EXTRACTA MATHEMATICAE doi:10.17398/2605-5686.37.2.261
Vol. **37**, Num. 2 (2022), 261 – 282 Available online October 26, 2022

Second derivative Lipschitz type inequalities for an integral transform of positive operators in Hilbert spaces

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Received July 6, 2022 Presented by M. Maestre
Accepted October 5, 2022

Abstract: For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1}\, d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. We show among others that, if $A \geq m_1 > 0$, $B \geq m_2 > 0$, then

$$\begin{aligned}
&\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) - D(\mathcal{D}(w,\mu))(A)(B-A)\| \\
&\qquad \leq \|B-A\|^2 \times \begin{cases} \frac{\mathcal{D}(w,\mu)(m_2) - \mathcal{D}(w,\mu)(m_1) - (m_2-m_1)\mathcal{D}'(w,\mu)(m_1)}{(m_2-m_1)^2} & \text{if } m_1 \neq m_2, \\ \frac{1}{2}\mathcal{D}''(w,\mu)(m) & \text{if } m_1 = m_2 = m, \end{cases}
\end{aligned}$$

where $D(\mathcal{D}(w,\mu))$ is the Fréchet derivative of $\mathcal{D}(w,\mu)$ as a function of operator and $\mathcal{D}''(w,\mu)$ is the second derivative of $\mathcal{D}(w,\mu)$ as a real function.

We also prove the norm integral inequalities for power $r \in (0,1]$ and A, $B \geq m > 0$,

$$\left\| \int_0^1 ((1-t)A + tB)^{r-1}\, dt - \left(\frac{A+B}{2}\right)^{r-1} \right\| \leq \frac{1}{24}(1-r)(2-r)\, m^{r-3}\, \|B-A\|^2$$

and

$$\left\| \frac{A^{r-1} + B^{r-1}}{2} - \int_0^1 ((1-t)A + tB)^{r-1}\, dt \right\| \leq \frac{1}{12}(1-r)(2-r)\, m^{r-3}\, \|B-A\|^2 .$$

Key words: operator monotone functions, operator convex functions, operator inequalities, midpoint inequality, trapezoid inequality.

MSC (2020): 47A63, 47A60.

ISSN: 0213-8743 (print), 2605-5686 (online)

1. Introduction

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T on H is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

In 1934, K. Löwner [14] had given a definitive characterization of operator monotone functions as follows, see for instance [4, pp. 144−145]:

Theorem 1. *A function* $f : [0,\infty) \to \mathbb{R}$ *is operator monotone in* $[0,\infty)$ *if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda) \tag{1.1}$$

where $b \geq 0$ *and a positive measure* μ *on* $[0,\infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0,1]$, see [12]. The function ln is also operator monotone on $(0,\infty)$. For other examples of operator monotone functions, see [10, 11].

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e., there is no constant $L > 0$ such that

$$\||A| - |B|\| \leq L \|A - B\|$$

for any A, $B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [7, 8] and Kato in [13], the following inequality holds

$$\||A| - |B|\| \leq \frac{2}{\pi} \|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.2}$$

for any $A, B \in \mathcal{B}(H)$ with $A \neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^*C)^{1/2}$ of an operator C, then the following inequality is true [1]

$$\||A| - |B|\|_{HS} \leq \sqrt{2}\,\|A - B\|_{HS} \tag{1.3}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A| - |B|\| \leq a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3\right) \tag{1.4}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2 .$$

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a)\, \|A - B\| \tag{1.5}$$

where f is an *operator monotone function* on $(0, \infty)$ and $A, B \geq a > 0$.

One of the problems in perturbation theory is to find bounds for $\|f(A) - f(B)\|$ in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [5, 9] and the references therein.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [4, p. 145]

$$t^r = \frac{\sin(r\pi)}{\pi} t \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.6}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.7}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda), \qquad t > 0\,, \tag{1.8}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.8) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda, \qquad t > 0\,. \tag{1.9}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t)\,, \qquad t > 0\,. \tag{1.10}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1)\mathcal{D}(w_{\ln})(t), \qquad t > 0\,. \tag{1.11}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\mu(\lambda)\,, \tag{1.12}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\lambda, \tag{1.13}$$

for $T > 0$.

In this paper, we show among others that, if $A \geq m_1 > 0$, $B \geq m_2 > 0$, then

$$\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) - D(\mathcal{D}(w,\mu))(A)(B-A)\|$$
$$\leq \|B-A\|^2 \times \begin{cases} \frac{\mathcal{D}(w,\mu)(m_2) - \mathcal{D}(w,\mu)(m_1) - (m_2-m_1)\mathcal{D}'(w,\mu)(m_1)}{(m_2-m_1)^2} & \text{if } m_1 \neq m_2, \\ \frac{1}{2}\mathcal{D}''(w,\mu)(m) & \text{if } m_1 = m_2 = m, \end{cases}$$

where $D(\mathcal{D}(w,\mu))$ is the Fréchet derivative of $\mathcal{D}(w,\mu)$ as a function of operator and $\mathcal{D}''(w,\mu)$ is the second derivative of $\mathcal{D}(w,\mu)$ as a real function.

We also prove the norm integral inequalities for power $r \in (0,1]$ and $A, B \geq m > 0$,

$$\left\| \int_0^1 ((1-t)A + tB)^{r-1} dt - \left(\frac{A+B}{2}\right)^{r-1} \right\|$$
$$\leq \frac{1}{24}(1-r)(2-r)m^{r-3}\|B-A\|^2$$

and

$$\left\| \frac{A^{r-1} + B^{r-1}}{2} - \int_0^1 ((1-t)A + tB)^{r-1} dt \right\|$$
$$\leq \frac{1}{12}(1-r)(2-r)m^{r-3}\|B-A\|^2.$$

2. Preliminary results

We have the following representation of the Fréchet derivative:

Lemma 1. *For all $A > 0$,*

$$D(\mathcal{D}(w,\mu))(A)(V) = -\int_0^\infty w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda) \tag{2.1}$$

for all $V \in S(H)$, the class of all selfadjoint operators on H.

Proof. By the definition of $\mathcal{D}(w,\mu)$ we have for t in a small open interval around 0 that

$$\begin{aligned}
&\mathcal{D}(w,\mu)\left(A+tV\right)-\mathcal{D}\left(w,\mu\right)\left(A\right)\\
&\quad=\int_0^\infty w\left(\lambda\right)\left[\left(\lambda+A+tV\right)^{-1}-\left(\lambda+A\right)^{-1}\right]d\mu\left(\lambda\right)\\
&\quad=\int_0^\infty w\left(\lambda\right)\left[\left(\lambda+A+tV\right)^{-1}\left(\lambda+A-\lambda-A-tV\right)\left(\lambda+A\right)^{-1}\right]d\mu\left(\lambda\right)\\
&\quad=-t\int_0^\infty w\left(\lambda\right)\left[\left(\lambda+A+tV\right)^{-1}V\left(\lambda+A\right)^{-1}\right]d\mu\left(\lambda\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{t\to 0}\frac{\mathcal{D}\left(w,\mu\right)\left(A+tV\right)-\mathcal{D}\left(w,\mu\right)\left(A\right)}{t}\\
&\qquad=-\lim_{t\to 0}\int_0^\infty w\left(\lambda\right)\left[\left(\lambda+A+tV\right)^{-1}V\left(\lambda+A\right)^{-1}\right]d\mu\left(\lambda\right)\\
&\qquad=-\int_0^\infty w\left(\lambda\right)\left[\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}\right]d\mu\left(\lambda\right)
\end{aligned}$$

and the identity (2.1) is obtained. ∎

The second Fréchet derivative can be represented as follows:

LEMMA 2. *For all* $A>0$,

$$\begin{aligned}
&D^2(\mathcal{D}(w,\mu))(A)(V,V)\\
&\quad=2\int_0^\infty w\left(\lambda\right)\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}d\mu\left(\lambda\right)
\end{aligned}\tag{2.2}$$

for all $V\in S\left(H\right)$.

Proof. We have by the definition of the Fréchet second derivative that

$$\begin{aligned}
&D^2\left(\mathcal{D}\left(w,\mu\right)\right)\left(A\right)\left(V,V\right)\\
&\qquad=\lim_{t\to 0}\frac{D\left(\mathcal{D}\left(w,\mu\right)\right)\left(A+tV\right)\left(V\right)-D\left(\mathcal{D}\left(w,\mu\right)\right)\left(A\right)\left(V\right)}{t}.
\end{aligned}$$

Observe, by (2.1), that we have for t in a small open interval around 0

$$\begin{aligned}
&D\left(\mathcal{D}\left(w,\mu\right)\right)\left(A+tV\right)\left(V\right)\\
&\qquad=-\int_0^\infty w\left(\lambda\right)\left(\lambda+A+tV\right)^{-1}V\left(\lambda+A+tV\right)^{-1}d\mu\left(\lambda\right),
\end{aligned}$$

which gives that

$$\begin{aligned}
&D\left(\mathcal{D}\left(w,\mu\right)\right)\left(A+tV\right)\left(V\right)-D\left(\mathcal{D}\left(w,\mu\right)\right)\left(A\right)\left(V\right)\\
&\quad=-\int_0^\infty w\left(\lambda\right)\left(\lambda+A+tV\right)^{-1}V\left(\lambda+A+tV\right)^{-1}d\mu\left(\lambda\right)\\
&\qquad\qquad+\int_0^\infty w\left(\lambda\right)\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}d\mu\left(\lambda\right)\\
&\quad=\int_0^\infty w(\lambda)\times\left[(\lambda+A)^{-1}V(\lambda+At)^{-1}-(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Define for $\lambda\geq 0$ and t as above,

$$U_{t,\lambda}:=\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}-\left(\lambda+A+tV\right)^{-1}V\left(\lambda+A+tV\right)^{-1}.$$

If we multiply both sides of $U_{t,\lambda}$ with $\lambda+A+tV$, the we get

$$\begin{aligned}
&(\lambda+A+tV)U_{t,\lambda}\left(\lambda+A+tV\right)\\
&\qquad=\left(\lambda+A+tV\right)\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}\left(\lambda+A+tV\right)-V\\
&\qquad=\left(1+tV\left(\lambda+A\right)^{-1}\right)V\left(1+t\left(\lambda+A\right)^{-1}V\right)-V\\
&\qquad=\left(V+tV\left(\lambda+A\right)^{-1}V\right)\left(1+t\left(\lambda+A\right)^{-1}V\right)-V\\
&\qquad=V+tV\left(\lambda+A\right)^{-1}V+tV\left(\lambda+A\right)^{-1}V\\
&\qquad\qquad+t^2V\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}V-V\\
&\qquad=2tV\left(\lambda+A\right)^{-1}V+t^2V\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}V\\
&\qquad=t\left[2V\left(\lambda+A\right)^{-1}V+tV\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}V\right].
\end{aligned}\tag{2.3}$$

If we multiply the equality by $(\lambda+A+tV)^{-1}$ both sides, we get for $t\neq 0$

$$\begin{aligned}
\frac{U_{t,\lambda}}{t}&=(\lambda+A+tV)^{-1}\left[2V(\lambda+A)^{-1}V+tV(\lambda+A)^{-1}V(\lambda+A)^{-1}V\right]\\
&\quad\times\left(\lambda+A+tV\right)^{-1}.
\end{aligned}\tag{2.4}$$

If we take the limit over $t\to 0$ in, then we get

$$\lim_{t\to 0}\left(\frac{U_{t,\lambda}}{t}\right)=2\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}.$$

Therefore, by the properties of limit under the sign of integral, we get

$$\begin{aligned}\lim_{t\to 0}&\frac{D\left(\mathcal{D}\left(w,\mu\right)\right)\left(A+tV\right)\left(V\right)-D\left(\mathcal{D}\left(w,\mu\right)\right)\left(A\right)\left(V\right)}{t}\\&=\int_0^\infty w\left(\lambda\right)\lim_{t\to 0}\left(\frac{U_{t,\lambda}}{t}\right)d\mu\left(\lambda\right)\\&=2\int_0^\infty w\left(\lambda\right)\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}d\mu\left(\lambda\right)\end{aligned}$$

and the representation (2.2) is obtained. ∎

Remark 1. One may ask if the above integral representation can be extended for higher derivative. The author thinks that is possible, however the calculations are more difficult to perform and are not presented here.

We have the following representation for the transform $\mathcal{D}(w,\mu)$:

THEOREM 2. *For all* $A,B>0$ *we have*

$$\begin{aligned}&\mathcal{D}\left(w,\mu\right)\left(B\right)\\&=\mathcal{D}\left(w,\mu\right)\left(A\right)-\int_0^\infty w\left(\lambda\right)\left(\lambda+A\right)^{-1}\left(B-A\right)\left(\lambda+A\right)^{-1}d\mu\left(\lambda\right)\\&\quad+2\int_0^1\left(1-t\right)\left[\int_0^\infty w\left(\lambda\right)\left(\lambda+\left(1-t\right)A+tB\right)^{-1}\left(B-A\right)\right.\\&\quad\left.\times\left(\lambda+\left(1-t\right)A+tB\right)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.\end{aligned}\tag{2.5}$$

Proof. We use the Taylor's type formula with integral remainder, see for instance [6, p. 112],

$$\begin{aligned}f\left(E\right)&=f\left(C\right)+D\left(f\right)\left(C\right)\left(E-C\right)\\&\quad+\int_0^1\left(1-t\right)D^2\left(f\right)\left(\left(1-t\right)C+tE\right)\left(E-C,E-C\right)dt\end{aligned}\tag{2.6}$$

that holds for functions f which are of class C^2 on an open and convex subset $\mathcal{O}$ in the Banach algebra $B\left(H\right)$ and $C,E\in\mathcal{O}$.

If we write (2.6) for $\mathcal{D}\left(w,\mu\right)$ and $A,B>0$, we get

$$\begin{aligned}\mathcal{D}\left(w,\mu\right)\left(B\right)&=\mathcal{D}\left(w,\mu\right)\left(A\right)+D\left(\mathcal{D}\left(w,\mu\right)\right)\left(A\right)\left(B-A\right)\\&\quad+\int_0^1\left(1-t\right)D^2\left(\mathcal{D}\left(w,\mu\right)\right)\left(\left(1-t\right)A+tB\right)\left(B-A,B-A\right)dt\end{aligned}$$

and by the representations (2.1) and (2.2) we obtain the desired result (2.5). ∎

3. Main results

We have the following Lipschitz type inequality:

THEOREM 3. *Assume that* $A \geq m_1 > 0$, $B \geq m_2 > 0$, *then*

$$\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) - D(\mathcal{D}(w,\mu))(A)(B-A)\| \tag{3.1}$$
$$\leq \|B-A\|^2 \times \begin{cases} \frac{\mathcal{D}(w,\mu)(m_2) - \mathcal{D}(w,\mu)(m_1) - (m_2-m_1)\mathcal{D}'(w,\mu)(m_1)}{(m_2-m_1)^2} & \text{if } m_1 \neq m_2\,, \\ \frac{1}{2}\mathcal{D}''(w,\mu)(m) & \text{if } m_1 = m_2 = m\,. \end{cases}$$

Proof. From (2.5) we get

$$\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) - D(\mathcal{D}(w,\mu))(A)(B-A)\|$$
$$\leq 2\int_0^1 (1-t)\left[\int_0^\infty w(\lambda)\left\|(\lambda + (1-t)A + tB)^{-1}(B-A)\right.\right. \tag{3.2}$$
$$\left.\left.\times (\lambda + (1-t)A + tB)^{-1}(B-A)(\lambda + (1-t)A + tB)^{-1}\right\| d\mu(\lambda)\right] dt$$
$$\leq 2\|B-A\|^2 \int_0^1 (1-t)\left(\int_0^\infty w(\lambda)\left\|(\lambda + (1-t)A + tB)^{-1}\right\|^3 d\mu(\lambda)\right) dt.$$

Assume that $m_2 > m_1$. Then

$$(1-t)A + tB + \lambda \geq (1-t)m_1 + tm_2 + \lambda\,,$$

which implies that

$$((1-t)A + tB + \lambda)^{-1} \leq ((1-t)m_1 + tm_2 + \lambda)^{-1}\,,$$

and

$$\left\|((1-t)A + tB + \lambda)^{-1}\right\|^3 \leq ((1-t)m_1 + tm_2 + \lambda)^{-3} \tag{3.3}$$

for all $t \in [0,1]$ and $\lambda \geq 0$.

Therefore, by integrating (3.3) we derive

$$\begin{aligned}
&\int_0^1 (1-t)\left(\int_0^\infty w(\lambda)\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^3 d\mu(\lambda)\right)dt\\
&\quad\le \int_0^1 (1-t)\left(\int_0^\infty w(\lambda)\left((1-t)m_1+tm_2+\lambda\right)^{-3} d\mu(\lambda)\right)dt \qquad (3.4)\\
&\quad=\frac{1}{(m_2-m_1)^2}\int_0^1 (1-t)\Big[\int_0^\infty w(\lambda)\left((1-t)m_1+tm_2+\lambda\right)^{-1}(m_2-m_1)\\
&\qquad\times\left((1-t)m_1+tm_2+\lambda\right)^{-1}(m_2-m_1)\\
&\qquad\times\left((1-t)m_1+tm_2+\lambda\right)^{-1}d\mu(\lambda)\Big]dt.
\end{aligned}$$

From (2.5) we have for $m_2>m_1$ that

$$\begin{aligned}
&\mathcal{D}(w,\mu)(m_2)-\mathcal{D}(w,\mu)(m_1)+(m_2-m_1)\int_0^\infty w(\lambda)(\lambda+m_1)^{-2}d\mu(\lambda)\\
&\quad=2\int_0^1 (1-t)\Big[\int_0^\infty w(\lambda)\left((1-t)m_1+tm_2+\lambda\right)^{-1}(m_2-m_1)\\
&\qquad\times\left((1-t)m_1+tm_2+\lambda\right)^{-1}(m_2-m_1) \qquad (3.5)\\
&\qquad\times\left((1-t)m_1+tm_2+\lambda\right)^{-1}d\mu(\lambda)\Big]dt.
\end{aligned}$$

Also

$$\int_0^\infty w(\lambda)(\lambda+m_1)^{-2}d\mu(\lambda)=-\mathcal{D}'(w,\mu)(m_1),$$

and then by (3.5) we get

$$\begin{aligned}
&\frac{1}{2(m_2-m_1)^2}\Big[\mathcal{D}(w,\mu)(m_2)-\mathcal{D}(w,\mu)(m_1)\\
&\qquad\qquad-(m_2-m_1)\mathcal{D}'(w,\mu)(m_1)\Big] \qquad (3.6)\\
&\quad=\frac{1}{(m_2-m_1)^2}\int_0^1 (1-t)\Big[\int_0^\infty w(\lambda)\left((1-t)m_1+tm_2+\lambda\right)^{-1}(m_2-m_1)\\
&\qquad\times((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)((1-t)m_1+tm_2+\lambda)^{-1}d\mu(\lambda)\Big]dt\,.
\end{aligned}$$

By utilizing (3.2) and (3.4) – (3.6) we derive (3.1).

The case $m_2<m_1$ goes in a similar way and we also obtain (3.1).

Assume that $m_2 = m_1 > 0$. Let $\epsilon > 0$. Then $B + \epsilon \geq m + \epsilon > m$. By the first inequality for $m_2 = m + \epsilon$ and $m_1 = m$, we have

$$\|\mathcal{D}(w,\mu)(B+\epsilon) - \mathcal{D}(w,\mu)(A) - D(\mathcal{D}(w,\mu))(A)(B+\epsilon-A)\| \tag{3.7}$$
$$\leq \|B+\epsilon-A\|^2 \frac{1}{\epsilon^2}\left[\mathcal{D}(w,\mu)(m+\epsilon) - \mathcal{D}(w,\mu)(m) - \epsilon\mathcal{D}'(w,\mu)(m)\right].$$

By Taylor's expansion theorem with the Lagrange remainder we have

$$\mathcal{D}(w,\mu)(m+\epsilon) - \mathcal{D}(w,\mu)(m) - \epsilon\mathcal{D}'(w,\mu)(m) = \frac{1}{2}\epsilon^2\mathcal{D}''(w,\mu)(\zeta_\epsilon)$$

with $m + \epsilon > \zeta_\epsilon > m$. Therefore

$$\lim_{\epsilon\to 0+}\frac{1}{\epsilon^2}\left[\mathcal{D}(w,\mu)(m+\epsilon) - \mathcal{D}(w,\mu)(m) - \epsilon\mathcal{D}'(w,\mu)(m)\right] = \frac{1}{2}\mathcal{D}''(w,\mu)(m)$$

and by taking the limit $\epsilon \to 0+$ in (3.7) then we get

$$\left\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) - D(\mathcal{D}(w,\mu))(A)(B-A)\right\|$$
$$\leq \frac{1}{2}\|B-A\|^2\mathcal{D}''(w,\mu)(m)$$

and the second part of (3.1) is proved. ∎

The case of operator monotone function is as follows:

Corollary 1. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function with $f(0) = 0$. If $A \geq m_1 > 0$, $B \geq m_2 > 0$, then*

$$\left\|f(B)B^{-1} - \left(2 - A^{-1}B\right)A^{-1}f(A) - A^{-1}D(f)(A)(B-A)\right\| \tag{3.8}$$
$$\leq \|B-A\|^2 \times \begin{cases} \frac{1}{(m_2-m_1)^2}\left[\frac{f(m_2)}{m_2} - \frac{f(m_1)}{m_1} - (m_2-m_1)\frac{f'(m_1)m_1 - f(m_1)}{m_1^2}\right] & \text{if } m_1 \neq m_2\,, \\ \frac{1}{2}\frac{f''(m)m^2 - 2mf'(m) + 2f(m)}{m^3} & \text{if } m_1 = m_2 = m\,. \end{cases}$$

Proof. We denote by ℓ the identity function $\ell(t) = t$, $t > 0$. By ℓ^{-1} we denote the function $\ell^{-1}(t) = t^{-1}$, $t > 0$. Using these notations we have

$$\mathcal{D}(\ell,\mu)(t) = \frac{f(t)}{t} - b\,, \qquad t > 0\,,$$

where $b \geq 0$ and μ is a positive measure on $(0,\infty)$.

The derivative of this function is

$$\mathcal{D}'(\ell,\mu)(t)=\frac{f'(t)\,t-f(t)}{t^2}\,,\qquad t>0\,,$$

and the second derivative

$$\begin{aligned}\mathcal{D}''(\ell,\mu)(t)&=\frac{(f'(t)\,t-f(t))'\,t^2-2t\,(f'(t)\,t-f(t))}{t^4}\\&=\frac{(f''(t)\,t+f'(t)-f'(t))\,t^2-2t\,(f'(t)\,t-f(t))}{t^4}\\&=\frac{f''(t)\,t^3-2t^2f'(t)+2tf(t)}{t^4}=\frac{f''(t)\,t^2-2tf'(t)+2f(t)}{t^3}\,.\end{aligned}$$

We have

$$\begin{aligned}&\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)-D\left(\ell^{-1}f\right)(A)(B-A)\\&\qquad=f(B)\,B^{-1}-f(A)\,A^{-1}\\&\qquad\quad-\left[D\left(\ell^{-1}\right)(A)\,(B-A)\,f(A)+\ell^{-1}(A)\,D(f)\,(A)\,(B-A)\right]\\&\qquad=f(B)\,B^{-1}-f(A)\,A^{-1}+A^{-1}(B-A)\,A^{-1}f(A)\\&\qquad\quad-A^{-1}D(f)\,(A)\,(B-A)\,,\end{aligned}$$

since, by using the definition of the Fréchet derivative,

$$D\left(\ell^{-1}\right)(A)\,(B-A)=-A^{-1}(B-A)\,A^{-1}.$$

Also

$$\begin{aligned}&\mathcal{D}(w,\mu)(m_2)-\mathcal{D}(w,\mu)(m_1)-(m_2-m_1)\,\mathcal{D}'(w,\mu)(m_1)\\&\qquad=\frac{f(m_2)}{m_2}-\frac{f(m_1)}{m_1}-(m_2-m_1)\,\frac{f'(m_1)\,m_1-f(m_1)}{m_1^2}\,.\end{aligned}$$

By making use of (3.1) we deduce (3.8). ∎

We consider the representation obtained from (1.9) for the operator $T>0$ and the power $r\in(0,1]$,

$$T^{r-1}=\mathcal{D}(\tilde{w}_r)(T)$$

with the kernel $\tilde{w}_r(\lambda):=\frac{\sin(r\pi)}{\pi}\lambda^{r-1}$, $r\in(0,1]$.

From (3.1) we get for $A \geq m_1 > 0$, $B \geq m_2 > 0$ and $r \in (0,1]$ that

$$\left\| B^{r-1} - A^{r-1} + \int_0^\infty \lambda^{r-1} (\lambda + A)^{-1} (B - A) (\lambda + A)^{-1} d\lambda \right\| \tag{3.9}$$
$$\leq \|B - A\|^2 \times \begin{cases} \frac{(1-r)(m_2-m_1)m_1^{r-2} - m_1^{r-1} + m_2^{r-1}}{(m_2-m_1)^2} & \text{if } m_1 \neq m_2\,, \\ \frac{1}{2}(1-r)(2-r)\, m^{r-3} & \text{if } m_1 = m_2 = m\,. \end{cases}$$

We have the following error bounds for operator Jensen's gap related to the n-tuple of positive operators $\mathbf{A} = (A_1, \ldots, A_n)$ and probability density n-tuple $\mathbf{p} = (p_1, \ldots, p_n)$,

$$J(\mathbf{A}, \mathbf{p}, \mathcal{D}(w,\mu)) := \sum_{k=1}^n p_k \mathcal{D}(w,\mu)(A_k) - \mathcal{D}(w,\mu)\left(\sum_{k=1}^n p_k A_k\right).$$

Theorem 4. *Assume that $A_i \geq m > 0$ for $i \in \{1, \ldots, n\}$ and consider the probability density n-tuple $\mathbf{p} = (p_1, \ldots, p_n)$, then*

$$\begin{aligned} \left\| J(\mathbf{A}, \mathbf{p}, \mathcal{D}(w,\mu)) \right\| &\leq \frac{1}{2} \mathcal{D}''(w,\mu)(m) \sum_{k=1}^n p_k \left\| A_k - \sum_{j=1}^n p_j A_j \right\|^2 \\ &\leq \frac{1}{2} \mathcal{D}''(w,\mu)(m) \sum_{k=1}^n \sum_{j=1}^n p_j p_k \|A_k - A_j\|^2 \\ &\leq \frac{1}{2} \mathcal{D}''(w,\mu)(m) \max_{k,j \in \{1,\ldots,n\}} \|A_k - A_j\|^2 . \end{aligned} \tag{3.10}$$

Proof. From (3.1) we get

$$\begin{aligned} &\left\| \mathcal{D}(w,\mu)(A_k) - \mathcal{D}(w,\mu)\left(\sum_{j=1}^n p_j A_j\right) \right. \\ &\qquad \left. - D(\mathcal{D}(w,\mu))\left(\sum_{j=1}^n p_j A_j\right)\left(A_k - \sum_{j=1}^n p_j A_j\right) \right\| \\ &\leq \frac{1}{2}\, \mathcal{D}''(w,\mu)(m) \left\| A_k - \sum_{j=1}^n p_j A_j \right\|^2 \end{aligned} \tag{3.11}$$

for all $k \in \{1, \ldots, n\}$.

If we multiply this inequality by $p_k \geq 0$ and sum over k from 1 to n, then we get

$$\sum_{k=1}^{n} \left\| p_k \mathcal{D}(w,\mu)(A_k) - p_k \mathcal{D}(w,\mu)\left(\sum_{j=1}^{n} p_j A_j\right) - D(\mathcal{D}(w,\mu))\left(\sum_{j=1}^{n} p_j A_j\right) p_k \left(A_k - \sum_{j=1}^{n} p_j A_j\right)\right\| \leq \frac{1}{2}\mathcal{D}''(w,\mu)(m) \sum_{k=1}^{n} p_k \left\| A_k - \sum_{j=1}^{n} p_j A_j \right\|^2 . \tag{3.12}$$

By making use of the triangle inequality for norms, we also have

$$\begin{aligned} &\sum_{k=1}^{n} \left\| p_k \mathcal{D}(w,\mu)(A_k) - p_k \mathcal{D}(w,\mu)\left(\sum_{j=1}^{n} p_j A_j\right) - D(\mathcal{D}(w,\mu))\left(\sum_{j=1}^{n} p_j A_j\right) p_k \left(A_k - \sum_{j=1}^{n} p_j A_j\right)\right\| \\ &\geq \left\| \sum_{k=1}^{n} p_k \mathcal{D}(w,\mu)(A_k) - \sum_{k=1}^{n} p_k \mathcal{D}(w,\mu)\left(\sum_{j=1}^{n} p_j A_j\right) - D(\mathcal{D}(w,\mu))\left(\sum_{j=1}^{n} p_j A_j\right)\left(\sum_{k=1}^{n} p_k A_k - \sum_{j=1}^{n} p_j A_j\right)\right\| \qquad (3.13) \\ &= \left\| \sum_{k=1}^{n} p_k \mathcal{D}(w,\mu)(A_k) - \mathcal{D}(w,\mu)\left(\sum_{j=1}^{n} p_j A_j\right)\right\| . \end{aligned}$$

By utilizing (3.12) and (3.13) we deduce the first part of (3.10). The rest is obvious. ∎

Remark 2. From (3.10) we can obtain the following norm inequalities for power $r \in (0,1]$,

$$\begin{aligned}
&\left\| \sum_{k=1}^{n} p_k A_k^{r-1} - \left(\sum_{k=1}^{n} p_k A_k \right)^{r-1} \right\| \qquad (3.14)\\
&\quad \le \frac{1}{2} (1-r)(2-r) m^{r-3} \sum_{k=1}^{n} p_k \left\| A_k - \sum_{j=1}^{n} p_j A_j \right\|^2 \\
&\quad \le \frac{1}{2} (1-r)(2-r) m^{r-3} \sum_{k=1}^{n} \sum_{j=1}^{n} p_j p_k \left\| A_k - A_j \right\|^2 \\
&\quad \le \frac{1}{2} (1-r)(2-r) m^{r-3} \max_{k,j\in\{1,\dots,n\}} \left\| A_k - A_j \right\|^2,
\end{aligned}$$

where $A_i \ge m > 0$ for $i \in \{1, \dots, n\}$ and the probability density n-tuple $\mathbf{p} = (p_1, \dots, p_n)$.

4. Midpoint and trapezoid inequalities

We have the following midpoint norm inequality:

Theorem 5. *If $A, B \ge m > 0$ for some constant m, then*

$$\left\| \int_0^1 \mathcal{D}(w,\mu)((1-t)A + tB)\,dt - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \right\| \le \frac{1}{24} \mathcal{D}''(w,\mu)(m) \|B-A\|^2 . \qquad (4.1)$$

Proof. From (3.1) we have for all $t \in [0,1]$ and $A, B \ge m > 0$,

$$\begin{aligned}
&\Bigg\| \mathcal{D}(w,\mu)((1-t)A+tB) - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\
&\qquad - D(\mathcal{D}(w,\mu))\left(\frac{A+B}{2}\right)\left((1-t)A + tB - \frac{A+B}{2}\right) \Bigg\| \\
&\le \frac{1}{2} \mathcal{D}''(w,\mu)(m) \left\| (1-t)A + tB - \frac{A+B}{2} \right\|^2 \qquad (4.2)\\
&= \frac{1}{2} \mathcal{D}''(w,\mu)(m) \left(t - \frac{1}{2} \right)^2 \|B-A\|^2 .
\end{aligned}$$

If we integrate this inequality, we get

$$\begin{aligned}\int_0^1 \bigg\| \mathcal{D}(w,\mu)((1-t)A+tB) - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\ - D(\mathcal{D}(w,\mu))\left(\frac{A+B}{2}\right)\left((1-t)A+tB-\frac{A+B}{2}\right)\bigg\| dt \\ \leq \frac{1}{2}\mathcal{D}''(w,\mu)(m)\|B-A\|^2 \int_0^1 \left(t-\frac{1}{2}\right)^2 dt \qquad (4.3) \\ = \frac{1}{24}\mathcal{D}''(w,\mu)(m)\|B-A\|^2 .\end{aligned}$$

Using the properties of norm and integral, we also have

$$\begin{aligned}\int_0^1 \bigg\| \mathcal{D}(w,\mu)((1-t)A+tB) - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\ - D(\mathcal{D}(w,\mu))\left(\frac{A+B}{2}\right)\left((1-t)A+tB-\frac{A+B}{2}\right)\bigg\| dt \\ \geq \bigg\| \int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \qquad (4.4) \\ - \left(\int_0^1 \left(t-\frac{1}{2}\right)dt\right) D(\mathcal{D}(w,\mu))\left(\frac{A+B}{2}\right)(B-A)\bigg\| \\ = \left\| \int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right\| .\end{aligned}$$

By employing (4.3) and (4.4) we derive the desired result (4.1). ∎

Corollary 2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function with $f(0)=0$. If $A,B\geq m>0$, then*

$$\begin{aligned}\left\| \int_0^1 ((1-t)A+tB)^{-1} f((1-t)A+tB)\,dt - \left(\frac{A+B}{2}\right)^{-1} f\left(\frac{A+B}{2}\right)\right\| \\ \leq \frac{f''(m)m^2-2mf'(m)+2f(m)}{24m^3}\|B-A\|^2 . \qquad (4.5)\end{aligned}$$

The proof follows by (4.1) for

$$\mathcal{D}(\ell,\mu)(t) = \frac{f(t)}{t} - b, \qquad t>0,$$

where $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

Remark 3. If $A, B \geq m > 0$, then for $r \in (0,1]$ we get by (4.5) that

$$\left\| \int_0^1 ((1-t) A + tB)^{r-1} dt - \left(\frac{A+B}{2}\right)^{r-1} \right\| \leq \frac{1}{24} (1-r)(2-r) m^{r-3} \|B-A\|^2 . \tag{4.6}$$

The trapezoid norm inequality will be our concern from now on.

For a continuous function f on $(0,\infty)$ and $A, B > 0$ we consider the auxiliary function $f_{A,B} : [0,1] \to \mathbb{R}$ defined by

$$f_{A,B}(t) := f((1-t) A + tB), \qquad t \in [0,1].$$

We have the following representations of the derivatives:

LEMMA 3. *Assume that the operator function generated by f is twice Fréchet differentiable in each $A > 0$, then for $B > 0$ we have that $f_{A,B}$ is twice differentiable on* $[0,1]$,

$$\frac{df_{A,B}(t)}{dt} = D(f)((1-t) A + tB)(B-A), \tag{4.7}$$

$$\frac{d^2 f_{A,B}(t)}{dt^2} = D^2(f)((1-t) A + tB)(B-A, B-A) \tag{4.8}$$

for $t \in [0,1]$, *where in* 0 *and* 1 *the derivatives are the right and left derivatives.*

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t+h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} \frac{f_{A,B}(t+h) - f(t)}{h} &= \frac{f((1-(t+h)) A + (t+h) B) - f((1-t) A + tB)}{h} \\ &= \frac{f((1-t) A + tB + h(B-A)) - f((1-t) A + tB)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} \frac{df_{A,B}(t)}{dt} &= \lim_{h\to 0} \frac{f_{A,B}(t+h) - f(t)}{h} \\ &= \lim_{h\to 0} \left[\frac{f((1-t) A + tB + h(B-A)) - f((1-t) A + tB)}{h}\right] \\ &= D(f)((1-t) A + tB)(B-A), \end{aligned}$$

which proves (4.7).

Similarly,

$$\frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt}-\frac{df_{A,B}(t)}{dt}\right]$$
$$=\frac{D(f)((1-(t+h))A+(t+h)B)(B-A)-D(f)((1-t)A+tB)(B-A)}{h}$$
$$=\frac{D(f)((1-t)A+tB+h(B-A))(B-A)-D(f)((1-t)A+tB)(B-A)}{h}$$

and by taking the limit over $h\to 0$, we get

$$\begin{aligned}\frac{d^2f_{A,B}(t)}{dt^2}&=\lim_{h\to 0}\left\{\frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt}-\frac{df_{A,B}(t)}{dt}\right]\right\}\\&=D^2(f)((1-t)A+tB)(B-A,B-A),\end{aligned}$$

which proves (4.8). ∎

For the transform $\mathcal{D}(w,\mu)(t)$ defined in the introduction, we consider the auxiliary function

$$\begin{aligned}\mathcal{D}(w,\mu)_{A,B}(t)&:=\mathcal{D}(w,\mu)((1-t)A+tB)\\&=\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\end{aligned}$$

where $A,B>0$ and $t\in[0,1]$.

Corollary 3. *For all $A,B>0$ and $t\in[0,1]$,*

$$\begin{aligned}\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}&=D(\mathcal{D}(w,\mu))((1-t)A+tB)(B-A)\qquad(4.9)\\&=-\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\\&\qquad\times(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\end{aligned}$$

and

$$\begin{aligned}&\frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}\\&\quad=D^2(\mathcal{D}(w,\mu))((1-t)A+tB)(B-A,B-A)\qquad(4.10)\\&\quad=2\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\\&\qquad\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda).\end{aligned}$$

We observe that if $f(t) = \mathcal{D}(w,\mu)(t)$, $t > 0$, in Lemma 3, then by the representations from Lemma 1 and Lemma 2 we obtain the desired equalities (4.9) and (4.10).

We have the following identity for the trapezoid rule:

Lemma 4. *For all $A, B > 0$ we have the identity*

$$\begin{aligned}\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2} &- \int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt \\ &= \int_0^1 t(1-t)\left[\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \qquad (4.11)\\ &\quad \left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,d\mu(\lambda)\right]dt\,.\end{aligned}$$

Proof. Using integration by parts for the Bochner integral, we have

$$\begin{aligned}\frac{1}{2}\int_0^1 & t(1-t)\frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt \\ &= \frac{1}{2}\left[t(1-t)\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}\bigg|_0^1 - \int_0^1(1-2t)\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}dt\right] \\ &= \int_0^1\left(t-\frac{1}{2}\right)\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}dt \\ &= \left(t-\frac{1}{2}\right)\mathcal{D}(w,\mu)_{A,B}(t)\bigg|_0^1 - \int_0^1\mathcal{D}(w,\mu)_{A,B}(t)\,dt \\ &= \frac{1}{2}\left[\mathcal{D}(w,\mu)_{A,B}(1)+\mathcal{D}(w,\mu)_{A,B}(0)\right] - \int_0^1\mathcal{D}(w,\mu)_{A,B}(t)\,dt\,,\end{aligned}$$

that gives the identity

$$\begin{aligned}\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2} &- \int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt \\ &= \frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt\,. \qquad (4.12)\end{aligned}$$

By (4.12) we have

$$\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt \tag{4.13}$$
$$= \int_0^1 t(1-t)\left[\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right.$$
$$\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt\,.$$

By making use of (4.10) and (4.13) we deduce (4.11). ∎

We can state now the corresponding trapezoid norm inequality:

Theorem 6. *If $A, B \geq m > 0$ for some constant m, then*

$$\left\|\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt\right\| \leq \frac{1}{12}\mathcal{D}''(w,\mu)(m)\|B-A\|^2\,. \tag{4.14}$$

Proof. By taking the norm in (4.11), we obtain

$$\left\|\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt\right\| \tag{4.15}$$
$$\leq \|B-A\|^2\int_0^1 t(1-t)\left(\int_0^\infty w(\lambda)\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^3 d\mu(\lambda)\right)dt.$$

Since $A, B \geq m > 0$, then for $\lambda \geq 0$ and $t \in [0,1]$,

$$\lambda+(1-t)A+tB \geq \lambda+m,$$

which implies that

$$(\lambda+(1-t)A+tB)^{-1} \leq (\lambda+m)^{-1}\,.$$

This implies that

$$\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^3 \leq (\lambda+m)^{-3}$$

for $\lambda \geq 0$ and $t \in [0,1]$.

By multiplying this inequality by $t(1-t)w(\lambda)\geq 0$ and integrating we get

$$\begin{aligned}\int_0^1 t(1-t)\left(\int_0^\infty w(\lambda)\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^3 d\mu(\lambda)\right)dt \\ \leq \left(\int_0^1 t(1-t)\,dt\right)\left(\int_0^\infty w(\lambda)(\lambda+m)^{-3}d\mu(\lambda)\right) \\ = \frac{1}{6}\int_0^\infty w(\lambda)(\lambda+m)^{-3}d\mu(\lambda).\end{aligned} \tag{4.16}$$

Taking the derivative over t twice in (1.8), we get

$$\mathcal{D}''(w,\mu)(t) := 2\int_0^\infty \frac{w(\lambda)}{(\lambda+t)^3}d\mu(\lambda), \qquad t>0,$$

that gives

$$\int_0^\infty w(\lambda)(\lambda+m)^{-3}d\mu(\lambda) = \frac{1}{2}\mathcal{D}''(w,\mu)(m)$$

and by (4.15) and (4.16) we derive (4.14). ∎

COROLLARY 4. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is an operator monotone function with* $f(0)=0$. *If* $A,B\geq m>0$, *then*

$$\begin{aligned}\left\|\frac{A^{-1}f(A)+B^{-1}f(B)}{2}-\int_0^1((1-t)A+tB)^{-1}f((1-t)A+tB)\,dt\right\| \\ \leq \frac{f''(m)m^2-2mf'(m)+2f(m)}{12m^3}\|B-A\|^2.\end{aligned} \tag{4.17}$$

The proof follows by (4.14) for

$$\mathcal{D}(\ell,\mu)(t) = \frac{f(t)}{t}-b, \qquad t>0,$$

where $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

Remark 4. If $A,B\geq m>0$, then for $r\in(0,1]$ we get by (4.5) that

$$\begin{aligned}\left\|\frac{A^{r-1}+B^{r-1}}{2}-\int_0^1((1-t)A+tB)^{r-1}dt\right\| \\ \leq \frac{1}{12}(1-r)(2-r)m^{r-3}\|B-A\|^2.\end{aligned} \tag{4.18}$$

Acknowledgements

The author would like to thank the anonymous referee for valuable comments that have been implemented in the final version of the manuscript.

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