EXTRACTA MATHEMATICAE
Vol. **38**, Num. 1 (2023), 85 – 104

doi:10.17398/2605-5686.38.1.85
Available online May 4, 2023

Estimating the number of limit cycles for one step perturbed homogeneous degenerate centers

M. MolaeiDerakhtenjani, O. RabieiMotlagh,
H.M. MohammadiNejad[@]

Department of Applied Mathematics, University of Birjand, Birjand, Iran

m.molaei@birjand.ac.ir , orabieimotlagh@birjand.ac.ir , hmohammadin@birjand.ac.ir

Received January 15, 2023
Accepted April 3, 2023

Presented by R. Prohens

Abstract: We consider a homogeneous degenerate center of order $2m+1$ and perturb it by a homogeneous polynomial of order $2m$. We study the Lyapunov constants around the origin to estimate the number of limit cycles. To do it, we classify the parameters and study their effect on the number of limit cycles. Finally, we find that the perturbed degenerate center without any condition has at least two limit cycles, and the number of the bifurcated limit cycles could reach $2m+3$.

Key words: Degenerate Center, Limit cycle, Lyapunov constant.

MSC (2020): 34C07, 34D10, 34D08.

1. Introduction

The 16th Hilbert problem is one of the 23 mathematical problems proposed by D. Hilbert in 1900 at the Second International Congress of Mathematical, cf. [10]. The second part of this problem is to find an upper bound for the number of limit cycles that bifurcates from planar polynomial ordinary differential systems. Since then, this problem has been studied by many authors, cf. [6, 11, 15, 16].

The weakened 16th Hilbert problem is a weaker version of this problem which was proposed by Arnold in 1977, cf. [1]. This problem is to find an upper bound for the number of bifurcated limit cycles from the period annulus of systems near Hamiltonian ones. D. Hilbert conjectured that his 16th problem could approach by perturbation techniques. Since then, some authors have been studying the number of limit cycles by perturbing the periodic orbits of a center. We remind that a fixed point of a system is called a center if it is surrounded by a neighborhood filled with periodic orbits. When the center is perturbed, the system may have limit cycles that bifurcate from some

[@] Corresponding author

ISSN: 0213-8743 (print), 2605-5686 (online)

of the periodic orbits of the center. Therefore, these studies could approach mathematicians to solve the 16th Hilbert problem. To this aim, some authors studied the perturbed center by applying the methods such as averaging theory, Melnikov function, or Poincaré map, cf. [4, 8, 14, 18].

There exist studies in which the authors used the Lyapunov constant to estimate the number of limit cycles, cf. [5, 9, 17]. Investigating the Taylor expansion of the corresponding Poincaré map is one way to compute the Lyapunov constant. In this case, the coefficients of Taylor expansion are the Lyapunov constants, cf. [13, 19]. The idea to obtain k small amplitude limit cycles is based on imposing conditions on Lyapunov constants $\mathcal{V}_i$ such that $\mathcal{V}_0 = \cdots = \mathcal{V}_{k-1} = 0$ and $\mathcal{V}_k \neq 0$, and then performing a suitable perturbation to have k small limit cycles.

Few papers consider this problem for degenerate centers. We remind that a center of a polynomial differential system is a degenerate center if, after applying a linear change of variables and a suitable time rescale, the system can be written as $\dot{x} = F_1(x,y)$ and $\dot{y} = F_2(x,y)$, where $F_1(x,y)$ and $F_2(x,y)$ are nonlinear polynomials. Authors in [2] used Melnikov functions to study the number of limit cycles for the perturbation of the degenerate center

$$\dot{x} = -y\left(\frac{x^2+y^2}{2}\right)^m, \qquad \dot{y} = x\left(\frac{x^2+y^2}{2}\right)^m, \qquad m \geq 1.$$

Authors in [12] used the averaging method of second order to study the perturbation of the cubic degenerate center

$$\dot{x} = -y\left(3x^2+y^2\right), \qquad \dot{y} = x\left(x^2-y^2\right),$$

and prove the existence of at most three limit cycles.

In this paper, we will study the number of limit cycles for the perturbed system

$$\dot{\xi} = X_{2m+1}(\xi) + \epsilon X_{2m}(\xi), \qquad \xi = (x,y), \qquad 0 < \epsilon \ll 1, \tag{1.1}$$

where $X_{2m+1} = (P_{2m+1}(x,y), Q_{2m+1}(x,y))$ is the homogeneous polynomial system with

$$P_{2m+1}(x,y) = \sum_{i=0}^{m} a_i x^{2i} y^{2(m-i)+1}, \qquad Q_{2m+1}(x,y) = \sum_{i=0}^{m} b_i x^{2i+1} y^{2(m-i)},$$

such that $a_i b_i < 0$ for all i, and $X_{2m} = (P_{2m}(x,y), Q_{2m}(x,y))$ with

$$P_{2m}(x,y) = \sum_{k=0}^{2m} \alpha_k x^k y^{2m-k}, \qquad Q_{2m}(x,y) = \sum_{k=0}^{2m} \beta_k x^k y^{2m-k}.$$

The origin is a symmetric degenerate center for the unperturbed homogeneous polynomial system (1.1), cf. [13, Theorem 1(II)].

Such systems are known as one step polynomial perturbations. In these systems, a homogeneous polynomial system is perturbed by another homogeneous polynomial term with one step bigger or smaller order. Such systems receive attention from mathematicians and physicists due to their role in nonlinear mechanics, cf. [3, 7]. Here, we aim to estimate the number of limit cycles for the perturbed system (1.1) by studying the Lyapunov constants. We compute the Lyapunov constants through the Taylor expansion of the Poincaré map

$$P(r_0,\epsilon) = r(2\pi, r_0, \epsilon) = \sum_{j=0}^{n} \frac{1}{j!}\epsilon^j \frac{\partial^j r}{\partial \epsilon^j}(2\pi, r_0, 0) + O(\epsilon^{n+1})$$

of the system (1.1). Here, (r,θ) shows the polar coordinates around the origin. The origin is assumed to be a perturbed symmetric degenerate center, and $r = r(\theta, r_0, \epsilon)$ is the solution of equation (1.1) such that $r(0, r_0, \epsilon) = r_0$ and $r(2\pi, r_0, 0) = r_0$. Then, we will study the Lyapunov constants by considering the coefficients of X_{2m}, α_ks and β_ks, as the parameters.

The paper is oriented as follows. In Section 2, we will compute $\left(\partial^j r/\partial\epsilon^j\right)(\theta, r_0, 0)$ for the perturbed system (1.1) and prove the equality (2.2). In Section 3, we will simplify $\left(\partial^j r/\partial\epsilon^j\right)(\theta, r_0, 0)$ for $\theta = 2\pi$ and classify the parameters into two types α_{2k+1}, β_{2k} and α_{2k}, β_{2k+1}. In Section 4, we will present our main results and estimate the number of limit cycles for the perturbed degenerate center (1.1) in Theorem 4.5. We will see that the one step perturbed degenerate center has at least two limit cycles, and the number of limit cycles can reach $2m+3$. At last, we will consider our results for a perturbed degenerate center of order 3.

2. Preliminaries

Consider the perturbed system (1.1). As [13, Section 3], the corresponding polar perturbed system is

$$\frac{dr}{d\theta} = rS(\theta) + \epsilon\, r\, \frac{C}{A(r\,A + \epsilon\, B)}, \tag{2.1}$$

where

$$S(\theta) = \frac{\langle n_\theta, X_{2m+1}(\theta)\rangle}{(n_\theta \wedge X_{2m+1}(\theta))}, \qquad A(\theta) = (n_\theta \wedge X_{2m+1}(\theta)),$$

$$B(\theta)=(n_\theta\wedge X_{2m}(\theta)),\qquad C(\theta)=(X_{2m}(\theta)\wedge X_{2m+1}(\theta)),$$

such that $n_\theta=(\cos(\theta),\sin(\theta))$, $X_{\mathbf{i}}(\theta)=(P_{\mathbf{i}}(\cos(\theta),\sin(\theta)),Q_{\mathbf{i}}(\cos(\theta),\sin(\theta))$ for $\mathbf{i}=2m,2m+1$. Also, the notations $\langle\ ,\ \rangle$ and $(\wedge)$ are respectively the inner and the wedge product of vectors in $\mathbb{R}^2$, i.e.,

$$\langle (a,b)\,,\,(c,d)\rangle = ac+bd,\qquad \Big((a,b)\,\wedge\,(c,d)\Big)=ad-bc.$$

Then by applying the relation [13, (20)], we obtain $\left(\partial^j r/\partial\epsilon^j\right)(\theta,r_0,0)$ for the polar perturbed system (2.1) as

$$\frac{\partial^j r}{\partial\epsilon^j}(\theta,r_0,0)=\\ j\,\chi(\theta)\sum_{k=0}^{j-1}\sum_{l=0}^{k}\binom{j-1}{k}\binom{k}{l}\int_0^\theta\chi^{-1}(\psi)\,\frac{\partial^{j-1-k}r}{\partial\epsilon^{j-1-k}}\,\mathcal{D}^l\mathsf{d}^{k-l}\Big(\frac{C}{A(r\,A+\epsilon\,B)}\Big)d\psi,$$

where $\chi(\theta)=\mathrm{Exp}\big(\int_0^\theta S(t)dt\big)$ is the fundamental matrix of the homogeneous part of (2.1). Also, $\mathcal{D}$ and d are the linear functional operators $\mathcal{D}F(g(x),x)=(\partial F/\partial g)\,(g(x),x)g'(x)$ and $\mathsf{d}F(g(x),x)=(\partial F/\partial x)\,(g(x),x)$, see [13, page 10] for more details. Next, we consider the above relation more precisely. For $l=0$, we have

$$\begin{aligned}
&j\,\chi(\theta)\int_0^\theta\chi^{-1}(\psi)\frac{\partial^{j-1}r}{\partial\epsilon^{j-1}}\left(\frac{C}{A(r\,A+\epsilon\,B)}\right)d\psi\\
&\qquad+\,j\,\chi(\theta)\sum_{k=1}^{j-1}\binom{j-1}{k}\binom{k}{0}\int_0^\theta\chi^{-1}(\psi)\,\frac{\partial^{j-1-k}r}{\partial\epsilon^{j-1-k}}\,\mathsf{d}^k\Big(\frac{C}{A(r\,A+\epsilon\,B)}\Big)d\psi\\
&=\,j\,\chi(\theta)\int_0^\theta\chi^{-1}(\psi)\,\frac{\partial^{j-1-k}r}{\partial\epsilon^{j-1-k}}\left(\frac{C}{A(r\,A+\epsilon\,B)}\right)d\psi\\
&\qquad+\,j\,\chi(\theta)\sum_{k=1}^{j-1}\binom{j-1}{k}\int_0^\theta\chi^{-1}(\psi)\,\frac{\partial^{j-1-k}r}{\partial\epsilon^{j-1-k}}\,\frac{C}{AB}\,\frac{(-1)^k k!}{\left(r\,\frac{A}{B}+\epsilon\right)^{k+1}}\,d\psi\\
&=\,j\,\chi(\theta)\sum_{k=0}^{j-1}\binom{j-1}{k}\int_0^\theta\chi^{-1}(\psi)\,\frac{\partial^{j-1-k}r}{\partial\epsilon^{j-1-k}}\,\frac{C}{AB}\,\frac{(-1)^k k!}{\left(r\,\frac{A}{B}+\epsilon\right)^{k+1}}\,d\psi.
\end{aligned}$$

So for $\epsilon=0$, we have

$$\sum_{k=0}^{j-1}(-1)^k j\,\chi(\theta)\binom{j-1}{k}\frac{k!}{r_0^{k+1}}\int_0^\theta\frac{\partial^{j-1-k}r}{\partial\epsilon^{j-1-k}}\,\frac{C}{A^2}\left(\frac{B}{A}\right)^k\frac{1}{\chi(\psi)^{k+2}}\,d\psi.$$

For the remaining (when $l \geq 1$), we have

$$j\,\chi(\theta)\sum_{k=1}^{j-1}\sum_{l=1}^{k}\binom{j-1}{k}\binom{k}{l}\int_0^\theta \chi^{-1}(\psi)\frac{\partial^{j-1-k}r}{\partial\epsilon^{j-1-k}}\,\mathcal{D}^l\mathsf{d}^{k-l}\left(\frac{C}{A(r\,A+\epsilon\,B)}\right)d\psi$$
$$= j\,\chi(\theta)\sum_{k=1}^{j-1}\sum_{l=1}^{k}\binom{j-1}{k}\binom{k}{l}\int_0^\theta \chi^{-1}(\psi)\frac{\partial^{j-1-k}r}{\partial\epsilon^{j-1-k}}\,\frac{C}{AB}\mathcal{D}^l\left(\frac{(-1)^{k-l}(k-l)!}{\left(r\,\frac{A}{B}+\epsilon\right)^{k-l+1}}\right)d\psi.$$

From the formula of Faà di Bruno, we have

$$\mathcal{D}^l\left(\frac{1}{\left(r\,\frac{A}{B}+\epsilon\right)^{k-l+1}}\right) = \sum_{t=1}^{l=1}\left(\frac{1}{\left(r\,\frac{A}{B}+\epsilon\right)^{k-l+1}}\right)^{(t)} B_{l,t}\left(\frac{\partial r}{\partial\epsilon},\frac{\partial^2 r}{\partial\epsilon^2},\ldots,\frac{\partial^{l-t+1}r}{\partial\epsilon^{l-t+1}}\right)$$
$$= \sum_{t=1}^{l=1}\frac{(-1)^t(k-l+t)!}{(k-l)!\left(r+\epsilon\,\frac{B}{A}\right)^{k-l+t+1}}\,B_{l,t}\left(\frac{\partial r}{\partial\epsilon},\frac{\partial^2 r}{\partial\epsilon^2},\ldots,\frac{\partial^{l-t+1}r}{\partial\epsilon^{l-t+1}}\right).$$

So for $l \geq 1$, we get

$$\sum_{k=1}^{j-1}\sum_{l=1}^{k}\sum_{t=1}^{l} j\,\chi(\theta)\binom{j-1}{k}\binom{k}{l}(-1)^{k-l+t}(k-l+t)!\int_0^\theta \chi^{-1}(\psi)$$
$$\frac{\partial^{j-1-k}r}{\partial\epsilon^{j-1-k}}\,\frac{C}{AB}\,\frac{1}{\left(r+\epsilon\,\frac{B}{A}\right)^{k-l+t+1}}B_{l,t}\left(\frac{\partial r}{\partial\epsilon},\frac{\partial^2 r}{\partial\epsilon^2},\ldots,\frac{\partial^{l-t+1}r}{\partial\epsilon^{l-t+1}}\right)d\psi.$$

Thus for $\epsilon = 0$, we have

$$\sum_{k=1}^{j-1}\sum_{l=1}^{k}\sum_{t=1}^{l} j\,\chi(\theta)\binom{j-1}{k}\binom{k}{l}(-1)^{k-l+t}\frac{(k-l+t)!}{r_0^{k-l+t+2}}\int_0^\theta \chi^{-1}(\psi)$$
$$\frac{\partial^{j-1-k}r}{\partial\epsilon^{j-1-k}}\,\frac{C}{A^2}\left(\frac{B}{A}\right)^{k-l}\frac{1}{\chi(\psi)^{k-l+t+2}}B_{l,t}\left(\frac{\partial r}{\partial\epsilon},\frac{\partial^2 r}{\partial\epsilon^2},\ldots,\frac{\partial^{l-t+1}r}{\partial\epsilon^{l-t+1}}\right)d\psi.$$

Finally, the following proposition is immediate.

PROPOSITION 2.1. *Consider the system* (2.1). *We have*

$$\frac{\partial^j r}{\partial\epsilon^j}(\theta,r_0,0) = \sum_{k=0}^{j-1}I_{(j,k)}(\theta) + \sum_{k=1}^{j-1}\sum_{l=1}^{k}\sum_{t=1}^{l}M_{(j,k,l,t)}(\theta), \tag{2.2}$$

where

$$
\begin{aligned}
&I_{(j,k)}(\theta) = \\
&\qquad (-1)^k j\,\chi(\theta)\binom{j-1}{k}\frac{k!}{r_0^{k+1}}\int_0^\theta \frac{\partial^{j-1-k} r}{\partial\epsilon^{j-1-k}}\,\frac{C}{A^2}\left(\frac{B}{A}\right)^k \frac{1}{\chi(\psi)^{k+2}}\,d\psi,
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
&M_{(j,k,l,t)}(\theta) = (-1)^{k-l+t} j\,\chi(\theta)\binom{j-1}{k}\binom{k}{l}\frac{(k-l+t)!}{r_0^{k-l+t+1}} \\
&\int_0^\theta \frac{\partial^{j-1-k} r}{\partial\epsilon^{j-1-k}}\,\frac{C}{A^2}\left(\frac{B}{A}\right)^{k-l}\frac{1}{\chi(\psi)^{k-l+t+2}}\,B_{l,t}\left(\frac{\partial r}{\partial\epsilon},\ldots,\frac{\partial^{l-t+1} r}{\partial\epsilon^{l-t+1}}\right)d\psi.
\end{aligned}
\tag{2.4}
$$

LEMMA 2.2. *Consider* (2.3) *and* (2.4). *We have*

R1: $I_{(j,0)} + M_{(j,j-1,j-1,1)} = 0$ *for all* j;

R2: $M(j,\,j-1,\,j-1,\,i+1) + \sum_{k=i}^{j-2} M(j,\,k,\,k,\,i) = 0$ *for all* $j, 1 \le i \le j-2$.

Furthermore, $\left(\partial^j r/\partial\epsilon^j\right)(\theta, r_0, 0)$ *is a* j*-th order homogeneous polynomial w.r.t.* α_k*s or* β_k*s.*

Proof. The R1 and R2 prove by substituting indexes in (2.3) and (2.4), and proportional use of $B_{l,t}$ in appendix 5.1. For the remain, consider (2.2). The proof is obvious by induction on j and the second condition of Faà di Bruno's formula. We note that the power of the parameters in $B(\theta)$ and $C(\theta)$ is one. ∎

In the following remark, we indicate the $\left(\partial^j r/\partial\epsilon^j\right)(\theta, r_0, 0)$ for $j = 1, 2, 3, 4$.

Remark 2.3.

$$
\begin{aligned}
\frac{\partial r}{\partial\epsilon}(\theta, r_0, 0) &= \chi(\theta)\int_0^\theta \frac{C}{\chi\,A^2}\,d\psi, \\
\frac{\partial^2 r}{\partial\epsilon^2}(\theta, r_0, 0) &= -\frac{2\,\chi(\theta)}{r_0}\int_0^\theta \frac{C\,B}{\chi^2\,A^3}\,d\psi, \\
\frac{\partial^3 r}{\partial\epsilon^3}(\theta, r_0, 0) &= \frac{6\,\chi(\theta)}{r_0^2}\int_0^\theta \frac{C\,B^2}{\chi^3\,A^4}\,d\psi + \frac{6\,\chi(\theta)}{r_0^2}\int_0^\theta \frac{\partial r}{\partial\epsilon}\frac{C\,B}{\chi^3\,A^3}\,d\psi, \\
\frac{\partial^4 r}{\partial\epsilon^4}(\theta, r_0, 0) &= -\frac{24\chi(\theta)}{r_0^3}\int_0^\theta \frac{C\,B^3}{\chi^4\,A^5}d\psi - \frac{48\chi(\theta)}{r_0^3}\int_0^\theta \frac{\partial r}{\partial\epsilon}\frac{C\,B^2}{\chi^4 A^4}d\psi \\
&\quad - \frac{24\chi(\theta)}{r_0^3}\int_0^\theta \left(\frac{\partial r}{\partial\epsilon}\right)^2\frac{C\,B}{\chi^4 A^3}d\psi + \frac{12\chi(\theta)}{r_0^2}\int_0^\theta \frac{\partial^2 r}{\partial\epsilon^2}\frac{C\,B}{\chi^3 A^3}d\psi.
\end{aligned}
$$

3. Lyapunov constant

In the following, we consider $\left(\partial^j r/\partial\epsilon^j\right)(\theta, r_0, 0)$ for $\theta = 2\pi$. We prove some points to simplify them and apply for some j. Consider the following lemma.

Lemma 3.1. ([13, Lemma 6]) *Let $f(\theta) > 0$ (< 0) be an even π-periodic function and i, j be nonnegative integers such that at least one of them is odd. Then*

$$\int_0^{2\pi} \frac{\cos^i(\theta)\sin^j(\theta)}{f(\theta)} d\theta = 0.$$

So we conclude the following lemma.

Lemma 3.2. *Let $f(\theta) > 0$ (< 0) be an even π-periodic function and i, j be odd nonnegative integers. Then*

$$\int_0^{\pi} \frac{\cos^i(\theta)\sin^j(\theta)}{f(\theta)} d\theta = 0.$$

Proof. We have

$$\begin{aligned}\int_0^{2\pi} \frac{\cos^i(\theta)\sin^j(\theta)}{f(\theta)} d\theta &= \int_0^{\pi} \frac{\cos^i(\theta)\sin^j(\theta)}{f(\theta)} d\theta + \int_{\pi}^{2\pi} \frac{\cos^i(\theta)\sin^j(\theta)}{f(\theta)} d\theta \\ &= \int_0^{\pi} \frac{\cos^i(\theta)\sin^j(\theta)}{f(\theta)} d\theta + \int_0^{\pi} \frac{(-1)^{i+j}\cos^i(\psi)\sin^j(\psi)}{f(\psi)} d\psi.\end{aligned}$$

Then by considering Lemma 3.1, we get

$$0 = \int_0^{2\pi} \frac{\cos^i(\theta)\sin^j(\theta)}{f(\theta)} d\theta = 2\int_0^{\pi} \frac{\cos^i(\theta)\sin^j(\theta)}{f(\theta)} d\theta.$$

∎

Next, we consider $X_{2m}(\theta) = (P_{2m}(\theta), Q_{2m}(\theta))$ as

$$P_{2m}(\theta) = \sum_{k=0}^{m} \alpha_{2k} \cos^{2k}(\theta)\sin^{2m-2k}(\theta) + \sum_{k=0}^{m-1} \alpha_{2k+1} \cos^{2k+1}(\theta)\sin^{2m-2k-1}(\theta),$$

$$Q_{2m}(\theta) = \sum_{k=0}^{m} \beta_{2k} \cos^{2k}(\theta)\sin^{2m-2k}(\theta) + \sum_{k=0}^{m-1} \beta_{2k+1} \cos^{2k+1}(\theta)\sin^{2m-2k-1}(\theta).$$

Now, by defining D_{2m} and $D_{2m}^{\perp}$ as

$$D_{2m}(\theta)=\left(\sum_{k=0}^{m-1}\alpha_{2k+1}\cos^{2k+1}(\theta)\sin^{2m-2k-1}(\theta),\sum_{k=0}^{m}\beta_{2k}\cos^{2k}(\theta)\sin^{2m-2k}(\theta)\right),$$

$$D_{2m}^{\perp}(\theta)=\left(\sum_{k=0}^{m}\alpha_{2k}\cos^{2k}(\theta)\sin^{2m-2k}(\theta),\sum_{k=0}^{m-1}\beta_{2k+1}\cos^{2k+1}(\theta)\sin^{2m-2k-1}(\theta)\right),$$

we have $X_{2m} = D_{2m} + D_{2m}^{\perp}$. This helps us to separate the parameters into two collections. One collection contains α_{2k+1}s and β_{2k}s, and the other collection contains α_{2k}s and β_{2k+1}s. Thus, we can decompose $B(\theta)$ and $C(\theta)$ as the following,

$$\begin{aligned} B(\theta)\ &:\quad (n_\theta \wedge D_{2m}(\theta)) + (n_\theta \wedge D_{2m}^{\perp}(\theta)) = \mathcal{B}(\theta) + \mathcal{B}^{\perp}(\theta),\\ C(\theta)\ &:\quad (D_{2m}(\theta)\wedge X_{2m+1}(\theta)) + (D_{2m}^{\perp}(\theta)\wedge X_{2m+1}(\theta)) = \mathcal{C}(\theta) + \mathcal{C}^{\perp}(\theta). \end{aligned}$$

By considering $\mathcal{B}(\theta)$ and $\mathcal{C}^{\perp}(\theta)$, we can see that the power of $\sin(\theta)$ is even, and the power of $\cos(\theta)$ is odd. Also by considering $\mathcal{B}^{\perp}(\theta)$ and $\mathcal{C}(\theta)$, we can see that the power of $\sin(\theta)$ is odd, and the power of $\cos(\theta)$ is even.

This classification helps us to use Lemma 3.1 and Lemma 3.2. We note that we only use this classification when the relations could simplify. Next, we introduce some particular cases.

Remark 3.3. Let $\mathcal{K}$ be a nonnegative integer and consider $C\,B^{\mathcal{K}}$. According to $C(\theta)$ and $B(\theta)$, we can see that the multiplication of $\sin(\theta)$ and $\cos(\theta)$ exists in $C\,B^{\mathcal{K}}$, for all $\mathcal{K}$. By applying an induction on $\mathcal{K}$, we have the following results:

- If $\mathcal{K}$ is even, then the $\sin(\theta)$ or $\cos(\theta)$ in $C\,B^{\mathcal{K}}$ has odd order.
- If $\mathcal{K}$ is odd, then the $\sin(\theta)$ and $\cos(\theta)$ in $C\,B^{\mathcal{K}}$ are both have an even or both have an odd order.

Remark 3.4. Consider (2.2) and note that $(\partial^0 r/\partial\epsilon^0)(\theta, r_0, 0) = \chi(\theta) r_0$. By substituting the suitable conditions, one of the terms of $(\partial^j r/\partial\epsilon^j)(2\pi, r_0, 0)$ for $k = j-1$ is

$$(-1)^{j-1}\frac{j!}{r_0^{j-1}}\int_0^{2\pi}\frac{C\,B^{j-1}}{A^{j+1}\chi^j(\psi)}d\psi.$$

By applying an induction on j and according to Lemma 3.1 and the above classification, this term is equal to zero when j is an odd integer number.

In the following proposition, we use these remarks to investigate $\left(\partial^j r/\partial\epsilon^j\right)(2\pi, r_0, 0)$ for $j = 1, 2, 3, 4$.

PROPOSITION 3.5. *Consider* $\left(\partial^j r/\partial\epsilon^j\right)(\theta, r_0, 0)$ *for* $j = 1, 2, 3, 4$. *We have*

$$\frac{\partial r}{\partial\epsilon}(2\pi, r_0, 0) = \int_0^{2\pi} \frac{\mathcal{C} + \mathcal{C}^\perp}{\chi A^2} d\psi = 0, \tag{3.1}$$

$$\frac{\partial^2 r}{\partial\epsilon^2}(2\pi, r_0, 0) = \frac{-2}{r_0} \int_0^{2\pi} \frac{\mathcal{C}^\perp\mathcal{B} + \mathcal{C}\mathcal{B}^\perp}{\chi^2 A^3}\, d\psi, \tag{3.2}$$

$$\frac{\partial^3 r}{\partial\epsilon^3}(2\pi, r_0, 0) = \frac{6}{r_0^2\chi(\pi)} \frac{\partial r}{\partial\epsilon}(\pi, r_0, 0) \int_0^{\pi} \frac{\mathcal{C}\mathcal{B} + \mathcal{C}^\perp\mathcal{B}^\perp}{\chi^2 A^3}\, d\psi, \tag{3.3}$$

and also,

$$\begin{aligned}
&\frac{\partial^4 r}{\partial\epsilon^4}(2\pi, r_0, 0) = \\
&\quad - \frac{24}{r_0^3} \int_0^{2\pi} \frac{(\mathcal{C}\mathcal{B} + \mathcal{C}^\perp\mathcal{B}^\perp)(2\mathcal{B}\mathcal{B}^\perp) + (\mathcal{C}\mathcal{B}^\perp + \mathcal{C}^\perp\mathcal{B})(\mathcal{B}^2 + {\mathcal{B}^\perp}^2)}{\chi^4 A^5} d\psi \\
&\quad + \left(\frac{-24}{r_0^3\chi^2(\pi)} \left(\frac{\partial r}{\partial\epsilon}(\pi) \right)^2 + \frac{12}{r_0^2\chi(\pi)} \frac{\partial^2 r}{\partial\epsilon^2}(\pi) \right) \int_0^{\pi} \frac{\mathcal{C}\mathcal{B} + \mathcal{C}^\perp\mathcal{B}^\perp}{\chi^2 A^3} d\psi \\
&\quad + \frac{48}{r_0^3\chi(\pi)} \frac{\partial r}{\partial\epsilon}(\pi) \int_0^{\pi} \frac{C\, B^2}{\chi^3 A^4} d\psi - \frac{96}{r_0^3} \int_0^{\pi} \frac{\partial r}{\partial\epsilon} \frac{C\, B^2}{\chi^4 A^4} d\psi \\
&\quad + \frac{48}{r_0^3\chi(\pi)} \frac{\partial r}{\partial\epsilon}(\pi) \int_0^{\pi} \frac{\partial r}{\partial\epsilon} \frac{C\, B}{\chi^3 A^3} d\psi \\
&\quad - \frac{48}{r_0^3} \int_0^{\pi} \left(\frac{\partial r}{\partial\epsilon} \right)^2 \frac{C\, B}{\chi^4 A^3} d\psi + \frac{24}{r_0^2} \int_0^{\pi} \frac{\partial^2 r}{\partial\epsilon^2} \frac{C\, B}{\chi^3 A^3} d\psi.
\end{aligned} \tag{3.4}$$

Where the relations (3.2), (3.3), *and* (3.4) *are not generally equal to zero.*

Proof. First, we note that $A(\theta)$ and $\chi(\theta)$ are π- periodic functions, and also $\chi(2\pi) = 1$, cf. [13]. Now by considering Lemma 3.1 and the above classification, we obviously obtain (3.2) and (3.3). Next, we consider $\left(\partial^3 r/\partial\epsilon^3\right)(2\pi, r_0, 0)$. According to remark 3.4, we get it as

$$\frac{\partial^3 r}{\partial\epsilon^3}(2\pi, r_0, 0) = \frac{6}{r_0^2} \int_0^{2\pi} \frac{C\, B}{\chi^3\, A^3} \frac{\partial r}{\partial\epsilon}\, d\psi,$$

where

$$\int_0^{2\pi} \frac{\partial r}{\partial \epsilon} \frac{C\,B}{\chi^3\,A^3} d\psi = \int_0^{\pi} \frac{\partial r}{\partial \epsilon} \frac{C\,B}{\chi^3\,A^3} d\psi + \int_0^{\pi} \frac{\partial r}{\partial \epsilon}(\phi+\pi, r_0, 0) \frac{C(\phi+\pi)\,B(\phi+\pi)}{\chi^3(\phi+\pi)\,A^3(\phi+\pi)} d\phi.$$

By considering $C(\theta)$, $B(\theta)$, and $(\partial r/\partial \epsilon)(\theta, r_0, 0)$, we get $C(\theta+\pi) = -C(\theta)$, $B(\theta+\pi) = -B(\theta)$, and

$$\frac{\partial r}{\partial \epsilon}(\theta+\pi, r_0, 0) = \frac{\chi(\theta)}{\chi(\pi)} \frac{\partial r}{\partial \epsilon}(\pi, r_0, 0) - \frac{\partial r}{\partial \epsilon}(\theta, r_0, 0).$$

Thus

$$\begin{aligned}\int_0^{2\pi} \frac{\partial r}{\partial \epsilon} \frac{C\,B}{\chi^3\,A^3} d\psi &= \int_0^{\pi} \frac{\partial r}{\partial \epsilon} \frac{C\,B}{\chi^3\,A^3} d\psi + \frac{1}{\chi(\pi)} \frac{\partial r}{\partial \epsilon}(\pi, r_0, 0) \int_0^{\pi} \frac{C\,B}{\chi^2\,A^3}\, d\phi \\ &\quad - \int_0^{\pi} \frac{\partial r}{\partial \epsilon}(\phi, r_0, 0) \frac{C\,B}{\chi^3\,A^3} d\phi \\ &= \frac{1}{\chi(\pi)} \frac{\partial r}{\partial \epsilon}(\pi, r_0, 0) \int_0^{\pi} \frac{(\mathcal{C}+\mathcal{C}^{\perp})(\mathcal{B}+\mathcal{B}^{\perp})}{\chi^2\,A^3}\, d\phi.\end{aligned}$$

Now by applying Lemma 3.2, we have

$$\int_0^{2\pi} \frac{C\,B}{\chi^3\,A^3} \frac{\partial r}{\partial \epsilon}\, d\psi = \frac{1}{\chi(\pi)} \frac{\partial r}{\partial \epsilon}(\pi, r_0, 0) \int_0^{\pi} \frac{\mathcal{C}\mathcal{B} + \mathcal{C}^{\perp}\mathcal{B}^{\perp}}{\chi^2\,A^3}\, d\psi.$$

So, (3.3) is immediate. Finally, by following the same process as above and also by considering

$$\frac{\partial^2 r}{\partial \epsilon^2}(\theta+\pi, r_0, 0) = \frac{\chi(\theta)}{\chi(\pi)} \frac{\partial^2 r}{\partial \epsilon^2}(\pi, r_0, 0) + \frac{\partial^2 r}{\partial \epsilon^2}(\theta, r_0, 0),$$

and Remark 2.3, we obtain (3.4). Finally, we note that by considering (3.2), (3.3) and (3.4), one easily concludes that they are not generally equal to zero. ∎

4. Estimating the number of limit cycles

In this section, we represent our main results.

Theorem 4.1. *Consider the perturbed system* (1.1).

1) *The system has at least two limit cycles.*
2) *The necessary condition that the system has more than two limit cycles is that the perturbed part has the two types of parameters, i.e., the parameters of* D_{2m}, α_{2k+1} *or* β_{2k}, *and the parameters of* $D_{2m}^{\perp}$, α_{2k} *or* β_{2k+1}, *for arbitrary* k.

We note that for simplicity, we consider the following notation in proof. Again by considering $\mathcal{K}$ as a nonnegative integer, assume

$$\mathfrak{C}^i\mathfrak{S}^j = \cos^{2\mathcal{K}+i}(\theta)\sin^{2m-2\mathcal{K}+j}(\theta).$$

As an example

$$\begin{aligned}\alpha_{2t}\,\mathfrak{C}\mathfrak{S} &= \alpha_{2t}\cos^{2t}(\theta)\sin^{2m-2t}(\theta),\\ \beta_{2k}\,\mathfrak{C}^1\mathfrak{S}^{-1} &= \beta_{2k}\cos^{2k+1}(\theta)\sin^{2m-2k-1}(\theta).\end{aligned}$$

Proof. 1) It is obvious by considering (3.1) and (3.2).

2) For this part, we consider (3.2) to find that under which necessary condition this relation could be zero. By substituting $\mathcal{C}$, $\mathcal{B}$, $\mathcal{C}^{\perp}$, and $\mathcal{B}^{\perp}$, we get

$$\mathcal{C}^{\perp}\mathcal{B}+\mathcal{C}\mathcal{B}^{\perp} = \big(D_{2m}^{\perp}\wedge X_{2m+1}\big)(n_\theta\wedge D_{2m}) + (D_{2m}\wedge X_{2m+1})\big(n_\theta\wedge D_{2m}^{\perp}\big).$$

Then by considering n_θ and X_{2m+1}, we have $\mathcal{C}^{\perp}\mathcal{B}+\mathcal{C}\mathcal{B}^{\perp}$ as

$$\begin{aligned}&P_{2m+1}(\theta)\Big(\sum\nolimits_{k=0}^{m}\alpha_{2k}\mathfrak{C}\mathfrak{S}^1\Big)\Big(\sum\nolimits_{k=0}^{m}\beta_{2k}\mathfrak{C}\mathfrak{S}\Big)\\
&\quad+P_{2m+1}(\theta)\Big(\sum\nolimits_{k=0}^{m-1}\alpha_{2k+1}\mathfrak{C}^1\mathfrak{S}\Big)\Big(\sum\nolimits_{k=0}^{m-1}\beta_{2k+1}\mathfrak{C}^1\mathfrak{S}^{-1}\Big)\\
&\quad-P_{2m+1}(\theta)\Big(\sum\nolimits_{k=0}^{m}\beta_{2k}\mathfrak{C}^1\mathfrak{S}\Big)\Big(\sum\nolimits_{k=0}^{m-1}\beta_{2k+1}\mathfrak{C}^1\mathfrak{S}^{-1}\Big)\\
&\quad-P_{2m+1}(\theta)\Big(\sum\nolimits_{k=0}^{m}\beta_{2k}\mathfrak{C}\mathfrak{S}\Big)\Big(\sum\nolimits_{k=0}^{m-1}\beta_{2k+1}\mathfrak{C}^2\mathfrak{S}^{-1}\Big)\\
&\quad+Q_{2m+1}(\theta)\Big(\sum\nolimits_{k=0}^{m}\alpha_{2k}\mathfrak{C}\mathfrak{S}\Big)\Big(\sum\nolimits_{k=0}^{m}\beta_{2k}\mathfrak{C}^1\mathfrak{S}\Big)\\
&\quad+Q_{2m+1}(\theta)\Big(\sum\nolimits_{k=0}^{m-1}\alpha_{2k+1}\mathfrak{C}^1\mathfrak{S}^{-1}\Big)\Big(\sum\nolimits_{k=0}^{m-1}\beta_{2k+1}\mathfrak{C}^2\mathfrak{S}^{-1}\Big)\\
&\quad-Q_{2m+1}(\theta)\Big(\sum\nolimits_{k=0}^{m}\alpha_{2k}\mathfrak{C}\mathfrak{S}\Big)\Big(\sum\nolimits_{k=0}^{m-1}\alpha_{2k+1}\mathfrak{C}^1\mathfrak{S}\Big)\\
&\quad-Q_{2m+1}(\theta)\Big(\sum\nolimits_{k=0}^{m}\alpha_{2k}\mathfrak{C}\mathfrak{S}^1\Big)\Big(\sum\nolimits_{k=0}^{m-1}\alpha_{2k+1}\mathfrak{C}^1\mathfrak{S}^{-1}\Big).\end{aligned}$$

As Lemma 2.2, we can see that the order of parameters in each term is two. Also, all terms appear as the production of parameters of D_{2m} and $D_{2m}^{\perp}$. Now let $0 \leq t \leq m$ be a fixed integer number and assume

$$\sum_{k=0}^{m} \alpha_{2k} \mathfrak{C}\mathfrak{S} = \alpha_{2t} \mathfrak{C}\mathfrak{S} + \sum_{k=0, k\neq t}^{m} \alpha_{2k} \mathfrak{C}\mathfrak{S}.$$

By considering the above relation, we can be zero (3.2) w.r.t. α_{2t}. In this case, we find

$$\alpha_{2t} = \frac{1}{\int_0^{2\pi} \frac{\Gamma(\psi)}{\chi^2\, A^3}\, d\psi} \int_0^{2\pi} \frac{\Lambda(\psi)}{\chi^2\, A^3}\, d\psi, \tag{4.1}$$

where $\Lambda(\psi)$ is the summation of all terms which are not dependent on the parameter α_{2t} (see appendix 5.2) and

$$\begin{aligned} \Gamma(\psi) &= (\sin(\psi) P_{2m+1}(\psi) + \cos(\psi) Q_{2m+1}(\psi)) \left(\sum_{k=0}^{m} \beta_{2k} \mathfrak{C}^{2t} \mathfrak{S}^{2m-2t} \right) \\ &\quad - 2Q_{2m+1}(\psi) \left(\sum_{k=0}^{m-1} \alpha_{2k+1} \mathfrak{C}^{2t+1} \mathfrak{S}^{2m-2t} \right). \end{aligned}$$

As we can see, (4.1) is well defined if the parameters α_{2k+1} or β_{2k} are not equal to zero for some k. It is worthwhile to note that we conclude the same result by considering the other parameters, α_{2t+1}, β_{2t}, and β_{2t+1}, to be zero (3.2). ∎

In the following, we study the perturbed system (1.1) to estimate the number of limit cycles. For this consideration, we have two points.

First, according to Theorem 4.1, the perturbed system (1.1) must have the two types of parameters to study the existence of more than two limit cycles. We can directly conclude the following lemma from Theorem 4.1.

LEMMA 4.2. *Consider the perturbed system* (1.1) *and assume that the perturbed part has two types of parameters. By considering the parameter* α_{2t} *as* (4.1) *for a fixed integer* t*, the perturbed system* (1.1) *has at least three limit cycles.*

In the following remark, we study the effect of assuming the specific parameter α_{2t} as (4.1) in considering the other Lyapunov constants.

Remark 4.3. Consider (4.1). We obtain that the parameters α_{2k} and β_{2k+1} appear in the numerator of (4.1) and the parameters α_{2k+1} and β_{2k} appear in the denominator of (4.1). Again we can see our classification of the parameters. Now by substituting (4.1) in the other Lyapunov constants, we can arrange the Lyapunov constants in the form of algebraic equations. The variables of these algebraic equations are the parameters that appear in the numerator of (4.1). The other parameters, the parameters in the denominator of (4.1), use to study the existence of zero for the algebraic equations. We emphasize that we conclude the same result by considering the other parameters to be zero $\left(\partial^2 r/\partial\epsilon^2\right)(2\pi)$. The only difference is the change of the two types of parameters from the numerator and denominator.

Second, according to Lemma 2.2, $\left(\partial^j r/\partial\epsilon^j\right)(2\pi, r_0, 0)$ is a j-th order homogeneous polynomial w.r.t. α_ks or β_ks. In the following lemma, we consider the simplified $\left(\partial^j r/\partial\epsilon^j\right)(2\pi, r_0, 0)$ for $j \geq 3$. We can see that these relations have terms in which the order of a specific parameter is equal to j.

LEMMA 4.4. *For $j \geq 3$, $\left(\partial^j r/\partial\epsilon^j\right)(2\pi, r_0, 0)$ contains a parameter with an exponent of the order of j.*

Proof. We note two points: first, the parameters α_{2k} and β_{2k+1} exist in $\mathcal{C}^{\perp}(\theta)$ and $\mathcal{B}^{\perp}(\theta)$; and the parameters α_{2k+1} and β_{2k} exist in $\mathcal{C}(\theta)$ and $\mathcal{B}(\theta)$. Second, for the existence of a term with the coefficient, for example α_{2k}^j, the term must be the j times production of $\mathcal{C}^{\perp}(\theta)$ or $\mathcal{B}^{\perp}(\theta)$. For $j = 3$, (3.3) simplifies as

$$\frac{6}{r_0^2\chi(\pi)}\int_0^{\pi}\frac{\mathcal{C}+\mathcal{C}^{\perp}}{\chi A^2}d\psi\int_0^{\pi}\frac{\mathcal{C}\mathcal{B}+\mathcal{C}^{\perp}\mathcal{B}^{\perp}}{\chi^2\, A^3}\,d\psi.$$

We can see that the relation has $\mathcal{C}\mathcal{C}\mathcal{B}$ and $\mathcal{C}^{\perp}\mathcal{C}^{\perp}\mathcal{B}^{\perp}$. For $j > 3$, see the following considerations. Let j be an even integer number and assume (2.3) for $k = j - 2$, i.e.,

$$I_{(j,j-2)}(\theta) = j\,\chi(\theta)\binom{j-1}{j-2}\frac{(j-2)!}{r_0^{j-1}}\int_0^{\theta}\frac{\partial r}{\partial\epsilon}\,\frac{C}{A^2}\left(\frac{B}{A}\right)^{j-2}\frac{1}{\chi(\psi)^j}\,d\psi.$$

By simplifying the $I_{(j,j-2)}(2\pi)$ even if $(\partial r/\partial\epsilon)(\pi)$ comes out of the integral, the CB^{j-2} does not decompose to $\mathcal{C}$, $\mathcal{C}^{\perp}$, $\mathcal{B}$, and $\mathcal{B}^{\perp}$. We note that the order of B is even. The result is obvious according to remark 3.3. So the relation has terms with $\mathcal{C}^2\mathcal{B}^{j-2}$ and also ${\mathcal{C}^{\perp}}^2{\mathcal{B}^{\perp}}^{j-2}$. Now let j be an odd integer number and assume (2.4) for $k = j-1$ and $l = t = 2$, i.e.,

$$M_{(j,j-1,2,2)}(\theta) = j\,\chi(\theta)\binom{j-1}{2}\frac{(j-1)!}{r_0^{j-1}}\int_0^\theta \frac{C}{A^2}\left(\frac{B}{A}\right)^{j-3}\frac{1}{\chi(\psi)^j}\left(\frac{\partial r}{\partial \epsilon}\right)^2 d\psi.$$

By following the same consideration as $I_{(j,j-2)}(2\pi)$, we conclude the result. ∎

Now we can conclude the following theorem.

THEOREM 4.5. *Consider the perturbed system* (1.1) *and assume that the perturbed part has two types of parameters. Then the number of the bifurcated limit cycles could reach* $2m+3$.

Proof. According to Theorem 4.1, the perturbed system (1.1) could have more than two limit cycles. We substitute the parameters α_{2t} as (4.1) and study the other Lyapunov constants. From Lemma 2.2, remark 4.3, and Lemma 4.4 each Lyapunov constant, i.e., $\left(\partial^j r/\partial\epsilon^j\right)(2\pi)$ for all j, is an algebraic equation of order j. Next, we use remark 4.3 to study the existence of zero for the algebraic equations. The number of the parameters which appear in the numerator of (4.1) is $2m+1$. So if we find the other parameters such that these algebraic equations have real roots, then the perturbed system (1.1) could have $2m+3$ limit cycles. ∎

In the following proposition, we consider the above results for the homogeneous degenerate center of order three.

Consider the perturbed system (1.1) for $m = 1$, i.e.,

$$\begin{aligned} \dot{x} &= a_0y^3 + a_1x^2y + \epsilon\left(\alpha_0y^2 + \alpha_1xy + \alpha_2x^2\right), \\ \dot{y} &= b_0xy^2 + b_1x^3 + \epsilon\left(\beta_0y^2 + \beta_1xy + \beta_2x^2\right). \end{aligned} \tag{4.2}$$

PROPOSITION 4.6. *Assume the perturbed system (4.2) such that* α_k *and* β_k, $k = 0,1,2$, *is not equal to zero and consider the following conditions.*

I. *Let the parameters* α_1, β_0, *and* β_2 *exist such that*

1) $\beta_2\mathcal{L}_{26} + \beta_0\mathcal{L}_{27} + \alpha_1\mathcal{L}_{29} \neq 0$,
2) $\mathcal{L}_{311}\mathcal{L}_{321} \neq 0$.

II. *Define the parameter*

$$\alpha_0 = \frac{\beta_1\left(\beta_2\mathcal{L}_{21} + \beta_0\mathcal{L}_{22} + \alpha_1\mathcal{L}_{25}\right) + \alpha_2\left(\beta_2\mathcal{L}_{23} + \beta_0\mathcal{L}_{24} + \alpha_1\mathcal{L}_{28}\right)}{\beta_2\mathcal{L}_{26} + \beta_0\mathcal{L}_{27} + \alpha_1\mathcal{L}_{29}}.$$

Respectively see Appendix (5.3) *and Appendix* (5.4) *for* $\mathcal{L}_{2i}$*s and* $\mathcal{L}_{3i}$*s. If conditions* I(1) *and* II *hold, then the perturbed system* (4.2) *has at least three limit cycles. If the conditions* I(1,2) *and* II *hold, then the perturbed system* (4.2) *has at least four limit cycles. Finally, if the conditions* I(1,2) *and* II *hold, and also the parameters* α_1, β_0, *and* β_2 *exist such that the quantic algebraic equation*

$$\begin{aligned}
&\alpha_2^4 \left(\mathcal{L}_{424} + \mathcal{L}_{434} + \mathcal{L}_{444} + \mathcal{L}_{464} + \mathcal{L}_{4523}\right) \\
&\quad + \alpha_2^3 \left(\mathcal{L}_{413} + \mathcal{L}_{423} + \mathcal{L}_{433} + \mathcal{L}_{443} + \mathcal{L}_{463} + \mathcal{L}_{473} + \mathcal{L}_{4523}\right) \\
&\quad + \alpha_2^2 \left(\mathcal{L}_{412} + \mathcal{L}_{422} + \mathcal{L}_{432} + \mathcal{L}_{442} + \mathcal{L}_{462} + \mathcal{L}_{472} + \mathcal{L}_{4522}\right) \\
&\quad + \alpha_2 \left(\mathcal{L}_{411} + \mathcal{L}_{421} + \mathcal{L}_{431} + \mathcal{L}_{441} + \mathcal{L}_{461} + \mathcal{L}_{471} + \mathcal{L}_{4521}\right) \\
&\quad + \mathcal{L}_{410} + \mathcal{L}_{420} + \mathcal{L}_{430} + \mathcal{L}_{440} + \mathcal{L}_{460} + \mathcal{L}_{470} + \mathcal{L}_{4520},
\end{aligned} \tag{4.3}$$

has a real root, then the perturbed system (4.2) *has at least five limit cycles, (the* $\mathcal{L}_{4i}$*s are the coefficients of* α_2^i, $i = 0, 1, 2, 3, 4$*).*

Proof. By applying (3.2) for the perturbed system (4.2) and simplifying it w.r.t. the parameters, we obtain it as

$$\begin{aligned}
&\beta_1\beta_2\mathcal{L}_{21} + \beta_0\beta_1\mathcal{L}_{22} + \alpha_2\beta_2\mathcal{L}_{23} + \alpha_2\beta_0\mathcal{L}_{24} + \alpha_1\beta_1\mathcal{L}_{25} \\
&\qquad + \alpha_0\beta_2\mathcal{L}_{26} + \alpha_0\beta_0\mathcal{L}_{27} + \alpha_1\alpha_2\mathcal{L}_{28} + \alpha_0\alpha_1\mathcal{L}_{29},
\end{aligned} \tag{4.4}$$

where $\mathcal{L}_{2i}$, $i = 1, \ldots, 9$, are the coefficients of parameters. Now we consider (4.4) as a quadratic algebraic equation and suppose that I(1) holds. We can easily see that this algebraic equation has a real root

$$\alpha_0 = \frac{\beta_1 \left(\beta_2\mathcal{L}_{21} + \beta_0\mathcal{L}_{22} + \alpha_1\mathcal{L}_{25}\right) + \alpha_2 \left(\beta_2\mathcal{L}_{23} + \beta_0\mathcal{L}_{24} + \alpha_1\mathcal{L}_{28}\right)}{\beta_2\mathcal{L}_{26} + \beta_0\mathcal{L}_{27} + \alpha_1\mathcal{L}_{29}}. \tag{4.5}$$

Thus according to this point that (3.3) is not generally equal to zero, the perturbed system (4.2) has at least three limit cycles. Next, we study the third Lyapunov constant, where we have the parameters α_0 as (4.5). For simplicity in computation, we consider (4.5) as

$$\alpha_0 = \beta_1\mathcal{A} + \alpha_2\mathcal{B},$$

where $\mathcal{A}$ and $\mathcal{B}$ obtain by considering (4.5). By computing (3.3) for $m = 1$ and simplifying it w.r.t. the parameters β_1 and α_2, we have

$$\begin{aligned}
&\beta_1^3\mathcal{L}_{311}\mathcal{L}_{321} + \beta_1^2 \left(\alpha_2\mathcal{L}_{312}\mathcal{L}_{321} + \alpha_2\mathcal{L}_{311}\mathcal{L}_{322} + \mathcal{L}_{311}\mathcal{L}_{320}\right) \\
&\qquad + \beta_1 \left(\alpha_2^2\mathcal{L}_{313}\mathcal{L}_{321} + \alpha_2^2\mathcal{L}_{312}\mathcal{L}_{322} + \alpha_2\mathcal{L}_{312}\mathcal{L}_{320} + \mathcal{L}_{310}\mathcal{L}_{321}\right) \\
&\qquad + \alpha_2^3\mathcal{L}_{313}\mathcal{L}_{322} + \alpha_2^2\mathcal{L}_{313}\mathcal{L}_{320} + \alpha_2\mathcal{L}_{310}\mathcal{L}_{322} + \mathcal{L}_{310}\mathcal{L}_{320},
\end{aligned} \tag{4.6}$$

where $\mathcal{L}_{3i}$s are the coefficients of the parameters. We note that $\mathcal{L}_{3i}$s are dependent on the parameters α_1, β_2, and β_0. As we can see (4.6) is the cubic algebraic equation w.r.t. the parameters β_1 and the parameters α_2. We consider it as an algebraic equation w.r.t. β_1. This equation has at least one real root if the coefficient of β_1, i.e., $\mathcal{L}_{311}\mathcal{L}_{321}$ is not equal to zero. Thus if I(1,2) and II hold, the perturbed system (4.2) has at least four limit cycles. For the last step, we study the fourth Lyapunov constant, (3.4), by substituting the parameters α_0 and β_1. Then by simplifying it w.r.t. the parameter α_2, we obtain (4.3). So if the parameters α_1, β_2, and β_0 exist such that the quantic algebraic equation (4.3) has a real root, then the perturbed system (4.2) has at least five limit cycles. ∎

5. Appendix

5.1. The formula of Faà di Bruno. Given two functions f and g, the generalization of the chain rule is known as Faà di Bruno's theorem.

$$\frac{d^n}{d^n x}(f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x))B_{n,k}\Big(g^{(1)}(x),\, g^{(2)}(x),\, \dots, g^{(n-k+1)}(x)\Big),$$

where $B_{n,k}$ are the Exponential Bell polynomials. The partial or incomplete exponential Bell polynomials are a triangular array of polynomials given by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{j_1!\, j_2! \,\dots\, j_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

Where the sum is taken over all sequences $j_1, j_2, j_3, \dots, j_{n-k+1}$ of nonnegative integers such that these two conditions are satisfied:

1. $j_1 + j_2 + j_3 + \cdots + j_{n-k+1} = k$,
2. $j_1 + 2\,j_2 + 3\,j_3 + \cdots + (n-k+1)\,j_{n-k+1} = n$.

For example

$$\begin{aligned}
B_{k,k}\big(x_1\big) &: \; j_1 = k,\\
B_{k+1,k}\big(x_1, x_2\big) &: \; j_1 = k-1,\; j_2 = 1,\\
B_{k+2,k}\big(x_1, x_2, x_3\big) &: \; j_1 = k-1,\; j_2 = 0,\; j_3 = 1,\\
&\; j_1 = k-2,\; j_2 = 2,\; j_3 = 0,
\end{aligned}$$

$$
\begin{aligned}
B_{k+3,k}\big(x_1,x_2,x_3,x_4\big) \;:\;\; & j_1=k-1,\; j_2=0,\; j_3=0,\; j_4=1,\\
& j_1=k-2,\; j_2=1,\; j_3=1,\; j_4=0,\\
& j_1=k-3,\; j_2=3,\; j_3=0,\; j_4=0.
\end{aligned}
$$

5.2. The relation $\Lambda(\theta)$ is

$$
\begin{aligned}
&\sum_{k=0,k\neq t}^{m} \alpha_{2k}\mathfrak{C}\mathfrak{S}\Bigg\{P_{2m+1}(\theta)\Bigg(\sum_{k=0}^{m}\beta_{2k}\mathfrak{C}\mathfrak{S}^{1}\Bigg) - Q_{2m+1}(\theta)\Bigg(\sum_{k=0}^{m-1}\alpha_{2k+1}\mathfrak{C}^{1}\mathfrak{S}\Bigg)\\
&\qquad -Q_{2m+1}(\theta)\Bigg(\sum_{k=0}^{m-1}\alpha_{2k+1}\mathfrak{C}^{1}\mathfrak{S}\Bigg) + Q_{2m+1}(\theta)\Bigg(\sum_{k=0}^{m}\beta_{2k}\mathfrak{C}^{1}\mathfrak{S}\Bigg)\Bigg\}\\
&\quad + P_{2m+1}(\theta)\Bigg(\sum_{k=0}^{m-1}\alpha_{2k+1}\mathfrak{C}^{1}\mathfrak{S}\Bigg)\Bigg(\sum_{k=0}^{m-1}\beta_{2k+1}\mathfrak{C}^{1}\mathfrak{S}^{-1}\Bigg)\\
&\quad - P_{2m+1}(\theta)\Bigg(\sum_{k=0}^{m}\beta_{2k}\mathfrak{C}^{1}\mathfrak{S}\Bigg)\Bigg(\sum_{k=0}^{m-1}\beta_{2k+1}\mathfrak{C}^{1}\mathfrak{S}^{-1}\Bigg)\\
&\quad - P_{2m+1}(\theta)\Bigg(\sum_{k=0}^{m}\beta_{2k}\mathfrak{C}\mathfrak{S}\Bigg)\Bigg(\sum_{k=0}^{m-1}\beta_{2k+1}\mathfrak{C}^{2}\mathfrak{S}^{-1}\Bigg)\\
&\quad + Q_{2m+1}(\theta)\Bigg(\sum_{k=0}^{m-1}\alpha_{2k+1}\mathfrak{C}^{1}\mathfrak{S}^{-1}\Bigg)\Bigg(\sum_{k=0}^{m-1}\beta_{2k+1}\mathfrak{C}^{2}\mathfrak{S}^{-1}\Bigg).
\end{aligned}
$$

5.3. The coefficients in the (4.4):

$$
\begin{aligned}
\mathcal{L}_{21} &= \frac{-2}{r_0}\int_0^{2\pi}\frac{-2P_3(\psi)\sin(\psi)\cos^4(\psi)}{\chi^2\,A^3}\,d\psi,\\
\mathcal{L}_{22} &= \frac{-2}{r_0}\int_0^{2\pi}\frac{-2P_3(\psi)\sin^3(\psi)\cos^2(\psi)}{\chi^2\,A^3}\,d\psi,\\
\mathcal{L}_{23} &= \frac{-2}{r_0}\int_0^{2\pi}\frac{P_3(\psi)\sin(\psi)\cos^4(\psi)+Q_3(\psi)\cos^5(\psi)}{\chi^2\,A^3}\,d\psi,\\
\mathcal{L}_{24} &= \frac{-2}{r_0}\int_0^{2\pi}\frac{P_3(\psi)\sin^3(\psi)\cos^2(\psi)+Q_3(\psi)\sin^2(\psi)\cos^3(\psi)}{\chi^2\,A^3}\,d\psi,
\end{aligned}
$$

$$\begin{aligned}
\mathcal{L}_{25} &= \frac{-2}{r_0}\int_0^{2\pi}\frac{P_3(\psi)\sin^3(\psi)\cos^2(\psi)+Q_3(\psi)\sin^2(\psi)\cos^3(\psi)}{\chi^2\,A^3}\,d\psi,\\
\mathcal{L}_{26} &= \frac{-2}{r_0}\int_0^{2\pi}\frac{P_3(\psi)\sin^3(\psi)\cos^2(\psi)+Q_3(\psi)\sin^2(\psi)\cos^3(\psi)}{\chi^2\,A^3}\,d\psi,\\
\mathcal{L}_{27} &= \frac{-2}{r_0}\int_0^{2\pi}\frac{P_3(\psi)\sin^5(\psi)+Q_3(\psi)\sin^4(\psi)\cos(\psi)}{\chi^2\,A^3}\,d\psi,\\
\mathcal{L}_{28} &= \frac{-2}{r_0}\int_0^{2\pi}\frac{-2Q_3(\psi)\sin^2(\psi)\cos^3(\psi)}{\chi^2\,A^3}\,d\psi,\\
\mathcal{L}_{29} &= \frac{-2}{r_0}\int_0^{2\pi}\frac{-2Q_3(\psi)\sin^4(\psi)\cos(\psi)}{\chi^2\,A^3}\,d\psi.
\end{aligned}$$

5.4. The coefficients in the (4.6). Consider $\alpha_0 = \mathcal{A}\beta_1 + \alpha_2\mathcal{B}$. We have

$$\begin{aligned}
\mathcal{L}_{310} = \frac{6}{r_0^2\chi(\pi)}\int_0^{\pi}\Bigg(&\frac{\alpha_1\beta_2P_3(\psi)\sin^2(\psi)\cos^3(\psi)+\alpha_1\beta_0P_3(\psi)\sin^4(\psi)\cos(\psi)}{\chi^2\,A^3}\\
&+\frac{\beta_0^2P_3(\psi)\sin^4(\psi)(-\cos(\psi))-\beta_2^2P_3(\psi)\cos^5(\psi)}{\chi^2\,A^3}\\
&+\frac{-2\beta_0\beta_2P_3(\psi)\sin^2(\psi)\cos^3(\psi)-\alpha_1^2Q_3(\psi)\sin^3(\psi)\cos^2(\psi)}{\chi^2\,A^3}\\
&+\frac{\alpha_1\beta_2Q_3(\psi)\sin(\psi)\cos^4(\psi)+\alpha_1\beta_0Q_3(\psi)\sin^3(\psi)+\cos^2(\psi)}{\chi^2\,A^3}\Bigg)d\psi,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{311} = \frac{6}{r_0^2\chi(\pi)}\int_0^{\pi}\Bigg(&\frac{\mathcal{A}P_3(\psi)\sin^4(\psi)\cos(\psi)-P_3(\psi)\sin^2(\psi)\cos^3(\psi)}{\chi^2\,A^3}\\
&+\frac{+\mathcal{A}Q_3(\psi)\sin^3(\psi)\cos^2(\psi)-\mathcal{A}^2Q_3(\psi)\sin^5(\psi)}{\chi^2\,A^3}\Bigg)d\psi,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{312} = \frac{6}{r_0^2\chi(\pi)}\int_0^{\pi}\Bigg(&\frac{P_3(\psi)\sin^2(\psi)\cos^3(\psi)+\mathcal{B}P_3(\psi)\sin^4(\psi)\cos(\psi)}{\chi^2\,A^3}\\
&+\frac{Q_3(\psi)\sin(\psi)\cos^4(\psi)-2\mathcal{A}Q_3(\psi)\sin^3(\psi)\cos^2(\psi)}{\chi^2\,A^3}\\
&+\frac{-2\mathcal{A}\mathcal{B}Q_3(\psi)\sin^5(\psi)+\mathcal{B}Q_3(\psi)\sin^3(\psi)\cos^2(\psi)}{\chi^2\,A^3}\Bigg)d\psi,
\end{aligned}$$

$$\mathcal{L}_{313} = \frac{6}{r_0^2\chi(\pi)}\int_0^{\pi}\left(\frac{-Q_3(\psi)\sin(\psi)\cos^4(\psi) - \mathcal{B}^2 Q_3(\psi)\sin^5(\psi)}{\chi^2\, A^3}\right.$$
$$\left. + \frac{-2\mathcal{B}Q_3(\psi)\sin^3(\psi)\cos^2(\psi)}{\chi^2\, A^3}\right)d\psi,$$

$$\mathcal{L}_{320} = \int_0^{\pi} \frac{-\beta_0 P_3(\psi)\sin^2(\psi) - \beta_2 P_3(\psi)\cos^2(\psi) + \alpha_1 Q_3(\psi)\sin(\psi)\cos(\psi)}{\chi A^2}\,d\psi,$$

$$\mathcal{L}_{321} = \int_0^{\pi} \frac{\mathcal{A}Q_3(\psi)\sin^2(\psi) - P_3(\psi)\sin(\psi)\cos(\psi)}{\chi A^2}\,d\psi,$$

$$\mathcal{L}_{322} = \int_0^{\pi} \frac{Q_3(\psi)\cos^2(\psi) + \mathcal{B}Q_3(\psi)\sin^2(\psi)}{\chi A^2}\,d\psi.$$

References

[1] V.I. Arnold, Loss of stability of self-oscillations close to resonance and versal deformations of equivariant vector fields, *Funct. Anal. Appl.* **11** (2) (1977), 85 – 92.

[2] A. Buică, J. Giné, J. Llibre, Bifurcation of limit cycles from a polynomial degenerate center, *Adv. Nonlinear Stud.* **10** (2010), 597 – 609.

[3] J. Chavarriga, J. Giné, M. Grau, Integrable systems via polynomial inverse integrating factors, *Bull. Sci. Math.* **126** (2002), 315 – 331.

[4] T. Chen, J. Llibre, Limit cycles of a second-order differential equation, *Appl. Math. Lett.* **88** (2019), 111 – 117.

[5] C. Christopher, Estimating limit cycle bifurcations from centers, in "Differential Equations with Symbolic Computation, 23 – 35, Trends in Mathematics, Birkhäuser, Basel, 2005.

[6] J. Francoise, H. He, D. Xiao, The number of limit cycles bifurcating from the period annulus of quasi-homogeneous Hamiltonian systems at any order, *J. Differential Equations* **276** (2021), 318 – 341.

[7] J. Giné, The center problem for a linear center perturbed by homogeneous polynomials, *Acta Math. Sin. (Engl. Ser.)* **22** (6) (2006), 1613 – 1620.

[8] J. Giné, L.F. Gouveia, J. Torregrosa, Lower bounds for the local cyclicity for families of centers, *J. Differential Equations* **275** (2021), 309 – 331.

[9] L.F. Gouveia, J. Torregrosa, Lower bounds for the local cyclicity of centers using high order developments and parallelization, *J. Differential Equations* **271** (2021), 447 – 479.

[10] D. Hilbert, Mathematical problems, *Bull. Amer. Math. Soc.* **8** (1902), 437 – 479. (M. Newton, Translation).

[11] B. Huang, L. Peng, Y. Cui, On the number of limit cycles bifurcating from a quartic reversible center, *Mediterr. J. Math.* **19** (2022), paper No. 220, 26 pp.

[12] J. Llibre, C. Pantazi, Limit cycles bifurcating from a degenerate center, *Math. Comput. Simulation* **120** (2016), 1 – 11.

[13] M. MolaeiDerakhtenjani, O. RabieiMotlagh, H.M. MohammadiNejad, A constructive approach to degenerate center problem, *Int. J. Dyn. Syst. Differ. Equ.* **12** (3) (2022), 247 – 266.

[14] R. Prohens, J. Torregrosa, New lower bounds for the Hilbert numbers using reversible centers, *Nonlinearity* **32** (2019), 331 – 355.

[15] J. Wang, L. Zhao, J. Zhou, On the number of limit cycles bifurcating from the linear center with an algebraic switching curve, *Qual. Theory Dyn. Syst.* **21** (2022), paper No. 87, 42 pp.

[16] Y. Xiong, M. Han, D. Xiao, The maximal number of limit cycles bifurcating from a Hamiltonian triangle quadratic systems, *J. Differential Equations* **280** (2021), 139 – 178.

[17] J. Yang, P. Yu, Nine limit cycles around a singular point by perturbing a cubic Hamiltonian system with a nilpotent center, *Appl. Math. Comput.* **298** (2017), 141 – 152.

[18] P. Yang, J. Yu, The number of limit cycles from a cubic center by the Melnikov function of any order, *J. Differential Equations* **268** (2020), 1463 – 1494.

[19] H. Żoladek, Eleven small limit cycles in a cubic vector field, *Nonlinearity* **8** (1995), 843 – 860.