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On Lie ideals satisfying certain differential identities in prime rings

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Abstract: Let R be a prime ring of characteristic not 2, L a nonzero square closed Lie ideal of R and let $F : R \to R$, $G : R \to R$ be generalized derivations associated with derivations $d : R \to R$, $g : R \to R$ respectively. In this paper, we study several conditions that imply that the Lie ideal is central. Moreover, it is shown that the assumption of primeness of R can not be removed.

Key words: Prime ring, derivation, generalized derivation, Lie ideal.

MSC (2020): 16N60, 16W25, 16R50.

1. Introduction

Throughout this paper R denotes an associative prime ring with center $Z(R)$. A ring R is called a prime ring if for any $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$. The symbol $[x, y]=xy - yx$ stands for the commutator operator for $x, y \in R$ and the symbol $x \circ y = xy + yx$ stands for the anti-commutator operator for $x, y \in R$. An additive subgroup L of R is said to be a Lie ideal of R if $[x, r] \in L$ for any $r \in R$ and $x \in L$. A Lie ideal L is said to be square closed if $u^2 \in L$ for all $u \in L$.

A map $d : R \to R$ is called a derivation, if $d(x + y) = d(x) + d(y)$ and $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive map $F : R \to R$ is said to be a generalized derivation of R, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Obviously, every

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derivation is a generalized derivation of R, but the converse is not necessarily true.

Several authors have studied many identities in prime and semiprime rings, involving derivations and generalized derivations, that imply commutativity of the ring. We refer the reader to [1, 2, 7, 8, 9, 11, 12, 15, 16, 18], where further references can be found.

The identities

(i) $F(xy) - xy \in Z(R)$ for all $x, y \in I$,

(ii) $F(xy) + xy \in Z(R)$ for all $x, y \in I$,

(iii) $F(xy) - yx \in Z(R)$ for all $x, y \in I$,

(iv) $F(xy) + yx \in Z(R)$ for all $x, y \in I$,

(v) $F(x)F(y) - xy \in Z(R)$ for all $x, y \in I$,

(vi) $F(x)F(y) + xy \in Z(R)$ for all $x, y \in I$,

when R is a prime ring, F is a generalized derivation of R associated with a non-zero derivation d and I is a non-zero two-sided ideal of R, were studied by Ashraf et al. in [2], proving that any of them implies the commutativity of R.

In a similar way, the identities (i) $F(x)F(y) - yx \in Z(R)$ and (ii) $F(x)F(y) + yx \in Z(R)$ for all x, y in some suitable subset of R were studied in [9] by Dhara et al. Recently, Tiwari et al. (see [18]) studied identities involving three summands and again obtained the commutativity of the prime ring R.

Identities involving the commutator and the anti-commutator have also been considered. In [4], Bell and Daif proved that if U is a nonzero right ideal of a semiprime ring R and d is a nonzero derivation of R such that $[d(x), d(y)] = [x, y]$ for all $x, y \in U$, then $U \subseteq Z(R)$. Ashraf et al. (see [3]) got the commutativity of a prime ring R satisfying any one of the following conditions:

(i) $d(x) \circ F(y) = 0$ for all $x, y \in I$,

(ii) $[d(x), F(y)] = 0$ for all $x, y \in I$,

(iii) $d(x) \circ F(y) = x \circ y$ for all $x, y \in I$,

(iv) $d(x) \circ F(y) + x \circ y = 0$ for all $x, y \in I$,

(v) $[d(x), F(y)] = [x, y]$ for all $x, y \in I$,

(vi) $[d(x), F(y)] + [x, y] = 0$ for all $x, y \in I$,

(vii) $d(x)F(y) \pm xy \in Z(R)$ for all $x, y \in I$, where I is a nonzero ideal of R and F is a generalized derivation of R associated with a nonzero derivation d. In [13], Shuliang studied the above identities for a square closed Lie ideal L of a prime ring R and obtained that either $d=0$ or $L \subseteq Z(R)$ and in [10], Dhara et al. studied the above identities in a semiprime ring.

In this paper we consider the following identities:

(i) $F(u) \circ v \pm d(u) \circ F(v) \pm u \circ v = 0$ for all $u, v \in L$,

(ii) $[F(u), v] \pm [d(u), F(v)] \pm [u, v] = 0$ for all $u, v \in L$,

(iii) $F([u, v]) \pm [d(u), F(v)] \pm [u, v] = 0$ for all $u, v \in L$,

(iv) $F(u \circ v) \pm [d(u), F(v)] \pm u \circ v = 0$ for all $u, v \in L$,

(v) $F(u)G(v) \pm d(u)F(v) \pm uv \in Z(R)$ for all $u, v \in L$,

(vi) $G(uv) \pm d(u)F(v) \pm F(vu) = 0$ for all $u, v \in L$,

(vii) $F(uv) \pm F(v)F(u) \pm uv \in Z(R)$ for all $u, v \in L$,

where L is a square closed Lie ideal in a prime ring R and F is a generalized derivation of R associated with a derivation d.

2. Preliminaries

In this paper R will denote always a prime associative ring of characteristic not equal to 2. This implies that for any element x in R, $2x = 0$ implies $x = 0$.

Let L be a square closed Lie ideal of R. Thus $u^2 \in L$ for all $u \in L$. Now for $u, v \in L$, $uv + vu = (u+v)^2 - u^2 - v^2 \in L$ and by definition of Lie ideal $uv - vu \in L$. Combining these two we get $2uv \in L$ for all $u, v \in L$.

The following identities shall be used very frequently throughout:

(1) $[xy, z] = x[y, z] + [x, z]y$ for all $x, y, z \in R$,

(2) $[x, yz] = y[x, z] + [x, y]z$ for all $x, y, z \in R$,

(3) $(x \circ yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$ for all $x, y, z \in R$,

(4) $(xy \circ z) = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$ for all $x, y, z \in R$.

Moreover in order to prove our results, we need the following facts:

Lemma 2.1. ([5, Lemma 2]) *If $L \nsubseteq Z(R)$ is a Lie deal of R, then $C_R(L) = Z(R)$.*

Lemma 2.2. ([5, Lemma 4]) *If $L \nsubseteq Z(R)$ is a Lie ideal of R and $aLb = 0$, then either $a = 0$ or $b = 0$.*

Lemma 2.3. ([5, Lemma 5]) *If d is a nonzero derivation of R and L a Lie ideal of R such that $d(L) = (0)$, then $L \subseteq Z(R)$.*

Lemma 2.4. ([9, Lemma 2.5]) *Let L be a nonzero Lie ideal of R and $V = \{u \in L \mid d(u) \in L\}$. Then V is also a nonzero Lie ideal of R. Moreover, if L is noncentral, then V is also noncentral.*

Lemma 2.5. ([14, Theorem 5]) *Let d be a nonzero derivation of R and L a nonzero Lie ideal of R such that $[u, d(u)] \in Z(R)$ for all $u \in L$. Then $L \subseteq Z(R)$.*

Lemma 2.6. ([17, Lemma 2.6]) *Let $L \not\subseteq Z(R)$ be a Lie ideal of R and $a, b \in R$ such that one of a, b is in L. If $aub + bua = 0$ for all $u \in L$, then $aub = bua = 0$ for all $u \in L$. Consequently, $a = 0$ or $b = 0$.*

Lemma 2.7. *Let d be a nonzero derivation of R and L a nonzero square closed Lie ideal of R. Suppose that $V = \{u \in L \mid d(u) \in L\}$. If $[d(u), d(v)] = 0$ for all $u \in L$ and $v \in V$, then $L \subseteq Z(R)$.*

Proof. If $L \subseteq Z(R)$, we are done. Thus on contrary, we assume that $L \not\subseteq Z(R)$. By Lemma 2.4, V is also noncentral Lie ideal of R such that $V \subseteq L$. We have that

$$[d(u), d(v)] = 0 \tag{1}$$

for all $u \in L$ and $v \in V$. Replacing u by $2uv$ and then using char $(R) \neq 2$, we have

$$[d(u)v + ud(v), d(v)] = 0 \tag{2}$$

for all $u \in L$ and $v \in V$. By using (1), we have

$$d(u)[v, d(v)] + [u, d(v)]d(v) = 0 \tag{3}$$

for all $u \in L$ and $v \in V$. Replacing u by $2wu$ in (3) and then using it, we have

$$d(w)u[v, d(v)] + [w, d(v)]ud(v) = 0 \tag{4}$$

for all $u, w \in L$ and $v \in V$. In particular,

$$d(v)u[v, d(v)] + [v, d(v)]ud(v) = 0 \tag{5}$$

for all $u \in L$ and $v \in V$. Invoking Lemma 2.6, we find $d(v)u[v, d(v)] = 0$ for all $v \in V$ and $u \in U$. Again by Lemma 2.2, $d(v) = 0$ or $[v, d(v)] = 0$. In any case we have $[v, d(v)] = 0$ for all $v \in V$. By Lemma 2.5, $V \subseteq Z(R)$, a contradiction. ∎

The following lemmas are the particular cases of [6, Theorem 1].

LEMMA 2.8. *Let d be a nonzero derivation of R and L a nonzero Lie ideal of R such that $u[[d(u), u], u] = 0$ for all $u \in L$. Then $L \subseteq Z(R)$.*

LEMMA 2.9. *Let F be a nonzero generalized derivation of R with associated nonzero derivation d and L a nonzero square closed Lie ideal of R such that $[F(u), u] = 0$ for all $u \in L$. Then $L \subseteq Z(R)$.*

3. Main results

THEOREM 3.1. *Let L be a nonzero square closed Lie ideal of R and F be a generalized derivation of R associated to nonzero derivation d of R. If $F(u) \circ v \pm d(u) \circ F(v) \pm u \circ v = 0$ for all $u, v \in L$, then $L \subseteq Z(R)$.*

Proof. We assume on the contrary that $L \nsubseteq Z(R)$. By hypothesis, we have

$$F(u) \circ v \pm d(u) \circ F(v) \pm u \circ v = 0 \tag{6}$$

for all $u, v \in L$. Replacing v by $2vw$ in (6), we obtain

$$2\{F(u) \circ vw \pm d(u) \circ (F(v)w + vd(w)) \pm u \circ (vw)\} = 0 \tag{7}$$

for all $u, v, w \in L$. Since characteristic of R is not 2, we have

$$\begin{aligned}(F(u) \circ v)w - v[F(u), w] \pm (d(u) \circ F(v))w \mp F(v)[d(u), w]\\ \pm d(u) \circ (vd(w)) \pm (u \circ v)w \mp v[u, w] = 0\end{aligned} \tag{8}$$

for all $u, v, w \in L$. Right multiplying (6) by w and then subtracting from (8), we get

$$-v[F(u), w] \mp F(v)[d(u), w] \pm d(u) \circ (vd(w)) \mp v[u, w] = 0 \tag{9}$$

for all $u, v, w \in L$.

Substituting v by $2pv$ in (9) and then using characteristic of R is not 2, we obtain

$$\begin{aligned} -pv[F(u), w] \mp F(pv)[d(u), w] \pm p(d(u) \circ (vd(w))) \\ \pm[d(u), p]vd(w) \mp pv[u, w] = 0 \end{aligned} \tag{10}$$

for all $u, v, w, p \in L$. Left multiplying (9) by p and then subtracting from (10), we get

$$\mp(F(pv) - pF(v))[d(u), w] \pm [d(u), p]vd(w) = 0 \tag{11}$$

for all $u, v, w, p \in L$. Replacing w by $2wt$ in (11) and then using characteristic of R is not 2 and (11), we obtain

$$\mp(F(pv) - pF(v))w[d(u), t] \pm [d(u), p]vwd(t) = 0 \tag{12}$$

for all $u, v, w, p, t \in L$.

Since $[q, d(s)] \in L$, thus $2[q, d(s)]w \in L$ for all $q, s, w \in L$. Replacing w by $2[q, d(s)]w$ in (12) and then using characteristic of R is not 2, we get

$$\mp(F(pv) - pF(v))[q, d(s)]w[d(u), t] \pm [d(u), p]v[q, d(s)]wd(t) = 0 \tag{13}$$

for all $u, v, w, p, q, s, t \in L$. Now re-writing the relation (11), we can write

$$\mp(F(pv) - pF(v))[q, d(s)] \pm [p, d(s)]vd(q) = 0 \tag{14}$$

for all $v, p, q, s \in L$.

By using (14) in (13), we have

$$\mp[p, d(s)]vd(q)w[d(u), t] \pm [d(u), p]v[q, d(s)]wd(t) = 0 \tag{15}$$

for all $u, v, w, p, q, s, t \in L$. We re-write it as

$$\mp[d(s), p]vd(q)w[d(u), t] \pm [d(u), p]v[d(s), q]wd(t) = 0 \tag{16}$$

for all $u, v, w, p, q, s, t \in L$.

Let $V = \{u \in L \mid d(u) \in L\}$. By Lemma 2.4, V is also noncentral Lie ideal of R such that $V \subseteq L$. Thus if $s \in V$, then $d(s) \in L$. Thus replacing p by $2d(s)p$ in (16) and then using characteristic of R is not 2 and the relation (16), we get

$$\pm[d(u), d(s)]pv[d(s), q]wd(t) = 0 \tag{17}$$

for all $u, v, w, p, q, t \in L$ and $s \in V$. By Lemma 2.2, for each $s \in V$, either $[d(u), d(s)] = 0$ for all $u \in L$ or $[d(s), q]wd(t) = 0$ for all $w \in L$.

Now we consider two subgroups

$$T_1 = \{s \in V \,:\, [d(u), d(s)] = 0 \text{ for all } u \in L\},$$
$$T_2 = \{s \in V \,:\, [d(s), q]wd(t) = 0 \text{ for all } q, w, t \in L\}.$$

It is very clear that T_1 and T_2 are two additive subgroups of V such that $T_1 \cup T_2 = V$. Since an additive subgroups can not be union of its two proper subgroups, we have either $T_1 = V$ or $T_2 = V$, that is, either $[d(u), d(s)] = 0$ for all $u \in L$ and $s \in V$ or $[d(s), q]wd(t) = 0$ for all $q, w, t \in L$ and $s \in V$.

If $[d(u), d(s)] = 0$ for all $u, s \in L$, then by Lemma 2.7, $L \subseteq Z(R)$, a contradiction.

On the other hand, if $[d(s), q]wd(t) = 0$ for all $q, w, t \in L$ and $s \in V$, then $[d(s), q]w[d(s), q] = 0$ for all $w \in L$ and $q, s \in V$. By Lemma 2.2, $[d(s), q] = 0$ for all $q, s \in V$. By Lemma 2.5, $V \subseteq Z(R)$, a contradiction. Thus the proof of the theorem is completed. ∎

THEOREM 3.2. *Let L be a nonzero square closed Lie ideal of R and F be a generalized derivation of R associated to nonzero derivation d of R. If $[F(u), v] \pm [d(u), F(v)] \pm [u, v] = 0$ for all $u, v \in L$, then $L \subseteq Z(R)$.*

Proof. We assume on the contrary that $L \not\subseteq Z(R)$. By hypothesis, we have

$$[F(u), v] \pm [d(u), F(v)] \pm [u, v] = 0 \tag{18}$$

for all $u, v \in L$. Replacing v by $2vw$ in (18), we obtain

$$2\{[F(u), vw] \pm [d(u), F(v)w + vd(w)] \pm [u, vw]\} = 0 \tag{19}$$

for all $u, v, w \in L$. Since characteristic of R is not 2, we have

$$\begin{aligned} [F(u), v]w + v[F(u), w] \pm ([d(u), F(v)]w + F(v)[d(u), w]) \\ \pm [d(u), vd(w)] \pm ([u, v]w + v[u, w]) = 0 \end{aligned} \tag{20}$$

for all $u, v, w \in L$. Right multiplying (18) by w and then subtracting from (20), we get

$$v[F(u), w] \pm F(v)[d(u), w]) \pm [d(u), vd(w)] \pm v[u, w] = 0 \tag{21}$$

for all $u, v, w \in L$.

Replacing v by $2pv$ in (21) and then usingcharacteristic of R is not 2, we obtain

$$\begin{aligned} pv[F(u), w] \pm F(pv)[d(u), w]) \pm p[d(u), vd(w)] \\ \pm[d(u), p]vd(w) \pm pv[u, w] = 0 \end{aligned} \tag{22}$$

for all $u, v, w, p \in L$. Left multiplying (21) by p and then subtracting from (22), we get

$$\pm(F(pv) - pF(v))[d(u), w]) \pm [d(u), p]vd(w) = 0 \tag{23}$$

for all $u, v, w, p \in L$. Using the same arguments used in the proof of Theorem 3.1 we get a contradiction, which proves this theorem. ∎

THEOREM 3.3. *Let L be a nonzero square closed Lie ideal of R and F be a generalized derivation of R associated to nonzero derivation d of R. If $F([u, v]) \pm [d(u), F(v)] \pm [u, v] = 0$ for all $u, v \in L$, then $L \subseteq Z(R)$.*

Proof. We assume on the contrary that $L \not\subseteq Z(R)$. By hypothesis, we have

$$F([u, v]) \pm [d(u), F(v)] \pm [u, v] = 0 \tag{24}$$

for all $u, v \in L$. Replacing u by $2uv$ in (24), we obtain

$$2(F([u, v])v + [u, v]d(v) \pm [d(u)v, F(v)] \pm [ud(v), F(v)] \pm [u, v]v) = 0 \tag{25}$$

for all $u, v \in L$. Since characteristic of R is not 2, we have

$$F([u, v])v + [u, v]d(v) \pm [d(u)v, F(v)] \pm [ud(v), F(v)] \pm [u, v]v = 0 \tag{26}$$

for all $u, v \in L$. Right multiplying (24) by v and then subtracting from (26), we get

$$[u, v]d(v) \pm d(u)[v, F(v)] \pm [ud(v), F(v)] = 0 \tag{27}$$

for all $u, v \in L$. Replacing u by $2vu$ in (27) and then using characteristic of R is not 2, we obtain

$$v[u, v]d(v) \pm (d(v)u + vd(u))[v, F(v)] \pm [vud(v), F(v)] = 0$$

that is,

$$\begin{aligned} v[u,v]d(v) \pm d(v)u[v,F(v)] \pm vd(u)[v,F(v)] \\ \pm v[ud(v),F(v)] \pm [v,F(v)]ud(v) = 0 \end{aligned} \tag{28}$$

for all $u,v \in L$. Left multiplying (27) by v and then subtracting from (28), we get

$$d(v)u[v,F(v)] \pm [v,F(v)]ud(v) = 0 \tag{29}$$

for all $u,v \in L$. Since $[v,F(v)] \in L$, thus $2u[v,F(v)] \in L$ and so $4u[v,F(v)]w \in L$ for all $u,v,w \in L$. Therefore, replacing u by $4u[v,F(v)]w$, where $w \in L$, in (29) and then since characteristic of R is not 2, we obtain

$$d(v)u[v,F(v)]w[v,F(v)] \pm [v,F(v)]u[v,F(v)]wd(v) = 0 \tag{30}$$

for all $u,v,w \in L$. Right multiplying (29) by $w[v,F(v)]$ and then subtracting from (30), we get

$$[v,F(v)]u([v,F(v)]wd(v) - d(v)w[v,F(v)]) = 0 \tag{31}$$

for all $u,v,w \in L$. By Lemma 2.2, for each $v \in V$, either $[v,F(v)] = 0$ or $[v,F(v)]wd(v) - d(v)w[v,F(v)] = 0$ for all $w \in L$. Since the first case implies the second case, we conclude that

$$[v,F(v)]wd(v) - d(v)w[v,F(v)] = 0 \tag{32}$$

for all $v,w \in L$. Now (32) and (29) together implies that

$$2d(v)u[v,F(v)] = 0$$

for all $u,v \in L$. Since characteristic of R is not 2, therefore $d(v)u[v,F(v)] = 0$ for all $u,v \in L$. By primeness of R, for each $v \in L$, it implies that either $d(v) = 0$ or $[v,F(v)] = 0$.

Let v be an element of L such that $d(v) = 0$. This gives $F(uv) = F(u)v$ for all $u \in L$. Hence, (24) gives that

$$F([v,u]) \pm [v,u] = 0 \tag{33}$$

for all $u \in L$. Replacing u by $2vu$ in (33) and using characteristic of R is not 2, we have

$$F(v)[v,u] \pm vd([v,u]) \pm v[v,u] = 0 \tag{34}$$

for all $u \in L$. Replacing u by $2vu$ in (34) and using characteristic of R is not 2, we have

$$F(v)v[v,u] \pm v^2 d([v,u]) \pm v^2[v,u] = 0 \tag{35}$$

for all $u \in L$. Left multiplying (34) by v and then subtracting from (35), we obtain

$$[F(v),v][v,u] = 0 \tag{36}$$

for all $u \in L$. Replacing u by $2tu$ in (36) and then using characteristic of R is not 2 and (36), we have

$$[F(v),v]t[v,u] = 0$$

for all $t,u \in L$. Since R is prime, $[F(v),v] = 0$ or $[v,U] = (0)$. By Lemma 2.1, $[v,U] = (0)$ implies $v \in Z(R)$.

Thus in any case, we have $[F(v),v] = 0$ for all $v \in L$. This implies by Lemma 2.9, $L \subseteq Z(R)$, a contradiction. ∎

THEOREM 3.4. *Let L be a nonzero square closed Lie ideal of R and F be a generalized derivation of R associated to nonzero derivation d of R. If $F(u \circ v) \pm [d(u),F(v)] \pm u \circ v = 0$ for all $u,v \in L$, then $L \subseteq Z(R)$.*

Proof. We assume on the contrary that $L \not\subseteq Z(R)$. We begin with the situation

$$F(u \circ v) \pm [d(u),F(v)] \pm u \circ v = 0 \tag{37}$$

for all $u,v \in L$. Replacing u by $2uv$ in (37), we obtain

$$\begin{aligned} 2(F(u \circ v)v + (u \circ v)d(v) \pm [d(u)v, F(v)] \\ \pm [ud(v),F(v)] \pm (u \circ v)v) = 0 \end{aligned} \tag{38}$$

for all $u,v \in L$. Since characteristic of R is not 2, we have

$$\begin{aligned} F(u \circ v)v + (u \circ v)d(v) \pm [d(u)v, F(v)] \\ \pm [ud(v),F(v)] \pm (u \circ v)v = 0 \end{aligned} \tag{39}$$

that is

$$\begin{aligned} F(u \circ v)v + (u \circ v)d(v) \pm [d(u),F(v)]v \\ \pm d(u)[v,F(v)] \pm [ud(v),F(v)] \pm (u \circ v)v = 0 \end{aligned} \tag{40}$$

for all $u, v \in L$. Right multiplying (37) by v and then subtracting from (40), we get

$$(u \circ v)d(v) \pm d(u)[v, F(v)] \pm [ud(v), F(v)] = 0 \tag{41}$$

for all $u, v \in L$. Replacing u by $2vu$ in (41) and then using characteristic of R is not 2, we obtain

$$\begin{aligned} v(u \circ v)d(v) \pm (vd(u) + d(v)u)[v, F(v)] \\ \pm v[ud(v), F(v)] \pm [v, F(v)]ud(v) = 0 \end{aligned} \tag{42}$$

for all $u, v \in L$. Left multiplying (41) by v and then subtracting from (42), we get

$$d(v)u[v, F(v)] + [v, F(v)]ud(v) = 0 \tag{43}$$

for all $u, v \in L$. Identity (43) coincides with identity (29) so arguing as in Theorem 3.3 we get

$$d(v)u[v, F(v)] = 0$$

for all $u, v \in L$. By primeness of R, for each $v \in L$, it implies that either $d(v) = 0$ or $[v, F(v)] = 0$. Let v be an element of L such that $d(v) = 0$. This gives $F(uv) = F(u)v$ for all $u \in L$. Hence, (37) gives that

$$F(v \circ u) \pm v \circ u = 0 \tag{44}$$

for all $u \in L$. Replacing u by $2vu$ in (44) and using characteristic of R is not 2, we have

$$F(v)(v \circ u) \pm vd(v \circ u) \pm v(v \circ u) = 0 \tag{45}$$

for all $u \in L$. Replacing u by $2vu$ in (45) and using characteristic of R is not 2, we have

$$F(v)v(v \circ u) \pm v^2 d(v \circ u) \pm v^2(v \circ u) = 0 \tag{46}$$

for all $u \in L$. Left multiplying (45) by v and then subtracting from (46), we obtain

$$[F(v), v](v \circ u) = 0 \tag{47}$$

for all $u \in L$. Replacing u by $2wu$ in (47) and then using characteristic of R is not 2, we have

$$0 = [F(v), v](v \circ wu) = [F(v), v]w[u, v] + [F(v), v](v \circ w)u$$

for all $u, w \in L$. Since $[F(v), v](v \circ w) = 0$ for all $v, w \in L$, we have $[F(v), v]w[u, v] = 0$ for all $u, v, w \in L$. Since R is prime, $[F(v), v] = 0$ or $[v, L] = (0)$. By Lemma 2.1, $[v, L] = (0)$ implies $v \in Z(R)$.

Thus in any case, we have $[F(v), v] = 0$ for all $v \in L$. This implies by Lemma 2.9, that $L \subseteq Z(R)$, which is a contradiction. ∎

THEOREM 3.5. *Let L be a nonzero square closed Lie ideal of R and F and G be two generalized derivations of R associated to derivations $d(\neq 0)$ and g of R respectively. If $F(u)G(v) \pm d(u)F(v) \pm uv \in Z(R)$ for all $u, v \in L$ and $d \pm g \neq 0$, then $L \subseteq Z(R)$.*

Proof. We assume on the contrary that $L \nsubseteq Z(R)$. By hypothesis, we have

$$F(u)G(v) \pm d(u)F(v) \pm uv \in Z(R) \tag{48}$$

for all $u, v \in L$. Replacing v by $2vw$ in (48), we have

$$2(F(u)G(v) \pm d(u)F(v) \pm uv)w + 2F(u)vg(w) \pm 2d(u)vd(w) \in Z(R) \tag{49}$$

for all $u, v, w \in L$. Since characteristic of R is not 2, we have

$$(F(u)G(v) \pm d(u)F(v) \pm uv)w + F(u)vg(w) \pm d(u)vd(w) \in Z(R) \tag{50}$$

for all $u, v, w \in L$. Commuting both sides of (50) with w, we obtain

$$\begin{aligned}&[(F(u)G(v) \pm d(u)F(v) \pm uv)w, w]\\&\quad+[F(u)vg(w) \pm d(u)vd(w), w] = 0\end{aligned} \tag{51}$$

for all $u, v, w \in L$. Using (48), we obtain

$$[F(u)vg(w) \pm d(u)vd(w), w] = 0 \tag{52}$$

for all $u, v, w \in L$. Replacing u by $2uw$ in (52) and then using characteristic of R is not 2, we have

$$\begin{aligned}&[F(u)wvg(w), w] \pm [d(u)wvd(w), w]\\&\quad+[ud(w)v(g(w) \pm d(w)), w] = 0\end{aligned} \tag{53}$$

for all $u, v, w \in L$. Replacing v by $2wv$ in (52) and using characteristic of R is not 2, we have

$$[F(u)wvg(w), w] \pm [d(u)wvd(w), w] = 0 \tag{54}$$

for all $u, v, w \in L$. Subtracting (54) from (53), we obtain

$$[ud(w)v(g(w) \pm d(w)), w] = 0$$

for all $u, v, w \in L$. Replacing u by $2tu$ and then using characteristic of R is not 2, it gives

$$[t, w]ud(w)v(g(w) \pm d(w)) = 0 \tag{55}$$

for all $u, v, w, t \in L$. Thus for each $w \in L$, either $[L, w] = (0)$ or $d(w) = 0$ or $d(w) \pm g(w) = 0$.

Now the first case i.e., $[L, w] = (0)$ implies $w \in Z(R)$ by Lemma 2.1. Then by (50), we have

$$F(u)vg(w) \pm d(u)vd(w) \in Z(R) \tag{56}$$

for all $u, v \in L$. Replacing u by $2ut$, where $t \in L$, we get

$$2(F(u)tvg(w) + ud(t)vg(w) \pm d(u)tvd(w) \pm ud(t)vd(w)) \in Z(R) \tag{57}$$

for all $u, v, t \in L$. Now we replace v by $2tv$ in (56) and obtain

$$2(F(u)tvg(w) \pm d(u)tvd(w)) \in Z(R) \tag{58}$$

for all $u, v, t \in L$. Subtracting (58) from (57) and then using characteristic of R is not 2, we have

$$ud(t)v(g(w) \pm d(w)) \in Z(R) \tag{59}$$

for all $u, v, t \in L$. Since $g(w) \pm d(w) \in Z(R)$ for $w \in Z(R)$, we have $0 = [ud(t)v(g(w) \pm d(w)), r] = [ud(t)v, r](g(w) \pm d(w))$ for all $u, v, t \in L$ and $r \in R$. Since center of a prime ring contains no divisor of zero, either $[ud(t)v, r] = 0$ for all $u, v, t \in L$ and $r \in R$ or $g(w) + d(w) = 0$. We consider the first case i.e., $[ud(t)v, r] = 0$ for all $u, v, t \in L$ and $r \in R$. Replacing u by $2su$, where $s \in L$, we obtain $0 = 2[sud(t)v, r] = 2s[ud(t)v, r] + 2[s, r]ud(t)v = 2[s, r]ud(t)v$ for all $u, v, t, s \in L$ and $r \in R$. By Lemma 2.2, either $[L, R] = (0)$ or $d(L) = (0)$. By Lemma 2.3 both cases give $L \subseteq Z(R)$, a contradiction.

Thus we conclude that for each $w \in L$, either $d(w) = 0$ or $d(w) \pm g(w) = 0$. But the sets $\{w \in L : d(w) = 0\}$ and $\{w \in L : d(w) \pm g(w) = 0\}$ are two additive subgroups of L whose union is L. By using the same argument used in Lemma 2.9 we get that either $d(L) = (0)$ or $(d \pm g)(L) = (0)$. Since L is noncentral, by Lemma 2.3, either $d = 0$ or $d \pm g = 0$, a contradiction. ∎

Theorem 3.6. *Let L be a nonzero square closed Lie ideal of R and F and G be generalized derivations of R associated to derivations d and g of R, respectively. If $d \neq 0$, $G(uv) \pm d(u)F(v) \pm F(vu) = 0$ for all $u, v \in L$, then $L \subseteq Z(R)$.*

Proof. We assume on the contrary that $L \nsubseteq Z(R)$. By hypothesis, we have

$$G(uv) \pm d(u)F(v) \pm F(vu) = 0 \tag{60}$$

for all $u, v \in L$. Replacing v by $2vu$ in the above relation and using $\mathrm{char}(R) \neq 2$, we get

$$(G(uv) \pm d(u)F(v) \pm F(vu))u + uvg(u) \pm d(u)vd(v) \pm vud(u) = 0. \tag{61}$$

Using (60) this gives

$$uvg(u) \pm d(u)vd(v) \pm vud(u) = 0 \tag{62}$$

for all $u, v \in L$. Again replacing v by $2uv$ and using characteristic of R is not 2, we have

$$u^2vg(u) \pm d(u)uvd(v) \pm uvud(u) = 0 \tag{63}$$

for all $u, v \in L$. Left multiplying (62) by u and subtracting from (63), we have

$$[d(u), u]vd(u) = 0 \tag{64}$$

for all $u, v \in L$. By primeness of R, for each $u \in L$, we have either $[d(u), u] = 0$ or $d(u) = 0$. Thus in each case, we have $[d(u), u] = 0$ for all $u \in L$. Now if $d \neq 0$, by Lemma 2.9, $[d(u), u] = 0$ for all $u \in L$ implies $L \subseteq Z(R)$, a contradiction. ∎

Theorem 3.7. *Let L be a nonzero square closed Lie ideal of R and F be a generalized derivation of R associated to the nonzero derivation d of R. If $F(uv) \pm F(v)F(u) \pm uv \in Z(R)$ for all $u, v \in L$, then $L \subseteq Z(R)$.*

Proof. We assume on the contrary that $L \nsubseteq Z(R)$. First we consider the relation

$$F(uv) + F(v)F(u) + uv \in Z(R) \tag{65}$$

for all $u, v \in L$. Replacing u with $2uw$ in (65) and using characteristic of R is not 2, we have

$$F(u)wv + ud(wv) + F(v)(F(u)w + ud(w)) + uwv \in Z(R) \tag{66}$$

for all $u, v, w \in L$.

Commuting with w, we have

$$\begin{aligned}[F(u)wv, w] + [ud(wv), w] + [F(v)F(u), w]w \\ +[F(v)ud(w) + uwv, w] = 0\end{aligned} \tag{67}$$

for all $u, v, w \in L$. From (65), we can write that $[F(uv)+F(v)F(u)+uv, w] = 0$ for all $u, v, w \in L$, that is, $[F(v)F(u), w] = -[F(uv)+uv, w]$ for all $u, v, w \in L$. Using this (67) reduces to

$$\begin{aligned}[F(u)wv, w] + [ud(wv), w] - [F(uv) + uv, w]w \\ +[F(v)ud(w) + uwv, w] = 0\end{aligned} \tag{68}$$

for all $u, v, w \in L$. Replacing v by w^2 in (68), we have

$$\begin{aligned}[F(u)w^3, w] + [ud(w^3), w] - [F(uw^2) + uw^2, w]w \\ +[F(w^2)ud(w) + uw^3, w] = 0\end{aligned} \tag{69}$$

that is,

$$[uw^2d(w), w] + [F(w^2)ud(w), w] = 0 \tag{70}$$

for all $u, w \in L$. Replacing u by $2wu$ in (68) and using characteristic of R is not 2, we have

$$\begin{aligned}[(F(w)u + wd(u))wv, w] + [wud(wv), w] \\ -[F(w)uv + wd(uv) + wuv, w]w + [F(v)wud(w) + wuwv, w] = 0\end{aligned}$$

that is,

$$\begin{aligned}[F(v)wud(w) + wd(u)wv + wud(wv) - wd(uv)w - wuvw + wuwv, w] \\ +[F(w)uwv, w] - [F(w)uv, w]w = 0\end{aligned}$$

for all $u, v, w \in L$. Assuming $v = w$, we have

$$[F(w)wud(w) + wud(w^2) + wd(u)w^2 - wd(uw)w, w] = 0 \tag{71}$$

this gives

$$[F(w)wud(w) + wud(w^2) - wud(w)w, w] = 0 \tag{72}$$

for all $u, w \in L$.

Subtracting (72) from (70), we get

$$[uw^2d(w), w] + [F(w^2)ud(w), w]$$
$$-[F(w)wud(w) + wud(w^2) - wud(w)w, w] = 0$$

for all $u, w \in L$. This reduces to

$$[uw^2d(w), w] + [wd(w)ud(w), w] - [wuwd(w), w] = 0 \tag{73}$$

for all $u, w \in L$. Now Replacing u by $2wu$ in (73) and using characteristic of R is not 2, we get

$$w[uw^2d(w), w] + [wd(w)wud(w), w] - w[wuwd(w), w] = 0 \tag{74}$$

for all $u, w \in L$. Left multiplying (73) by w and then subtracting from (74), we get

$$[w[d(w), w]ud(w), w] = 0 \tag{75}$$

for all $u, w \in L$. Again replacing u by $2uw$ in the above relation and using characteristic of R is not 2, we get

$$[w[d(w), w]uwd(w), w] = 0 \tag{76}$$

for all $u, w \in L$. Now right multiplying (75) by w and then subtracting from (76), we obtain

$$[w[d(w), w]u[d(w), w], w] = 0 \tag{77}$$

and hence

$$[w[d(w), w]uw[d(w), w], w] = 0 \tag{78}$$

for all $u, w \in L$. This implies

$$w[d(w), w]uw[d(w), w]w - w^2[d(w), w]uw[d(w), w] = 0 \tag{79}$$

for all $u, w \in L$. Since L is a Lie ideal of R, $[d(w), w] \in L$ for all $w \in L$ and so $2[d(w), w]x \in L$, $4w[d(w), w]x \in L$ and $8uw[d(w), w]x \in L$ for all $u, w, x \in L$. Hence, we can replace u with $8uw[d(w), w]x$ in (79) and then using characteristic of R is not 2 we obtain,

$$\begin{aligned} &w[d(w), w]uw[d(w), w]xw[d(w), w]w \\ &-w^2[d(w), w]uw[d(w), w]xw[d(w), w] = 0 \end{aligned} \tag{80}$$

for all $u, w, x \in L$. Using (79), (80) gives

$$\begin{aligned} &w[d(w), w]uw^2[d(w), w]xw[d(w), w] \\ &-w[d(w), w]uw[d(w), w]wxw[d(w), w] = 0 \end{aligned} \tag{81}$$

that is

$$w[d(w), w]u[w[d(w), w], w]xw[d(w), w] = 0 \tag{82}$$

for all $u, w, x \in L$. This implies

$$w[[d(w), w], w]uw[[d(w), w], w]xw[[d(w), w], w] = 0$$

for all $u, w, x \in L$. By primeness of R, $w[[d(w), w], w] = 0$ for all $w \in L$. Then by Lemma 2.8, $L \subseteq Z(R)$ if $d \neq 0$ which is a contradiction.

The remaining identities can be proved in a similar way. ∎

Now we present an example which shows that the primeness hypothesis in the theorems is not superfluous.

Example 3.1. Let $\mathbb{Z}$ be the ring of integers. Consider

$$R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\} \qquad \text{and} \qquad L = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} : b \in \mathbb{Z} \right\}.$$

Clearly, R is a ring under the usual addition and multiplication of matrices and L is a nonzero square closed Lie ideal of R. But for

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = (0),$$

R is not prime ring.

Define the maps on R as follows:

$$F\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 2b & 0 \end{pmatrix}, \qquad d\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2b & 0 \end{pmatrix};$$

$$G\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ b-c & c \end{pmatrix}, \qquad g\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a-c & 0 \end{pmatrix}.$$

Then F and G are generalized derivations of R associated with nonzero derivations d and g respectively. Moreover, the conditions of Theorem 3.3, Theorem 3.4, Theorem 3.5, Theorem 3.6 and Theorem 3.7 are satisfied with F, G and d, but $L \nsubseteq Z(R)$. Hence the primeness assumption can not be removed.

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