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Smooth 2-homogeneous polynomials on the plane with a hexagonal norm

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Abstract: Motivated by the classifications of extreme and exposed 2-homogeneous polynomials on the plane with the hexagonal norm $\|(x,y)\| = \max\{|y|, |x| + \frac{1}{2}|y|\}$ (see [15, 16]), we classify all smooth 2-homogeneous polynomials on $\mathbb{R}^2$ with the hexagonal norm.

Key words: The Krein-Milman theorem, smooth points, extreme points, exposed points, 2-homogeneous polynomials on the plane with the hexagonal norm.

MSC (2020): 46A22.

1. Introduction

One of the main results about smooth points is known as "the Mazur density theorem". Recall that the Mazur density theorem ([9, p. 71]) says that the set of all the smooth points of a solid closed convex subset of a separable Banach space is a residual subset of its boundary. We denote by B_E the closed unit ball of a real Banach space E and also by E^* the dual space of E. We recall that a point $x \in B_E$ is said to be an *extreme point* of B_E if the equation $x = \frac{1}{2}(y+z)$ for some $y, z \in B_E$ implies that $x = y = z$. A point $x \in B_E$ is called an *exposed point* of B_E if there is an $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. A point $x \in B_E$ is called a *smooth point* of B_E if there is a unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. We denote by $\operatorname{ext} B_E, \exp B_E$ and $\operatorname{sm} B_E$ the set of extreme points, the set of exposed points and the set of smooth points of B_E, respectively. For $n \in \mathbb{N}$, we denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \cdots, x_n)|$. A n-linear form T is symmetric if $T(x_1, \ldots, x_n) = T(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every permutation σ on $\{1, 2, \ldots, n\}$. We denote by $\mathcal{L}_s(^nE)$ the Banach space of all continuous symmetric n-linear

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forms on E. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a unique $T \in \mathcal{L}_s(^nE)$ such that $P(x) = T(x, \cdots, x)$ for every $x \in E$. In this case it is convenient to write $T = \check{P}$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [7].

Choi *et al.* [2, 3, 4, 5] initiated and characterized the smooth points, extreme points and exposed points of the unit balls of $\mathcal{P}(^2l_1^2), \mathcal{P}(^2l_2^2)$ and $\mathcal{P}(^2c_0)$. Kim [10] and Choi and Kim [6] classified the exposed 2-homogeneous polynomials on $\mathcal{P}(^2l_p^2)$ $(1 \leq p \leq \infty)$. Kim *et al.* [17] characterized the exposed 2-homogeneous polynomials on Hilbert spaces. Kim [11, 12, 14] classified the smooth points, extreme points and exposed points of the unit ball of $\mathcal{P}(^2d_*(1,w)^2)$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm of weight w. For some applications of the classification of the extreme points of the unit ball of $\mathcal{P}(^2d_*(1,w)^2)$, Kim [13] investigated polarization and unconditional constants of $\mathcal{P}(^2d_*(1,w)^2)$. Thus we fully described the geometry of the unit ball of $\mathcal{P}(^2d_*(1,w)^2)$. We refer to [1, 8, 18, 19] and references therein for some recent work about extremal properties of homogeneous polynomials on some classical Banach spaces.

We will denote by $P(x, y) = ax^2 + by^2 + cxy$ a 2-homogeneous polynomial on a real Banach space of dimension 2 for some $a, b, c \in \mathbb{R}$. Let $0 < w < 1$ be fixed. We denote $\mathbb{R}^2_{h(w)} = \mathbb{R}^2$ with the hexagonal norm of weight w by

$$\|(x, y)\|_{h(w)} := \max\big\{|y|\,,\, |x| + (1 - w)|y|\big\}.$$

Throughout the paper we will denote $\mathbb{R}^2_{h(\frac{1}{2})}$ by $\mathcal{H}$. Kim [15, 16] classified the extreme and exposed points of the unit ball of $\mathcal{P}(^2\mathcal{H})$ as follows:

$$\text{(a)} \qquad \operatorname{ext} B_{\mathcal{P}(^2\mathcal{H})} = \Big\{ \pm y^2, \pm\Big(x^2 + \frac{1}{4}y^2 \pm xy\Big)\,, \pm\Big(x^2 + \frac{3}{4}y^2\Big)\,, \\ \pm\Big[x^2 + \Big(\frac{c^2}{4} - 1\Big)y^2 \pm cxy\Big]\ (0 \leq c \leq 1)\,, \\ \pm\Big[ax^2 + \Big(\frac{a + 4\sqrt{1-a}}{4} - 1\Big)y^2 \\ \pm (a + 2\sqrt{1-a})xy\Big]\ (0 \leq a \leq 1)\Big\};$$

$$\text{(b)} \qquad \exp B_{\mathcal{P}(^2\mathcal{H})} = \operatorname{ext} B_{\mathcal{P}(^2\mathcal{H})}.$$

In this paper we classify $\mathrm{sm}\, B_{\mathcal{P}(^2\mathcal{H})}$ using the classifications of $\mathrm{ext}\, B_{\mathcal{P}(^2\mathcal{H})}$ and $\exp B_{\mathcal{P}(^2\mathcal{H})}$.

2. Results

THEOREM 2.1. ([15]) *Let $P(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2\mathcal{H})$ with $a \geq 0$, $c \geq 0$ and $a^2 + b^2 + c^2 \neq 0$. Then:*

Case 1: $c < a$.

If $a \leq 4b$, then

$$\begin{aligned}\|P\| &= \max\left\{a, b, \left|\frac{1}{4}a + b\right| + \frac{1}{2}c, \frac{4ab - c^2}{4a}, \frac{4ab - c^2}{2c + a + 4b}, \frac{4ab - c^2}{|2c - a - 4b|}\right\}\\ &= \max\left\{a, b, \left|\frac{1}{4}a + b\right| + \frac{1}{2}c\right\}.\end{aligned}$$

If $a > 4b$, then $\|P\| = \max\left\{a, |b|, \left|\frac{1}{4}a + b\right| + \frac{1}{2}c, \frac{|c^2-4ab|}{4a}\right\}$.

Case 2: $c \geq a$.

If $a \leq 4b$, then $\|P\| = \max\left\{a, b, \left|\frac{1}{4}a + b\right| + \frac{1}{2}c, \frac{|c^2-4ab|}{2c+a+4b}\right\}$.

If $a > 4b$, then $\|P\| = \max\left\{a, |b|, \left|\frac{1}{4}a + b\right| + \frac{1}{2}c, \frac{c^2-4ab}{2c-a-4b}\right\}$.

Note that if $\|P\| = 1$, then $|a| \leq 1, |b| \leq 1, |c| \leq 2$.

THEOREM 2.2. ([15, 16])

$$\begin{aligned}\mathrm{ext}\, B_{\mathcal{P}(^2\mathcal{H})} &= \exp B_{\mathcal{P}(^2\mathcal{H})}\\ &= \Bigg\{ \pm y^2, \pm\left(x^2 + \frac{1}{4}y^2 \pm xy\right), \pm\left(x^2 + \frac{3}{4}y^2\right),\\ &\qquad \pm\left[x^2 + \left(\frac{c^2}{4} - 1\right)y^2 \pm cxy\right] \ (0 \leq c \leq 1),\\ &\qquad \pm\Bigg[ax^2 + \left(\frac{a + 4\sqrt{1-a}}{4} - 1\right)y^2\\ &\qquad\qquad \pm (a + 2\sqrt{1-a})xy\Bigg] \ (0 \leq a \leq 1)\Bigg\}.\end{aligned}$$

By the Krein-Milman theorem, a convex function (like a functional norm, for instance) defined on a convex set (like the unit ball of a finite dimensional polynomial space) attains its maximum at one extreme point of the convex set.

THEOREM 2.3. ([16]) *Let $f \in \mathcal{P}(^2\mathcal{H})^*$ with $\alpha = f(x^2)$, $\beta = f(y^2)$, $\gamma = f(xy)$. Then*

$$\|f\| = \max\Big\{|\beta|, \Big|\alpha + \frac{1}{4}\beta\Big| + |\gamma|, \Big|\alpha + \frac{3}{4}\beta\Big|, \Big|\alpha + \Big(\frac{c^2}{4} - 1\Big)\beta\Big| + c|\gamma| \ (0 \le c \le 1),$$
$$\Big|a\alpha + \Big(\frac{a + 4\sqrt{1-a}}{4} - 1\Big)\beta\Big| + (a + 2\sqrt{1-a})|\gamma| \ (0 \le a \le 1)\Big\}.$$

Proof. It follows from Theorem 2.2 and the fact that $\|f\| = \sup_{P \in \text{ext}\,\mathbf{B}} |f(P)|$, where $\mathbf{B} := B_{\mathcal{P}(^2\mathcal{H})}$. ∎

Note that if $\|f\| = 1$, then $|\alpha| \le 1$, $|\beta| \le 1$, $|\gamma| \le \frac{1}{2}$.

Remark. Let $P(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2\mathcal{H})$ with $\|P\| = 1$. Then the following are equivalent:

(1) P is smooth;

(2) $-P(x, y) = -ax^2 - by^2 - cxy$ is smooth;

(3) $P(x, -y) = ax^2 + by^2 - cxy$ is smooth.

As a consequence of the previous remark, our attention can be restricted to polynomials $Q(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2\mathcal{H})$ with $a \ge 0$, $c \ge 0$.

We are in position to prove the main result of this paper.

THEOREM 2.4. *Let $P(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2\mathcal{H})$ with $a \ge 0$, $c \ge 0$, $\|P\| = 1$. Then P is a smooth point of the unit ball of $\mathcal{P}(^2\mathcal{H})$ if and only if one of the following mutually exclusive conditions holds:*

(1) $a = 0\,,\ 0 < |b| < 1\,;$

(2) $a = 1\,,\ b = -\frac{3}{4}\,,\ \frac{1}{4}\,,\ c < 1\,;$

(3) $a = 1\,,\ -1 < b < -\frac{3}{4}\,,\ b - \frac{c}{2} > -\frac{5}{4}\,,\ \frac{c^2}{4} - b < 1\,;$

(4) $a = 1\,,\ -\frac{3}{4} < b < \frac{1}{4}\,;$

(5) $a = 1\,,\ \frac{1}{4} \le b\,,\ b + \frac{c}{2} < \frac{3}{4}\,;$

(6) $0 < a < 1\,,\ b = 0\,;$

(7) $0 < a < 1\,,\ c \le a\,,\ 0 \ne 4b < a\,;$

(8) $0 < a < 1\,,\ 0 < c \le a < 4b\,;$

(9) $0 < a < 1\,,\ 4b = a < c\,;$

(10) $0 < a < 1\,,\ 0 \neq 4b < a < c\,,\ c \neq a + 2\sqrt{1-a}\,;$

(11) $0 < a < 1\,,\ a < 4b\,,\ a < c\,.$

Proof. Let $Q(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2\mathcal{H})$ with $a \geq 0$, $c \geq 0$ and $\|Q\| = 1$.

Case 1: $a = 0$.

Note that if $b = 0$ or ± 1, then Q is not smooth. In fact, if $b = 0$, then $Q = 2xy$. For $j = 1, 2$, let $f_j \in \mathcal{P}(^2\mathcal{H})^*$ be such that

$$f_1\big(x^2\big) = \frac{1}{4}\,,\ \ f_1\big(y^2\big) = 1\,,\ \ f_1(xy) = \frac{1}{2}\,,\ \ f_2\big(x^2\big) = 0 = f_2\big(y^2\big)\,,\ \ f_2(xy) = \frac{1}{2}\,.$$

By Theorem 2.3, $f_j(Q) = 1 = \|f_j\|$ for $j = 1, 2$. Thus Q is not smooth. If $b = \pm 1$, then $P = \pm y^2$. For $j = 1, 2$, let $f_j \in \mathcal{P}(^2\mathcal{H})^*$ be such that

$$f_1\big(x^2\big) = \pm\frac{1}{4}\,, \qquad f_1\big(y^2\big) = \pm 1\,, \qquad f_1(xy) = \pm\frac{1}{2}\,,$$
$$f_2\big(x^2\big) = 0 = f_2(xy)\,, \qquad f_2\big(y^2\big) = \pm 1\,.$$

By Theorem 2.3, $f_j(Q) = 1 = \|f_j\|$ for $j = 1, 2$. Thus Q is not smooth.

Claim: if $a = 0$, $0 < |b| < 1$, then Q is smooth.

Without loss of generality, we may assume that $0 < b < 1$. By Theorem 2.1, $1 = \|Q\| = b + \frac{1}{2}c$. Thus $c = 2(1-b)$, so $0 < c < 2$. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. Notice that $1 = b\beta + c\gamma$. We will show that $\alpha = \frac{1}{4}$, $\beta = 1$, $\gamma = \frac{1}{2}$. Since $0 < b < 1$, $0 < c < 2$, we can choose $\delta > 0$ such that

$$0 < 2(1-b) + t = c + t < 2\,, \qquad 0 < b - \frac{1}{2}t < 1\,,$$

for all $t \in (-\delta, \delta)$. Let $Q_t(x,y) = \Big(b - \frac{1}{2}t\Big)y^2 + (c+t)xy$ for all $t \in (-\delta, \delta)$. By Theorem 2.1, $\|Q_t\| = 1$ for all $t \in (-\delta, \delta)$. For all $t \in (-\delta, \delta)$,

$$1 = b\beta + c\gamma \geq f(Q_t) = \left(b - \frac{1}{2}t\right)\beta + (c+t)\gamma\,,$$

which shows that $t\Big(\gamma - \frac{1}{2}\beta\Big) \leq 0$, for all $t \in (-\delta, \delta)$. Thus $\gamma = \frac{1}{2}\beta$. Since $1 = f(Q) = b\beta + c\gamma = 2\gamma$, we have $\beta = 1$, $\gamma = \frac{1}{2}$. By Theorem 2.3, $1 \geq$

$\left|\alpha+\frac{1}{4}\beta\right|+|\gamma|=\left|\alpha+\frac{1}{4}\right|+\frac{1}{2}$, so

$$-\frac{3}{4}\leq\alpha\leq\frac{1}{4}\,. \tag{1}$$

By Theorem 2.3, for $0\leq\tilde{c}\leq 1$,

$$1\geq\left|\alpha+\left(\frac{\tilde{c}^2}{4}-1\right)\right|+\frac{\tilde{c}}{2}=-\left(\alpha+\left(\frac{\tilde{c}^2}{4}-1\right)\right)+\frac{\tilde{c}}{2}\,,$$

which implies that

$$4\alpha\geq\sup_{0\leq\tilde{c}\leq 1}(2\tilde{c}-\tilde{c}^2)=1\,. \tag{2}$$

By (1) and (2), $\alpha=\frac{1}{4}$. Therefore, Q is smooth.

CASE 2: $a=1$.

If $b=-1$, then $Q=x^2-y^2$. For $j=1,2$, let $f_j\in\mathcal{P}(^2\mathcal{H})^*$ be such that

$$f_1(x^2)=1\,,\qquad f_1(y^2)=0=f_1(xy)\,,$$
$$f_2(x^2)=0=f_2(xy)\,,\qquad f_2(y^2)=-1\,.$$

By Theorem 2.3, $f_j(Q)=1=\|f_j\|$ for $j=1,2$. Hence, Q is not smooth.

Claim: if $\left(a=1,\ b=-\frac{3}{4},\ \frac{1}{4},\ c<1\right)$, $\left(a=1,\ -1<b<\frac{1}{4},\ b\neq-\frac{3}{4}\right)$ or $\left(a=1,\ \frac{1}{4}\leq b,\ b+\frac{c}{2}<\frac{3}{4}\right)$, then Q is smooth.

Note that if $a=1, b=-\frac{3}{4}$, then $c\leq 1$. Note also that if $a=1, b=-\frac{3}{4}, c=1$, then Q is not smooth.

Suppose that $a=1$, $b=-\frac{3}{4}$, $c<1$. Let $f\in\mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q)=1=\|f\|$. Then $1=\alpha-\frac{3}{4}\beta+c\gamma$. We will show that $\alpha=1$, $\beta=\gamma=0$. Since $0\leq c<1$ and by Theorem 2.1, we can choose $\delta>0$ such that $\|R_u\|=\|S_v\|=1$ for all $u,v\in(-\delta,\delta)$, where

$$R_u(x,y)=x^2-\frac{3}{4}y^2+(c+u)xy\,,$$
$$S_v(x,y)=x^2-\left(\frac{3}{4}+v\right)y^2+cxy\in\mathcal{P}(^2\mathcal{H})\,.$$

It follows that, for all $u,v\in(-\delta,\delta)$,

$$1=\alpha-\frac{3}{4}\beta+c\gamma\geq f(R_u)=\alpha-\frac{3}{4}\beta+(c+u)\gamma\,,$$
$$1=\alpha-\frac{3}{4}\beta+c\gamma\geq f(S_v)=\alpha-\left(\frac{3}{4}+v\right)\beta+c\gamma\,,$$

which shows that $\alpha = 1$, $\beta = \gamma = 0$. Therefore, Q is smooth. By a similar argument, if $a = 1$, $b = \frac{1}{4}$, $c < 1$, then Q is smooth.

Suppose that $a = 1$, $-1 < b < \frac{1}{4}$, $b \neq -\frac{3}{4}$. Let $a = 1$, $-1 < b < -\frac{3}{4}$. We will show that $c < 1$. If not, then $1 \leq c \leq 2$. By Theorem 2.1, $b - \frac{c}{2} \geq -\frac{5}{4}$, $\frac{c^2-4b}{2c-1-4b} \leq 1$, which shows that $c = 1$, $b \geq -\frac{3}{4}$. This is a contradiction. Hence, by Theorem 2.1, $b - \frac{c}{2} \geq -\frac{5}{4}$, $\frac{c^2}{4} - b \leq 1$. We claim that if

$$a = 1\,, \qquad -1 < b < -\frac{3}{4}\,, \qquad b - \frac{c}{2} > -\frac{5}{4}\,, \qquad \frac{c^2}{4} - b < 1\,,$$

then Q is smooth. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. Then, $1 = \alpha + b\beta + c\gamma$. We will show that $\alpha = 1$, $\beta = \gamma = 0$. By Theorem 2.1, we can choose $\delta > 0$ such that $\|R_u\| = \|S_v\| = 1$ for all $u, v \in (-\delta, \delta)$, where

$$R_u(x, y) = x^2 + by^2 + (c + u)xy\,,$$
$$S_v(x, y) = x^2 + (b + v)y^2 + cxy \in \mathcal{P}(^2\mathcal{H})\,.$$

Thus $\alpha = 1$, $\beta = \gamma = 0$. Therefore, Q is smooth.

Note that if

$$a = 1\,, \qquad -1 < b < -\frac{3}{4}\,, \qquad b - \frac{c}{2} \geq -\frac{5}{4}\,, \qquad \frac{c^2}{4} - b = 1\,,$$

then Q is not smooth letting $f_j \in \mathcal{P}(^2\mathcal{H})^*$ be such that

$$f_1(x^2) = 1\,, \qquad f_1(y^2) = 0 = f_1(xy)\,,$$
$$f_2(x^2) = -\frac{c^2}{4}\,, \qquad f_2(y^2) = -1\,, \qquad f_2(xy) = \frac{c}{2}\,.$$

Thus $x^2 + (\frac{c^2}{4} - 1)y^2 + cxy$ $(0 \leq c \leq 1)$ is not smooth.

Note also that if

$$a = 1\,, \qquad -1 < b < -\frac{3}{4}\,, \qquad b - \frac{c}{2} = -\frac{5}{4}\,, \qquad \frac{c^2}{4} - b \leq 1\,,$$

then Q is not smooth letting $f_j \in \mathcal{P}(^2\mathcal{H})^*$ be such that

$$f_1(x^2) = 1\,, \qquad f_1(y^2) = 0 = f_1(xy)\,,$$
$$f_2(x^2) = -\frac{1}{4}\,, \qquad f_2(y^2) = -1\,, \qquad f_2(xy) = \frac{1}{2}\,.$$

Let $a = 1$, $-\frac{3}{4} < b < \frac{1}{4}$. We will show that Q is smooth. First, suppose that $-\frac{3}{4} < b < 0$. Since $\|Q\| = 1$, by Theorem 2.1, we have $0 \leq c \leq 1$. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. Then $1 = \alpha + b\beta + c\gamma$. We will show that $\alpha = 1$, $\beta = 0 = \gamma$. Since $-\frac{3}{4} < b < 0$, By Theorem 2.1, we can choose $\delta > 0$ such that $\|R_u\| = \|S_v\| = 1$ for all $u, v \in (-\delta, \delta)$, where

$$R_u(x,y) = x^2 + (b+u)y^2 + cxy\,,$$
$$S_v(x,y) = x^2 + by^2 + (c+v)xy \in \mathcal{P}(^2\mathcal{H})\,.$$

Thus $\alpha = 1$, $\beta = 0 = \gamma$. Hence, Q is smooth.

Suppose that $c = 1$. Then $1 = \alpha + \gamma$, $\alpha \geq 0$, $\gamma \geq 0$. By Theorem 2.3,

$$\begin{aligned} 1 &\geq \sup_{0\leq \tilde{a}\leq 1} \tilde{a}\alpha + (\tilde{a} + 2\sqrt{1-\tilde{a}})\gamma \\ &= \sup_{0\leq \tilde{a}\leq 1} 2\sqrt{1-\tilde{a}}(1-\alpha) + \tilde{a} \;=\; 1 + (1-\alpha)^2\,, \end{aligned}$$

which implies that $\alpha = 1$. Therefore, $\alpha = 1$, $\beta = 0 = \gamma$. We have shown that if $0 < c \leq 1$, then Q is smooth. Suppose that $c = 0$. Since $1 = \alpha + b\beta$, $\beta = 0$, we have $\alpha = 1$. By Theorem 2.3, $1 \geq \left|\alpha + \frac{1}{4}\beta\right| + |\gamma| = 1 + \gamma$, which shows that $\gamma = 0$. Hence, Q is smooth.

Suppose that $0 \leq b < \frac{1}{4}$. Since $\|Q\| = 1$, by Theorem 2.1, $0 \leq c \leq 1$.

Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. We will show that $\alpha = 1$, $\beta = 0 = \gamma$. Since $1 = f(Q) = \alpha + b\beta + c\gamma$, we have $\alpha > 0$. Indeed, if $\alpha \leq 0$, then

$$1 \leq b\beta + c\gamma \leq b|\beta| + c|\gamma| < \frac{1}{4} + \frac{1}{2} = \frac{3}{4}\,,$$

which is a contradiction. We also claim that $\alpha + \frac{1}{4}\beta \geq 0$. If not, then $\alpha < \frac{1}{4}|\beta| \leq \frac{1}{4}$, which implies that

$$\frac{3}{4} < 1 - \alpha = b\beta + c\gamma \leq b|\beta| + c|\gamma| < \frac{3}{4}\,,$$

which is a contradiction. Note that

$$\alpha + b\beta = 1 - c\gamma \geq 1 - c|\gamma| \geq 1 - \frac{c}{2} \geq \frac{1}{2}\,.$$

By Theorem 2.3,

$$\alpha + \frac{1}{4}\beta + |\gamma| = \left|\alpha + \frac{1}{4}\beta\right| + |\gamma| \leq 1 = \alpha + b\beta + c\gamma \leq \alpha + b\beta + c|\gamma|\,,$$

which shows that

$$\left(\frac{1}{4} - b\right)\beta \le (c-1)|\gamma| \le 0\,.$$

Hence, $\beta \le 0$. By Theorem 2.3, for all $0 \le \tilde{c} \le 1$, it follows that

$$\begin{aligned}\alpha + \left(1 - \frac{\tilde{c}^2}{4}\right)|\beta| + \tilde{c}|\gamma| &= \left|\alpha + \left(\frac{\tilde{c}^2}{4} - 1\right)\beta\right| + \tilde{c}|\gamma| \\ &\le 1 = \alpha + b\beta + c\gamma \\ &\le \alpha + b\beta + c|\gamma| \;=\; \alpha - b|\beta| + c|\gamma|\,,\end{aligned}$$

which implies that

$$\left(1 - \frac{\tilde{c}^2}{4} + b\right)|\beta| \le (c - \tilde{c})|\gamma| \qquad (0 \le \tilde{c} \le 1)\,.$$

Thus

$$\left(1 - \frac{c^2}{4} + b\right)|\beta| = \lim_{\tilde{c}\to c-}\left(1 - \frac{\tilde{c}^2}{4} + b\right)|\beta| \le \lim_{\tilde{c}\to c-}(c - \tilde{c})|\gamma| = 0\,,$$

so $\beta = 0$. Since $1 = f(Q) = \alpha + c\gamma$, we have $\gamma \ge 0$. By Theorem 2.3,

$$\tilde{a}\alpha + \big(\tilde{a} + 2\sqrt{1-\tilde{a}}\big)\gamma \le 1 = \alpha + c\gamma \qquad (0 \le \tilde{a} \le 1)\,,$$

which implies that

$$(\tilde{a} - c + 2\sqrt{1-\tilde{a}})\gamma \le (1-\tilde{a})\alpha \qquad (0 \le \tilde{a} \le 1)\,. \tag{3}$$

If $c < 1$, then

$$(1-c)\gamma = \lim_{\tilde{a}\to 1-}\big(\tilde{a} - c + 2\sqrt{1-\tilde{a}}\big)\gamma \le \lim_{\tilde{a}\to 1-}(1-\tilde{a})\alpha = 0\,,$$

so $\gamma = 0$. Therefore, $\alpha = 1$, $\beta = 0$. Suppose that $c = 1$. By (3),

$$(\tilde{a} - 1 + 2\sqrt{1-\tilde{a}})\gamma \le (1-\tilde{a})\alpha \qquad (0 \le \tilde{a} \le 1)\,,$$

which implies that

$$2\gamma = \lim_{\tilde{a}\to 1-}\big(2 - \sqrt{1-\tilde{a}}\big)\gamma \le \Big(\lim_{\tilde{a}\to 1-}\sqrt{1-\tilde{a}}\Big)\alpha = 0\,,$$

so $\gamma = 0$. Therefore, $\alpha = 1$, $\beta = 0 = \gamma$. Hence, Q is smooth.

Suppose that $a = 1$, $\frac{1}{4} \le b$. Since $\|Q\| = 1$, we have $b + \frac{c}{2} \le \frac{3}{4}$. If $b + \frac{c}{2} = \frac{3}{4}$, then Q is not smooth letting $f_j \in \mathcal{P}(^2\mathcal{H})^*$ be such that

$$f_1(x^2) = \frac{1}{4}, \qquad f_1(y^2) = 1, \qquad f_1(xy) = \frac{1}{2},$$
$$f_2(x^2) = 1, \qquad f_2(y^2) = 0 = f_2(xy).$$

Let $b + \frac{c}{2} < \frac{3}{4}$. Note that if $b = \frac{1}{4}$, then $Q = x^2 + \frac{1}{4}y^2 + cxy$ for $0 \le c < 1$. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. Then $\alpha = 1$, $\beta = 0 = \gamma$. Thus Q is smooth.

Suppose that $a = 1$, $\frac{1}{4} < b$. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. Then $\alpha = 1$, $\beta = 0 = \gamma$. Thus Q is smooth.

Case 3: $0 < a < 1$.

Suppose that $b = 0$. We will show that $c > a$. If not, then $\|Q\| < 1$, which is a contradiction. Hence, $c > a$. We claim that Q is smooth. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. We will show that $\alpha = \frac{1}{c^2}$, $\beta = \frac{4(1-a)}{c^2}$, $\gamma = \frac{2(c-1)}{c^2}$. Note that $\frac{1}{4}a + \frac{1}{2}c < 1$, $0 < c < 2$. We may choose $\delta > 0$ such that $\|R_u\| = \|S_v\| = 1$ for all $u, v \in (-\delta, \delta)$, where

$$R_u(x, y) = \big(a + u(2 - 2c - u)\big)x^2 + (c + u)xy,$$
$$S_v(x, y) = \left(a + \frac{4(a-1)v}{1 - 4v}\right)x^2 + vy^2 + cxy \in \mathcal{P}(^2\mathcal{H}).$$

Then $\gamma = 2(c-1)\alpha$, $\beta = 4(1-a)\alpha$. It follows that

$$1 = a\alpha + c\gamma = c(2 - c)\alpha + c(2c - 2)\alpha = c^2\alpha,$$

proving that $\alpha = \frac{1}{c^2}$, $\beta = \frac{4(1-a)}{c^2}$, $\gamma = \frac{2(c-1)}{c^2}$. Thus Q is smooth.

Suppose that $b \neq 0$. Let $c \le a$. Suppose that $c \le a \le 4b$. Notice that if $a = 4b$, then $\|Q\| < 1$. Hence, Q is not smooth.

Suppose that $a < 4b$. Then, $0 < b \le 1$. If $b = 1$, then $\|Q\| > 1$, which is impossible. We claim that if $c = a$, $0 < b < 1$, then Q is smooth. Let $0 < b < 1$. By Theorem 2.1, $1 = \|Q\| = \frac{3}{4}a + b$. Therefore,

$$Q = ax^2 + \left(1 - \frac{3}{4}a\right)y^2 + axy$$

for $0 < a < 1$. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. Then $1 = a\alpha + \left(1 - \frac{3}{4}a\right)\beta + a\gamma$. We will show that $\alpha = \frac{1}{4}$, $\beta = 1$, $\gamma = \frac{1}{2}$. We can choose $\delta > 0$ such that $\|R_u\| = \|S_v\| = 1$ for all $u, v \in (-\delta, \delta)$, where

$$R_u(x,y) = ax^2 + \left(1 - \frac{3a}{4} + u\right)y^2 + (a-2u)xy\,,$$
$$S_v(x,y) = (a-2v)x^2 + \left(1 - \frac{3a}{4}\right)y^2 + (a+v)xy \in \mathcal{P}(^2\mathcal{H})\,.$$

Then $\beta = 2\gamma$, $\gamma = 2\alpha$. Therefore, $\alpha = \frac{1}{4}$, $\beta = 1$, $\gamma = \frac{1}{2}$. Thus Q is smooth.

Notice that if $0 = c < a < 4b$, then Q is not smooth letting $f_j \in \mathcal{P}(^2\mathcal{H})^*$ be such that

$$f_1(x^2) = \frac{1}{4} = f_2(x^2)\,, \qquad f_1(y^2) = 1 = f_2(y^2)\,,$$
$$f_1(xy) = \frac{1}{2}\,, \qquad f_2(xy) = 0\,.$$

Claim: if $0 < c < a < 4b$, then Q is smooth.

By Theorem 2.1, $1 = \|Q\| = \frac{1}{4}a + b + \frac{1}{2}c$. Thus $0 < b < 1$. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. We will show that $\alpha = \frac{1}{4}, \beta = 1, \gamma = \frac{1}{2}$. We choose $\delta > 0$ such that $\|R_{u,v}\| = 1$ for all $u, v \in (-\delta, \delta)$, where

$$R_{u,v}(x,y) = (a+u)x^2 + (b+v)y^2 + \left(c - \frac{1}{2}u - 2v\right)xy \in \mathcal{P}(^2\mathcal{H})\,.$$

Thus $\alpha = \frac{1}{4}$, $\beta = 1$, $\gamma = \frac{1}{2}$. Therefore, Q is smooth.

Claim: if $c \le a, 4b < a$, then Q is smooth.

Suppose that $c = a$, $4b < a$. By Theorem 2.1, $1 = \|Q\| = \left|\frac{1}{4}a + b\right| + \frac{1}{2}a$. Notice that $\frac{1}{4}a + b < 0$. Thus

$$Q = ax^2 + \left(\frac{1}{4}a - 1\right)y^2 + axy$$

for $0 < a < 1$. We will show that Q is smooth. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. We will show that $\alpha = -\frac{1}{4}$, $\beta = -1$, $\gamma = \frac{1}{2}$. Choose $0 < \delta < 1$ such that

$$0 < a + 2v < a + v < 1\,, \qquad \frac{(a+v)^2 - 4a\left(\frac{1}{4}a - 1\right)}{2(a+v) - a - 4\left(\frac{1}{4}a - 1\right)} < 1$$

for all $v \in (-\delta, 0)$. Let

$$R_v = (a+2v)x^2 + \left(\frac{1}{4}a - 1\right)y^2 + (a+v)xy$$

for $v \in (-\delta, 0)$. By Theorem 2.1, $1 = \|R_v\|$. Thus $\gamma \geq -2\alpha$. Choose $0 < \delta_1 < 1$ such that

$$0 < a+v < 1\,, \qquad \frac{a^2 - 4(a+v)\big(\frac{1}{4}a - 1 - \frac{1}{4}v\big)}{2a - (a+v) - 4\big(\frac{1}{4}a - 1 - \frac{1}{4}v\big)} < 1$$

for all $v \in (-\delta_1, 0)$. Let

$$S_v = (a+v)x^2 + \left(\frac{1}{4}a - 1 - \frac{1}{4}v\right)y^2 + axy$$

for $v \in (-\delta_1, 0)$. By Theorem 2.1, $1 = \|S_v\|$. Thus $\alpha \geq \frac{1}{4}\beta$. Choose $0 < \delta_2 < 1$ such that

$$\frac{(a+2v)^2 - 4a\big(\frac{1}{4}a - 1 + v\big)}{2(a+2v) - a - 4\big(\frac{1}{4}a - 1 + v\big)} < 1$$

for all $v \in (0, \delta_2)$. Let

$$W_u = ax^2 + \left(\frac{1}{4}a - 1 + u\right)y^2 + (a+2u)xy$$

for $u \in (0, \delta_2)$. By Theorem 2.1, $1 = \|W_u\|$. Thus $\beta \leq -2\gamma$. Let $\beta = -1 + \epsilon$ for some $0 \leq \epsilon < 1$. By Theorem 2.3, it follows that

$$\begin{aligned}
1 &\geq \sup_{0 \leq c \leq 1} \left| \alpha + \left(\frac{c^2}{4} - 1\right)(-1+\epsilon) \right| + c\gamma \\
&= \sup_{0 \leq c \leq 1} -\frac{1}{4}(c - 2\gamma)^2 + \gamma^2 - \gamma + \frac{5}{4} + \epsilon\left(\frac{1}{a} - \frac{5}{4} + \frac{c^2}{4}\right) \\
&\geq \max\left\{ \gamma^2 - \gamma + \frac{5}{4} + \epsilon\left(\frac{1}{a} - \frac{5}{4} + \frac{(2\gamma)^2}{4}\right), \right. \\
&\qquad \left. -\frac{1}{4}(1 - 2\gamma)^2 + \gamma^2 - \gamma + \frac{5}{4} + \epsilon\left(\frac{1}{a} - 1\right) \right\} \\
&= \max\left\{ \left(\gamma - \frac{1}{2}\right)^2 + 1 + \epsilon\left(\frac{1}{a} - \frac{5}{4} + \gamma^2\right), 1 + \epsilon\left(\frac{1}{a} - 1\right) \right\} \\
&\geq 1 + \epsilon\left(\frac{1}{a} - 1\right) \geq 1\,,
\end{aligned}$$

which shows that $\epsilon = 0 = (\gamma - \frac{1}{2})^2$. Thus $\alpha = -\frac{1}{4}$, $\beta = -1$, $\gamma = \frac{1}{2}$. Hence, Q is smooth.

Suppose that $c < a$, $4b < a$. Note that $-1 \le b < 0$. If $b = -1$, then $Q = ax^2 - y^2$. We will show that it is smooth. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. Notice that $\alpha = 0$, $\beta = -1$, $\gamma = 0$. Hence, Q is smooth.

Let $-1 < b < 0$. Then $c > 0$.

Claim: $1 = \frac{|c^2-4ab|}{4a} = \frac{c^2-4ab}{4a}$.

First, suppose that $\frac{1}{4}a \ge |b|$. Then $\left|\frac{1}{4}a+b\right| + \frac{1}{2}c = \frac{1}{4}a+b+\frac{1}{2}c < a < 1$. By Theorem 2.1, $1 = \|Q\| = \frac{|c^2-4ab|}{4a}$. Let $\frac{1}{4}a < |b|$. Notice that $\left|\frac{1}{4}a + b\right| + \frac{1}{2}c < \frac{c^2+4a|b|}{4a}$, so $1 = \frac{|c^2-4ab|}{4a} = \frac{c^2-4ab}{4a}$. Suppose that $0 < c < 1$. We will show that Q is smooth. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. We will show that $\alpha = -\frac{c^2}{4a^2}$, $\beta = -1$, $\gamma = \frac{c}{2a}$. We choose $\delta > 0$ such that $\|R_v\| = \|S_w\| = 1$ for all $v, w \in (-\delta, \delta)$, where

$$R_v(x,y) = \left(a - \frac{av}{1+b+v}\right)x^2 + (b+v)y^2 + cxy\,,$$
$$S_w(x,y) = ax^2 + \left(b + \frac{w(2c+w)}{4a}\right)y^2 + (c+w)xy \in \mathcal{P}(^2\mathcal{H})\,.$$

Notice that $\beta = \frac{a}{1+b}\alpha$, $\gamma = -\frac{c}{2a}\beta$. Therefore, $\alpha = -\frac{c^2}{4a^2}$, $\beta = -1$, $\gamma = \frac{c}{2a}$. Hence, Q is smooth.

Suppose that $c = 0$. Then $Q = ax^2 - y^2$ for $0 < a < 1$, which is smooth.

Suppose that $c > a$.

Claim: if $c > a = 4b$, then Q is smooth.

Notice that $Q = ax^2 + \frac{a}{4}y^2 + (2-a)xy$. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. By the previous arguments, $\alpha = \frac{1}{4}, \beta = 1, \gamma = \frac{1}{2}$. Thus Q is smooth.

Claim: if $c > a > 4b$, $c \ne a + 2\sqrt{1-a}$, then Q is smooth.

By Theorem 2.1, $-1 < b < \frac{1}{4}$, $0 < c < 2$. Notice that

$$\left|\frac{1}{4}a + b\right| + \frac{1}{2}c < 1 \quad \text{and} \quad \frac{c^2 - 4ab}{2c - a - 4b} = 1\,,$$

or

$$\frac{c^2 - 4ab}{2c - a - 4b} < 1 \quad \text{and} \quad \left|\frac{1}{4}a + b\right| + \frac{1}{2}c = 1\,.$$

First, suppose that $\left|\frac{1}{4}a+b\right|+\frac{1}{2}c<1$, $\frac{c^2-4ab}{2c-a-4b}=1$. Let $f\in\mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q)=1=\|f\|$. We will show that $\alpha=\frac{(c-4b)^2}{(2c-a-4b)^2}$, $\beta=\frac{4(c-a)^2}{(2c-a-4b)^2}$, $\gamma=\frac{2(c-a)(c-4b)}{(2c-a-4b)^2}$. We may choose $\delta>0$ such that

$$0<1-4b-4v\,,\qquad 0<a+\frac{4(a-1)v}{1-4b-4v}<1\,,\qquad -1<b+v<\frac{1}{4}\,,$$
$$4(b+v)<a+\frac{4(a-1)v}{1-4b-4v}<c\,,\qquad \left|\frac{1}{4}\Big(a+\frac{4(a-1)v}{1-4b-4v}\Big)+b+v\right|+\frac{1}{2}c<1$$

for all $v\in(-\delta,\delta)$. Let

$$R_v(x,y)=\Big(a+\frac{4(a-1)v}{1-4b-4v}\Big)x^2+(b+v)y^2+cxy$$

for all $v\in(-\delta,\delta)$. By Theorem 2.1,

$$\|R_v\|=\frac{c^2-4\Big(a+\frac{4(a-1)v}{1-4b-4v}\Big)(b+v)}{2c-\Big(a+\frac{4(a-1)v}{1-4b-4v}\Big)-4(b+v)}=1$$

for all $v\in(-\delta,\delta)$. Notice that

$$\beta=\frac{4(1-a)}{1-4b}\alpha\,. \tag{4}$$

We may choose $\epsilon>0$ such that

$$-1<b+\frac{w(2c-2+w)}{4(a-1)}<\frac{1}{4}\,,\qquad 4\Big(b+\frac{w(2c-2+w)}{4(a-1)}\Big)<a<c+w<2\,,$$
$$\left|\frac{1}{4}a+b+\frac{w(2c-2+w)}{4(a-1)}\right|+\frac{1}{2}(c+w)<1$$

for all $w\in(-\epsilon,\epsilon)$. Let

$$S_w(x,y)=ax^2+\Big(b+\frac{w(2c-2+w)}{4(a-1)}\Big)y^2+(c+w)xy$$

for all $w\in(-\epsilon,\epsilon)$. By Theorem 2.1,

$$\|S_w\|=\frac{(c+w)^2-4a\Big(b+\frac{w(2c-2+w)}{4(a-1)}\Big)}{2(c+w)-a-4\Big(b+\frac{w(2c-2+w)}{4(a-1)}\Big)}=1$$

for all $w \in (-\epsilon, \epsilon)$. Notice that $\gamma = \frac{(c-1)}{2(1-a)}\beta$ and by (4), $\gamma = \frac{2(c-1)}{1-4b}\alpha$. It follows that

$$1 = a\alpha + b\beta + c\gamma = \alpha\left(a + \frac{4b(1-a)}{1-4b} + \frac{2c(c-1)}{1-4b}\right) = \alpha\left(\frac{2c-a-4b}{1-4b}\right),$$

which implies that $\alpha = \frac{1-4b}{2c-a-4b}$ and $\frac{1-4b}{2c-a-4b} = \frac{(c-4b)^2}{(2c-a-4b)^2}$. Therefore,

$$\alpha = \frac{(c-4b)^2}{(2c-a-4b)^2}, \qquad \beta = \frac{4(c-a)^2}{(2c-a-4b)^2}, \qquad \gamma = \frac{2(c-a)(c-4b)}{(2c-a-4b)^2}.$$

Thus Q is smooth.

Suppose that $\frac{c^2-4ab}{2c-a-4b} < 1$, $\left|\frac{1}{4}a + b\right| + \frac{1}{2}c = 1$. Note that $\frac{1}{4}a + b \neq 0$. First, suppose that $\frac{1}{4}a + b > 0$. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. We will show that $\alpha = \frac{1}{4}$, $\beta = 1$, $\gamma = \frac{1}{2}$. We choose $\delta > 0$ such that

$$R_u(x,y) = (a+u)x^2 + \left(b - \frac{1}{4}u\right)y^2 + cxy,$$

$$S_v(x,y) = ax^2 + \left(b - \frac{v}{2}\right)y^2 + (c+v)xy \in \mathcal{P}(^2\mathcal{H})$$

for all $u, v \in (-\delta, \delta)$. Notice that $\gamma = \frac{1}{2}\beta$, $\gamma = 2\alpha$. Thus $\alpha = \frac{1}{4}$, $\beta = 1$, $\gamma = \frac{1}{2}$. Hence, Q is smooth.

Next, suppose that $\frac{1}{4}a + b < 0$. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. By the previous argument, $\alpha = -\frac{1}{4}$, $\beta = -1$, $\gamma = \frac{1}{2}$. Thus Q is smooth.

Suppose that $c > a > 4b$, $c = a + 2\sqrt{1-a}$. We will show that Q is not smooth. By Theorem 2.1, $1 = \|Q\| \geq \frac{\left(a+2\sqrt{1-a}\right)^2 - 4ab}{2\left(a+2\sqrt{1-a}\right) - a - 4b}$. Thus $-1 < b \leq \frac{a+4\sqrt{1-a}}{4} - 1 < 0$, so $\frac{1}{4}a + b < 0$. Since

$$1 \geq \left|\frac{1}{4}a + b\right| + \frac{1}{2}c = -\left(\frac{1}{4}a + b\right) + \frac{1}{2}c,$$

which implies that $b \geq \frac{a+4\sqrt{1-a}}{4} - 1$, so $b = \frac{a+4\sqrt{1-a}}{4} - 1$ and

$$Q = ax^2 + \left(\frac{a+4\sqrt{1-a}}{4} - 1\right)y^2 + \left(a + 2\sqrt{1-a}\right)xy \qquad (0 < a < 1).$$

For $j = 1, 2$, let $f_j \in \mathcal{P}(^2\mathcal{H})^*$ be such that

$$f_1(x^2) = -\frac{1}{4}, \quad f_1(y^2) = -1, \quad f_1(xy) = \frac{1}{2}, \quad f_2(x^2) = \frac{\left(2 - \sqrt{1-a}\right)^2}{4},$$

$$f_2(y^2) = 1 - a, \qquad f_2(xy) = \frac{\sqrt{1-a}\left(2 - \sqrt{1-a}\right)}{2}.$$

Clearly $f_j(Q) = 1 = \|f_1\|$ for $j = 1, 2$. We claim that $\|f_2\| = 1$. Indeed, for $P = a'x^2 + b'y^2 + c'xy \in \mathcal{P}(^2\mathcal{H})$, we have

$$\begin{aligned}\delta_{\left(\frac{2-\sqrt{1-a}}{2},\sqrt{1-a}\right)}(P) &= P\left(\frac{2-\sqrt{1-a}}{2}, \sqrt{1-a}\right)\\ &= a'\left(\frac{2-\sqrt{1-a}}{2}\right)^2 + b'\left(\sqrt{1-a}\right)^2 + c'\left(\frac{2-\sqrt{1-a}}{2}\right)\sqrt{1-a}\\ &= f_2(P)\,,\end{aligned}$$

which implies that $f_2 = \delta_{\left(\frac{2-\sqrt{1-a}}{2},\sqrt{1-a}\right)}$. Thus

$$\|f_2\| = \left\|\delta_{\left(\frac{2-\sqrt{1-a}}{2},\sqrt{1-a}\right)}\right\| \le \left\|\left(\frac{2-\sqrt{1-a}}{2}, \sqrt{1-a}\right)\right\|_{h(\frac{1}{2})} = 1\,.$$

Since $f_2(Q) = 1$, $\|f_2\| = 1$. Therefore, Q is not smooth.

Claim: if $c > a$, $a < 4b$, then Q is smooth.

By Theorem 2.1, $0 < b < 1$, $0 < c < 2$. Let $f \in \mathcal{P}(^2\mathcal{H})^*$ be such that $f(Q) = 1 = \|f\|$. By the previous arguments, $\alpha = \frac{1}{4}$, $\beta = 1$, $\gamma = \frac{1}{2}$. Thus Q is smooth.

Therefore, we complete the proof. ∎

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