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Construction of Hom-pre-Jordan algebras and Hom-J-dendriform algebras

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Abstract: The aim of this work is to introduce and study the notions of Hom-pre-Jordan algebra and Hom-J-dendriform algebra which generalize Hom-Jordan algebras. Hom-pre-Jordan algebras are regarded as the underlying algebraic structures of the Hom-Jordan algebras behind the Rota-Baxter operators and $\mathcal{O}$-operators introduced in this paper. Hom-pre-Jordan algebras are also analogues of Hom-pre-Lie algebras for Hom-Jordan algebras. The anti-commutator of a Hom-pre-Jordan algebra is a Hom-Jordan algebra and the left multiplication operator gives a representation of a Hom-Jordan algebra. On the other hand, a Hom-J-dendriform algebra is a Hom-Jordan algebraic analogue of a Hom-dendriform algebra such that the anti-commutator of the sum of the two operations is a Hom-pre-Jordan algebra.

Key words: Hom-Jordan algebra, Hom-pre-Jordan algebra, Hom-J-dendriform algebra, $\mathcal{O}$-operator.

MSC (2020): 17A15, 17C10, 17C50.

Introduction

In order to study periodicity phenomena in algebraic K-theory, J.-L. Loday introduced, in 1995, the notion of dendriform algebra (see [9]). Dendriform algebras are algebras with two operations, which dichotomize the notion of associative algebra. Later the notion of tridendriform algebra were introduced by Loday and Ronco in their study of polytopes and Koszul duality (see [8]). In 2003 and in order to determine the algebraic structure behind a pair of commuting Rota-Baxter operators (on an associative algebra), Aguiar and Loday introduced the notion of quadri-algebra [1]. We refer to this kind of algebras as Loday algebras. Thus, it is natural to consider the Jordan algebraic analogue of Loday algebras as well as their Lie algebraic analogue.

Jordan algebras were introduced in the context of axiomatic quantum mechanics in 1932 by the physicist P. Jordan and appeared in many areas of

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mathematics such as differential geometry, Lie theory, physics and analysis (see [3, 7, 14] for more details). The Jordan algebraic analogues of Loday algebras were considered. Indeed, the notion of pre-Jordan algebra as a Jordan algebraic analogue of a pre-Lie algebra was introduced in [6]. A pre-Jordan algebra is a vector space A with a bilinear multiplication $\cdot$ such that the product $x \circ y = x \cdot y + y \cdot x$ endows A with the structure of a Jordan algebra, and the left multiplication operator $L(x) : y \mapsto x \cdot y$ defines a representation of this Jordan algebra on A. In other words, the product $x \cdot y$ satisfies the following identities:

$$\begin{aligned}
&(x \circ y) \cdot (z \cdot u) + (y \circ z) \cdot (x \cdot u) + (z \circ x) \cdot (y \cdot u) \\
&\qquad\qquad = z \cdot [(x \circ y) \cdot u] + x \cdot [(y \circ z) \cdot u] + y \cdot [(z \circ x) \cdot u], \\
&x \cdot [y \cdot (z \cdot u)] + z \cdot [y \cdot (x \cdot u)] + [(x \circ z) \circ y] \cdot u \\
&\qquad\qquad = z \cdot [(x \circ y) \cdot u] + x \cdot [(y \circ z) \cdot u] + y \cdot [(z \circ x) \cdot u].
\end{aligned}$$

In order to find a dendriform algebra whose anti-commutator is a pre-Jordan algebra, Hou and Bai introduced the notion of J-dendriform algebra [5]. They are, also related to pre-Jordan algebras in the same way as pre-Jordan algebras are related to Jordan algebras. They showed that an $\mathcal{O}$-operator (specially a Rota-Baxter operator of weight zero) on a pre-Jordan algebra or two commuting Rota-Baxter operators on a Jordan algebra give a J-dendriform algebra. In addition, they considered the relationships between J-dendriform algebras and Loday algebras especially quadri-algebras.

Hom-type algebras have been investigated by many authors. In general, Hom-type algebras are a kind of algebras in which the usual identities defining the structure are twisted by homomorphisms. Such algebras appeared in 1990s in examples of q-deformations of Witt and Virasoro algebras. Motivated by these examples and their generalization, Hartwig, Larsson and Silvestrov introduced and studied Hom-Lie algebras in [4]. The notion of Hom-Jordan algebras was first introduced by A. Makhlouf in [11] with a connection to Hom-associative algebras and then D. Yau modified slightly the definition in [15] and established their relationships with Hom-alternative algebras.

We aim in this paper to introduce and study Hom-pre-Jordan algebras and Hom-J-dendriform algebras generalizing pre-Jordan algebras and J-dendriform algebras. The anti-commutator of a Hom-pre-Jordan algebra is a Hom-Jordan algebra and the left multiplication operators give a representation of this Hom-Jordan algebra, which is the beauty of such a structure. Similarly, a Hom-J-dendriform algebra gives rise to a Hom-pre-Jordan algebra and a Hom-Jordan

algebra in the same way as a Hom-dendriform algebra gives rise to a Hom-pre-Lie algebra and a Hom-Lie algebra (see [10]).

The paper is organized as follows. In Section 1, we recall some basic facts about Hom-Jordan algebras. In Section 2, we introduce the notions of Hom-pre-Jordan algebra and bimodule of a Hom-pre-Jordan algebra. We provide some properties and develop some construction theorems. In Section 3, we introduce the notion of Hom-J-dendriform algebra and study some of their fundamental properties in terms of $\mathcal{O}$-operators of pre-Jordan algebras.

Throughout this paper $\mathbb{K}$ is a field of characteristic 0 and all vector spaces are over $\mathbb{K}$. We refer to a Hom-algebra as a tuple (A, μ, α), where A is a vector space, μ is a multiplication and α is a linear map. It is said to be regular if α is invertible. A Hom-associator with respect to a Hom-algebra is a trilinear map as_α defined for all $x, y, z \in A$ by $as_\alpha(x, y, z) = (xy)\alpha(z) - \alpha(x)(yz)$. We denote for simplicity the multiplication and composition by concatenation when there is no ambiguity.

1. Basic results on Hom-Jordan algebras

In this section, we recall some basics about Hom-Jordan algebras introduced in [15] and introduce the notion of a representation of a Hom-Jordan algebra.

Definition 1.1. A *Hom-Jordan algebra* is a Hom-algebra $(A, \circ, \alpha)$ satisfying the following conditions

$$x \circ y = y \circ x\,, \tag{1.1}$$

$$as_\alpha\big(x \circ x, \alpha(y), \alpha(x)\big) = 0\,, \tag{1.2}$$

for all $x, y \in A$.

Remark 1.1. Since the characteristic of $\mathbb{K}$ is 0, condition (1.2) is equivalent to the following identity (for all $x, y, z, u \in A$)

$$\circlearrowleft_{x,y,z} as_\alpha(x \circ y, \alpha(u), \alpha(z)) = 0\,, \tag{1.3}$$

or equivalently,

$$\begin{aligned}((x \circ y) \circ \alpha(u)) \circ \alpha^2(z) + ((y \circ z) \circ \alpha(u)) \circ \alpha^2(x) + ((z \circ x) \circ \alpha(u)) \circ \alpha^2(y)\\ = \alpha(x \circ y)(\alpha(u) \circ \alpha(z)) + \alpha(y \circ z)(\alpha(u) \circ \alpha(x)) + \alpha(z \circ x)(\alpha(u) \circ \alpha(y))\,.\end{aligned} \tag{1.4}$$

Definition 1.2. Let $(A, \circ, \alpha)$ be a Hom-Jordan algebra and V be a vector space. Let $\rho\colon A \to gl(V)$ be a linear map and $\phi\colon V \to V$ be an algebra morphism. Then (V, ρ, ϕ) is called a *representation (or a module) of* $(A, \circ, \alpha)$ if for all $x, y, z \in A$

$$\phi\rho(x) = \rho(\alpha(x))\phi\,, \tag{1.5}$$

$$\begin{aligned}\rho(\alpha^2(x))\rho(y\circ z)\phi + \rho(\alpha^2(y))\rho(z\circ x)\phi + \rho(\alpha^2(z))\rho(x\circ y)\phi \\ = \rho(\alpha(x)\circ\alpha(y))\rho(\alpha(z))\phi + \rho(\alpha(y)\circ\alpha(z))\rho(\alpha(x))\phi \\ + \rho(\alpha(z)\circ\alpha(x))\rho(\alpha(y))\phi\,,\end{aligned} \tag{1.6}$$

$$\begin{aligned}\rho((x\circ y)\circ\alpha(z))\phi^2 + \rho(\alpha^2(x))\rho(\alpha(z))\rho(y) + \rho(\alpha^2(z))\rho(\alpha(y))\rho(x) \\ = \rho(\alpha(x)\circ\alpha(y))\rho(\alpha(z))\phi + \rho(\alpha(y)\circ\alpha(z))\rho(\alpha(x))\phi \\ + \rho(\alpha(z)\circ\alpha(x))\rho(\alpha(y))\phi.\end{aligned} \tag{1.7}$$

Proposition 1.1. *Let $(A, \circ, \alpha)$ be a Hom-Jordan algebra, then (V, ρ, ϕ) is a representation of A if and only if there exists a Hom-Jordan algebra structure on the direct sum $A \oplus V$ given by*

$$(x + u) * (y + v) = x \circ y + \rho(x)v + \rho(y)u\,, \quad \forall\, x, y \in A\,,\ u, v \in V. \tag{1.8}$$

We denote it by $A \ltimes_{\rho,\phi} V$ or simply $A \ltimes V$.

Example 1.1. Let $(A, \circ, \alpha)$ be a Hom-Jordan algebra. Let $ad\colon A \to gl(A)$ be a map defined by $ad(x)(y) = x \circ y = y \circ x$, for all $x, y \in A$. Then (A, ad, α) is a representation of $(A, \circ, \alpha)$ called the adjoint representation of A.

Definition 1.3. Let $(A, \circ, \alpha)$ be a Hom-Jordan algebra and (V, ρ, ϕ) be a representation. A linear map $T\colon V \to A$ is called an *$\mathcal{O}$-operator of A associated to ρ* if it satisfies

$$T\phi = \alpha T, \tag{1.9}$$

$$T(u) \circ T(v) = T\big(\rho(T(u))v + \rho(T(v))u\big), \quad \forall\, u, v \in V. \tag{1.10}$$

An $\mathcal{O}$-operator on A associated to the adjoint representation (A, ad, α) is called a *Rota-Baxter operator of weight zero*. Hence, a Rota-Baxter operator on a Hom-Jordan algebra $(A, \circ, \alpha)$ is a linear map $R\colon A \to A$ satisfying

$$R\alpha = \alpha R\,, \tag{1.11}$$

$$R(x) \circ R(y) = R\big(R(x) \circ y + x \circ R(y)\big), \quad \forall\, x, y \in A\,. \tag{1.12}$$

2. Hom-pre-Jordan algebras

In this section, we generalize the notion of pre-Jordan algebra introduced in [6] to the Hom case and study the relationships with Hom-Jordan algebras, Hom-dendriform algebras and Hom-pre-alternative algebras in terms of $\mathcal{O}$-operators of Hom-Jordan algebras.

2.1. Definition and basic properties

Definition 2.1. A *Hom-pre-Jordan algebra* is a Hom-algebra $(A, \cdot, \alpha)$ satisfying, for any $x, y, z, u \in A$, the following identities

$$\begin{aligned}[\alpha(x) \circ \alpha(y)] \cdot [\alpha(z) \cdot \alpha(u)] + [\alpha(y) \circ \alpha(z)] \cdot [\alpha(x) \cdot \alpha(u)] \\ + [\alpha(z) \circ \alpha(x)] \cdot [\alpha(y) \cdot \alpha(u)] \qquad (2.1) \\ = \alpha^2(x) \cdot [(y \circ z) \cdot \alpha(u)] + \alpha^2(y) \cdot [(z \circ x) \cdot \alpha(u)] + \alpha^2(z) \cdot [(x \circ y) \cdot \alpha(u)],\end{aligned}$$

$$\begin{aligned}[(x \circ z) \circ \alpha(y)] \cdot \alpha^2(u) + \alpha^2(x) \cdot [\alpha(y) \cdot (z \cdot u)] + \alpha^2(z) \cdot [\alpha(y) \cdot (x \cdot u)] \qquad (2.2) \\ = \alpha^2(x) \cdot [(y \circ z) \cdot \alpha(u)] + \alpha^2(y) \cdot [(z \circ x) \cdot \alpha(u)] + \alpha^2(z) \cdot [(x \circ y) \cdot \alpha(u)],\end{aligned}$$

where $x \circ y = x \cdot y + y \cdot x$. When α is an algebra morphism, the Hom-pre-Jordan algebra $(A, \cdot, \alpha)$ will be called *multiplicative*.

Remark 2.1. Equations (2.1) and (2.2) are equivalent to the following equations (for any $x, y, z, u \in A$) respectively

$$\begin{aligned}(x, y, z, u)^1_\alpha + (y, z, x, u)^1_\alpha + (z, x, y, u)^1_\alpha \qquad (2.3) \\ + (y, x, z, u)^1_\alpha + (x, z, y, u)^1_\alpha + (z, y, x, u)^1_\alpha = 0\,,\end{aligned}$$

$$\begin{aligned}as_\alpha(\alpha(x), \alpha(y), z \cdot u) - as_\alpha(x \cdot z, \alpha(y), \alpha(u)) + (y, z, x, u)^2_\alpha \qquad (2.4) \\ +(y, x, z, u)^2_\alpha + as_\alpha(\alpha(z), \alpha(y), x \cdot u) - as_\alpha(z \cdot x, \alpha(y), \alpha(u)) = 0\,,\end{aligned}$$

where

$$(x, y, z, u)^1_\alpha = [\alpha(x) \cdot \alpha(y)] \cdot [\alpha(z) \cdot \alpha(u)] - \alpha^2(x) \cdot [(y \cdot z) \cdot \alpha(u)],$$

$$(x, y, z, u)^2_\alpha = [\alpha(x) \cdot \alpha(y)] \cdot [\alpha(z) \cdot \alpha(u)] - [\alpha(x) \cdot (y \cdot z)] \cdot \alpha^2(u).$$

Remark 2.2. Any Hom-associative algebra is a Hom-pre-Jordan algebra.

PROPOSITION 2.1. *Let $(A, \cdot, \alpha)$ be a Hom-pre-Jordan algebra. Then the product given by*

$$x \circ y = x \cdot y + y \cdot x \tag{2.5}$$

defines a Hom-Jordan algebra structure on A, which is called the associated Hom-Jordan algebra of $(A, \cdot, \alpha)$ and $(A, \cdot, \alpha)$ is also called a compatible Hom-pre-Jordan algebra structure on the Hom-Jordan algebra $(A, \circ, \alpha)$.

Proof. Let $x, y, z, u \in A$, it is easy to show that

$$\begin{aligned}((x \circ y) \circ \alpha(u)) \circ \alpha^2(z) + ((y \circ z) \circ \alpha(u)) \circ \alpha^2(x) + ((z \circ x) \circ \alpha(u)) \circ \alpha^2(y)\\ = (\alpha(x) \circ \alpha(y))(\alpha(u) \circ \alpha(z)) + (\alpha(y) \circ \alpha(z))(\alpha(u) \circ \alpha(x))\\ + (\alpha(z) \circ \alpha(x))(\alpha(u) \circ \alpha(y))\end{aligned}$$

if and only if $l_1 + l_2 + l_3 + l_4 = r_1 + r_2 + r_3 + r_4$, where

$$\begin{aligned}
l_1 &= \circlearrowleft_{x,y,z} \alpha^2(x) \cdot [(y \circ z) \cdot \alpha(u)],\\
l_2 &= [(x \circ y) \circ \alpha(u)] \cdot \alpha^2(z) + \alpha^2(x) \cdot [\alpha(u) \cdot (y \cdot z)] + \alpha^2(y) \cdot [\alpha(u) \cdot (x \cdot z)],\\
l_3 &= [(x \circ z) \circ \alpha(u)] \cdot \alpha^2(y) + \alpha^2(x) \cdot [\alpha(u) \cdot (z \cdot y)] + \alpha^2(z) \cdot [\alpha(u) \cdot (x \cdot u)],\\
l_4 &= [(y \circ z) \circ \alpha(u)] \cdot \alpha^2(x) + \alpha^2(y) \cdot [\alpha(u) \cdot (z \cdot x)] + \alpha^2(z) \cdot [\alpha(u) \cdot (y \cdot x)],
\end{aligned}$$

and

$$\begin{aligned}
r_1 &= \circlearrowleft_{x,y,z} (\alpha(x) \circ \alpha(y)) \cdot (\alpha(z) \cdot \alpha(u)),\\
r_2 &= \circlearrowleft_{x,y,u} (\alpha(x) \circ \alpha(y)) \cdot (\alpha(u) \cdot \alpha(z)),\\
r_3 &= \circlearrowleft_{x,z,u} (\alpha(x) \circ \alpha(z)) \cdot (\alpha(u) \cdot \alpha(y)),\\
r_4 &= \circlearrowleft_{y,z,u} (\alpha(y) \circ \alpha(z)) \cdot (\alpha(u) \cdot \alpha(x)).
\end{aligned}$$

Now using Definition 2.1, we can easily see that $l_i = r_i$, for $i = 1, \ldots, 4$. ∎

EXAMPLE 2.1. Consider the 2-dimensional vector space A generated by the basis $\{e_1, e_2\}$ and define the multiplication

$\cdot$	e_1	e_2
e_1	e_1	0
e_2	0	0

and the linear map

$$\alpha(e_1) = e_1\,, \qquad \alpha(e_2) = 0\,.$$

Then $(A, \cdot, \alpha)$ is a Hom-pre-Jordan algebra. According to the above proposition, the associated Hom-Jordan algebra $(A, \circ, \alpha)$ is given by

$\circ$	e_1	e_2
e_1	$2e_1$	0
e_2	0	0

.

The following conclusion can be obtained straightforwardly using the previous proposition.

PROPOSITION 2.2. *Let $(A, \cdot, \alpha)$ be a Hom-algebra. Then $(A, \cdot, \alpha)$ is a Hom-pre-Jordan algebra if and only if $(A, \circ, \alpha)$ defined by equation (2.5) is a Hom-Jordan algebra and (A, L, α) is a representation of $(A, \circ, \alpha)$, where L denotes the left multiplication operator on A.*

Proof. Straightforward. ∎

PROPOSITION 2.3. *Let $(A, \circ, \alpha)$ be a Hom-Jordan algebra and (V, ρ, ϕ) be a representation. If T is an $\mathcal{O}$-operator associated to ρ, then $(V, *, \phi)$ is a Hom-pre-Jordan algebra, where*

$$u * v = \rho(T(u))v, \qquad \forall\, u, v \in V. \tag{2.6}$$

Therefore there exists an associated Hom-Jordan algebra structure on V given by equation (2.5) and T is a homomorphism of Hom-Jordan algebras. Moreover, $T(V) = \{T(v) | v \in V\} \subset A$ is a Hom-Jordan subalgebra of $(A, \circ, \alpha)$ and there is an induced Hom-pre-Jordan algebra structure on $T(V)$ given by

$$T(u).T(v) = T(u * v), \qquad \forall\, u, v \in V. \tag{2.7}$$

The corresponding associated Hom-Jordan algebra structure on $T(V)$ given by equation (2.5) is just a Hom-Jordan subalgebra of $(A, \circ, \alpha)$ and T is a homomorphism of Hom-pre-Jordan algebras.

Proof. Let $u, v, w, a \in V$ and set $x = T(u)$, $y = T(v)$, $z = T(w)$ and $u \bullet v = u * v + v * u$. Note first that $T(u \bullet v) = T(u) \circ T(v)$. Then

$$\begin{aligned}(\phi(u) \bullet \phi(v)) * (\phi(w) * \phi(a)) &= \rho(T(\rho(T(\phi(u) \bullet \phi(v))))) \rho(T(\phi(w))) \phi(a) \\ &= \rho(T(\phi(u)) \circ T(\phi(v))) \rho(T(\phi(w))) \phi(a) \\ &= \rho(\alpha(x) \circ \alpha(y)) \rho(\alpha(z)) \phi(a),\end{aligned}$$

$$\begin{aligned}
\phi^2(u) * [(v \bullet w) * \phi(a)] &= \rho(T(\phi^2(u)))\rho(T(v \bullet w))\phi(a) \\
&= \rho(T(\phi^2(u)))\rho(T(v) \circ T(w))\phi(a) \\
&= \rho(\alpha^2(x))\rho(y \circ z)\phi(a), \\
\phi^2(u) * [\phi(v) * (w * a)] &= \rho(T(\phi^2(u)))\rho(T(\phi(v)))\rho(T(w))a \\
&= \rho(\alpha^2(x))\rho(\alpha(y))\rho(z)a, \\
[(u \bullet v) \bullet \phi(w)] * \phi^2(a) &= \rho(T([(u \bullet v) \bullet \phi(w)]))\phi^2(a) \\
&= \rho([T(u \bullet v) \circ T(\phi(w))])\phi^2(a) \\
&= \rho([(T(u) \circ T(v)) \circ T(\phi(w))])\phi^2(a) \\
&= \rho([(x \circ y) \circ \alpha(z)])\phi^2(a).
\end{aligned}$$

Hence,

$$\begin{aligned}
&(\phi(u) \bullet \phi(v)) * (\phi(w) * \phi(a)) + (\phi(v) \bullet \phi(w)) * (\phi(u) * \phi(a)) \\
&\qquad + (\phi(w) \bullet \phi(u)) * (\phi(v) * \phi(a)) \\
&\quad = \rho(\alpha(x) \circ \alpha(y))\rho(\alpha(z))\phi(a) + \rho(\alpha(y) \circ \alpha(z))\rho(\alpha(x))\phi(a) \\
&\qquad + \rho(\alpha(z) \circ \alpha(x))\rho(\alpha(y))\phi(a) \\
&\quad = \rho(\alpha^2(x))\rho(y \circ z)\phi(a) + \rho(\alpha^2(y))\rho(z \circ x)\phi(a) \\
&\qquad + \rho(\alpha^2(z))\rho(x \circ y)\phi(a) + \phi^2(u) * [(v \bullet w) * \phi(a)] \\
&\qquad + \phi^2(v) * [(w \bullet u) * \phi(a)] + \phi^2(w) * [(u \bullet v) * \phi(a)],
\end{aligned}$$

and

$$\begin{aligned}
&[(u \bullet v) \bullet \phi(w)] * \phi^2(a) + \phi^2(u) * [\phi(w) * (v * a)] + \phi^2(w) * [\phi(v) * (u * a)] \\
&\quad = \rho([(x \circ y) \circ \alpha(z)])\phi^2(a) + \rho(\alpha^2(x))\rho(\alpha(z))\rho(y)a \\
&\qquad + \rho(\alpha^2(z))\rho(\alpha(y))\rho(x)a \\
&\quad = \rho(\alpha^2(x))\rho(y \circ z)\phi(a) + \rho(\alpha^2(y))\rho(z \circ x)\phi(a) \\
&\qquad + \rho(\alpha^2(z))\rho(x \circ y)\phi(a) + \phi^2(u) * [(v \bullet w) * \phi(a)] \\
&\qquad + \phi^2(v) * [(w \bullet u) * \phi(a)] + \phi^2(w) * [(u \bullet v) * \phi(a)].
\end{aligned}$$

Therefore, $(V, *, \phi)$ is a Hom-pre-Jordan algebra. The other conclusions follow immediately. ∎

An obvious consequence of Proposition 2.3 is the following construction of a Hom-pre-Jordan algebra in terms of a Rota-Baxter operator (of weight zero) of a Hom-Jordan algebra.

COROLLARY 2.1. *Let $(A, \circ, \alpha)$ be a Hom-Jordan algebra and R be a Rota-Baxter operator (of weight zero) on A. Then there is a Hom-pre-Jordan algebra structure on A given by*

$$x \cdot y = R(x) \circ y, \qquad \forall\, x, y \in A\,.$$

Proof. Straightforward. ∎

EXAMPLE 2.2. Let $\{e_1, e_2\}$ be a basis of a 2-dimensional vector space A over $\mathbb{K}$. The following product $\circ$ and the linear map α define, for any scalar a, a Hom-Jordan algebra on A:

$\circ$	e_1	e_2
e_1	$2e_1$	$2ae_2$
e_2	$2ae_2$	0

$$, \qquad \alpha(e_1) = e_1\,, \quad \alpha(e_2) = ae_2\,.$$

Define the linear map $R\colon A \to A$ with respect to the basis $\{e_1, e_2\}$ by

$$R(e_1) = be_2\,, \qquad R(e_2) = 0\,.$$

Then R is a Rota-Baxter operator on the Hom-Jordan algebra $(A, \circ, \alpha)$, where a and b are parameters in $\mathbb{K}$. Using Corollary 2.1, there is a Hom-pre-Jordan algebra structure, with respect the same twist map α, given by the following multiplication table

$\cdot$	e_1	e_2
e_1	$2abe_2$	0
e_2	0	0

.

EXAMPLE 2.3. Let $\{e_1, e_2, e_3\}$ be a basis of a 3-dimensional vector space A over $\mathbb{K}$. The following product $\circ$ and the linear map α define the following Hom-Jordan algebras over $\mathbb{K}$.

$\circ$	e_1	e_2	e_3
e_1	ae_1	ae_2	be_3
e_2	ae_2	ae_2	$\frac{b}{2}e_3$
e_3	be_3	$\frac{b}{2}e_3$	0

$$, \qquad \alpha(e_1) = ae_1, \quad \alpha(e_2) = ae_2, \quad \alpha(e_3) = be_3,$$

where a and b are parameters in $\mathbb{K}$. Let R be the operator defined with respect to the basis $\{e_1, e_2, e_3\}$ by

$$R(e_1) = \lambda_1 e_3, \qquad R(e_2) = \lambda_2 e_3, \qquad R(e_3) = 0,$$

where λ_1 and λ_2 are parameters in $\mathbb{K}$. Then we can easily check that R is a Rota-Baxter operator on A. Now, using Corollary 2.1, there is a Hom-pre-Jordan algebra structure on A, with the same twist map and a multiplication given by $x \cdot y = R(x) \circ y$ for all $x, y \in A$, that is

$\cdot$	e_1	e_2	e_3
e_1	$\lambda_1 b e_3$	$\lambda_1 \frac{b}{2} e_3$	0
e_2	$\lambda_2 b e_3$	$\lambda_2 \frac{b}{2} e_3$	0
e_3	0	0	0

.

Corollary 2.2. *Let $(A, \circ, \alpha)$ be a Hom-Jordan algebra. Then there exists a compatible Hom-pre-Jordan algebra structure on A if and only if there exists an invertible $\mathcal{O}$-operator of $(A, \circ, \alpha)$.*

Proof. Let $(A, \cdot, \alpha)$ be a Hom-pre-Jordan algebra and $(A, \circ, \alpha)$ be the associated Hom-Jordan algebra. Then the identity map $id\colon A \to A$ is an invertible $\mathcal{O}$-operator of $(A, \circ, \alpha)$ associated to (A, ad, α).

Conversely, suppose that there exists an invertible $\mathcal{O}$-operator T of $(A, \circ, \alpha)$ associated to a representation (V, ρ, ϕ), then by Proposition 2.3, there is a Hom-pre-Jordan algebra structure on $T(V) = A$ given by

$$T(u) \cdot T(v) = T(\rho(T(u))v), \qquad \text{for all } u, v \in V.$$

If we set $T(u) = x$ and $T(v) = y$, then we obtain

$$x \cdot y = T(\rho(x)T^{-1}(y)), \qquad \text{for all } x, y \in A.$$

It is a compatible Hom-pre-Jordan algebra structure on $(A, \circ, \alpha)$. Indeed,

$$\begin{aligned} x \cdot y + y \cdot x &= T\big(\rho(x)T^{-1}(y) + \rho(y)T^{-1}(x)\big) \\ &= T(T^{-1}(x)) \circ T(T^{-1}(y)) = x \circ y. \end{aligned}$$

∎

The following result reveals the relationship between Hom-pre-Jordan algebras, Hom-pre-alternative algebras and so Hom-dendriform algebras. We recall the following definitions introduced in [12, 10].

Definition 2.2. A *Hom-pre-alternative algebra* is a quadruple $(A, \prec, \succ, \alpha)$, where $\prec, \succ\colon A \otimes A \to A$ and $\alpha\colon A \to A$ are linear maps satisfying

$$(x \succ y) \prec \alpha(z) - a(x) \succ (y \prec z) + (y \prec x) \prec \alpha(z) - a(y) \prec (x \star z) = 0, \tag{2.8}$$

$$(x \succ y) \prec \alpha(z) - a(x) \succ (y \prec z) + (x \star z) \succ \alpha(y) - a(x) \succ (z \succ y) = 0, \tag{2.9}$$

$$(x \prec y) \prec \alpha(z) - a(x) \prec (y \star z) + (x \prec z) \prec \alpha(y) - a(x) \prec (z \star y) = 0, \tag{2.10}$$

$$(x \star y) \succ \alpha(z) - a(x) \succ (y \succ z) + (y \star x) \succ \alpha(z) - a(y) \succ (x \succ z) = 0, \tag{2.11}$$

for all $x, y, z \in A$, where $x \star y = x \prec y + x \succ y$.

Definition 2.3. A *Hom-dendriform algebra* is a quadruple $(A, \prec, \succ, \alpha)$, where $\prec, \succ\colon A \otimes A \to A$ and $\alpha : A \to A$ are linear maps satisfying

$$(x \succ y) \prec \alpha(z) - a(x) \succ (y \prec z) = 0, \tag{2.12}$$

$$(x \prec y) \prec \alpha(z) - a(x) \prec (y \star z) = 0, \tag{2.13}$$

$$(x \star y) \succ \alpha(z) - a(x) \succ (y \succ z) = 0, \tag{2.14}$$

for all $x, y, z \in A$, where $x \star y = x \prec y + x \succ y$.

Proposition 2.4. *Let $(A, \prec, \succ, \alpha)$ be a Hom-pre-alternative algebra. Then the product given by*

$$x \cdot y = x \succ y + y \prec x, \qquad \forall\, x, y \in A,$$

defines a Hom-pre-Jordan algebra structure on A.

Proof. Let $x, y, z, u \in A$, set $x \star y = x \prec y + x \succ y$ and $x \circ y = x \cdot y + y \cdot x = x \star y + y \star x$. We will just prove the identity (2.1). One has

$$\begin{aligned}
&\circlearrowleft_{x,y,z}\big([\alpha(x) \circ \alpha(y)] \cdot [\alpha(z) \cdot \alpha(u)] - \alpha^2(x) \cdot [(y \circ z) \cdot \alpha(u)]\big) \\
&\quad = \circlearrowleft_{x,y,z} \big([\alpha(x) \circ \alpha(y)] \succ [\alpha(z) \succ \alpha(u)] + [\alpha(x) \circ \alpha(y)] \succ [\alpha(u) \prec \alpha(z)] \\
&\qquad + [\alpha(z) \succ \alpha(y)] \prec [\alpha(x) \circ \alpha(y)] + [\alpha(u) \prec \alpha(z)] \prec [\alpha(x) \circ \alpha(y)] \\
&\qquad - \alpha^2(x) \succ [(y \circ z) \succ \alpha(u)] - \alpha^2(x) \succ [\alpha(u) \prec (y \circ z)] \\
&\qquad - [(y \circ z) \succ \alpha(u)] \prec \alpha^2(x) - [\alpha(u) \prec (y \circ z)] \prec \alpha^2(x)\big) \\
&\quad = \circlearrowleft_{x,y,z} \big([(x \circ y) \circ \alpha(z)] \succ \alpha^2(u) + [\alpha(u) \prec \alpha(z)] \prec [\alpha(x) \circ \alpha(y)] \\
&\qquad - [\alpha(u) \prec (y \circ z)] \prec \alpha^2(x)\big).
\end{aligned}$$

Since $(A, \star, \alpha)$ is a Hom-alternative algebra (see [12]), we have

$$\circlearrowleft_{x,y,z} \big([(x \circ y) \circ \alpha(z)] \succ \alpha^2(u)\big) = 0.$$

In addition using the fact that $(A, \prec, \succ, \alpha)$ is a Hom-pre-alternative algebra, then we obtain

$$\begin{aligned}\circlearrowleft_{x,y,z} \big([(x \circ y) \circ \alpha(z)] \succ \alpha^2(u) + [\alpha(u) \prec \alpha(z)] \prec [\alpha(x) \circ \alpha(y)] \\ - [\alpha(u) \prec (y \circ z)] \prec \alpha^2(x)\big) = 0.\end{aligned}$$

The identity (2.2) can be obtained similarly. ∎

Since any Hom-dendriform algebra is a Hom-pre-alternative algebra, we obtain the following conclusion.

Corollary 2.3. *Let $(A, \prec, \succ, \alpha)$ be a Hom-dendriform algebra. Then the product given by*

$$x \cdot y = x \succ y + y \prec x, \qquad \forall\, x, y \in A,$$

defines a Hom-pre-Jordan algebra structure on A.

2.2. Bimodules and $\mathcal{O}$-operators In this section, we introduce and study bimodules of Hom-pre-Jordan algebras.

Definition 2.4. Let $(A, \cdot, \alpha)$ be a Hom-pre-Jordan algebra and V be a vector space. Let $l, r\colon A \to gl(V)$ be two linear maps and $\phi \in gl(V)$. Then (V, l, r, ϕ) is called a *bimodule of A* if the following conditions hold (for any $x, y, z \in A$):

$$\phi l(x) = l(\alpha(x))\phi, \qquad \phi r(x) = r(\alpha(x))\phi, \tag{2.15}$$

$$\begin{aligned}l(\alpha^2(x))l(y \circ z)\phi + l(\alpha^2(y))l(z \circ x)\phi + l(\alpha^2(z))l(x \circ y)\phi \\ = l(\alpha(x) \circ \alpha(y))l(\alpha(z))\phi + l(\alpha(y) \circ \alpha(z))l(\alpha(x))\phi \\ + l(\alpha(z) \circ \alpha(x))l(\alpha(y))\phi,\end{aligned} \tag{2.16}$$

$$\begin{aligned}l((x \circ z) \circ \alpha(y))\phi^2 + l(\alpha^2(x))l(\alpha(z))l(y) + l(\alpha^2(z))l(\alpha(y))l(x) \\ = l(\alpha(x) \circ \alpha(y))l(\alpha(z))\phi + l(\alpha(y) \circ \alpha(z))l(\alpha(x))\phi \\ + l(\alpha(z) \circ \alpha(x))l(\alpha(y))\phi,\end{aligned} \tag{2.17}$$

$$\begin{aligned}
&\phi\big(l(x\circ y)r(z)+r(x\cdot z)l(y)+r(y\cdot z)r(x)+r(x\cdot z)r(y)+r(y\cdot z)l(x)\big)\\
&\quad= l(\alpha^2(x))r(\alpha(z))l(y)+l(\alpha^2(y))r(\alpha(z))r(x)+r[(x\circ y)\alpha(z)]\phi^2\\
&\qquad\quad + l(\alpha^2(y))r(\alpha(z))l(x)+l(\alpha^2(x))r(\alpha(z))r(y),
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
&\phi\big(r(z\cdot y)l(x)+r(x\cdot y)r(z)+l(x\circ z)r(y)+r(x\cdot y)l(z)+r(z\cdot y)r(x)\big)\\
&\quad= \big(l(\alpha^2(x))r(z\cdot y)+r(\alpha^2(y))r(x\circ z)\\
&\qquad\quad + r(\alpha^2(y))l(x\circ z)+l(\alpha^2(z))r(x\cdot y)\big)\phi,
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
&\phi\big(l(x\cdot y)r(z)+r(x\cdot z)l(y)+r(y\cdot z)r(x)+l(y\cdot x)r(z)\\
&\qquad\qquad + r(x\cdot z)r(y)\ +r(y\cdot z)l(x)\big)\\
&\quad= l(\alpha^2(x))l(\alpha(y))r(z)+r(\alpha^2(z))l(\alpha(y))r(x)\\
&\qquad\quad + r(\alpha^2(z))r(\alpha(y))r(x)+r(\alpha^2(z))l(\alpha(y))l(x)\\
&\qquad\quad + r[\alpha(y)\cdot(x\cdot z)]\phi^2+r(\alpha^2(z))r(\alpha(y))l(x),
\end{aligned} \tag{2.20}$$

where $x\circ y = x\cdot y + y\cdot x$.

PROPOSITION 2.5. *Let $(A,\cdot,\alpha)$ be a Hom-pre-Jordan algebra, V be a vector space, $l,r\colon A\to gl(V)$ be linear maps and $\phi\in gl(V)$. Then (V,l,r,ϕ) is a bimodule of A if and only if the direct sum $A\oplus V$ (as vector spaces) turns into a Hom-pre-Jordan algebra (semidirect sum) by defining the multiplication in $A\oplus V$ as*

$$(x+u)*(y+v) = x\cdot y + l(x)v + r(y)u, \qquad \forall\, x,y\in A,\ u,v\in V.$$

We denote it by $A\ltimes^{\alpha,\phi}_{l,r} V$ or simply $A\ltimes V$.

PROPOSITION 2.6. *Let (V,l,r,ϕ) be a bimodule of a Hom-pre-Jordan algebra $(A,\cdot,\alpha)$ and $(A,\circ,\alpha)$ be its associated Hom-Jordan algebra. Then*

(a) *(V,l,ϕ) is a representation of $(A,\circ,\alpha)$,*

(b) *$(V,l+r,\phi)$ is a representation of $(A,\circ,\alpha)$.*

Proof. (a) Follows immediately from equations (2.16)-(2.17). For (b), by Proposition 2.5, $A\ltimes^{\alpha,\phi}_{l,r}V$ is a Hom-pre-Jordan algebra. Consider its associated

Hom-Jordan algebra $(A \oplus V, \widetilde{\circ}, \alpha + \phi)$, we have

$$\begin{aligned}(x+u)\widetilde{\circ}(y+v) &= (x+u) * (y+v) + (y+v) * (x+u)\\ &= x \cdot y + l(x)v + r(y)u + y \cdot x + l(y)u + r(x)v\\ &= x \circ y + (l+r)(x)v + (l+r)(y)u\,.\end{aligned}$$

According to Proposition 1.1, we deduce that $(V, l+r, \phi)$ is a representation of $(A, \circ, \alpha)$. ∎

Definition 2.5. Let $(A, \cdot, \alpha)$ be a Hom-pre-Jordan algebra and (V, l, r, ϕ) be a bimodule. A linear map $T\colon V \to A$ is called an *$\mathcal{O}$-operator of* $(A, \cdot, \alpha)$ *associated to* (V, l, r, ϕ) if

$$T\phi = \alpha T, \tag{2.21}$$

$$T(u) \cdot T(v) = T\big(l(T(u))v + r(T(v))u\big), \qquad \forall\, u, v \in V. \tag{2.22}$$

In particular, a Rota-Baxter operator (of weight zero) on a Hom-pre-Jordan algebra $(A, \cdot, \alpha)$ is a linear map $R : A \to A$ satisfying

$$R\alpha = \alpha R, \tag{2.23}$$

$$R(x) \cdot R(y) = R\big(R(x) \cdot y + x \cdot R(y)\big), \qquad \forall\, x, y \in A. \tag{2.24}$$

Remark 2.3. If T is an $\mathcal{O}$-operator of a Hom-pre-Jordan algebra $(A, \cdot, \alpha)$ associated to (V, l, r, ϕ), then T is an $\mathcal{O}$-operator of its associated Hom-Jordan algebra $(A, \circ, \alpha)$ associated to $(V, l+r, \phi)$.

3. Hom-J-dendriform algebras

In this section, we introduce the notion of Hom-J-dendriform algebra and discuss the relationship with Hom-pre-Jordan algebras.

Definition 3.1. A *Hom-J-dendriform algebra* is a quadruple $(A, \prec, \succ, \alpha)$, where A is a vector space equipped with a linear map $\alpha\colon A \to A$ and two products denoted by $\prec, \succ\colon A \otimes A \to A$ satisfying the following identities (for any $x, y, z, u \in A$)

$$\begin{aligned}\alpha(x \circ y) \succ \alpha(z \succ u) &+ \alpha(y \circ z) \succ \alpha(x \succ u) + \alpha(z \circ x) \succ \alpha(y \succ u)\\ &= \alpha^2(x) \succ [(y \circ z) \succ \alpha(u)] + \alpha^2(y) \succ [(z \circ x) \succ \alpha(u)]\\ &\quad + \alpha^2(z) \succ [(x \circ y) \succ \alpha(u)],\end{aligned} \tag{3.1}$$

$$\begin{aligned}\alpha(x\circ y)\succ\alpha(z\succ u)+\alpha(y\circ z)\succ\alpha(x\succ u)+\alpha(z\circ x)\succ\alpha(y\succ u)\\ =\alpha^2(x)\succ[\alpha(y)\succ(z\succ u)]+\alpha^2(z)\succ[\alpha(y)\succ(x\succ u)]\\ +[\alpha(y)\circ(z\circ x)]\succ\alpha^2(u),\end{aligned}\tag{3.2}$$

$$\begin{aligned}\alpha(x\circ y)\succ\alpha(z\prec u)+\alpha(x\cdot z)\prec\alpha(y\diamond u)+\alpha(y\cdot z)\prec\alpha(x\diamond u)\\ =\alpha^2(x)\succ[\alpha(z)\prec(y\diamond u)]+\alpha^2(y)\succ[\alpha(z)\prec(x\diamond u)]\\ +[(x\circ y)\cdot\alpha(z)]\prec\alpha^2(u),\end{aligned}\tag{3.3}$$

$$\begin{aligned}\alpha(z\cdot y)\prec\alpha(x\diamond u)+\alpha(x\cdot y)\prec\alpha(z\diamond u)+\alpha(x\circ z)\succ\alpha(y\prec u)\\ =\alpha^2(x)\succ[(z\cdot y)\prec\alpha(u)]+\alpha^2(z)\succ[(x\cdot y)\prec\alpha(u)]\\ +\alpha^2(y)\prec[(x\circ z)\diamond\alpha(u)],\end{aligned}\tag{3.4}$$

$$\begin{aligned}\alpha(x\circ y)\succ\alpha(z\prec u)+\alpha(x\cdot z)\succ\alpha(y\diamond u)+\alpha(y\cdot z)\prec\alpha(x\diamond u)\\ =\alpha^2(x)\succ[\alpha(y)\succ(z\prec u)]+\alpha^2(z)\prec[\alpha(y)\diamond(z\diamond u)]\\ +[\alpha(y)\cdot(x\cdot z)]\prec\alpha^2(u),\end{aligned}\tag{3.5}$$

where

$$x\cdot y=x\succ y+y\prec x,\tag{3.6}$$

$$x\diamond y=x\succ y+x\prec y,\tag{3.7}$$

$$x\circ y=x\cdot y+y\cdot x=x\diamond y+y\diamond x.\tag{3.8}$$

Remark 3.1. Let $(A,\prec,\succ,\alpha)$ be a Hom-J-dendriform algebra. If $\prec:=0$ then $(A,\succ,\alpha)$ is a Hom-pre-Jordan algebra.

PROPOSITION 3.1. *Let $(A,\prec,\succ,\alpha)$ be a Hom-J-dendriform algebra.*

(a) *The product given by equation* (3.6) *defines a Hom-pre-Jordan algebra $(A,\cdot,\alpha)$, called the associated vertical Hom-pre-Jordan algebra.*

(b) *The product given by equation* (3.7) *defines a Hom-pre-Jordan algebra $(A,\diamond,\alpha)$, called the associated horizontal Hom-pre-Jordan algebra.*

(c) *The associated vertical and horizontal Hom-pre-Jordan algebras $(A,\cdot,\alpha)$ and $(A,\diamond,\alpha)$ have the same associated Hom-Jordan algebra $(A,\circ,\alpha)$ defined by equation* (3.8)*, called the associated Hom-Jordan algebra of $(A,\prec,\succ,\alpha)$.*

Proof. We will just prove (a). Let $x, y, z, u \in A$

$$\begin{aligned}
&[\alpha(x)\circ\alpha(y)]\cdot[\alpha(z)\cdot\alpha(u)] + [\alpha(y)\circ\alpha(z)]\cdot[\alpha(x)\cdot\alpha(u)] \\
&\qquad + [\alpha(z)\circ\alpha(x)]\cdot[\alpha(y)\cdot\alpha(u)] \\
&= [\alpha(x)\circ\alpha(y)]\succ[\alpha(z)\succ\alpha(u)] + [\alpha(y)\circ\alpha(z)]\succ[\alpha(x)\succ\alpha(u)] \\
&\quad + [\alpha(z)\circ\alpha(x)]\succ[\alpha(y)\succ\alpha(u)] + [\alpha(x)\circ\alpha(y)]\succ[\alpha(u)\prec\alpha(z)] \\
&\quad + [\alpha(x)\cdot\alpha(u)]\prec[\alpha(y)\diamond\alpha(z)] + [\alpha(y)\cdot\alpha(u)]\prec[\alpha(x)\diamond\alpha(z)] \\
&\quad + [\alpha(y)\circ\alpha(z)]\succ[\alpha(u)\prec\alpha(x)] + [\alpha(z)\cdot\alpha(u)]\prec[\alpha(y)\diamond\alpha(x)] \\
&\quad + [\alpha(y)\cdot\alpha(u)]\prec[\alpha(z)\diamond\alpha(x)] + [\alpha(z)\circ\alpha(x)]\succ[\alpha(u)\prec\alpha(y)] \\
&\quad + [\alpha(x)\cdot\alpha(u)]\prec[\alpha(z)\diamond\alpha(y)] + [\alpha(z)\cdot\alpha(u)]\prec[\alpha(x)\diamond\alpha(y)] \\
&= \alpha^2(x)\succ[(y\circ z)\succ\alpha(u)] + \alpha^2(y)\succ[(z\circ x)\succ\alpha(u)] \\
&\quad + \alpha^2(z)\succ[(x\circ y)\succ\alpha(u)] + \alpha^2(x)\succ[\alpha(u)\prec(y\diamond z)] \\
&\quad + \alpha^2(y)\succ[\alpha(u)\prec(x\diamond z)] + [(x\circ y)\cdot\alpha(u)]\prec\alpha^2(x) \\
&\quad + \alpha^2(z)\succ[\alpha(u)\prec(y\diamond x)] + \alpha^2(y)\succ[\alpha(u)\prec(z\diamond x)] \\
&\quad + [(z\circ y)\cdot\alpha(u)]\prec\alpha^2(x) + \alpha^2(z)\succ[\alpha(u)\prec(x\diamond y)] \\
&\quad + \alpha^2(x)\succ[\alpha(u)\prec(z\diamond y)] + [(z\circ x)\cdot\alpha(u)]\prec\alpha^2(y) \\
&= \alpha^2(x)\cdot[(y\circ z)\cdot\alpha(u)] + \alpha^2(y)\cdot[(z\circ x)\cdot\alpha(u)] + \alpha^2(z)\cdot[(x\circ y)\cdot\alpha(u)].
\end{aligned}$$

Similarly, we get (2.2). ∎

PROPOSITION 3.2. *Let $(A, \prec, \succ, \alpha)$ be a Hom-J-dendriform algebra. Then $(A, L_\succ, R_\prec, \alpha)$ is a bimodule of its associated horizontal Hom-pre-Jordan algebra $(A, \diamond, \alpha)$.*

Proof. We check equation (2.16) and equation (2.19). Indeed, for any $x, y, z, u \in A$, we have

$$\begin{aligned}
&L_\succ(\alpha^2(x))L_\succ(y\circ z)\alpha(u) + L_\succ(\alpha^2(y))L_\succ(z\circ x)\alpha(u) \\
&\qquad + L_\succ(\alpha^2(z))L_\succ(x\circ y)\alpha(u) \\
&= \alpha^2(x)\succ[(y\circ z)\succ\alpha(u)] + \alpha^2(y)\succ[(z\circ x)\succ\alpha(u)] \\
&\qquad + \alpha^2(z)\succ[(x\circ y)\succ\alpha(u)] \\
&= \alpha(x\circ y)\succ\alpha(z\succ u) + \alpha(y\circ z)\succ\alpha(x\succ u) + \alpha(z\circ x)\succ\alpha(y\succ u) \\
&= L_\succ(\alpha(x)\circ\alpha(y))L_\succ(\alpha(z))\alpha(u) + L_\succ(\alpha(y)\circ\alpha(z))L_\succ(\alpha(x))\alpha(u) \\
&\qquad + L_\succ(\alpha(z)\circ\alpha(x))L_\succ(\alpha(y))\alpha(u).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\alpha\Big(&R_{\prec}(z\diamond y)L_{\succ}(x)u + R_{\prec}(x\diamond y)R_{\prec}(z)u + L_{\succ}(x\circ z)R_{\prec}(y)u \\
&\qquad + R_{\prec}(x\diamond y)L_{\succ}(z)u + R_{\prec}(z\diamond y)R_{\prec}(x)u\Big) \\
&= \alpha(x\succ u)\prec\alpha(z\diamond y) + \alpha(u\prec z)\prec\alpha(x\diamond y) + \alpha(x\circ z)\succ\alpha(u\prec y) \\
&\qquad + \alpha(z\succ u)\prec\alpha(x\diamond y) + \alpha(u\prec x)\prec\alpha(z\diamond y) \\
&= \alpha(x\cdot u)\prec\alpha(z\diamond y) + \alpha(z\cdot u)\prec\alpha(x\diamond y) + \alpha(x\circ z)\succ\alpha(u\prec y) \\
&= \alpha^2(x)\succ[\alpha(u)\prec(z\diamond y)] + \alpha^2(z)\succ[\alpha(u)\prec(x\diamond y)] \\
&\qquad + [(x\circ z)\cdot\alpha(u)]\prec\alpha^2(y) \\
&= L_{\succ}(\alpha^2(x))R_{\prec}(z\diamond y)\alpha(u) + L_{\succ}(\alpha^2(z))R_{\prec}(x\diamond y)\alpha(u) \\
&\qquad + R_{\prec}(\alpha^2(y))L_{\succ}(x\circ z)\alpha(u) + R_{\prec}(\alpha^2(y))R_{\prec}(x\circ z)\alpha(u).
\end{aligned}$$

Other identities can be proved using similar computations. ∎

PROPOSITION 3.3. *Let $(A,\prec,\succ)$ be a J-dendriform algebra and $\alpha\colon A\to A$ be an algebra morphism. Then $(A,\prec_\alpha,\succ_\alpha,\alpha)$ is a Hom-J-dendriform algebra, where for any $x,y\in A$*

$$x\prec_\alpha y = \alpha(x)\prec\alpha(y), \qquad x\succ_\alpha y = \alpha(x)\succ\alpha(y).$$

Proof. Straightforward. ∎

EXAMPLE 3.1. Let A be a three dimensional vector space with basis $\{e_1, e_2, e_3\}$. Then $(A,\cdot)$ is a pre-Jordan algebra, where the formal characteristic matrix is given by

$\cdot$	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	$-e_2$	e_2	e_3
e_3	$-e_3$	0	0

.

Let $R\colon A\to A$ be the linear map defined with respect to the basis $\{e_1,e_2,e_3\}$ by the matrix

$$\begin{pmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{12} & r_{13} \\ 0 & r_{32} & -r_{12} \end{pmatrix} \qquad \text{with } r_{12}^2 + r_{13}r_{32} = 0\,.$$

It is easy to check that R is a Rota-Baxter operator on A, see [13]. Therefore, it induces a J-dendriform algebra structure on A given by

$\prec$	e_1	e_2	e_3
e_1	0	$r_{12}e_1 + r_{12}e_2 + r_{32}e_3$	$r_{13}e_1 + r_{13}e_2 - r_{12}e_3$
e_2	0	$r_{32}e_3$	$-r_{12}e_3$
e_3	0	$-r_{12}e_3$	$-r_{13}e_3$

,

$\succ$	e_1	e_2	e_3
e_1	0	0	0
e_2	$r_{12}e_1 + r_{12}e_2 + r_{32}e_3$	$r_{32}e_2$	$2r_{12}e_3$
e_3	$r_{13}e_1 - r_{13}e_2 + r_{12}e_3$	0	$2r_{13}e_3$

.

Consider, now the linear map $\alpha\colon A \to A$ defined by

$$\alpha(e_1) = e_1, \qquad \alpha(e_2) = e_2, \qquad \alpha(e_3) = \lambda e_3,\ \lambda \neq 0\,.$$

It is easy to check that α is a morphism of J-dendriform algebras. Then according to Proposition 3.3, $(A, \prec_\alpha, \succ_\alpha, \alpha)$ is a Hom-J-dendriform algebra.

Proposition 3.4. *Let $(A, \prec, \succ, \alpha)$ be a Hom-J-dendriform algebra. Define two bilinear products $\prec^t, \succ^t$ on A by*

$$x \prec^t y = y \prec x, \qquad x \succ^t y = y \succ x, \qquad \forall\, x, y \in A\,. \tag{3.9}$$

Then $(A, \prec^t, \succ^t, \alpha)$ is a Hom-J-dendriform algebra called the transpose of A. Moreover, its associated horizontal Hom-pre-Jordan algebra is the associated vertical Hom-pre-Jordan algebra $(A, \cdot, \alpha)$ of $(A, \prec, \succ, \alpha)$ and its associated vertical Hom-pre-Jordan algebra is the associated horizontal Hom-pre-Jordan algebra $(A, \diamond, \alpha)$ of $(A, \prec, \succ, \alpha)$.

Proof. Note first that

$$\begin{aligned}
x \cdot^t y &= x \succ^t y + y \prec^t x = x \succ y + x \prec y = x \diamond y,\\
x \diamond^t y &= x \succ^t y + x \prec^t y = x \succ y + y \prec x = x \cdot y,\\
x \circ^t y &= x \succ^t y + x \prec^t y + y \succ^t x + y \prec^t x\\
&= x \succ y + y \prec x + y \succ x + x \prec y = x \circ y.
\end{aligned}$$

Therefore we can easily check (equation (3.1))^t=(equation (3.1)), (equation (3.2))^t=(equation (3.2)), (equation (3.3))^t=(equation (3.4)), (equation (3.4))^t =(equation (3.3)) and (equation (3.5))^t=(equation (3.5)). Hence $(A, \prec^t, \succ^t, \alpha)$ is a Hom-J-dendriform algebra. ∎

PROPOSITION 3.5. *Let $(A, \cdot, \alpha)$ be a Hom-pre-Jordan algebra and (V, l, r, ϕ) be a bimodule. Let $T\colon V \to A$ be an $\mathcal{O}$-operator of A associated to (V, l, r, ϕ). Then there exists a Hom-J-dendriform algebra structure on V given by*

$$u \prec v = r(T(u))v, \qquad u \succ v = l(T(u))v, \qquad \forall\, u, v \in V. \tag{3.10}$$

Therefore, there is a Hom-pre-Jordan algebra on V given by equation (3.6) as the associated vertical Hom-pre-Jordan algebra of $(V, \prec, \succ)$ and T is a homomorphism of Hom-pre-Jordan algebras. Moreover, $T(V) = \{T(v) \,|\, v \in V\} \subset A$ is a Hom-pre-Jordan subalgebra of $(A, \cdot, \alpha)$, and there is an induced Hom-J-dendriform algebra structure on $T(V)$ given by

$$\begin{aligned} T(u) \prec T(v) &= T(u \prec v), \\ T(u) \succ T(v) &= T(u \succ v), \end{aligned} \qquad \forall\, u, v \in V. \tag{3.11}$$

Furthermore, its corresponding associated vertical Hom-pre-Jordan algebra structure on $T(V)$ is just the subalgebra of the Hom-pre-Jordan $(A, \cdot, \alpha)$, and T is a homomorphism of Hom-J-dendriform algebras.

Proof. For any $a, b, c, u \in V$, we set

$$T(a) = x, \quad T(b) = y \quad \text{and} \quad T(c) = z.$$

Then

$$\begin{aligned} &\phi(a \circ b) \succ \phi(c \succ u) \\ &\quad = (\phi(a) \succ \phi(b) + \phi(b) \prec \phi(a) + \phi(b) \succ \phi(a) \\ &\qquad\qquad + \phi(a) \prec \phi(b)) \succ (\phi(c) \succ \phi(u)) \\ &\quad = \big(l(T(\phi(a)))\phi(b) + r(T(\phi(b)))\phi(a) + l(T(\phi(b)))\phi(a) \\ &\qquad\qquad + r(T(\phi(a)))\phi(b)\big) \succ l(T(\phi(c)))\phi(u) \\ &\quad = l(T(l(T(\phi(a)))\phi(b) + r(T(\phi(b)))\phi(a) + l(T(\phi(b)))\phi(a) \\ &\qquad\qquad + r(T(\phi(a)))\phi(b)))l(T(\phi(c)))\phi(u) \\ &\quad = l(T(\phi(a)) \cdot T(\phi(b)) + T(\phi(b)) \cdot T(\phi(a)))l(T(\phi(c)))\phi(u) \\ &\quad = l(\alpha(x) \circ \alpha(y))l(\alpha(z))\phi(u) \end{aligned}$$

and

$$\begin{aligned}
\phi^2(a)\succ[(b\circ c)\succ\phi(u)]&\\
&=\phi^2(a)\succ[(b\succ c+c\prec b+c\succ b+b\prec c)\succ\phi(u)]\\
&=\phi^2(a)\succ l(T(l(T(b))c+r(T(c))b+l(T(c))b+r(T(b))c))\phi(u)\\
&=\phi^2(a)\succ l(y\circ z)\phi(u)\\
&=l(T(\phi^2(a)))l(y\circ z)\phi(u)\\
&=l(\alpha^2(x))l(y\circ z)\phi(u).
\end{aligned}$$

Hence

$$\begin{aligned}
\phi(a\circ b)\succ\phi(c\succ u)+\phi(b\circ c)\succ\phi(a\succ u)+\phi(c\circ a)\succ\phi(b\succ u)&\\
&=l(\alpha(x)\circ\alpha(y))l(\alpha(z))\phi(u)+l(\alpha(y)\circ\alpha(z))l(\alpha(x))\phi(u)\\
&\qquad+l(\alpha(z)\circ\alpha(x))l(\alpha(y))\phi(u)\\
&=l(\alpha^2(x))l(y\circ z)\phi(u)+l(\alpha^2(y))l(z\circ x)\phi(u)+l(\alpha^2(z))l(x\circ y)\phi(u)\\
&=\phi^2(a)\succ[(b\circ c)\succ\phi(u)]+\phi^2(b)\succ[(c\circ a)\succ\phi(u)]\\
&\qquad+\phi^2(c)\succ[(a\circ b)\succ\phi(u)].
\end{aligned}$$

Therefore, equation (3.1) holds. Using a similar computation, equations (3.2)–(3.5) hold. Then $(V,\prec,\succ,\phi)$ is a Hom-J-dendriform algebra. The other conclusions can be checked similarly. ∎

COROLLARY 3.1. *Let $(A,\cdot,\alpha)$ be a Hom-pre-Jordan algebra and R be a Rota-Baxter operator (of weight zero) on A. Then the products, given by*

$$x\prec y=y\cdot R(x),\qquad x\succ y=R(x)\cdot y,\qquad \forall\, x,y\in A$$

define a Hom-J-dendriform algebra on A with the same twist map.

THEOREM 3.1. *Let $(A,\cdot,\alpha)$ be a Hom-pre-Jordan algebra. Then there is a Hom-J-dendriform algebra such that $(A,\cdot,\alpha)$ is the associated vertical Hom-pre-Jordan algebra if and only if there exists an invertible $\mathcal{O}$-operator of $(A,\cdot,\alpha)$.*

Proof. Suppose that $(A,\prec,\succ,\alpha)$ is a Hom-J-dendriform algebra with respect to $(A,\cdot,\alpha)$. Then the identity map $id\colon A\to A$ is an $\mathcal{O}$-operator of $(A,\cdot,\alpha)$ associated to $(A,L_\succ,L_\prec,\alpha)$, where, for any $x,y\in A$,

$$L_\succ(x)(y)=x\succ y\qquad\text{and}\qquad L_\prec(x)(y)=x\prec y.$$

Conversely, let $T\colon V \to A$ be an $\mathcal{O}$-operator of $(A, \cdot, \alpha)$ associated to a bimodule (V, l, r, ϕ). By Proposition 3.5, there exists a Hom-J-dendriform algebra on $T(V) = A$ given by

$$\begin{aligned} T(u) \prec T(v) &= T(r(T(u))v), \\ T(u) \succ T(v) &= T(l(T(u))v), \end{aligned} \qquad \forall\, u, v \in V.$$

By setting $x = T(u)$ and $y = T(v)$, we get

$$x \prec y = T(r(x)T^{-1}(y)) \qquad \text{and} \qquad x \succ y = T(l(x)T^{-1}(y)).$$

Finally, for any $x, y \in A$, we have

$$\begin{aligned} x \succ y + y \prec x &= T(r(x)T^{-1}(y)) + T(l(x)T^{-1}(y)) \\ &= T(r(x)T^{-1}(y) + l(x)T^{-1}(y)) \\ &= T(T^{-1}(x)) \cdot T(T^{-1}(y)) \;=\; x \cdot y. \end{aligned}$$

∎

Lemma 3.1. *Let R_1 and R_2 be two commuting Rota-Baxter operators (of weight zero) on a Hom-Jordan algebra $(A, \circ, \alpha)$. Then R_2 is a Rota-Baxter operator (of weight zero) on the Hom-pre-Jordan algebra $(A, \cdot, \alpha)$, where*

$$x \cdot y = R_1(x) \circ y, \qquad \forall\, x, y \in A.$$

Proof. For any $x, y \in A$, we have

$$\begin{aligned} R_2(x) \cdot R_2(y) &= R_1(R_2(x)) \circ R_2(y) \\ &= R_2(R_1(R_2(x)) \circ y + R_1(x) \circ R_2(y)) \\ &= R_2(R_2(x) \cdot y + x \cdot R_2(y)). \end{aligned}$$

This finishes the proof. ∎

Corollary 3.2. *Let R_1 and R_2 be two commuting Rota-Baxter operators (of weight zero) on a Hom-Jordan algebra $(A, \circ, \alpha)$. Then there exists a Hom-J-dendriform algebra structure on A given by*

$$x \prec y = R_1(y) \circ R_2(x), \quad x \succ y = R_1R_2(x) \circ y, \qquad \forall\, x, y \in A. \tag{3.12}$$

Proof. By Lemma 3.1, R_2 is Rota-Baxter operator of weight zero on $(A, \cdot, \alpha)$, where

$$x \cdot y = R_1(x) \circ y.$$

Then, applying Corollary 3.1, there exists a Hom-J-dendriform algebra structure on A given by

$$x \prec y = R_1(y) \circ R_2(x), \quad x \succ y = R_1 R_2(x) \circ y, \qquad \forall\, x, y \in A.$$

∎

We end this section by discussing some adjunctions between the categories of considered non-associative algebras.

Let **HomRBJ** be the category of Rota-Baxter Hom-Jordan algebras in which objects are quadruples of the form $(A, \circ, \alpha, R)$. Let **HomRBpJ** be the category of Rota-Baxter Hom-pre-Jordan algebras in which objects are quadruples of the form $(A, \cdot, \alpha, R)$. Notice that the morphisms are defined in a natural way, that is maps which are compatible with the multiplication, the twist maps and Rota-Baxter operators. The category of Hom-pre-Jordan algebras is denoted by **HompJ** and that of Hom-J-dendriform algebras by **HomJdend**.

THEOREM 3.2. 1. *There is an adjoint pair of functors*

$$U_{HP} : \mathbf{HompJ} \rightleftarrows \mathbf{HomRBJ} : HP, \tag{3.13}$$

in which the right adjoint is given by

$$HP(A, \circ, \alpha, R) = (A, \cdot, \alpha) \in \mathbf{HompJ}$$

with

$$x \cdot y \;=\; R(x) \circ y \tag{3.14}$$

for $x, y \in A$.

2. *There is an adjoint pair of functors*

$$U_{HD} : \mathbf{HomJdend} \rightleftarrows \mathbf{HomRBpJ} : HD, \tag{3.15}$$

in which the right adjoint is given by

$$HD(A, \cdot, \alpha, R) = (A, \prec, \succ, \alpha) \in \mathbf{HomJdend}$$

with

$$x \prec y \;=\; x \cdot R(y) \quad \text{and} \quad x \succ y \;=\; R(x) \cdot y \tag{3.16}$$

for $x, y \in A$.

Proof. The proof is based on Corollaries 2.1, 2.2, 2.3, 3.1 and Proposition 3.1. ∎

The following result says that, a Rota-Baxter Hom-pre-Jordan algebra can be given a new Hom-pre-Jordan structure.

COROLLARY 3.3. *Let* $(A, \cdot, \alpha, R)$ *be an object in* **HomRBpJ**. *Define a multiplication on* A *by*

$$x * y = x \cdot R(y) + R(x) \cdot y$$

for $x, y \in A$. *Then* $A' = (A, *, \alpha)$ *is a Hom-pre-Jordan algebra and* $R(x * y) = R(x) \cdot R(y)$.

Remark 3.2. Following [2] and considering the operads of Hom-Jordan algebras, Hom-pre-Jordan algebras and Hom-J-dendriform algebras, we have that the operad of Hom-Jordan algebras is the successor of the operad of Hom-pre-Jordan algebras and the operad of Hom-J-dendriform algebras is the successor of the operad of Hom-pre-Jordan algebras.

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