1.1\], this means that \( \mathcal{R}(A) + \mathcal{N}(A^d) \) is a closed subspace of \( X \) and so \( A \in QF(d). \) From equality (3.3), we obtain that \( X = W \oplus V = W \oplus \mathcal{N}(A_0) \oplus L = \mathcal{N}(A_0) \oplus W \oplus L \) and therefore we get \( (\mathcal{N}(A) \cap \mathcal{R}(A^d)) \) and \( (\mathcal{R}(A) + \mathcal{N}(A^d)) \) are complemented subspaces of \( X. \)

**Theorem 3.2.** Let \( A \in \mathcal{C}(X) \) be such that \( \rho(A) \neq \emptyset. \) If \( A \) is a generalized B-Fredholm operator, then there exists an integer \( n \in \mathbb{N} \) such that \( \mathcal{R}(A^n) \) is closed and such that the operator \( A_n \) is of Saphar type.

**Proof.** Assume that \( A \) is a generalized B-Fredholm operator, and let \( d = \text{dis}(A). \) Then \( \mathcal{R}(A^d) \) is closed. Consider the operator \( A_d : \mathcal{R}(A^d) \to \mathcal{R}(A^d). \) If \( A \) is a generalized B-Fredholm operator, then \( \mathcal{N}(A) \cap \mathcal{R}(A^d) \) and \( \mathcal{R}(A) + \mathcal{N}(A^d) \) are complemented subspaces in \( X \) and therefore there exist two closed subspaces \( L \) and \( M \) such that
\[
X = [\mathcal{N}(A) \cap \mathcal{R}(A^d)] \oplus L, \quad (3.5)
\]
\[
X = [\mathcal{R}(A) + \mathcal{N}(A^d)] \oplus M. \quad (3.6)
\]
Hence, using equality (3.5), we get \( \mathcal{R}(A^d) = \mathcal{N}(A_d) \oplus (L \cap \mathcal{R}(A^d)), \) which means that \( \mathcal{N}(A_d) \) is a complemented subspace. We have \( \mathcal{R}(A_d) = \mathcal{R}(A^{d+1}) \) is a closed subspace, because \( A \) is a quasi-Fredholm of degree \( d. \) Now, it remains to show that \( \mathcal{R}(A_d) \) is complemented. Since \( \rho(A) \neq \emptyset, \) from \[7, Lemma 1.1\], we have \( X = \mathcal{D}(A^d) + \mathcal{R}(A). \) Then, we get
\[
A^d(X) = A^d(\mathcal{D}(A^d) + \mathcal{R}(A))
\]
\[
\subseteq A^d(\mathcal{D}(A^d)) + A^d(\mathcal{R}(A)) \subseteq A^d(\mathcal{D}(A^d)) = \mathcal{R}(A^d).
\]
Thus, we get \( \mathcal{R}(A^d) = A^d(X) \) and from equality (3.6) we have
\[
\mathcal{R}(A^d) = A^d(X) = A^d(\mathcal{R}(A) + \mathcal{N}(A^d)) \oplus A^d(M) = \mathcal{R}(A^{d+1}) \oplus A^d(M).
\]
From the Neubauer Lemma \[6, Proposition 2.1.1\] we obtain that \( A^d(M) \) is a closed subspace. Therefore, we obtain that \( \mathcal{R}(A_d) \) is complemented. Accordingly, the operator \( A_d \) is of Saphar type. \( \blacksquare \)
We notice that if \( A \in \mathcal{C}(X) \) is a densely defined linear operator, then the adjoint operator \( A^* \) exists, belongs to \( \mathcal{C}(X^*) \) and is a densely defined linear operator, where \( X^* \) is the dual space of \( X \). If \( M \) is a subspace of \( X \), then by \( M^\perp \) we denote the annihilator of \( M \) as a subspace of \( X^* \). Clearly, \( M^\perp \) is a closed subspace of \( X^* \).

**Proposition 3.1.** Let \( A \in \mathcal{C}(X) \) be densely defined. If \( A \) is a generalized B-Fredholm operator, then \( A^* \) is a generalized B-Fredholm operator.

**Proof.** If \( A \) is a generalized B-Fredholm operator, then it is a quasi-Fredholm of degree \( d \in \mathbb{N} \). Then, it follows from [6, Proposition 3.3.5] that \( A^* \in \mathcal{QF}(d) \). Hence, we get

\[
\mathcal{R}(A) + \mathcal{N}(A^d) = [\mathcal{N}(A^*) \cap \mathcal{R}(A^*)^d]^\perp,
\]

\[
\mathcal{R}(A^*) + \mathcal{N}(A^*)^d = [\mathcal{N}(A) \cap \mathcal{R}(A)^d]^\perp.
\]

Since \( \mathcal{N}(A) \cap \mathcal{R}(A^d) \) and \( \mathcal{N}(A^d) + \mathcal{R}(A) \) are complemented in \( X \), then we get \( \mathcal{N}(A^*) \cap \mathcal{R}(A^*)^d \) and \( \mathcal{R}(A^*) + \mathcal{N}(A^*)^d \) are complemented in \( X^* \). This prove that \( A^* \) is a generalized B-Fredholm operator.

4. **Generalized B-Fredholm spectrum**

In this section, we define and we study an essential spectrum related to the class of unbounded generalized B-Fredholm operators named the generalized B-Fredholm spectrum, which is defined as follows:

**Definition 4.1.** Let \( A \in \mathcal{C}(X) \). The generalized B-Fredholm spectrum of \( A \) is defined by:

\[
\sigma^g_{bf}(A) := \{ \lambda \in \mathbb{C} : A - \lambda I \notin \mathcal{G}(X) \}
\]

and the generalized B-Fredholm set of \( A \) is defined by

\[
\rho^g_{bf}(A) = \mathbb{C} \setminus \sigma^g_{bf}(A).
\]

**Proposition 4.1.** Let \( A \in \mathcal{C}(X) \) be such that \( \rho(A) \neq \emptyset \). Then, the generalized B-Fredholm spectrum \( \sigma^g_{bf}(A) \) of \( A \) is a closed subset of \( \mathbb{C} \) contained in the usual spectrum \( \sigma(A) \) of \( A \).
Proof. If $\lambda \notin \sigma(A)$ then $A - \lambda I$ is invertible and therefore $A - \lambda I$ is a B-Fredholm operator. Hence from Remark 3.1 we get $\lambda \notin \sigma^0_B(A)$. If $\alpha \notin \sigma^0_B(A)$, then $A - \alpha I$ is a generalized B-Fredholm operator. Set $S = A - \alpha I$. By Theorem 3.1 there exist two closed subspaces $M$ and $N$ invariant under $A$ of $X$ such that $X = M \oplus N$ and $S = S/M \oplus S/N$, where $S/M$ is of type Saphar and $S/N$ is a nilpotent operator. Since $S/M$ is of Saphar type, then from [14, Theorem 2] there exists an open disc $D(0, \varepsilon)$ centered at 0 such that $S/M - \lambda I$ is of Saphar type, for all $\lambda \in D(0, \varepsilon) \setminus \{0\}$. As $S/N$ is a nilpotent operator, then we get $S/M - \lambda I$ is invertible, for all $\lambda \neq 0$. Then, using Theorem 3.1 we get $S - \lambda I$ is a generalized B-Fredholm operator, for all $\lambda \in D(\alpha, \varepsilon) \setminus \{\alpha\}$. So $\rho^g_B(A)$ is open in $\mathbb{C}$ or equivalently $\sigma^g_B(A)$ is a closed subset of $\mathbb{C}$. 

Remark 4.1. Note that, the generalized B-Fredholm spectrum can be empty. For example, if $A$ is a nilpotent operator, then $\sigma^g_B(A) = \emptyset$ and since $\sigma^g_B(A) \subseteq \sigma_B(A)$, then we get $\sigma^g_B(A) = \emptyset$, where $\sigma_B(A) = \{\lambda \in \mathbb{C}, A - \lambda I$ is not B-Fredholm\}, is the B-Fredholm spectrum.

Proposition 4.2. Let $A \in \mathcal{C}(X)$ be a closed invertible operator with a dense domain. Then,

$$\sigma^g_B(A) = \{\lambda^{-1} : \lambda \in \sigma^g_B(A^{-1}) \setminus \{0\}\}.$$ 

Proof. Using the relations proved in [5 Proposition 3.3 and Proposition 3.4], we obtain that $A - \lambda I$ is a generalized B-Fredholm operator if and only if $A^{-1} - \lambda^{-1} I$ is also, for all $\lambda \neq 0$. 

Corollary 4.1. Let $A, T \in \mathcal{C}(X)$ be two closed invertible operators with a dense domain and such that $AT = TA$. If the bounded operator $A^{-1} - T^{-1}$ is of power finite-rank, then

$A - \lambda I$ is generalized B-Fredholm $\Rightarrow$ $T - \lambda I$ is quasi-Fredholm, for all $\lambda \neq 0$.

Proof. Let $\lambda \neq 0$. If the operator $A - \lambda I$ is generalized B-Fredholm, then from Proposition 4.2 we have the same also for the bounded operator $A^{-1} - \lambda^{-1} I$. Since $A^{-1} - T^{-1}$ is of power finite-rank operator, then it follows from [12 Theorem 3.1] and Lemma 2.2 that, $T^{-1} - \lambda^{-1} I = T^{-1} - \lambda^{-1} I - A^{-1} + A^{-1}$ is a bounded quasi-Fredholm operator, and therefore from [5 Theorem 3.6] the operator $T - \lambda I$ is also quasi-Fredholm.
Corollary 4.2. Let $A, T \in \mathcal{C}(X)$ be two closed invertible operators with a dense domain and such that $AT = TA$. If the bounded operator $A^{-1} - T^{-1}$ is nilpotent, then

$$A - \lambda I \text{ is generalized } B\text{-Fredholm} \quad \Rightarrow \quad T - \lambda I \text{ is quasi-Fredholm, for all } \lambda \neq 0.$$ 

References