EXTRACTA MATHEMATICAE
Vol. **38**, Num. 1 (2023), 105 – 123

doi:10.17398/2605-5686.38.1.105
Available online May 31, 2023

The character variety of one relator groups

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Received October 26, 2022
Accepted April 23, 2023

Presented by A. Turull

Abstract: We consider some families of one relator groups arising as fundamental groups of 3-dimensional manifolds, and calculate their character varieties in $\mathrm{SL}(2,\mathbb{C})$. Then we give simple geometrical descriptions of such varieties, and determine the number of their irreducible components. Our paper relates to the work of Baker-Petersen, Qazaqzeh and Morales-Marcén on the character variety of certain classes of one relator groups, but we use different methods based on the concept of palindrome presentations of given groups.

Key words: Finitely generated group, torus link, torus bundle, character variety, $\mathrm{SL}(2,\mathbb{C})$ representation, Kauffman bracket skein module.

MSC (2020): 20C15, 57M25, 57M27.

1. Introduction

Let G be a finitely presented group. A *representation* of G is a group homomorphism from G to $\mathrm{SL}(2,\mathbb{C})$. Two representations are said to be *conjugate* if they differ by an inner automorphism of $\mathrm{SL}(2,\mathbb{C})$. A representation is *reducible* if it is conjugate to a representation into upper triangular matrices. Otherwise, the representation is called *irreducible*. The *character variety* of G is the set of conjugacy classes of representations of G into $\mathrm{SL}(2,\mathbb{C})$. The character variety of G is a closed algebraic subset of $\mathbb{C}^n$ for some n (see [8, 17]).

The character variety of the fundamental group of any hyperbolic 3-manifold contains some topological informations about the structure of the given manifold (see [8, 25]). A general equation form for such character varieties does not exist in the literature. However, they have been calculated for many classes of (hyperbolic) 3–manifolds.

Representations of two-bridge knot groups have been investigated in [3, 11, 23]. Character varieties of pretzel links and twisted Whitehead links have been determined in [27]. Recursive formulas for the character varieties of twist knots can be found in [13]. A very different method to determine the

ISSN: 0213-8743 (print), 2605-5686 (online)

character variety of twist knot groups has been proposed in [5]. The results are obtained by using special presentations of the knot groups, whose relators are *palindromes* (see [4]). This means that the relators read the same forwards or backwards as words in the generators.

In this paper we propose a method to determine the character variety of a class of torus links which is different to that developed in [21]. Our method reduces the computations presented in the quoted paper, and permits to give an easy geometrical description of the character varieties of these torus links. Using such a description we also give simplified proofs of some algebraic results obtained in [21]. The method is then applied to the fundamental group of once-punctured torus bundles. Such manifolds can be obtained by $(n+2,1)$ Dehn filling on one boundary component of the Whitehead link (WL) exterior. Using the concept of palindrome word, we give a geometrical description of the character varieties of such torus bundles. This relates to the main result of [1], using very different techniques for computing character varieties. As a further new result, we then derive the character varieties of another family of bordered 3-manifolds, arising from $(6n+2, 2n+1)$ Dehn filling on one boundary component of the WL exterior.

2. Technical preliminaries

We think of $\mathrm{SL}(2,\mathbb{C})$ as the 2×2 complex matrices of determinant 1 in the set of 2×2 complex matrices $\mathcal{M}(2,\mathbb{C})$. It is known that every matrix $A \in \mathcal{M}(2,\mathbb{C})$ splits as the direct sum of a scalar multiple of the identity matrix plus a trace zero matrix. In particular, we can write $A = A^+ + A^- = \alpha I_2 + A^-$, with $\sigma(A) = 2\,\alpha$ and $\sigma(A^-) = 0$, where $\sigma(A)$ denotes the *trace* of the matrix A and I_2 denotes the 2×2 identity matrix. So we can write $A = \alpha + A^-$.

For $A, B \in \mathcal{M}(2,\mathbb{C})$, set

$$A^+ = \alpha, \qquad B^+ = \beta, \qquad (A^- B^-)^+ = \gamma,$$

where α, β and γ represent complex numbers or scalar diagonal matrices depending on the context.

We define two families of polynomials, which naturally arise from computing the n-th powers of a matrix $A \in \mathrm{SL}(2,\mathbb{C})$. Write $A = \alpha + A^-$ as above, and

$$A^n = f_n(\alpha) + g_n(\alpha)\, A^-, \tag{2.1}$$

where $\sigma(A) = 2\,\alpha \in \mathbb{C}$. The polynomial f_n can be expressed in terms of g_n and g_{n-1}.

LEMMA 2.1. *With the above notations, we have*

$$f_n(\alpha) \,=\, \alpha\, g_n(\alpha) \,-\, g_{n-1}(\alpha). \tag{2.2}$$

Proof. Since $(A^-)^2 = \alpha^2 - 1$ from [5, Lemma 2.1(3)], it follows that

$$\begin{aligned} A^n = AA^{n-1} &= (\alpha + A^-)\,[f_{n-1}(\alpha) + g_{n-1}(\alpha)A^-] \\ &= \alpha f_{n-1}(\alpha) + (\alpha^2 - 1)g_{n-1}(\alpha) + [f_{n-1}(\alpha) + \alpha g_{n-1}(\alpha)]A^-. \end{aligned}$$

Equating this formula and (2.1) yields

$$f_n(\alpha) = \alpha f_{n-1}(\alpha) + (\alpha^2 - 1)g_{n-1}(\alpha) \tag{2.3}$$

and

$$g_n(\alpha) = f_{n-1}(\alpha) + \alpha g_{n-1}(\alpha). \tag{2.4}$$

Multiplying (2.4) by α, we get

$$\alpha g_n(\alpha) = \alpha f_{n-1}(\alpha) + \alpha^2 g_{n-1}(\alpha).$$

Using the last expression, we can eliminate $\alpha f_{n-1}(\alpha)$ from (2.3), that is,

$$f_n(\alpha) = \alpha g_n(\alpha) - \alpha^2 g_{n-1}(\alpha) + (\alpha^2 - 1)g_{n-1}(\alpha),$$

which gives (2.2). ∎

Moreover, we can derive the recursive expressions of f_n and g_n.

LEMMA 2.2. *The families of polynomials $\{f_n\}$ and $\{g_n\}$ are defined by the recurrence formulas*

$$g_n(\alpha) = \,2\,\alpha\, g_{n-1}(\alpha) \,-\, g_{n-2}(\alpha) \tag{2.5}$$

and

$$f_n(\alpha) = \,2\,\alpha\, f_{n-1}(\alpha) \,-\, f_{n-2}(\alpha) \tag{2.6}$$

for every $n \geq 1$, with the initial values $g_{-1}(\alpha) = -1$ and $g_0(\alpha) = 0$, $f_{-1}(\alpha) = \alpha$ and $f_0(\alpha) = 1$, respectively.

Proof. Substituting the expression of f_{n-1} from (2.2) into (2.4) yields

$$g_n(\alpha) = \alpha g_{n-1}(\alpha) - g_{n-2}(\alpha) + \alpha g_{n-1}(\alpha),$$

which gives (2.5).

Multiplying by α the formula of f_{n-1} from (2.2), we get

$$\alpha f_{n-1}(\alpha) = \alpha^2 g_{n-1}(\alpha) - \alpha g_{n-2}(\alpha).$$

Using the last expression, we can eliminate $\alpha^2 g_{n-1}(\alpha)$ from (2.3), that is,

$$f_n(\alpha) = 2\,\alpha\, f_{n-1}(\alpha) + \alpha g_{n-2}(\alpha) - g_{n-1}(\alpha).$$

By (2.5) written for $n-1$, we get

$$\begin{aligned} f_n(\alpha) &= 2\,\alpha\, f_{n-1}(\alpha) + \alpha g_{n-2}(\alpha) - [2\,\alpha\, g_{n-2}(\alpha) - g_{n-3}(\alpha)] \\ &= 2\,\alpha\, f_{n-1}(\alpha) - \alpha g_{n-2}(\alpha) + g_{n-3}(\alpha) \\ &= 2\,\alpha\, f_{n-1}(\alpha) - [\alpha g_{n-2}(\alpha) - g_{n-3}(\alpha)]. \end{aligned}$$

This implies (2.6) as the expression inside the brackets is precisely $f_{n-2}(\alpha)$ by (2.2). ∎

Lemma 2.3. *The following identities*

$$g_n^2(\alpha) \;=\; 1 \;+\; g_{n-1}(\alpha)\, g_{n+1}(\alpha)$$

and

$$2\, g_n(\alpha)\, \alpha \;-\; g_n^2(\alpha) \;=\; [g_{n+1}(\alpha) \;-\; 1]\,[1 \;-\; g_{n-1}(\alpha)]$$

hold.

Proof. The first formula is proved by induction on n. If $n = 0,\,1,\,2$, then $g_0^2 = 1 + g_{-1}\, g_1 = 0$, $g_1^2 = 1 + g_0\, g_2 = 1$, and $g_2^2(\alpha) = 1 + g_1\, g_3 = 4\alpha^2$, respectively, as $g_{-1} = -1$, $g_0 = 0$, $g_1 = 1$, $g_2(\alpha) = 2\alpha$, and $g_3(\alpha) = 4\alpha^2 - 1$. Using the inductive hypothesis and (2.5), we get

$$\begin{aligned} 1 + g_{n-1}(\alpha)\, g_{n+1}(\alpha) &= 1 + g_{n-1}(\alpha)\,[2\alpha\, g_n(\alpha) \;-\; g_{n-1}(\alpha)] \\ &= 1 + 2\alpha\, g_n(\alpha)\, g_{n-1}(\alpha) \;-\; g_{n-1}^2(\alpha) \\ &= 1 \;+\; 2\alpha\, g_n(\alpha)\, g_{n-1}(\alpha) \;-\; 1 \;-\; g_{n-2}(\alpha)\, g_n(\alpha) \\ &= g_n(\alpha)[2\alpha g_{n-1}(\alpha) \;-\; g_{n-2}(\alpha)] = g_n^2(\alpha). \end{aligned}$$

For the second equality, we have

$$\begin{aligned} [g_{n+1}(\alpha) \;-\; 1]\,[1 \;-\; g_{n-1}(\alpha)] &= g_{n+1}(\alpha) \;-\; g_{n+1}(\alpha)\, g_{n-1}(\alpha) \;+\; g_{n-1}(\alpha) \;-\; 1 \\ &= g_{n+1}(\alpha) \;+\; 1 \;-\; g_n^2(\alpha) \;-\; 1 \;+\; g_{n-1}(\alpha) \\ &= 2\,\alpha\, g_n(\alpha) \;-\; g_{n-1}(\alpha) \;-\; g_n^2(\alpha) \;+\; g_{n-1}(\alpha) \\ &= 2\,\alpha\, g_n(\alpha) \;-\; g_n^2(\alpha) \end{aligned}$$

by using the first equality of the statement and the recursive formula of $g_n(\alpha)$ in (2.5). ∎

The polynomials $\{g_n\}$ are related to the n-th Chebyshev polynomial of the first kind $S_n(x)$ (see [14]), that is, $g_n(\alpha) = S_{n-1}(2\,\alpha)$. Furthermore, we also have $g_n(\alpha) = F_n(2\,\alpha)$, where F_n denotes the n-th Fibonacci polynomial (see, for example, [1, 26]). Finally, g_n relates with the Hilden-Lozano-Montesinos polynomial p_n (see [10]) by the formula $g_{n+1}(\alpha) = p_n(2\alpha)$.

Further algebraic properties of polynomials f_n and g_n have been described in [5, Proposition 2.3].

Through the paper we also need the following result:

Lemma 2.4. *Let $\{a, b\}$ be a set of generators of a 2-generator group G, and let ρ be an irreducible representation of G into $SL(2,\mathbb{C})$. Setting $A = \rho(a)$ and $B = \rho(b)$, the set $\mathcal{B} = \{I_2, A^-, B^-, (A^-B^-)^-\}$ is a basis for the 4-dimensional vector space $\mathcal{M}(2,\mathbb{C})$.*

For a proof see, for example, [12, Lemma 1.2]. Furthermore, we implicitly use the well-known fact that a representation of a group with two generators a and b is determined by the traces of these generators and of their product ab (see, for example, [9]).

3. Torus links

Let $C(2n)$ denote the rational link in Conway's normal form (see [15, p. 24]), which is the torus link depicted in Figure 1. It is the closure of the braid σ_1^{2n}, where σ_1 is the standard generator of the braid group B_2 on two strands. Equivalently, it is the closure of the braid $(\sigma_{2n-1}\,\sigma_{2n-2}\,\cdots\,\sigma_1)^2$ with $\sigma_1,\ \sigma_2, \ldots, \sigma_{2n-1}$ being the standard generators of the braid group B_{2n} on $2n$ strands. Note that the torus link $C(2n)$ is given by $T(2n, 2)$ according to Rolfsen's notation [24].

Theorem 3.1. *The character variety of the torus link $C(2n)$, $n \geq 1$, is defined by the equation*

$$(AB\,-\,BA)\,g_n(\alpha)\,=\,0.$$

The first factor determines the character variety for abelian representations into $SL(2,\mathbb{C})$, and the second factor determines the character variety for non-abelian representations of the link group G_n.

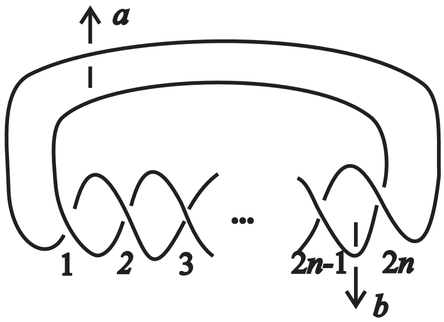

Figure 1: The torus link $C(2n)$, $n \geq 1$.

Proof. Let G_n denote the fundamental group of the exterior of $C(2n)$ in the oriented 3-sphere $\mathbb{S}^3$, i.e., $G_n = \pi_1(\mathbb{S}^3 \backslash C(2n))$. The group G_n admits the finite presentation $\langle a, b : (ab)^n = (ba)^n \rangle$. We provide a geometric interpretation of the generators of G_n by representing them in Figure 1. Setting $u = ab$ and $v = b$ (hence $a = uv^{-1}$ and $b = v$), we get the finite presentation $\langle u, v : u^n v = v u^n \rangle$. Sending u and v to the matrices A and B, respectively, the last relation gives $A^n B = B A^n$ in $\mathrm{SL}(2, \mathbb{C})$. For $n \geq 1$, we have

$$\begin{aligned} A^n B &= [f_n(\alpha) + g_n(\alpha) A^-]\,(\beta + B^-) \\ &= \beta f_n(\alpha) I_2 + \beta g_n(\alpha) A^- + f_n(\alpha) B^- + g_n(\alpha) A^- B^- \end{aligned}$$

and

$$\begin{aligned} B A^n &= (\beta + B^-)\,[f_n(\alpha) + g_n(\alpha) A^-] \\ &= \beta f_n(\alpha) I_2 + \beta g_n(\alpha) A^- + f_n(\alpha) B^- + g_n(\alpha) B^- A^-. \end{aligned}$$

Computing the difference gives

$$A^n B - B A^n = (A^- B^- - B^- A^-)\, g_n(\alpha)$$

hence

$$A^n B - B A^n = (AB - BA)\, g_n(\alpha)$$

as $A^- B^- - B^- A^- = AB - BA$. This produces the defining relations of the character variety of $C(2n)$ (or G_n). ∎

The techniques used in the above proof are different from those employed by Qazaqzeh in [21, Theorem 1.2]. For a given representation ρ of the group

$G_n = \langle a, b \,:\, (ab)^n = (ba)^n \rangle$ into $\mathrm{SL}(2,\mathbb{C})$, the cited author denotes by $\mathrm{tr}(x)$ the trace of $\rho(x)$, for any word x in the generators a and b. Then $\mathrm{tr}(a)$, $\mathrm{tr}(b)$ and $\mathrm{tr}(ab)$ are abbreviated by t_1, t_2 and t_3, respectively. His result states that the defining polynomial of the character variety of G_n is given by

$$\mathrm{tr}\left((ab)^n a^{-1} b^{-1}\right) \,-\, \mathrm{tr}\left((ba)^{n-1}\right) \,=\, (t_3^2 + t_2^2 + t_1^2 - t_3 t_2 t_1 - 4)\, S_{n-1}(t_3),$$

where the first (resp. second) factor on the right side determines the character variety for abelian (resp. nonabelian) representations. Here $S_k(x)$ is the kth Chebyshev polynomial of the first kind, defined recursively by $S_0(x) = 1$, $S_1(x) = x$ and $S_k(x) = xS_{k-1}(x) - S_{k-2}(x)$. The proof of this formula is given by induction on n, using the trace identities and the recursive definition of the Chebyshev polynomials.

The same elementary methods in the proof of Theorem 3.1 can be used to obtain the defining polynomial of the character variety of a class of torus knots from [20] and the characters of certain families of one relator groups from [18, 19, 22]. Namely, the authors in [18, 19] consider the group $G = \langle x, y : x^m = y^n \rangle$ with m and n nonzero integers, and compute the number of irreducible components of the character variety of G in $\mathrm{SL}(2,\mathbb{C})$. A defining polynomial of the $\mathrm{SL}(2,\mathbb{C})$ character variety of the torus knot of type $(m, 2)$ has been provided by Oller-Marcén in [20].

Recurrence formulas based on (generalized) Fibonacci polynomials have been proposed in [26, Theorem 7 and Theorem 11] to derive HOMFLY polynomials (and hence Alexander-Conway polynomials and Jones polynomials) of torus links $C(2n)$. Generalized Fibonacci polynomials can be related to our classes of polynomials $\{f_n\}$ and $\{g_n\}$, as remarked above.

For $n = 1$, $g_n(\alpha) = 1$, hence the equation in Theorem 3.1 reduces to $AB = BA$, which determines the character variety for abelian representations into $\mathrm{SL}(2,\mathbb{C})$. So in the sequel, we discuss the case $C(2n+2)$ with n being ≥ 1. Theorem 3.1 directly gives an easy geometrical description of the character variety of such torus links.

THEOREM 3.2. *In the complex 3-space (X, Y, Z) the character variety for nonabelian representations of the torus link $C(2n + 2)$ consists of the union of n horizontal planes of the form $Z_k = 2\cos[k\pi/(n + 1)]$, for $1 \leq k \leq n$.*

Proof. We set $Z = 2\,\alpha = \sigma(A)$, $X = 2\,\beta = \sigma(B) = \sigma(B^{-1})$, and $Y = \sigma(AB^{-1})$. From the relation

$$AB^{-1} = (\alpha \,+\, A^-)\,(\beta \,-\, B^-) = \alpha\,\beta\, I_2 \,+\, \beta\, A^- \,-\, \alpha\, B^- \,-\, A^- B^-,$$

it follows that

$$Y = \sigma(AB^{-1}) = 2\alpha\beta - 2\gamma$$

as $\sigma(A^-) = \sigma(B^-) = 0$ and $\sigma(A^-B^-) = 2\gamma$. The roots of the second factor $g_n(Z/2) = 0$ are given by $Z_k = 2\cos[k\pi/(n+1)]$ for any $1 \leq k \leq n$. See [5, Proposition 2.3(9)] and [10, Proposition 1.3]. ∎

Using the Chesebro formula for $g_{n+1}(\alpha)$ (see [7]), we can give a different expression for the defining equation in Theorem 3.2.

Corollary 3.3. *In the complex* 3*-space* (X, Y, Z) *the character variety for nonabelian representations of the torus link* $C(2n+2)$ *is defined by the equation*

$$\left[\frac{(Z + \sqrt{Z^2 - 4})^{n+1} - (Z - \sqrt{Z^2 - 4})^{n+1}}{2^{n+1}\sqrt{Z^2 - 4}}\right] = 0$$

for $-2 < Z < 2$ *(real number).*

To illustrate geometrically the support of the character variety in Theorem 3.2 and Corollary 3.3 we explicitly discuss the cases $n = 1, \ldots, 5$.

If $n = 1$, there is one horizontal plane of the form $Z = 2\cos(\pi/2) = 0$ from Theorem 3.2. The equation of the second factor in Corollary 3.3 becomes $Z = 0$.

If $n = 2$, there are two horizontal planes with equations $Z = 2\cos(\pi/3) = 1$ and $Z = 2\cos(2\pi/3) = -1$ (see Theorem 3.2). The equation of the second factor in Corollary 3.3 becomes $Z^2 - 1 = 0$.

If $n = 3$, there are three horizontal planes with equations $Z = 2\cos(\pi/4) = \sqrt{2}$, $Z = 2\cos(\pi/2) = 0$, and $Z = 2\cos(3\pi/4) = -\sqrt{2}$. The equation of the second factor in Corollary 3.3 becomes $Z(Z^2 - 2) = 0$.

If $n = 4$, there are four horizontal planes with equations $Z = 2\cos(\pi/5) = (1 + \sqrt{5})/2$, $Z = 2\cos(2\pi/5) = (\sqrt{5} - 1)/2$, $Z = 2\cos(3\pi/5) = (1 - \sqrt{5})/2$, and $Z = 2\cos(4\pi/5) = (-1 - \sqrt{5})/2$. The equation of the second factor in Corollary 3.3 becomes $Z^4 - 3Z^2 + 1 = 0$, which has the four roots $\pm(1 \pm \sqrt{5})/2$, as requested.

If $n = 5$, there are five horizontal planes with equations $Z = 2\cos(\pi/6) = \sqrt{3}$, $Z = 2\cos(\pi/3) = 1$, $Z = 2\cos(\pi/2) = 0$, $Z = 2\cos(2\pi/3) = -1$, and $Z = 2\cos(5\pi/6) = -\sqrt{3}$. The equation of the second factor in Corollary 3.3 becomes $Z(Z^2 - 1)(Z^2 - 3) = 0$, which has the above roots.

As remarked in [24, Example 10], the genus of the torus link $C(2n+2)$ is n, which precisely coincides with the number of horizontal planes in the character variety of $C(2n+2)$, i.e., the degree of the polynomial $g_{n+1}(Z/2)$.

Since the character varieties of G_n and G_m have different number of irreducible components if $n \neq m$, we derive the following well-known result (see [21, Corollary 1.3]).

COROLLARY 3.4. *The groups G_n and G_m are isomorphic if and only if $n = m$.*

Note that Corollary 3.4 also follows from the theory of Seifert manifolds since the torus link complement $C(2n)$ is a Seifert fiber space with one exceptional fiber.

Let M be an oriented compact 3-manifold. Then the *Kauffman bracket skein module* $\mathcal{K}(M)$ of M is defined to be the quotient of the module freely generated by equivalence classes of framed links in M over $\mathbb{Z}[t, t^{-1}]$ by the smallest submodule containing Kauffman relations (see [2] for more details). The topological meaning of this module has been explained in [2] for $t = -1$. More precisely, setting $t = -1$ and tensoring such a module with $\mathbb{C}$ produces a natural algebra structure, denoted $\mathcal{K}_{-1}(M)$, over $\mathbb{C}$. Furthermore, this algebra is canonically isomorphic to the coordinate ring of the character variety of $\pi_1(M)$ after factoring it by its nilradical (see [2, Theorem 10]). Then Theorem 3.2 allows to give a simplified proof of Theorem 1.4 from [21].

THEOREM 3.5. *Let M denote the exterior of $C(2n+2)$, $n \geq 1$, in the oriented 3-sphere, $\mathcal{K}(M)$ the Kauffman bracket skein module of M, and N the $(t+1)$-torsion submodule of $\mathcal{K}(M)$. Then the quotient $\mathcal{K}(M)/N$ is a free module over $\mathbb{Z}[t, t^{-1}]$ with a basis $\mathcal{B} = \{x^i y^j z^k : i, j \geq 0,\ 0 \leq k \leq n\}$, where x, y, and z represent the conjugacy classes of uv^{-1}, v, and u in the presentation $\langle u, v : u^n v = v u^n \rangle$ of $\pi_1(M)$, respectively.*

Proof. By Theorem 3.2 the coordinate ring of the character variety of $\pi_1(M)$ admits the basis $\mathcal{B}$ (over $\mathbb{C}$) indicated in the statement. In fact, the horizontal planes $Z = 2\cos[k\pi/(n+1)]$, $1 \leq k \leq n$, plus the neutral element for $k = 0$, give $n+1$ conjugacy classes of the statement. By [21] the quotient of $\mathcal{K}_{-1}(M)$ over its nilradical is isomorphic (over $\mathbb{C}$) to $\mathcal{K}_{-1}(M)$. Hence $\mathcal{B}$ is linearly independent (over $\mathbb{C}$) in $\mathcal{K}_{-1}(M)$. Then it is a basis for $\mathcal{K}(M)/N$. ∎

For a description of $\mathcal{K}(M)$, when M is the exterior of a 2-bridge link, we refer to [16].

4. Once-punctured torus bundles

Let us consider the once-punctured torus bundles with tunnel number one, that is, the once-punctured torus bundles that arise from filling one boundary component of the Whitehead link (WL) exterior. See Figure 2.

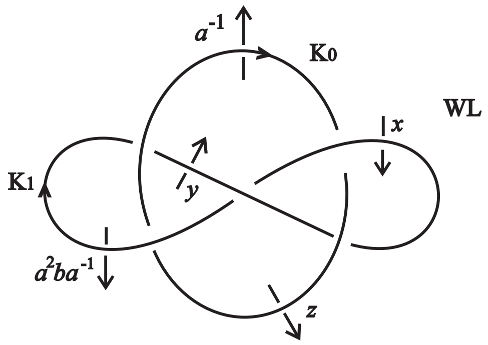

Figure 2: A planar projection of the Whitehead link.

The character varieties of such manifolds have been determined in [1]. Using the concept of palindrome word, we compute the defining polynomials of these character varieties with different techniques with respect to [1]. Up to homeomorphism, the monodromy of the once-punctured torus bundle $M_n = (T \times I)/Q_n$ is $Q_n = \tau_{c_1}\,\tau_{c_2}^{n+2}$, where c_1 and c_2 are curves forming a basis for the fiber T (a torus) and τ_c means a right-handed Dehn twist about the curve c. Here $I = [0,1]$. The manifold M_n can be obtained by $(n+2,1)$ Dehn filling on one boundary component of the WL exterior, and it is the exterior of a certain genus one fibered knot in the lens space $L(n+2,1)$. It is known that M_n is hyperbolic if and only if $|n| > 2$, contains an essential torus (i.e., is toroidal) if and only if $|n| = 2$, and is a Seifert fiber space if and only if $|n| \leq 1$. See, for example, [1, Lemma 2.8].

By [1, Lemma 2.5], the fundamental group $\pi_1(M_n)$ is isomorphic to

$$\Gamma_n \,=\, \langle a, b :\, a^{-n} \,=\, b^{-1}\,a\,b^2\,a\,b^{-1}\,\rangle. \tag{4.1}$$

We provide a geometric interpretation of the generators of Γ_n by representing them in Figure 2. We choose meridians μ_0, μ_1 and longitudes λ_0, λ_1 on the oriented components K_0, K_1 of WL, respectively, (see Figure 2) such that $[\mu_i, \lambda_i] = 1$, for $i = 0, 1$, and $\lambda_i \sim 0$ in $\mathbb{S}^3 \backslash K_i$. Then we have $\mu_0 = a^{-1}$, $\mu_1 = a^2ba^{-1}$, $\lambda_0 = xab^{-1}a^{-2}$ and $\lambda_1 = az$, where x and z are represented in Figure 2. The Wirtinger presentation of the group $\pi(WL) = \pi_1(\mathbb{S}^3 \backslash WL)$ has generators a, b, x, y and z and relations $ya^{-1} = aba^{-1}$, $z = ab^{-1}a^{-1}ba^{-1}$, $yx = a^2ba^{-1}y$ and $xz = a^{-1}x$. Then we obtain the relation $xz = a^{-1}b^{-1}ab^2$ after doing the appropriate elimination. Eliminating $x = b^{-1}ab^2$, $y = ab$ and $z = ab^{-1}a^{-1}ba^{-1}$ yields a finite presentation for $\pi(WL)$ with generators a and b and relation

$$b^{-1}ab^2ab^{-1}a^{-1}ba^{-1} = a^{-1}b^{-1}ab^2. \tag{4.2}$$

A presentation for Γ_n can be obtained from that of $\pi(WL)$ by adding the surgery relation

$$\mu_0^{-(n+2)} \lambda_0 = 1 \tag{4.3}$$

where $\mu_0 = a^{-1}$ and $\lambda_0 = xab^{-1}a^{-2} = b^{-1}ab^2ab^{-1}a^{-2}$. Substituting these formulas into (4.3) gives

$$a^{n+2}b^{-1}ab^2ab^{-1}a^{-2} = 1$$

hence

$$a^nb^{-1}ab^2ab^{-1} = 1$$

which is equivalent to the relation in (4.1). Now (4.2) is a consequence of the relation in (4.1), so it can be dropped. In fact, we have the following sequences of Tietze transformations:

$$\begin{aligned}(b^{-1}ab^2ab^{-1})a^{-1}ba^{-1} &= a^{-1}b^{-1}ab^2,\\ a^{-n}a^{-1}ba^{-1} &= a^{-1}b^{-1}ab^2,\\ a^{-n} &= b^{-1}ab^2ab^{-1},\end{aligned}$$

which is the relation of Γ_n.

THEOREM 4.1. *For every $n \in \mathbb{Z}$, let M_n be the once-punctured torus bundle of tunnel number one, and $\Gamma_n = \pi_1(M_n)$. In the complex plane (X, Z), the defining equation of the character variety of Γ_n is given by*

$$[g_{n+1}(Z/2) - 1]\,[X^2 - 1 + g_{n-1}(Z/2)] = 0.$$

In the hyperbolic case $|n| > 2$, the character variety for nonabelian representations of Γ_n (or M_n) consists of the hyperelliptic curve given by

$$X^2 + g_{n-1}(Z/2) - 1 = 0$$

and a finite number of horizontal lines (counted with their multiplicities) of the form $Z = Z_k$, where Z_k is a root of the equation $g_{n+1}(Z/2) - 1 = 0$.

Proof. From the relation in (4.1), or equivalently $ba^{-n}b = ab^2a$, sending a and b to the matrices A and B, respectively, gives the relation in $\mathrm{SL}(2,\mathbb{C})$

$$B\,A^{-n}\,B \,=\, A\,B^2\,A$$

which is palindrome in the left and right sides. Set $A = \alpha + A^-$ and $B = \beta + B^-$. As a direct application of the Cayley-Hamilton theorem, the formula

$$A^{-n} \,=\, f_n(\alpha) \,-\, g_n(\alpha)\,A^-$$

holds. By direct calculations on palindrome words, it follows

$$B\,A^{-n}\,B \,=\, q_0\,I_2 \,+\, q_1\,A^- \,+\, q_2\,B^-$$

where

$$\begin{aligned} q_0 &= (2\,\beta^2 \,-\, 1)\,f_n(\alpha) \,-\, 2\,\beta\,\gamma\,g_n(\alpha)\,, \\ q_1 &= -\,g_n(\alpha)\,, \\ q_2 &= 2\,\beta\,f_n(\alpha) \,-\, 2\,\gamma\,g_n(\alpha)\,, \end{aligned}$$

with $A^+ = \alpha$, $B^+ = \beta$ and $(A^-B^-)^+ = \gamma$, i.e., $\sigma(A) = 2\,\alpha$, $\sigma(B) = 2\,\beta$ and $\sigma(A^-B^-) = 2\,\gamma$.

As above, by direct computations on palindromes, we have

$$A\,B^2\,A \,=\, q_0^{'}\,I_2 \,+\, q_1^{'}\,A^- \,+\, q_2^{'}\,B^-$$

where

$$\begin{aligned} q_0^{'} &= (2\,\alpha^2 \,-\, 1)\,(2\,\beta^2 \,-\, 1) \,+\, 4\,\alpha\,\beta\,\gamma\,, \\ q_1^{'} &= 2\,\alpha\,(2\,\beta^2 \,-\, 1) \,+\, 4\,\beta\,\gamma\,, \\ q_2^{'} &= 2\,\beta\,. \end{aligned}$$

Equating $q_i = q_i^{'}$, $i = 0, 1, 2$, gives the defining polynomials of the character variety for Γ_n (or M_n). From $q_2 = q_2^{'}$ we derive an expression of γ in terms of α and β. So the representation (up to conjugacy) is only determined by the traces $\sigma(A) = 2\alpha$ and $\sigma(B) = 2\beta$. Substituting the cited expression of γ into $q_1 = q_1^{'}$ yields the defining equation of the character variety. In fact, $q_0 = q_0^{'}$

is a consequence of the other equations. Thus the character variety of Γ_n has equation

$$g_n^2(\alpha) + 2\alpha(2\beta^2 - 1)\, g_n(\alpha) + 4\beta^2\,[f_n(\alpha) - 1] = 0.$$

We can express $f_n(\alpha)$ in terms of $g_n(\alpha)$ and $g_{n-1}(\alpha)$. Multiply out gives the equation

$$g_n^2(\alpha) + 2\alpha(4\beta^2 - 1)\, g_n(\alpha) - 4\beta^2\,[g_{n-1}(\alpha) + 1] = 0.$$

Set $Z = 2\alpha \in \mathbb{C}$ and $X = 2\beta \in \mathbb{C}$. Then we get

$$g_n^2(Z/2) + Z(X^2 - 1)\, g_n(Z/2) - X^2\,[g_{n-1}(Z/2) + 1] = 0$$

or, equivalently,

$$g_n^2(Z/2) + [g_n(Z/2)\, Z - g_{n-1}(Z/2) - 1]\, X^2 - g_n(Z/2)\, Z = 0$$

hence

$$g_n^2(Z/2) + [g_{n+1}(Z/2) - 1]\, X^2 - g_n(Z/2)\, Z = 0.$$

By Lemma 2.3, the defining equation of the character variety of Γ_n is given by the first formula in the statement. The last sentence of the theorem follows from [5, Proposition 2.3]. ∎

Since $g_n(Z/2) = F_n(Z)$, Theorem 4.1 relates to Theorem 5.1 from Baker and Petersen [1] in the sense that we obtain a similar hyperelliptic curve. More precisely, these authors prove that if $|n| > 2$, then there is a unique canonical component of the $\mathrm{SL}(2,\mathbb{C})$ character variety of M_n, and it is birational to the hyperelliptic curve given by $w^2 = -\widehat{h}_n(y)\,\widehat{\ell}_n(y)$ in the complex plane (w, y), where the polynomials $\widehat{h}_n$ and $\widehat{\ell}_n$ are specific factors of Fibonacci polynomials. If n is not congruent to 2 (mod 4), this is the only component of the $\mathrm{SL}(2;\mathbb{C})$ character variety which contains the characters of an irreducible representation. If $n \equiv 2 \pmod 4$, there is an additional component which is isomorphic to $\mathbb{C}$. If n is not equal to -2, all the components consisting of characters of reducible representations are isomorphic to affine conics (including lines) and consist of characters of abelian representations. However, the methods used by the cited authors (based on the invariant theory) are similar to those developed by Qazaqzeh in [21] for the class of torus links.

To illustrate geometrically the support of the character variety in Theorem 4.1 we explicitly discuss the hyperbolic cases $n = 3, \ldots, 6$.

If $n = 3$, the equation $X^2 + g_2(Z/2) - 1 = 0$ becomes $X^2 + Z - 1 = 0$ as $g_2(\alpha) = 2\alpha = Z$. Furthermore, the equation $g_4(Z/2) - 1 = 0$ becomes

$$Z^3 - 2Z - 1 = (Z + 1)\,(Z^2 - Z - 1) = 0$$

as $g_4(\alpha) = 8\alpha^3 - 4\alpha = Z^3 - 2Z$. Then, in the complex plane (X, Z), the character variety of Γ_3 (or M_3) consists of the parabola $Z = 1 - X^2$ and the union of three horizontal lines with equations $Z = -1$ and $Z = (1 \pm \sqrt{5})/2$.

If $n = 4$, the equation $X^2 + g_3(Z/2) - 1 = 0$ becomes $X^2 + Z^2 - 2 = 0$ as $g_3(\alpha) = 4\alpha^2 - 1 = Z^2 - 1$. Furthermore, the equation $g_5(Z/2) - 1 = 0$ becomes $Z^4 - 3Z^2 = Z^2(Z^2 - 3)$ as $g_5(\alpha) = 16\alpha^4 - 12\alpha^2 + 1 = Z^4 - 3Z^2 + 1$. So, in the complex plane (X, Z), the character variety of Γ_4 (or M_4) consists of the ellipse $X^2 + Z^2 = 2$ and the union of four (counted with their multiplicities) horizontal lines with equations $Z = 0$ (counted twice) and $Z = \pm\sqrt{3}$.

If $n = 5$, the equation $X^2 + g_4(Z/2) - 1 = 0$ becomes $X^2 + Z^3 - 2Z - 1 = 0$ or, equivalently, $X^2 + (Z + 1)\,(Z^2 - Z - 1) = 0$. The equation $g_6(Z/2) - 1 = 0$ becomes $Z^5 - 4Z^3 + 3Z - 1 = 0$ as

$$g_6(\alpha) = 32\,\alpha^5 - 32\,\alpha^3 + 6\,\alpha = Z^5 - 4\,Z^3 + 3\,Z.$$

Thus, in the complex plane (X, Z), the character variety of Γ_5 (or M_5) consists of the elliptic cubic (in fact, the Newton divergent parabola) of equation $X^2 = -Z^3 + 2Z + 1$ and the union of five horizontal lines with equations of the form $Z = Z_k$, where Z_k is a root of

$$Z^5 - 4Z^3 + 3Z - 1 = (Z^2 + Z - 1)\,(Z^3 - Z^2 - 2Z + 1) = 0.$$

From the first factor we get $Z_{1,2} = (-1 \pm \sqrt{5})/2$. The equation

$$Z^3 - Z^2 - 2Z + 1 = 0$$

becomes $x^3 + px + q = 0$ with $p = -\frac{7}{3}$ and $q = \frac{7}{27}$ by using the transformation $Z = x + \frac{1}{3}$. Since $\Delta = \frac{q^2}{4} + \frac{p^3}{27} = -\frac{49}{108} < 0$, there are three real roots $x_1 = 2a$, $x_2 = -a - b\sqrt{3}$ and $x_3 = -a + b\sqrt{3}$, where $a + ib = \sqrt[3]{w}$ and $w = -\frac{q}{2} + i\sqrt{\Delta}$.

If $n = 6$, the equation $X^2 + g_5(Z/2) - 1 = 0$ becomes $X^2 + Z^4 - 3Z^2 = 0$. The equation $g_7(Z/2) - 1 = 0$ becomes

$$Z^6 - 5Z^4 + 6Z^2 - 2 = (Z^2 - 1)\,(Z^4 - 4Z^2 + 2) = 0$$

as

$$g_7(\alpha) = 64\,\alpha^6 - 80\,\alpha^4 + 24\,\alpha^2 - 1 = Z^6 - 5Z^4 + 6Z^2 - 1.$$

Thus, in the complex plane (X, Z), the character variety of Γ_6 consists of the hyperelliptic quartic $X^2 = -Z^4 + 3Z^2$ and the union of six horizontal lines with equations of the form $Z = Z_k$, where Z_k takes on the values ± 1 and $\pm\sqrt{2 \pm \sqrt{2}}$.

5. Cusped manifolds from Dehn fillings

For every $n \geq 0$, let N_n be the one-cusped 3-manifold obtained by performing a $(6n+2, 2n+1)$ Dehn filling on one boundary component of the WL exterior, leaving the other component open. See Figure 3. It is known that N_n is hyperbolic for every $n \geq 1$.

Among all fillings of one cusp of the Whitehead exterior we focus on the $(6n+2, 2n+1)$ fillings since their fundamental group has a simple palindrome presentation. See (5.1) below. However, the proposed techniques for computing character varieties of such manifolds can also be applied in the general case.

By [6, Proposition 4.1] the fundamental group $\pi_1(N_n)$ is isomorphic to

$$\begin{aligned} \Lambda_n &= \langle\, a,\, b \,:\, a\,b\,a \,=\, (b^3\, a^{-3})^{2n}\, b^3 \,\rangle \\ &= \langle\, a,\, b \,:\, a\,b\,a \,=\, b^3\,(a^{-3}\, b^3)^{2n} \,\rangle. \end{aligned} \tag{5.1}$$

We provide a geometric interpretation of the generators of Λ_n in Figure 3.

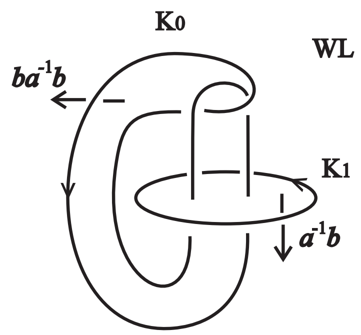

Figure 3: Another planar projection of the Whitehead link.

Theorem 5.1. *For every $n \geq 0$, let N_n be the one-cusped 3-manifold obtained by $(6n+2, 2n+1)$ Dehn filling on one boundary component of the*

WL exterior, and let $\Lambda_n = \pi_1(N_n)$*. In the complex* 3*-space* (X, Y, Z)*, the character variety of the group* Λ_n *(or* N_n*) is determined by the equations*

$$Y + (Z^2 - 1)g_{2n}(\delta) = 0\,,$$
$$(X^2 - 1)g_{2n+1}(\delta) \, - \, 1 = 0\,,$$

where δ *is given by*

$$2\delta = X^3Z^3 - 2XZ^3 - 2X^3Z - X^2YZ^2 + X^2Y + YZ^2 + 5XZ - Y.$$

Proof. From the relation in (5.1), sending a and b to the matrices A and B, respectively, gives the relation in $\mathrm{SL}(2, \mathbb{C})$

$$A\,B\,A \, = \, (B^3\,A^{-3})^{2n}\,B^3,$$

which is palindrome in the left and right sides. By direct computations on palindromes, we obtain

$$A\,B\,A = \bar{q}_0\,I_2 \, + \, \bar{q}_1\,A^- \, + \, \bar{q}_2\,B^-$$

where

$$\bar{q}_0 = (2\alpha^2 - 1)\beta \, + \, 2\alpha\gamma\,,$$
$$\bar{q}_1 = 2\alpha\beta + 2\gamma\,,$$
$$\bar{q}_2 = 1\,,$$

with $A^+ = \alpha$, $B^+ = \beta$ and $(A^-B^-)^+ = \gamma$, as usual.

Define $L = B^3A^{-3}$. Then $L = \delta + L^-$, where $\sigma(L) = 2\delta$. We get

$$L = p_0I_2 + p_1A^- + p_2B^- + p_3A^-B^-$$

where

$$p_0 = (4\alpha^3 - 3\alpha)(4\beta^3 - 3\beta) - 2(4\alpha^2 - 1)(4\beta^2 - 1)\gamma\,,$$
$$p_1 = -(4\alpha^2 - 1)(4\beta^3 - 3\beta)\,,$$
$$p_2 = (4\alpha^3 - 3\alpha)(4\beta^2 - 1)\,,$$
$$p_3 = (4\alpha^2 - 1)(4\beta^2 - 1)\,.$$

Since $\sigma(A^-) = \sigma(B^-) = 0$ and $\sigma(A^-B^-) = 2\gamma$, we obtain

$$\delta = p_0 + \gamma p_3. \tag{5.2}$$

It follows that

$$L^- = -\gamma p_3 I_2 + p_1 A^- + p_2 B^- + p_3 A^- B^- \tag{5.3}$$

and

$$L^- B^- = (\beta^2 - 1)p_2 I_2 + (\beta^2 - 1)p_3 A^- - \gamma p_3 B^- + p_1 A^- B^- . \tag{5.4}$$

Using (5.3) and (5.4) we obtain

$$\begin{aligned}(B^3\, A^{-3})^{2n}\, B^3 = L^{2n} B^3 &= [f_{2n}(\delta) + g_{2n}(\delta)\, L^-]\, [4\beta^3 - 3\beta + (4\beta^2 - 1)B^-] \\ &= \bar{q}_0^{'} I_2 + \bar{q}_1^{'} A^- + \bar{q}_2^{'} B^-\end{aligned}$$

where

$$\begin{aligned}\bar{q}_0^{'} &= (4\beta^3 - 3\beta) f_{2n}(\delta) + [(4\beta^2 - 1)(\beta^2 - 1)p_2 - (4\beta^3 - 3\beta)\gamma p_3] g_{2n}(\delta)\,, \\ \bar{q}_1^{'} &= -(4\alpha^2 - 1) g_{2n}(\delta)\,, \\ \bar{q}_2^{'} &= (4\beta^2 - 1) f_{2n}(\delta) + [(4\beta^3 - 3\beta)p_2 - (4\beta^2 - 1)\gamma p_3] g_{2n}(\delta)\,.\end{aligned}$$

By (2.2) and (5.2) and using the above expressions of p_2 and p_3 in terms of α and β, the polynomial $\bar{q}_2^{'}$ becomes

$$\begin{aligned}\bar{q}_2^{'} &= (4\beta^2 - 1)[\delta g_{2n}(\delta) - g_{2n-1}(\delta)] + (4\beta^2 - 1)\delta g_{2n}(\delta) \\ &= 2\delta(4\beta^2 - 1) g_{2n}(\delta) - (4\beta^2 - 1) g_{2n-1}(\delta) \\ &= (4\beta^2 - 1)\, g_{2n+1}(\delta).\end{aligned}$$

By Lemma 2.4, equating $\bar{q}_i = \bar{q}_i^{'}$, $i = 0, 1, 2$, gives the equations of the character variety of the group Λ_n. We see that $\bar{q}_0 = \bar{q}_0^{'}$ is a consequence of the other two equations.

We set $Z = 2\alpha = \sigma(A)$, $X = 2\beta = \sigma(B)$, and $Y = \sigma(AB) = 2\alpha\beta + 2\gamma$. Solving α, β and γ as functions of X, Y and Z and substituting into p_0 and p_3, equation (5.2) becomes the formula of 2δ given in the statement. Expressing $\bar{q}_1 = \bar{q}_1^{'}$ and $\bar{q}_2 = \bar{q}_2^{'}$ in terms of X, Y and Z yields the first two equations in the statement of the theorem. ∎

Acknowledgements

Work performed under the auspices of the scientific group G.N.S.A.G.A. of the C.N.R (National Research Council) of Italy and partially supported by the MUR (Ministero dell'Universitá e della Ricerca) of Italy within the project *Strutture Geometriche, Combinatoria e loro Applicazioni*, and by a research grant FAR 2022 of the University of Modena and Reggio Emilia. The authors would like to thank the anonymous referee for his/her useful comments and suggestions, which improved the final version of the paper.

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