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The Arens-Calderon theorem for commutative topological algebras

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Abstract: A theorem of Arens and Calderon states that if A is a commutative Banach algebra with Jacobson radical $\mathrm{Rad}(A)$, and if $a_0, \ldots, a_n \in A$ with $a_0 \in \mathrm{Rad}(A)$ and a_1 an invertible element of A, then there exists $y \in \mathrm{Rad}(A)$ such that $\sum_{k=0}^{n} a_k y^k = 0$. In this paper, we give extensions of this result to commutative non-normed topological algebras, as this is vital for extending an embedding theorem of Allan in [2] regarding the embedding of the formal power series algebra $\mathbb{C}[[X]]$ into a commutative Banach algebra.

Key words: Formal power series; theorem of Arens and Calderon; commutative topological algebra; functional calculus.

MSC (2020): 46H05, 46H020, 46H30.

1. Introduction

In [16], one of the main questions raised is whether for a given topological algebra $A[\tau]$ and an element $x \in A$, there exists a unital homomorphism $\psi\colon \mathbb{C}[[X]] \to A$ such that $\psi(X) = x$. Here, $\mathbb{C}[[X]]$ denotes the algebra of all complex formal power series with indeterminate X. This question was answered positively by G.R. Allan in [2, Theorem 2] for the case where A is a commutative Banach algebra, and x is in the (Jacobson) radical of A for which there exists a natural number $m \geq 1$ such that $\{0\} \neq Ax^m \subset \overline{Ax^{m+1}}$. The proof of Allan's result in [2] is entirely algebraic, except in an application of the following theorem of Arens and Calderon, which relies on A being a Banach algebra: Let A be a commutative Banach algebra with identity and Jacobson radical $\mathrm{Rad}(A)$. If $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in A$ are such that $a_0 \in \mathrm{Rad}(A)$ and a_1 is invertible, then there exists $y \in \mathrm{Rad}(A)$ such that $\sum_{k=0}^{n} a_k y^k = 0$.

This observation immediately yields a strategy for answering the above

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question by only having to extend the above result of Arens and Calderon to general commutative topological algebras. This is precisely the purpose of this manuscript. Namely, we extend the above result of Arens and Calderon to commutative Fréchet locally m-convex algebras having weakly compact character space (see Theorem 4.2 below), the algebras in Theorem 5.2 and to all Mackey-complete Gelfand Mazur Q-algebras with continuous inversion and nonempty character space (see Corollary 5.10 below). The proof of the original theorem of Arens and Calderon for commutative Banach algebras in the first paragraph above relies heavily on complex analysis of several variables (see [11, Lemma 3.2.8]) and the proofs of our extended Arens-Calderon theorems alluded to above will therefore also rely on several complex variable function theory.

This paper is organized as follows: Section 2 contains all background material required in order to prove the main results of this paper. Section 3 gives a detailed exposition of the proof of the Arens-Calderon theorem for commutative Banach algebras, as given in [11, Lemma 3.2.8], as this proof gives us the strategy to prove the main results mentioned in the previous paragraph, given in Sections 4 and 5 of this manuscript.

2. Preliminaries

All algebras are assumed to have an identity element 1*, unless stated otherwise.* A *topological algebra* $A[\tau]$ is a complex algebra equipped with a topology τ making A a Hausdorff topological vector space, such that the multiplication on A is separately continuous. We say that a topological algebra is a *Fréchet algebra* if it is complete and metrizable. A *locally convex algebra* is a topological algebra which is locally convex as a topological vector space. A *Q-algebra* will mean a topological algebra in which the set of invertible elements G_A of A is an open subset of A.

We say that a topological algebra $A[\tau]$ is a *locally m-convex* algebra if it its topology is defined by a directed family of submultiplicative seminorms. By a submultiplicative seminorm, we mean a seminorm p such that $p(xy) \leq p(x)p(y)$ for all $x, y \in A$. Observe that every Banach algebra is a locally m-convex algebra and that every locally m-convex algebra is a locally convex algebra. Let $A[\tau]$ be a locally m-convex algebra defined by a family of submultiplicative seminorms $\{p_\gamma : \gamma \in \Gamma\}$. For every $\gamma \in \Gamma$, let $N_\gamma = \{x \in A : p_\gamma(x) = 0\}$. Then A/N_γ is a normed algebra with respect to the seminorm $\dot{p}_\gamma$, where $\dot{p}_\gamma(x + N_\gamma) = p_\gamma(x)$ for all $x \in A$, and its completion with respect to the seminorm $\dot{p}_\gamma$ is denoted by A_γ. It can be shown that $A = \lim_{\leftarrow}(A/N_\gamma)$ up to topological isomorphism, where $\lim_{\leftarrow}(A/N_\gamma)$ denotes the inverse limit of the normed algebras A/N_γ, $\gamma \in \Gamma$. If, in addition, A is complete, then $A = \lim_{\leftarrow} A_\gamma$ up to topological isomorphism (we refer to [9, p. 15-16] for details). If A is complete, then $A = \lim_{\leftarrow} A_\gamma$ will be referred to as the *Arens-Michael decomposition* of A, and an element $a = (a_\gamma)_\gamma$ will refer to a typical element in the Arens-Michael decomposition of A.

A *Gelfand-Mazur algebra* is a topological algebra $A[\tau]$ such that A/M is topological isomorphic to $\mathbb{C}$ for all $M \in \mathcal{M}(A)$, where $\mathcal{M}(A)$ denotes the set of all closed two-sided ideals of A which are maximal as left or right ideals of A. If $\mathcal{M}(A) = \emptyset$, then $A[\tau]$ is trivially a Gelfand-Mazur algebra. For any topological algebra $A[\tau]$, we let $X(A)$ denote the set of all continuous characters of A, i.e., the set of all continuous nonzero multiplicative linear functionals of A. If $A[\tau]$ is a Gelfand-Mazur algebra with $\mathcal{M}(A) \neq \emptyset$, then $M \in \mathcal{M}(A)$ if and only if M is the kernel of some $\phi \in X(A)$. The reader is referred to [1] for a more thorough investigation of Gelfand-Mazur algebras.

The *spectrum* of an element x in an algebra A is the set

$$\mathrm{Sp}_A(x) = \{\lambda \in \mathbb{C} : x - \lambda 1 \notin G_A\}.$$

Also, we let $\rho_A(x) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}_A(x)\}$, the *spectral radius* of x.

More generally, if $x_1, \ldots, x_n \in A$, then the *joint spectrum* $\mathrm{Sp}_A(x_1, \ldots, x_n)$ of $(x_1, \ldots, x_n) \in A^n$ is the complement in $\mathbb{C}^n$ of the following set:

$$\Big\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \, : \, \exists\, b_1, \ldots, b_n \in A \text{ such that } \sum\nolimits_{k=1}^n b_k(\lambda_k 1 - x_k) = 1\Big\}.$$

For an open subset U of $\mathbb{C}^n$, we let $\mathrm{Hol}(U)$ denote the set of all analytic functions on U. If Δ is a compact subset of $\mathbb{C}^n$, we define $\mathrm{Hol}(\Delta)$ as the direct limit of $\mathrm{Hol}(U)$ for all open subsets U of $\mathbb{C}^n$ with $\Delta \subset U$.

Let $b_1, \ldots, b_n \in A$. An open subset V of $\mathbb{C}^n$ is said to be an *elementary resolvent set* for $(b_1, \ldots, b_n) \in A^n$ if there exist $q_1, q_2, \ldots, q_n \in \mathrm{Hol}(V)$ such that $\sum_{k=1}^n q_k(\lambda)(\lambda_k 1 - b_k) = 1$ for all $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in V$. The *resolvent set* of $(b_1, \ldots, b_n) \in A^n$ is the union of all elementary resolvent sets for $(b_1, \ldots, b_n) \in A^n$ and is an open subset of $\mathbb{C}^n$. The complement $\tau(b_1, \ldots, b_n, A)$ in $\mathbb{C}^n$ of the resolvent set of $(b_1, \ldots, b_n) \in A^n$ is called the *analytic joint spectrum* of $(b_1, \ldots, b_n) \in A^n$ and is a closed subset of $\mathbb{C}^n$. We refer to [4] for further details.

The proof of the following proposition is the same as that of [6, Proposition 19.8].

Proposition 2.1. *Let $A[\tau]$ be a commutative Gelfand-Mazur algebra with $X(A) \neq \emptyset$. Let $b_1, \ldots, b_n \in A$. Then*

$$\mathrm{Sp}_A(b_1, \ldots, b_n, A) = \{\phi(b_1), \ldots, \phi(b_n) : \phi \in X(A)\}.$$

3. The Arens-Calderon theorem for commutative Banach algebras

The Arens-Calderon Theorem for commutative Banach algebras is the following result.

Theorem 3.1. ([11, Lemma 3.2.8]) *Let A be a commutative Banach algebra with identity, and let $a_0, a_1, \ldots, a_n \in A$ with $a_0 \in \mathrm{Rad}(A)$ and a_1 invertible in A. Then there exists $y \in \mathrm{Rad}(A)$ such that $a_0 + a_1 y + \cdots + a_n y^n = 0$.*

Theorem 3.1 was originally proved in [5, Theorem 7.3], which is a more general result. The proofs of Theorem 3.1 in [5] and [11] involve complex analysis of several variables. It would be interesting to know if one can extend Theorem 3.1 to commutative Fréchet locally convex algebras with identity. We therefore give a detailed proof of Theorem 3.1, as given in [11], as this serves as the foundation on which to extend the result. Observe that the result is trivial if the algebra A is semi-simple (for then $a_0 = 0$, and therefore $y = 0 \in \mathrm{Rad}(A)$ is a solution). Without loss of generality, we may assume that $a_1 = 1$.

Proof. (*of Theorem* 3.1) We give the proof as in [11, Lemma 3.2.8]. Without loss of generality, we may assume that $a_1 = 1$. Consider the equation $z_0 + w + z_2 w^2 + \cdots + z_n w^n = 0$, where $z_0, z_2, \ldots, z_n, w$ are complex variables. By the implicit function theorem [11, Theorem 2.1.2], there is a unique analytic solution to the previous equation, say w, where w is analytic in a neighbourhood of 0 in $\mathbb{C}^n$ and $w = 0$ when $z_0 = z_2 = \cdots = z_n = 0$. The function w can be written as $w(z) = \sum_\alpha c_\alpha z^\alpha$, where $z = (z_0, z_2, \ldots, z_n)$, $\alpha = (\alpha_0, \alpha_2, \ldots, \alpha_n)$. Also, there exists $r > 0$ such that $\sum_\alpha |c_\alpha| r^{|\alpha|} < \infty$, where $|\alpha| = \alpha_0 + \alpha_2 + \cdots + \alpha_n$.

Let $w(a) = \Sigma_{\alpha \neq 0} c_\alpha a^\alpha$, where $a = (a_0, a_2, \ldots, a_n)$. We show that $w(a)$ is absolutely convergent. Choose $R > 1$ such that $\|a_j\| < R^{j-1}$ for all $2 \leq j \leq n$. If $\alpha = (\alpha_0, \alpha_2, \ldots, \alpha_n)$ with $L(\alpha) > 0$, then

$$\begin{aligned}\|a^\alpha\| &= \|a_0^{\alpha_0} a_2^{\alpha_2} \cdots a_n^{\alpha_n}\| \leq \|a_0^{\alpha_0}\| \cdot \|a_2^{\alpha_2}\| \cdots \|a_n^{\alpha_n}\| \\ &\leq \|a_0^{\alpha_0}\| \cdot \|a_2\|^{\alpha_2} \cdots \|a_n\|^{\alpha_n} < \|a_0^{\alpha_0}\| (R^{2-1})^{\alpha_2} \cdots (R^{n-1})^{\alpha_n} \\ &= \|a_0^{\alpha_0}\| R^{\alpha_2 + \cdots + (n-1)\alpha_n} < \|a_0^{\alpha_0}\| R^{\alpha_2 + \cdots + (n-1)\alpha_n + L(\alpha)} = \|a_0^{\alpha_0}\| R^{\alpha_0}.\end{aligned}$$

Since $a_0 \in \mathrm{Rad}(A)$, we get that $\rho_A(a_0) = 0$, and therefore $\lim_{\alpha_0 \to \infty} \|a_0^{\alpha_0}\|^{\frac{1}{\alpha_0}} = 0$. Without loss of generality, we may assume that $r < 1$. Since $\|a_0^{\alpha_0}\|^{\frac{1}{\alpha_0}} \to 0$ for α_0 large enough, we have that $\|a_0^{\alpha_0}\|^{\frac{1}{\alpha_0}} \leq \frac{r^2}{R}$. Hence $\|a_0^{\alpha_0}\| \cdot R^{\alpha_0} \leq r^{2\alpha_0}$. Since $r < 1$, we have that $r^{\alpha_0} \leq r^{\alpha_2 + \cdots + \alpha_n}$ for α_0 large enough. We therefore get that $r^{2\alpha_0} \leq r^{|\alpha|}$. It follows that $\|a^\alpha\| < \|a_0^{\alpha_0}\| \cdot R^{\alpha_0} \leq r^{|\alpha|}$. Therefore $w(a)$ is absolutely convergent.

Since all terms in the series are in $\mathrm{Rad}(A)$, and since $\mathrm{Rad}(A)$ is closed (since all maximal ideals are closed), it follows that $w(a) \in \mathrm{Rad}(A)$. By what we have in the first paragraph of the proof, and by the fact that A is commutative, it follows that $y = w(a)$ is a solution to $a_0 + a_1 y + \cdots + a_n y^n = 0$. ∎

4. The locally m-convex case

The main result of this section is Theorem 4.2 below. To prove this result, we require the analytic functional calculus for complete locally m-convex barrelled topological algebras, which is [8, Lemma 0.2]. The proof of the latter result in [8] has some detail missing, and in [16, Theorem 3.13] we gave a proof of this result in the one variable case for which all Banach algebras appearing in the Arens-Michael decomposition of the algebra are semi-simple. We made the assumption of semi-simplicity on all of these Banach algebras in order to circumvent the usage of [8, Lemma 0.1]. We therefore begin this section by giving a detailed proof of [8, Lemma 0.1], which therefore completes the proof in [8] of the analytic functional calculus result that we require, namely, [8, Lemma 0.2].

Theorem 4.1. ([8, Lemma 0.1]) *Let A be a complete locally m-convex barrelled topological algebra. The following are equivalent:*

(i) *G_A is open.*

(ii) *$X(A)$ (the character space of A) is a compact set of A' (the dual space of A) in the weak topology.*

(iii) *The topology of A is determined by a family $(p_\gamma)_{\gamma \in \Gamma}$ of algebra semi-norms, such that for all $\gamma \in \Gamma$, $X(A)$ is homeomorphic to $X(A_\gamma)$ when these spaces are endowed with the weak topologies.*

Proof. (i) ⇒ (ii): Since A is unital and if we suppose that $X(A) \neq \emptyset$, then the implication follows by [9, Theorem 6.11].

(ii) $\Rightarrow$ (iii): Let $X(A)$ be compact with respect to the weak topology. We first show that $X(A)$ is bounded with respect to the topology of simple convergence, i.e., with respect to the weak topology. Towards this purpose, let U be an open subset of A' containing 0, say

$$U = \{\psi \in A' : |\psi(x_j)| < \varepsilon,\ \varepsilon > 0,\ x_1, x_2, \dots, x_n \in A\}.$$

Now for every $\phi \in X(A)$, we consider the open set

$$U(\phi) = \{\psi \in A' : |(\psi - \phi)(x_j)| < \delta,\ j = 1, 2, \dots, n\}, \qquad \text{for } \delta > 0.$$

Since $X(A) \subseteq \cup_{\phi \in X(A)} U(\phi)$ and given that $X(A)$ is weakly compact by hypothesis, we get that there are $\phi_1, \phi_2, \dots, \phi_m \in X(A)$ such that

$$X(A) \subseteq \cup_{i=1}^{m} U(\phi_i).$$

Let $0 < \omega < \varepsilon$. For every $i \in 1, 2, \dots, m$, there exists $\lambda_i > 0$ such that $|\phi_i(x_j)| < \lambda_i \cdot \omega$ for all $j = 1, 2, \dots, n$.

Consider $\lambda = \max\{\lambda_1, \lambda_2, \dots, \lambda_m, \frac{\delta}{\varepsilon - \omega}\}$.

Let $\phi \in X(A)$. Then $\phi \in U(\phi_i)$ for some $i \in \{1, 2, \dots, m\}$. Hence, for all $j = 1, 2, \dots, n$, we have that

$$|\phi(x_j)| \ \leq \ |(\phi - \phi_i)(x_j)| + |\phi_i(x_j)| \ < \ \delta + \lambda \cdot \omega \ \leq \ \lambda \cdot \varepsilon.$$

So $\frac{1}{\lambda}\phi \in U$ and thus $\phi \in \lambda U$. Therefore, $X(A) \subseteq \lambda U$, thus $X(A)$ is bounded with respect to the weak topology.

Then, since A is barrelled, by [13, Theorem 4.2] we have that $X(A)$ is equicontinuous. Hence, given a 0-neighborhood V in $\mathbb{C}$, say $V = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, there exists a neighborhood U of 0 in A, say $U = \{a \in A : p_{\gamma_0}(a) < \delta\}$, for some $\gamma_0 \in \Gamma$ and $\delta > 0$ such that $\phi(U) \subseteq V$ for all $\phi \in X(A)$. Equivalently, we have that

$$X(A) \subseteq \{a \in A : \ p_{\gamma_0}(a) < \delta\}^{\circ}.$$

Hence, if $a \in N_{\gamma_0}$, then for every $\phi \in X(A)$, we get that $|\phi(na)| < 1$, for all $n \in \mathbb{N}$ and hence $\phi(a) = 0$. So we conclude that $N_{\gamma_0} \subseteq \mathrm{Ker}(\phi)$.

Thus every $\phi \in X(A)$ induces a character on A/N_{γ_0}, say $\phi_{\gamma_0} : A/N_{\gamma_0} \to \mathbb{C}$, such that $\phi_{\gamma_0}(a + N_{\gamma_0}) := \phi(a)$, which is continuous with respect to $\dot{p}_{\gamma_0}$: $A \cong \leftarrow_{\gamma} A_{\gamma}$ up to topological $*$-isomorphism: If $a_\lambda + N_{\gamma_0} \to a + N_{\gamma_0}$ with respect to $\dot{p}_{\gamma_0}$, then $p_{\gamma_0}(a_\lambda - a) \to 0$. Therefore, if λ is large enough, it follows from the above that $a_\lambda - a \in U$, and hence $\phi(a_\lambda - a) \in V$. Therefore

$\phi_{\gamma_0}((a_\lambda + N_{\gamma_0}) - (a + N_{\gamma_0})) \in V$. Since V is, without loss of generality, an arbitrary neighbourhood of $0 \in \mathbb{C}$, we get that ϕ_{γ_0} is continuous. The map ϕ_{γ_0} is therefore a character on the Banach algebra A_γ and is continuous.

The same holds for every $\gamma \geq \gamma_0$, i.e., we get characters $\phi_\gamma : A/\ker p_\gamma \to \mathbb{C}$. Now for all $\gamma, \delta \geq \gamma_0$, it is easily seen that the maps $h'_{\delta\gamma} : A'_\gamma \to A'_\delta$ map $X(A_\gamma)$ bijectively onto $X(A_\delta)$. This bijection is also a homeomorphism, since both spaces involved are Hausdorff and compact. Since $X(A) \simeq \varinjlim X(A_\gamma)$, the result follows.

(iii) $\Rightarrow$ (i): By the assumption $X(A) \simeq X(A_\gamma)$ for all $\gamma \in \Gamma$, an element $x \in A$ is invertible in A if and only if $\phi(x) \neq 0$ for all $\phi \in X(A)$. Also, by $X(A) \simeq X(A_\gamma)$,

$$X(A) \subset \big\{\phi \in A' : |\phi(b)| < 1 \text{ for all } b \in V\big\},$$

where $V = \{b \in A : p_\gamma(b) < 1\}$, for some $\gamma \in \Gamma$.

Then $1 + \frac{1}{2}V \subseteq G_A$. Indeed if $\omega \in V$, then $|\phi(\omega)| < 1$ $(*)$, for all ϕ in $X(A)$. Therefore $\phi\big(1 + \frac{1}{2}\omega\big) = 1 + \frac{1}{2}\phi(\omega) \neq 0$ (for otherwise we would get that $\phi(\omega) = -2$, a contradiction to $(*)$). Therefore $1 + \frac{1}{2}\omega \in G_A$.

Now let $x \in G_A$. By continuity of multiplication, there is a 0-neighborhood in A, say Ω, such that $x^{-1}\Omega \subseteq \frac{1}{2}V$. Hence $\Omega \subseteq \frac{1}{2}xV$. So we have the following:

$$\begin{aligned} 1 + \frac{1}{2}V \subseteq G_A \quad &\Rightarrow \quad x^{-1}\Big(x + \frac{1}{2}xV\Big) \subseteq G_A \quad \Rightarrow \\ x + \frac{1}{2}xV \subseteq xG_A \subseteq G_A \quad &\Rightarrow \quad x + \Omega \subseteq G_A. \end{aligned}$$

Therefore for $x \in G_A$, there is a neighbourhood $x + \Omega$ of x in A such that $x + \Omega \subseteq G_A$, which shows that G_A is open. ∎

The proof of Theorem 4.2 below is based on the proof of Theorem 3.1. However, in the proof of Theorem 3.1, $w(a)$ is regarded as a series which is shown to be convergent. In the non-normed case, it is not immediately apparent as to why the series converges. For this reason, in the proof of Theorem 4.2 below, $w(a)$ is considered to be the functional calculus of $w(z)$.

THEOREM 4.2. *Let $A[\tau]$ be a commutative Fréchet locally m-convex algebra with weakly compact character space. Let $a_0 \in \mathrm{Rad}(A)$, a_1 invertible and $a_2, \ldots, a_n \in A$. Then there exists $y \in \mathrm{Rad}(A)$ such that*

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

Proof. We first note that $A[\tau]$ is barrelled, as it is a Fréchet locally convex algebra. Without loss generality, $a_1 = 1$. Consider the equation $z_0+w+z_2w^2+\cdots+z_nw^n = 0$, where $z_0, z_2, \ldots, z_n, w$ are complex variables. By the implicit function theorem [11, Theorem 2.1.2], there is a unique analytic solution to the previous equation, say w, where w is analytic in a neighbourhood of 0 in $\mathbb{C}^n$ and $w = 0$ when $z_0 = z_2 = \cdots = z_n = 0$. The function w can be written as $w(z) = \sum_\alpha c_\alpha z^\alpha$, where $z = (z_0, z_2, \ldots, z_n)$, $\alpha = (\alpha_0, \alpha_2, \ldots, \alpha_n)$. Also, there exists $r > 0$ such that $\sum_\alpha |c_\alpha| r^{|\alpha|} < \infty$, where $|\alpha| = \alpha_0 + \alpha_2 + \cdots + \alpha_n$.

Now consider the equation

$$a_0 + a_1 y + \cdots + a_n y^n = 0. \tag{4.1}$$

Let $w(a)$ be the analytic functional calculus of $w(z)$. Then $w(a)$ is a solution to the above equation, where $a = (a_0, a_2, \ldots, a_n)$.

We now show that $w(a) \in \mathrm{Rad}(A)$. Note that $w(a)$ is of the form $\theta : \mathrm{Hol}(K) \to A$, where $K = \mathrm{Sp}_A(a_0, a_2, \ldots, a_n)$ and $\theta(w) = w(a)$ and $\theta(z_i) = a_i$ for all $0 \leq i \leq n$ (by [8, Lemma 0.2]).

We have to show that $w(a)$ is well defined, i.e., we have to show that the domain of the analytic function $z \in \mathbb{C}^n \mapsto w(z)$ contains K. Let $x \in A$. Since $X(A)$ is weakly compact, there exists $M_x > 0$ such that $|\hat{x}(\phi)| \leq M_x$ for all $\phi \in X(A)$. Let

$$M = \left(M_{a_0}^2 + M_{a_2}^2 + \cdots + M_{a_n}^2\right)^{\frac{1}{2}}.$$

Then

$$\left(|\phi(a_0)|^2 + |\phi(a_2)|^2 + \cdots + |\phi(a_n)|^2\right)^{\frac{1}{2}} \leq M$$

for all $\phi \in X(A)$. By Proposition 2.1 (in particular, for any complete commutative locally m-convex algebra),

$$K = \left\{(\phi(a_0), \phi(a_2), \ldots, \phi(a_n)) \,:\, \phi \in X(A)\right\}.$$

There exists $k > 0$ such that $\frac{M}{k} < r$. Now

$$\frac{1}{k^2}\left(|\phi(a_0)|^2 + |\phi(a_2)|^2 + \cdots + |\phi(a_n)|^2\right) \leq \frac{M^2}{k^2} < r^2.$$

Therefore

$$\left|\phi\Big(\frac{1}{k}a_0\Big)\right|^2 + \left|\phi\Big(\frac{1}{k}a_2\Big)\right|^2 + \cdots + \left|\phi\Big(\frac{1}{k}a_n\Big)\right|^2 < r^2,$$

and hence

$$\left(\left|\phi\Big(\frac{1}{k}a_0\Big)\right|^2 + \left|\phi\Big(\frac{1}{k}a_2\Big)\right|^2 + \cdots + \left|\phi\Big(\frac{1}{k}a_n\Big)\right|^2\right)^{\frac{1}{2}} < r,$$

for all $\phi \in X(A)$. One can therefore change the original equation to

$$\frac{1}{k}z_0 + \frac{1}{k}w + \frac{1}{k}z_2w^2 + \cdots + \frac{1}{k}z_nw^n = 0,$$

as this equation has precisely the same solutions as the original equation. Therefore $z \in \mathbb{C}^n \mapsto w(z)$ is the unique analytic solution to the last displayed (second) equation referred to above. Observe that

$$\mathrm{Sp}_A\left(\frac{1}{k}a_0, \frac{1}{k}a_2, \ldots, \frac{1}{k}a_n\right) = \left\{\phi\Big(\frac{1}{k}a_0\Big), \phi\Big(\frac{1}{k}a_2\Big), \ldots, \phi\Big(\frac{1}{k}a_n\Big) \,:\, \phi \in X(A)\right\}.$$

Therefore

$$\mathrm{Sp}_A\left(\frac{1}{k}a_0, \frac{1}{k}a_2, \ldots, \frac{1}{k}a_n\right)$$

is inside $B(0,r)$, and therefore $\mathrm{Sp}_A\left(\frac{1}{k}a_0, \frac{1}{k}a_2, \ldots, \frac{1}{k}a_n\right)$ is inside an open set on which w is analytic. We therefore obtain that $w\big(\frac{1}{k}a_0, \frac{1}{k}a_2, \ldots, \frac{1}{k}a_n\big)$ is the solution to the second polynomial equation, and therefore the original equation (4.1). We may therefore assume, without loss of generality, that the domain of the analytic function $z \in \mathbb{C}^n \mapsto w(z)$ contains K.

By the proof of [8, Lemma 0.2] in [8],

$$K = \mathrm{Sp}_A(a_0, a_2, \ldots, a_n) = \mathrm{Sp}_{A_\gamma}\left((a_0)_\gamma, (a_2)_\gamma, \ldots, (a_n)_\gamma\right)$$

for all γ, where $A = \lim_{\leftarrow} A_\gamma$ is the Arens-Michael decomposition of A. Observe that, here, we use the fact that the character space of A is weakly compact (see Theorem 4.1). Then we have the Banach algebra analytic functional calculus $w_\gamma : \mathrm{Hol}(K) \to A_\gamma$ of $w(z)$. For every γ, we have that $w_\gamma(z_i) = (a_i)_\gamma$ and $w(a) = \lim_{\leftarrow} w_\gamma(a)$ (by the proof of [8, Lemma 0.2]). Now

$$\left(a_0 + y + a_2y^2 + \cdots + a_ny^n\right)_\gamma = (a_0)_\gamma + y_\gamma + (a_2)_\gamma y_\gamma^2 + \cdots + (a_n)_\gamma y_\gamma^n = 0,$$

where $y = (y_\gamma)_\gamma \in \lim_{\leftarrow} A_\gamma = A$. For every γ, note that $(a_1)_\gamma$ is invertible in A_γ.

Observe that also for every γ, $(a_0)_\gamma \in \mathrm{Rad}(A_\gamma)$: By [17, Proposition 11.2 and Corollary 11.6], we get that $\mathrm{Sp}_{A_\gamma}((a_0)_\gamma) \subseteq \mathrm{Sp}_A(a_0) = \{0\}$. So $(a_0)_\gamma \in \mathrm{Rad}(A_\gamma)$, by [17, Corollary 11.6].

Therefore, by the Arens-Calderon theorem for Banach algebras (Theorem 3.1) , $w_\gamma(a_\gamma) \in \mathrm{Rad}(A_\gamma)$ (by the proof of Theorem 3.1 above, $w_\gamma(a_\gamma)$ is a series and all corresponding series terms are in $\mathrm{Rad}(A_\gamma)$. Hence $w_\gamma(a_\gamma) \in \mathrm{Rad}(A_\gamma)$, as $\mathrm{Rad}(A_\gamma)$ is closed by [17, Proposition 10.14]). Hence $\rho_\gamma(w_\gamma(a_\gamma)) = 0$ for all γ, where $\rho_\gamma(w_\gamma(a_\gamma))$ denotes the spectral radius of $w_\gamma(a_\gamma)$ for all $\gamma \in \Gamma$. Therefore $\rho_A(w(a)) = 0$, and hence $w(a) \in \mathrm{Rad}(A)$. ∎

The following corollary follows immediately from the above result and the remarks following [16, Theorem 3.10]. It is also a straight consequence of [3, Theorem 7].

Corollary 4.3. *Let $A[\tau]$ be a commutative Fréchet locally m-convex algebra with identity and having weakly compact character space. Let $x \in \mathrm{Rad}(A)$ be such that $\{0\} \neq Ax^m \subseteq \overline{Ax^{m+1}}$ for some $m \geq 1$. Then there is a unital injective algebra homomorphism $\psi : \mathbb{C}[[X]] \to A$ such that $\psi(X) = x$.*

5. The general case: beyond the m-convex case

In [4, Theorem 4.6], and the middle of page 425 in [4], an analytic functional calculus is given for all topological algebras $A[\tau]$ of the following type: $A[\tau]$ is a unital commutative locally convex algebra such that every compact subset of A is contained in a convex compact subset of A.

In [4, p. 406], it is noted, without proof or reference, that every Banach algebra is an algebra of this type. We therefore give a proof of this fact, for sake of completeness, in the next observation, as it appears not be very well known.

Proposition 5.1. *Every compact subset of a Banach space X is contained in a compact convex subset of X.*

Proof. Let K be a compact subset of X. By [12, Proposition 1.e.2], there is a sequence (x_n) in X such that $x_n \to 0$ with respect to the norm topology on X, and $K \subseteq \overline{\mathrm{conv}}(x_n)$, the closed convex hull of (x_n). We show that $\overline{\mathrm{conv}}(x_n)$ is a compact subset of X.

First consider a sequence (u_n) in $\mathrm{conv}(x_n)$, the convex hull of (x_n). Then $u_n = \sum_{k=1}^{m_n} \lambda_{k,n} y_{k,n}$, where $\lambda_{k,n} \in \mathbb{C}$, $0 \leq \lambda_{k,n} \leq 1$, $\sum_{k=1}^{m_n} \lambda_{k,n} = 1$ and $(y_{k,n})$ is a subsequence of (x_n). We show that it has a convergent subsequence converging to an element in $\overline{\mathrm{conv}}(x_n)$.

Observe that $0 \leq \|u_n\| \leq \sum_{k=1}^{m_n} \lambda_{k,n} \|y_{k,n}\|$. Let $\epsilon > 0$. Note that $y_{k,n} \to 0$ as $n \to \infty$, for all $k \in \mathbb{N}$. Therefore, for all $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that $\|y_{k,N_k}\| < \epsilon$. Therefore

$$0 \leq \|u_{N_k}\| \leq \sum_{k=1}^{m_{N_k}} \lambda_{k,N_k} \epsilon \leq \epsilon.$$

This yields a subsequence (u_{N_k}) of (u_n) with $u_{N_k} \to 0$. Now $0 \in \overline{\mathrm{conv}}(x_n)$, as

$\overline{\mathrm{conv}}(x_n)$ is closed, (u_{N_k}) is in $\overline{\mathrm{conv}}(x_n)$ and $u_{N_k} \to 0$. Therefore (u_n) has a subsequence which converges to an element in $\overline{\mathrm{conv}}(x_n)$.

Now let (v_m) be a sequence in $\overline{\mathrm{conv}}(x_n)$. Then there is a subsequence $(v_{j,m})$ of (v_m) such that $v_{j,m} \to v_m$ as $j \to \infty$ for all $m \in \mathbb{N}$. From what is proved above, it follows that there is a subsequence (v_{j_k,m_k}) of $(v_{j,n})$ such that $(v_{j_k,m_k}) \to 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\|v_{j_k,m_k}\| < \epsilon$ for all $j_k, m_k \geq N$. So $\|v_{m_k}\| = \lim_{j_k} \|v_{j_k,m_k}\| \leq \epsilon$ if $m_k \geq N$. So $v_{m_k} \to 0 \in \overline{\mathrm{conv}}(x_n)$. Therefore X is compact. ∎

THEOREM 5.2. *Let $A[\tau]$ be a unital commutative locally convex algebra such that every compact subset of A is contained in a convex compact subset of A. Assume also that the following conditions are satisfied.*

(i) *For each $b_1, \ldots, b_n \in A$, $(\lambda 1 - b_i)^{-1} \to 0$ uniformly as $|\lambda| \to \infty$.*

(ii) *For each $b_1, \ldots, b_n \in A$, $\tau(b_1, \ldots, b_n, A) \subseteq \Delta$ for some compact subset Δ of $\mathbb{C}^n$. Here, $\tau(b_1, \ldots, b_n, A)$ denotes the topological joint spectrum of $b_1, \ldots, b_n$ in A, as defined in the paragraph just before Proposition 2.1.*

(iii) $\mathrm{Rad}(A)$ *is closed.*

Let $a_0 \in \mathrm{Rad}(A)$, a_1 invertible and $a_2, \ldots, a_n \in A$. Then there exists $y \in \mathrm{Rad}(A)$ such that $a_0 + a_1 y + \cdots + a_n y^n = 0$.

Proof. We follow the same argument and notation as in the proof of Theorem 3.1. Let $w(z) = \sum_{\alpha \neq 0} c_\alpha z^\alpha$. There exists $r > 0$ such that $\sum_{\alpha \neq 0} |c_\alpha| r^{|\alpha|} < \infty$. We start off by showing, without loss of generality, that $z \in \mathbb{C}^n \mapsto w(z)$ is analytic on a neighbourhood of $\tau(a_0, a_2, \ldots, a_n, A)$. By definition of analytic joint spectrum, it follows easily that

$$\tau\Big(\frac{1}{k}a_0, \frac{1}{k}a_2, \ldots, \frac{1}{k}a_n, A\Big) = \frac{1}{k}\tau(a_0, a_2, \ldots, a_n, A)$$

for all $k > 0$. By (ii), $\tau(a_0, a_2, \ldots, a_n, A)$ is a bounded subset of $\mathbb{C}^n$, and therefore, there exists $M > 0$ such that $\|z\| \leq M$ for all $z \in \tau(a_0, a_2, \ldots, a_n, A)$. Let $k_1 > 0$ such that $\frac{M}{k_1} < r$. Then

$$\frac{1}{k_1} z \in \tau\Big(\frac{1}{k_1}a_0, \frac{1}{k_1}a_2, \ldots, \frac{1}{k_1}a_n, A\Big)$$

if $z \in \tau(a_1, a_2, \ldots, a_n, A)$. Therefore

$$\tau\Big(\frac{1}{k_1}a_0, \frac{1}{k_1}a_2, \ldots, \frac{1}{k_1}a_n, A\Big) \subseteq B(0, r).$$

Take Δ to be $\tau\big(\frac{1}{k_1}a_0, \frac{1}{k_1}a_2, \ldots, \frac{1}{k_1}a_n, A\big)$. Then Δ is a compact subset of $\mathbb{C}^n$, as it is bounded and closed in $\mathbb{C}^n$ by definition of analytic joint spectrum. By changing to the equation

$$\frac{1}{k_1}z_0 + \frac{1}{k_1}w + \frac{1}{k_1}z_2w^2 + \cdots + \frac{1}{k_1}z_nw^n = 0,$$

we can reason as in the proof of Theorem 4.2 to obtain, without loss of generality, that $z \in \mathbb{C}^n \mapsto w(z)$ is analytic on a neighbourhood of $\tau(a_0, a_2, \ldots, a_n, A)$.

It now follows that we may define the analytic functional calculus $w(a) = \sum_{\alpha \neq 0} c_\alpha a^\alpha$ of a, as defined in [4, Theorem 4.6], where $a = (a_0, a_2, \ldots, a_n)$: $w(a) = J_\Delta(w)$ as in [4, Theorem 4.6]. Furthermore, by [4, Theorem 4.6], J_Δ is a continuous homomorphism. Therefore the above series for $w(a)$ converges, i.e., $w(a)$ is well defined.

Now $a^\alpha \in \mathrm{Rad}(A)$ by the same reason as that given in the proof of Theorem 3.1 above. Therefore, all terms in the series for $w(a)$ above belong to $\mathrm{Rad}(A)$. Consequently, since $\mathrm{Rad}(A)$ is closed, the series converges to an element in $\mathrm{Rad}(A)$, i.e., $w(a) \in \mathrm{Rad}(A)$. Now $y = w(a) \in \mathrm{Rad}(A)$ is the solution to $a_0 + a_1y + \cdots + a_ny^n = 0$. ∎

The following corollary follows immediately from the above result and the remarks following [16, Theorem 3.10].

Corollary 5.3. *Let $A[\tau]$ be a unital commutative locally convex algebra such that every compact subset of A is contained in a convex compact subset of A. Assume also that the following conditions are satisfied.*

(i) *For each $b_1, \ldots, b_n \in A$, $(\lambda 1 - b_i)^{-1} \to 0$ uniformly as $|\lambda| \to \infty$.*

(ii) *For each $b_1, \ldots, b_n \in A$, $\tau(b_1, \ldots, b_n, A) \subseteq \Delta$ for some compact subset Δ of $\mathbb{C}^n$.*

(iii) $\mathrm{Rad}(A)$ *is closed.*

Let $x \in \mathrm{Rad}(A)$ be such that $\{0\} \neq Ax^m \subseteq \overline{Ax^{m+1}}$ for some $m \geq 1$. Then there is a unital injective algebra homomorphism $\psi : \mathbb{C}[[X]] \to A$ such that $\psi(X) = x$.

In light of the above observations, it would be interesting to know when $\mathrm{Rad}(A)$ is closed. Below, we give an answer to this question, and the proof is similar to that of [17, Proposition 10.14].

Proposition 5.4. *If* $A[\tau]$ *is a unital commutative Gelfand-Mazur algebra with* $X(A) \neq \emptyset$ *and*

$$G_A = \{x \in A : \phi(x) \neq 0 \text{ for all } \phi \in X(A)\},$$

then $\mathrm{Rad}(A)$ *is closed.*

Proof. $\mathrm{Rad}(A) \subseteq B := \cap\{M : M \text{ a closed maximal ideal of } A\}$. Now let $x \in B$. Then $x \in \mathrm{Ker}(\phi)$ for all $\phi \in X(A)$, as $A[\tau]$ is a Gelfand-Mazur algebra. So $\phi(x) = 0$ for all $\phi \in X$. Hence $\phi(1 + xy) = \phi(1) + \phi(x)\phi(y) = 1$ for all $\phi \in X(A)$ for all $y \in A$. Therefore $\phi(1+xy) \neq 0$ for all $\phi \in X(A)$ and for all $y \in A$. Therefore, by hypothesis, $1 + xy \in G_A$ for all $y \in A$, and hence $x \in \mathrm{Rad}(A)$. So $B \subseteq \mathrm{Rad}(A)$ and hence $\mathrm{Rad}(A) = \cap\{M : M \text{ a closed maximal ideal of } A\}$. Therefore $\mathrm{Rad}(A)$ is closed. ∎

For the next corollary, we recall that every Fréchet locally convex algebra is a Gelfand-Mazur algebra [1].

Corollary 5.5. *If* $A[\tau]$ *is a unital commutative Fréchet locally convex algebra with* $X(A) \neq \emptyset$ *and*

$$G_A = \{x \in A : \phi(x) \neq 0 \text{ for all } \phi \in X(A)\},$$

then $\mathrm{Rad}(A)$ *is closed.*

Remark 5.6. The assumption in Theorem 4.2 that the unital commutative Frechet locally m-convex algebra $A[\tau]$ have weakly compact character space leads automatically to the fact that $K = \mathrm{Sp}_A(a_1, \ldots, a_n)$ is compact, and therefore bounded. This is not always true if $A[\tau]$ is just an m-convex algebra with character space not necessarily weakly compact. The reason is that then $A[\tau]$ is not necessarily a Q-algebra, by [8, Lemma 0.1], so that K need not be bounded, and hence not compact. In [4, Theorem 5.3], the assumption that $A[\tau]$ have weakly compact character space is removed, with no other assumptions added to the hypothesis.

The following result generalizes Theorem 4.2, in that we drop the assumption of a weakly compact character space, assuming that every compact subset is contained in a compact convex subset of the algebra.

Theorem 5.7. *Let* $A[\tau]$ *be a commutative Fréchet locally* m*-convex algebra which satisfies conditions* (i) *and* (ii) *in Theorem* 5.2, *and such that* *every compact subset of A is contained in a compact convex subset of A. Let $a_0 \in \mathrm{Rad}(A)$, a_1 invertible and $a_2, \dots, a_n \in A$. Then there exists $y \in \mathrm{Rad}(A)$ such that*

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

Proof. By [17, Proposition 10.14], $\mathrm{Rad}(A)$ is closed. We now follow the same thread and notation as in the proof of Theorem 4.2. Let Δ be a compact subset of $\mathbb{C}^n$. By the proof of Theorem 5.2, we may assume, without loss of generality, that the analytic function given by $w(z) = \Sigma_\alpha c_\alpha z^\alpha$ (as in the proof of Theorem 4.2) is analytic on a neighbourhood U containing a compact subset Δ of $\mathbb{C}^n$, and that Δ contains $\tau(a_0, a_2, \dots, a_n, A)$. We observe that $w(a)$ can be taken to be the image of $w(z) = \Sigma_\alpha c_\alpha z^\alpha$ via the analytic functional calculus given in [4, Theorem 5.3], namely,

$$J_\Delta \colon \mathrm{Hol}(U, A) \to A, \quad J_\Delta(w) = w(a),$$

where $\mathrm{Hol}(U, A)$ denotes the set of all A-valued analytic functions on an open set U in $\mathbb{C}^n$. Note that $z \mapsto w(z)$ is in $\mathrm{Hol}(U)$ for some open subset U of $\mathbb{C}^n$, and can be considered to be in $\mathrm{Hol}(U, A)$: note that $\mathbb{C}$ is naturally embedded into A by observing that $\mathbb{C} \cdot 1 \subseteq A$. In this regard, one also needs the fact that there is a family of seminorms defining the topology τ of A, all of which map the identity element of A to that of $\mathbb{C}$. Since $A[\tau]$ is Fréchet, the existence of such a family of seminorms can be assumed. The rest of the proof remains the same as that of Theorem 4.2. ∎

Corollary 5.8. *Let $A[\tau]$ be a commutative Fréchet locally m-convex algebra with identity, which satisfies conditions* (i) *and* (ii) *in Theorem* 5.2. *Let $x \in \mathrm{Rad}(A)$ be such that $\{0\} \neq Ax^m \subseteq \overline{Ax^{m+1}}$ for some $m \geq 1$. Then there is a unital injective algebra homomorphism $\psi : \mathbb{C}[[X]] \to A$ such that $\psi(X) = x$.*

Theorem 5.9. ([7, Theorem 4.8]) *Let $A[\tau]$ be a Mackey-complete commutative Q-algebra with continuous inversion. For every open subset U of $\mathbb{C}^n$, let*

$$A_U = \big\{(x_1, x_2, \dots, x_n) \in A^n \,:\, \mathrm{Sp}_A(x_1, x_2, \dots, x_n) \subseteq U\big\}.$$

There is a unique family of maps $\theta_{A,U} \colon \mathrm{Hol}(U) \times A_U \to A$, where U is any open subset of $\mathbb{C}^n$, such that for every $a \in A_U$, $\theta_{A,U} \colon f \mapsto \theta_{A,U}(f, a)$ is a continuous unital algebra homomorphism which maps the j^{th}-coordinate function $\zeta \in \mathbb{C}^n \mapsto \zeta_j$ to x_j.

The above theorem generalizes the analytic functional calculus for Fréchet commutative locally m-convex algebras with weakly compact character space, as in [8, Lemma 0.2]: we recall here that weakly compact character space implies that $A[\tau]$ is a Q-algebra, by Theorem 4.1, and all complete m-convex algebras have continuous inversion. Also, every Fréchet locally convex algebra is Mackey-complete (see, for instance, [10, p. 13]).

Furthermore, in Q-algebras with continuous inversion, one has always has that the joint spectrum of a finite number of elements in the algebra is compact [7, Section 2].

Corollary 5.10. *Let $A[\tau]$ be a Mackey-complete commutative Q-algebra with continuous inversion, which is also a Gelfand-Mazur algebra with nonempty character space. Let $a_0 \in \mathrm{Rad}(A)$, a_1 invertible and $a_2, \dots, a_n \in A$. Then there exists $y \in \mathrm{Rad}(A)$ such that $a_0 + a_1 y + \cdots + a_n y^n = 0$.*

Proof. Since $A[\tau]$ is a Q-algebra, all maximal ideals are closed, and therefore $\mathrm{Rad}(A)$ is closed. By the same argument given in the proof of Theorem 4.2, we get that $z \in \mathbb{C}^n \mapsto w(z)$, as in the proof of Theorem 4.2, is analytic on a neighbourhood of the joint spectrum $\mathrm{Sp}_A(a_0, a_2, \dots, a_n)$ of $a = (a_0, a_2, \dots, a_n)$. Here, we need the facts that $A[\tau]$ is a Gelfand-Mazur algebra with nonempty character space, in order to be able to apply Proposition 2.1 to get that

$$\mathrm{Sp}_A(a_0, a_2, \dots, a_n) = \big\{\phi(a_0), \phi(a_2), \dots, \phi(a_n) \,:\, \phi \in X(A)\big\}.$$

The result now follows from Theorem 5.9 (instead of [4, Theorem 4.6]) and the rest of the proof of Theorem 5.2. ∎

Corollary 5.11. *Let $A[\tau]$ be a unital Mackey-complete commutative Q-algebra with continuous inversion, which is also a Gelfand-Mazur algebra with nonempty character space. Let $x \in \mathrm{Rad}(A)$ be such that $\{0\} \neq Ax^m \subseteq \overline{Ax^{m+1}}$ for some $m \geq 1$. Then there is a unital injective algebra homomorphism $\psi : \mathbb{C}[[X]] \to A$ such that $\psi(X) = x$.*

The class of mb-algebras, as defined in [15, p. 531-532], is a commutative locally convex algebra $A[\tau]$ which is a directed union of Banach algebras $\{A_{\alpha:\alpha\in\Lambda}\}$, such that every bounded subset of A is a bounded subset of some A_α (see the proposition on p. 532 in [15]). For every mb-algebra $A[\tau]$, and every $x_1, \dots, x_n \in A$, there exists a bounded unital algebra homomorphism $\theta\colon \mathrm{Hol}(\mathrm{Sp}_A(x_1, \dots, x_n)) \to A$ which maps the coordinate function z_i to x_i for all $1 \leq i \leq n$ (by the proposition on p. 532 in [15]). The above homomorphism is only known to be bounded and not necessarily continuous. It is therefore not so immediate if one can obtain similar results of an Arens-Calderon nature for *mb*-algebras.

An analytic functional calculus for commutative *b*-algebras has been obtained by L. Waelbroeck, which is [14, Chapter VI, Proposition 4]. A commutative *b*-algebra is a commutative locally algebra $A[\tau]$ of which every element in the topological boundedness is a completant subset of A. Here, and from here on, we refer to [14, Chapter 2] for all unexplained concepts appearing in this definition. Furthermore, the multiplication of a commutative *b*-algebra is bounded and a product of bounded sets is bounded (see [14, Chapter II, Proposition 4] and its proof for details). To be more specific, the analytic functional calculus in the setting of a commutative *b*-algebra is the following result [14, Chapter VI, Proposition 4]: If $A[\tau]$ is a unital commutative *b*-algebra, then there exists a unital algebra homomorphism $\mathrm{Hol}(\mathrm{Sp}_r(a_1, \ldots, a_n)) \to A$ which maps the projection maps z_i of $(z_1, \ldots, z_n)$ to a_i, where all a_i are regular elements of A, i.e., $(\lambda 1 - a_i)^{-1} \to 0$ as $|\lambda| \to \infty$, for all $1 \leq i \leq n$. Here, $\mathrm{Sp}_r(a_1, \ldots, a_n)$ denotes the regular joint spectrum of $(a_1, \ldots, a_n)$. It follows from the proof of [14, Chapter VI, Proposition 4] that the above algebra homomorphism is bounded, but it appears not necessarily to be continuous. It is again not so immediate to obtain similar results of an Arens-Calderon nature for commutative *b*-algebras.

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