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Dynamics of products of nonnegative matrices

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Abstract: The aim of this manuscript is to understand the dynamics of products of nonnegative matrices. We extend a well known consequence of the Perron-Frobenius theorem on the periodic points of a nonnegative matrix to products of finitely many nonnegative matrices associated to a word and later to products of nonnegative matrices associated to a word, possibly of infinite length.

Key words: Products of nonnegative matrices, common eigenvectors, common periodic points, orbits of infinite matrix products.

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1. Introduction

Given a family $\mathcal{F}$ of functions on a set Ω, an element $w_0 \in \Omega$ is said to be a common fixed point for $\mathcal{F}$ if $f(w_0) = w_0$ for all $f \in \mathcal{F}$. The existence and computation of such a point has been a topic of interest among several mathematicians, for instance see [2, 11]. Of particular interest is when the collection is a multiplicative semigroup or a group $\mathcal{M}$ of matrices, where a more general question on the existence of common eigenvectors arises. A classic example of a multiplicative semigroup of matrices is the collection of matrices whose entries are nonnegative real numbers. In a recent work, Bernik *et al.* [3] determined certain conditions that ensures the existence of a common fixed point and more generally the existence of a common eigenvector for such a collection $\mathcal{M}$. The existence of common eigenvectors for a collection of matrices is in itself a nontrivial question and plays a major role in many problems in matrix analysis. For recent results on periods and periodic points of iterations of sub-homogeneous maps on a proper polyhedral cone, we refer the reader to [1] (for instance, see Theorem 4.2) and the references cited therein.

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We work throughout with the field $\mathbb{R}$ of real numbers. Let $M_n(\mathbb{R})$ denote the real vector space of $n \times n$ matrices. The subset of $M_n(\mathbb{R})$ consisting of matrices whose entries are nonnegative real numbers (such a matrix is usually called a *nonnegative matrix*) is denoted by $M_n(\mathbb{R}_+)$. For any matrix $A \in M_n(\mathbb{R})$, we denote and define *the spectrum, the spectral radius* and *the norm* of A respectively, as follows:

$$\begin{aligned} \operatorname{spec}(A) = {} & \text{the set of all eigenvalues of } A\text{, some of which may be} \\ & \text{complex numbers;} \\ \rho(A) = {} & \max\big\{|\lambda| \,:\, \lambda \in \operatorname{spec}(A)\big\}; \\ \|A\| = {} & \text{the operator norm of } A\text{, induced by the Euclidean} \\ & \text{norm of } \mathbb{R}^n. \end{aligned}$$

For any $N < \infty$, we fix a finite collection of matrices, $\big\{A_1, A_2, \ldots, A_N : A_r \in M_n(\mathbb{R}_+)\big\}$ and define the following *discrete dynamical system*: for $x_0 \in \mathbb{R}^n$, define

$$x_{j+1} := A_{\omega_j} x_j\,, \qquad \text{for } \omega_j \in \big\{1, 2, \ldots, N\big\}. \tag{1.1}$$

That is, from a point x_j at time $t = j$, we arrive at the point x_{j+1} at time $t = j + 1$ in the iteration of any generic point in $\mathbb{R}^n$, by randomly choosing one of the matrices from the above mentioned finite collection and the action by the chosen matrix. Observe that in order to achieve proper meaning to the above mentioned iterative scheme, one expects to understand nonhomogeneous products of matrices.

Recall that given a self map f on a topological space X, an element $x \in X$ is called a *periodic point* of f if there exists a positive integer q such that $f^q(x) = x$. In such a case, the smallest such integer q that satisfies $f^q(x) = x$ is called the *period* of the periodic point x. The starting point of this work is the following consequence of the Perron-Frobenius theorem, as can be found in [8, Theorem B.4.7].

THEOREM 1.1. *Let $A \in M_n(\mathbb{R}_+)$ with $\rho(A) \le 1$. Then, there exists a positive integer q such that for every $x \in \mathbb{R}^n$ with $\big(\|A^k x\|\big)_{k\in\mathbb{N}}$ bounded, we have*

$$\lim_{k\to\infty} A^{kq} x \,=\, \xi_x\,,$$

where ξ_x is a periodic point of A whose period divides q.

The spectral radius condition in Theorem 1.1 can be dispensed with, only at the cost of looking at the orbits of points in the positive cone $\mathbb{R}^n_+$. An illustration to this effect, can be found in [2, page 321].

We are interested in a generalization of Theorem 1.1, when the matrix A in the above theorem is replaced by a product of the matrices A_r's, possibly an infinite one, drawn from the finite collection of nonnegative matrices, $\{A_1, \dots, A_N\}$. Besides generalizing Theorem 1.1 as described above, we also bring out the existence of common periodic points for the said collection of matrices.

This manuscript is organized as follows: In Section 2, we introduce basic notations, however only as much necessary to state the main results of this paper, namely Theorem 2.1 and Theorem 2.3. In Section 3, we familiarise the readers with some results from the literature, on adequate conditions to impose on a collection of matrices that ensures the existence of common eigenvectors. In Section 4, we prove Theorem 2.1 and highlight a special case of the theorem as Corollary 4.2, when the collection of matrices satisfy an additional hypothesis. We follow this with Section 5 where we write a few examples, that illustrate the theorems. In Section 6, we focus on words of infinite length based on the finite collection of matrices that we have considered so far and write the proof of Theorem 2.3.

2. Main results

In this section, we introduce some notations, explain the underlying settings of the main results and state our main results of this paper. As explained in the introductory section, we fix a finite set of nonnegative matrices $\{A_1, \dots, A_N\}$, $N < \infty$. For any finite $M \in \mathbb{N}$ and $p \in \mathbb{N}$, we denote the set of all p-lettered words on the set of first M positive integers by

$$\Sigma_M^p := \{\omega = (\omega_1 \omega_2 \cdots \omega_p) \,:\, \omega_r \in \{1, \dots, M\}\}.$$

For any p-lettered word $\omega := (\omega_1 \omega_2 \cdots \omega_p) \in \Sigma_N^p$, we define the (finite) matrix product

$$A_\omega := A_{\omega_p} \times A_{\omega_{p-1}} \times \cdots \times A_{\omega_2} \times A_{\omega_1}. \tag{2.1}$$

A key hypothesis in our first theorem assumes the existence of a nontrivial set of common eigenvectors, say $E = \{v_1, v_2, \dots, v_d\}$ for the given collection of matrices, $\{A_1, \dots, A_N\}$. These common eigenvectors may be vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$. A sufficient condition that ensures the existence of common eigenvectors for the given collection is to demand the collection to be partially commuting, quasi-commuting or a Laffey pair when $N = 2$, or the collection to be quasi-commuting when $N \geq 3$. Each of these terms is explained in Section 3. Define

$$\begin{aligned}\mathcal{LC}(E) = \big\{\alpha_1 v_1 + \cdots + \alpha_d v_d : \;& \alpha_j \in \mathbb{C} \text{ satisfying} \\ & \alpha_{s_1} = \overline{\alpha_{s_2}} \text{ for all } v_{s_1} = \overline{v_{s_2}} \\ & \text{and } \alpha_j \in \mathbb{R} \text{ otherwise}\big\}.\end{aligned} \tag{2.2}$$

We now state our first result in this article.

THEOREM 2.1. *Let $\{A_1, A_2, \ldots, A_N\}$, $N < \infty$, be a collection of $n \times n$ matrices with nonnegative entries, each having spectral radius* 1*. Assume that the collection satisfies at least one of the following conditions, that ensures the existence of a nontrivial set of common eigenvectors.*

1. *If $N = 2$, then the collection is either partially commuting, quasi-commuting or a Laffey pair.*
2. *If $N \geq 3$, then the collection is quasi-commuting.*

Let E denote the set of all common eigenvectors of the collection of matrices. For any finite p, let $\omega \in \Sigma_N^p$ and A_ω be the matrix associated to the word ω. Then, for any vector $x \in \mathcal{LC}(E)$, there exists an integer $q_\omega \geq 1$ such that

$$\lim_{k\to\infty} A_\omega^{kq_\omega} x = \xi_{(x,\omega)}\,, \tag{2.3}$$

where $\xi_{(x,\omega)}$ is a periodic point of A_ω, whose period divides q_ω. Moreover, when $p \geq N$ and ω is such that for all $1 \leq r \leq N$, there exists $1 \leq j \leq p$ such that $\omega_j = r$, the integer q_ω and the limiting point $\xi_{(x,\omega)} \in \mathbb{R}^n$ are independent of the choice of ω.

Careful readers may have already observed that, subject to the spectral radius condition as found in the hypothesis of Theorem 2.1, we have $\mathcal{LC}(E) \subseteq \{x \in \mathbb{R}^n : \sup \|A_\omega^k x\| < \infty\}$. However, since the spectral radius of A_ω can not be determined in general, we are forced to only work with vectors in $\mathcal{LC}(E)$. Nevertheless, when the collection of matrices satisfy the spectral radius condition and are simultaneously diagonalizable, the above two sets coincide and is equal to $\mathbb{R}^n$. We will look at examples of this kind, later in Section 5. We now illustrate the case when the above set inclusion is proper.

Consider the following pair of non-commuting, diagonalizable nonnegative matrices both having spectral radius 1, with one common eigenvector, namely e_1.

$$A_1 = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 1 \end{bmatrix}.$$

Consider the two-lettered word $\omega = 12 \in \Sigma_2^2$. Then, the spectrum of A_ω, given by $\left\{\frac{2+\sqrt{3}}{3}, 1, \frac{2-\sqrt{3}}{3}\right\}$ has corresponding eigenvectors

$$u = \big(0, 1+\sqrt{3}, -1\big), \quad e_1, \quad v = \big(0, 2-\sqrt{3}, -1\big)$$

that form a basis for $\mathbb{R}^3$. Since the largest eigenvalue is larger than unity, we observe that the sequence $\big(\|A_\omega^k u\|\big)_{k\in\mathbb{N}}$ is unbounded. Thus, $\lim\limits_{k\to\infty} A_\omega^k u$ does not exist. Further, in this case, we note that

$$\begin{aligned}\mathcal{LC}(E) \;=\; \{\alpha_1 e_1 \,:\, \alpha \in \mathbb{R}\} \subsetneq & \left\{x \in \mathbb{R}^3 \,:\, \sup_k \left\|A_\omega^k x\right\| < \infty\right\} \\ =& \;\{\alpha_1 e_1 + \alpha_2 v \,:\, \alpha_1, \alpha_2 \in \mathbb{R}\}\,.\end{aligned}$$

We now denote the interior of the nonnegative orthant of $\mathbb{R}^n$ by $(\mathbb{R}_+^n)^\circ$, a convex cone and define the logarithm map and the exponential map, that appear frequently in nonlinear Perron-Frobenius theory as follows: $\log : (\mathbb{R}_+^n)^\circ \to \mathbb{R}^n$ and $\exp : \mathbb{R}^n \to (\mathbb{R}_+^n)^\circ$ by

$$\log(x) \;=\; (\log x_1, \ldots, \log x_n) \qquad \text{and} \qquad \exp(x) \;=\; (e^{x_1}, \ldots, e^{x_n})\,.$$

As one may expect, these functions act as inverses of each other in the interior of $\mathbb{R}_+^n$. More on these functions and their uses in nonlinear Perron-Frobenius theory can be found in the monograph [8]. A nonnegative matrix, when viewed as a linear map on $\mathbb{R}^n$, preserves the partial order induced by $\mathbb{R}_+^n$. A map f defined on a cone in $\mathbb{R}^n$ is said to be *subhomogeneous* if for every $\lambda \in [0,1]$, we have $\lambda f(x) \le f(\lambda x)$ for every x in the cone and *homogeneous* if $f(\lambda x) = \lambda f(x)$ for every nonnegative λ and every x in the cone. It is then easy to verify that the function $f := \exp \circ A \circ \log$ is a well-defined subhomogeneous map on $(\mathbb{R}_+^n)^\circ$. We now state a corollary to Theorem 2.1 for an appropriate subhomogeneous map, f_ω.

COROLLARY 2.2. *Let* $\{A_1, \ldots, A_N\}$, $N < \infty$ *be a set of* $n \times n$ *matrices satisfying all the hypotheses in Theorem* 2.1. *For any finite* p, *let* $\omega \in \Sigma_N^p$ *and* A_ω *be the matrix associated with the word* ω. *Consider the function* $f_\omega \colon (\mathbb{R}_+^n)^\circ \longrightarrow (\mathbb{R}_+^n)^\circ$ *given by* $f_\omega = \exp \circ A_\omega \circ \log$. *Then, for any* $y = e^x \in (\mathbb{R}_+^n)^\circ$ *where* $x \in \mathcal{LC}(E)$, *there exists an integer* $q \ge 1$ *such that*

$$\lim_{k\to\infty} f_\omega^{kq} y \;=\; \eta_y\,, \tag{2.4}$$

where η_y *is a periodic point of* f_ω, *whose period divides* q.

Our final theorem in this paper concerns the orbit of some $x \in \mathbb{R}^n$ under the action of some infinitely long word, whose letters belong to $\{A_1, \ldots, A_N\}$. In order to make our lives simpler, we shall assume that the given collection of matrices are pairwise commuting, with each matrix being diagonalizable over $\mathbb{C}$. This ensures the existence of n linearly independent (over $\mathbb{C}$) common eigenvectors $E = \{v_1, \ldots, v_n\}$ for the given collection. Let the first κ of these common eigenvectors correspond to eigenvalues of modulus 1 for every matrix A_r, in the collection. Observe, in this case that $\mathcal{LC}(E) = \mathbb{R}^n$. Further, for any vector $x \in \mathbb{R}^n$ given by $x = \sum_{s=1}^{n} \alpha_s v_s$ obeying the conditions mentioned in Equation (2.2), we define the support of the vector x as

$$I(x) \,=\, \mathrm{supp}(x) \,=\, \big\{1 \le s \le n \,:\, \alpha_s \neq 0\big\}.$$

We now give a brief overview of the space of infinite-lettered words on finitely many letters. According the discrete metric on the set of letters $\{1, 2, \ldots, N\}$ using the Kronecker delta function, one can topologize Σ_N^p with the appropriate product metric. When $p = \infty$, notice that the basis for the topology on the space of infinite-lettered words on N symbols, namely Σ_N^∞, is given by the cylinder sets that fixes the set of initial finite coordinates, i.e., given any $\omega \in \Sigma_N^p$ for some $p \in \mathbb{Z}^+$, the corresponding cylinder set is given by

$$\big[\omega_1\omega_2\cdots\omega_p\big] \,=\, \big\{\tau \in \Sigma_N^\infty \,:\, \tau_j = \omega_j \text{ for } 1 \le j \le p\big\}.$$

For more details on the spaces Σ_N^p or Σ_N^∞, one may refer [6].

For any p-lettered word $\omega \in \Sigma_N^p$, we denote by $\overline{\omega}$, the infinite-lettered word obtained by concatenating ω with itself, infinitely many times, i.e., $\overline{\omega} = (\omega\,\omega\,\cdots)$. Under the topology defined on Σ_N^∞, one may observe that

$$\Sigma_N^\infty \;=\; \overline{\bigcup_{p \ge 1} \big\{\overline{\omega} \,:\, \omega \in \Sigma_N^p\big\}}.$$

We know, from Theorem 2.1, that upon satisfying the necessary technical conditions, $\lim\limits_{k\to\infty} A_\omega^{kq} x = \xi_x$, whenever $x \in \mathcal{LC}(E)$. Thus, the following definition makes sense. Let

$$\widetilde{A_\omega} \,:=\, A_{\overline{\omega}} \,=\, (A_\omega^q)^k \quad \text{as } k \to \infty\,.$$

However, since Theorem 2.1 only asserts ξ_x to be a periodic point whose period divides q, we shall consider the map $\widetilde{A_\omega} : \mathbb{R}^n \to (\mathbb{R}^n)^q$. The precise action of $\widetilde{A_\omega}$ on points in $\mathbb{R}^n$ is given by

$$\widetilde{A_\omega}(x) \,=\, \big(\xi_x, A_\omega\xi_x, \ldots, A_\omega^{q-1}\xi_x\big)\,. \tag{2.5}$$

Let $\tau = \big(\tau_1\,\tau_2\,\tau_3\,\cdots\big) \in \Sigma_N^\infty$ be any arbitrary infinite lettered word that encounters all the N letters within a finite time, say m. It is easy to observe that the sequence

$$\Big(\overline{(\tau_1\,\cdots\,\tau_m)}, \overline{(\tau_1\,\cdots\,\tau_{m+1})}, \cdots\Big) \qquad \text{converges to } \tau,$$

in the topology on Σ_N^∞, as described above. For any $p \geq m$, denote by $\overline{\tau^{[p]}}$, the infinite-lettered word $\overline{(\tau_1\,\cdots\,\tau_p)}$ that occurs in the sequence, written above that converges to any given $\tau \in \Sigma_N^\infty$. Moreover, from the discussion above, we have that

$$\widetilde{A_{\tau^{[p]}}}x \;=\; \Big(\xi_x, A_{\tau^{[p]}}\xi_x, \ldots, A_{\tau^{[p]}}^{q-1}\xi_x\Big)\,.$$

Notice that the first component of the vector in $(\mathbb{R}^n)^q$ is always ξ_x for all $p \geq m$. Further, we define for every $r \in \big\{1, \ldots, N\big\}$,

$$\Phi_{(\tau,r)}(p) \;=\; \#\ \text{of}\ A_r\ \text{in}\ A_{\tau^{[p]}}.$$

We now state our final theorem in this paper.

THEOREM 2.3. *Let $\big\{A_1, \ldots, A_N\big\}$, $N < \infty$, be a collection of $n \times n$ simultaneously diagonalizable nonnegative matrices each having spectral radius at most* 1. *Suppose $\tau \in \Sigma_N^\infty$ encounters all the N letters within a finite time, say m. Then, for any $x \in \mathbb{R}^n$, there exists an increasing sequence $\{p_\gamma\}_{\gamma \geq 1}$ (depending on x), of positive integers and a finite collection of positive integers $\big\{\Lambda_{(r,s)}\big\}$ for $1 \leq r \leq N$ and $1 \leq s \leq \kappa$ such that*

$$\sum_{r=1}^{N} \Lambda_{(r,s)}\left[\Phi_{(\tau,r)}\big(p_{\gamma_k}\big) - \Phi_{(\tau,r)}\big(p_{\gamma_{k'}}\big)\right] \;\equiv\; 0 \pmod{q}, \qquad \text{for all } s \in I(x) \cap \{1, \ldots, \kappa\},$$

where p_{γ_k} and $p_{\gamma_{k'}}$ are any two integers from the sequence $\{p_\gamma\}$.

The above result may appear to be explaining an arithmetic property in a paper that deals with random dynamical systems generated by finitely many matrices; however, the authors urge the readers to note the following. As explained earlier, we know that $\widetilde{A_{\tau^{[p]}}} : \mathbb{R}^n \to (\mathbb{R}^n)^q$. Observe that $\lim\limits_{p\to\infty} \widetilde{A_{\tau^{[p]}}}x$ does not necessarily exist. However, from the proof of Theorem 2.3, we will obtain the following corollary:

COROLLARY 2.4. *For each $x \in \mathbb{R}^n$, there exists an increasing sequence $\{p_\gamma\}_{\gamma \geq 1}$ (depending on x) such that $\left\{\widetilde{A_{\tau[p_\gamma]}x}\right\}_{\gamma \geq 1}$ is a constant sequence and therefore, $\lim_{\gamma \to \infty} \widetilde{A_{\tau[p_\gamma]}}x$ exists.*

3. Common eigenvectors for a collection of matrices

A key ingredient in our main results in this work is the existence of a nontrivial set of common eigenvectors for a given collection $\{A_1, \ldots, A_N\}$ of matrices. It is a well known result that if every matrix in the collection is diagonalizable over $\mathbb{C}$ with the collection commuting pairwise, there is a common similarity matrix that puts all the matrices in a diagonal form. A collection of non-commuting matrices may or may not have common eigenvectors. The question as to which collections of matrices possess common eigenvectors is extremely nontrivial. In what follows, we give a brief account of this question that is essential for this work. We begin with the following definition.

DEFINITION 3.1. A collection $\{A_1, \ldots, A_N\}$ of matrices is said to be *quasi-commuting* if for each pair (r, r') of indices, both A_r and $A_{r'}$ commute with their (additive) commutator $[A_r, A_{r'}] := A_r A_{r'} - A_{r'} A_r$.

A classical result of McCoy [5, Theorem 2.4.8.7] says the following:

THEOREM 3.2. *Let $\{A_1, \ldots, A_N\}$ be a collection of $n \times n$ matrices. The following statements are equivalent.*

1. *For every polynomial $p(t_1, \ldots, t_N)$ in N non-commuting variables $t_1, \ldots, t_N$ and every $r, r' = 1, \ldots, N$, $p(A_1, \ldots, A_N)[A_r, A_{r'}]$ is nilpotent.*
2. *There is a unitary matrix U such that $U^* A_r U$ is upper triangular for every $r = 1, \ldots, N$.*
3. *There is an ordering $\lambda_1^{(r)}, \ldots, \lambda_n^{(r)}$ of the eigenvalues of each of the matrices A_r, $1 \leq r \leq N$ such that for any polynomial $p(t_1, \ldots, t_N)$ in N non-commuting variables, the eigenvalues of $p(A_1, \ldots, A_N)$ are $p\left(\lambda_s^{(1)}, \ldots, \lambda_s^{(N)}\right)$, $s = 1, \ldots, n$.*

If the matrices and the polynomials are over the real field, then all calculations may be carried out over $\mathbb{R}$, provided all the matrices have eigenvalues in $\mathbb{R}$. It turns out that a sufficient condition that guarantees any of the above three statements is when the collection of matrices is quasi-commutative (see

Drazin *et al.* [4]). Moreover, the first statement implies that the collection $\{A_1, \ldots, A_N\}$ has common eigenvectors. There are also other classes of matrices which possess common eigenvectors.

A pair (A_1, A_2) of matrices is said to *partially commute* if they have common eigenvectors. Moreover, two matrices A_1 and A_2 partially commute *iff* the *Shemesh subspace* $\mathcal{N} = \bigcap_{k,l=1}^{n-1} \ker\left(\left[A_1^k, A_2^l\right]\right)$ is a nontrivial maximal invariant subspace of A_1 and A_2 over which both A_1 and A_2 commute (see Shemesh [10]). The number of linearly independent common eigenvectors of the pair cannot exceed the dimension of $\mathcal{N}$. A pair (A_1, A_2) of matrices is called a *Laffey pair* if $\operatorname{rank}([A_1, A_2]) = 1$. It can be shown that such a pair of matrices partially commute, but do not commute.

4. Proof of Theorem 2.1

In this section, we prove Theorem 2.1, after stating a theorem due to Frobenius. Suppose A is an irreducible matrix in $M_n(\mathbb{R}_+)$ such that there are exactly κ eigenvalues of modulus $\rho(A)$. This integer κ is called the *index of imprimitivity* of A. If $\kappa = 1$, the matrix A is said to be *primitive*. If $\kappa > 1$, the matrix is said to be *imprimitive*.

Theorem 4.1. ([12, Theorem 6.18]) *Let A be an irreducible nonnegative matrix with its index of imprimitivity equal to κ. If $\lambda_1, \ldots, \lambda_\kappa$ are the eigenvalues of A of modulus $\rho(A)$, then $\lambda_1, \ldots, \lambda_\kappa$ are the distinct κ-th roots of $[\rho(A)]^\kappa$.*

Proof of Theorem 2.1. Recall that $E = \{v_1, \ldots, v_d\}$ is a set of d common eigenvectors of the matrices $A_1, \ldots, A_N$ that satisfies $A_r v_s = \lambda_{(r,s)} v_s$, where $\lambda_{(r,s)}$ is an eigenvalue of the matrix A_r corresponding to the eigenvector v_s, $1 \le s \le d$. Observe that for any p-lettered word $\omega = (\omega_1 \cdots \omega_p)$, we have

$$A_\omega v_s = \lambda_{(\omega_p,s)} \cdots \lambda_{(\omega_1,s)} v_s = \lambda_{(\omega,s)} v_s\,, \qquad \text{where } \lambda_{(\omega,s)} = \lambda_{(\omega_p,s)} \cdots \lambda_{(\omega_1,s)}.$$

We now rearrange the common eigenvectors $\{v_1, \ldots, v_d\}$ as

$$\{v_1, \ldots, v_\kappa, v_{\kappa+1}, \ldots, v_d\},$$

where κ is defined as

$$\kappa = \#\left\{v_s : A_r v_s = \lambda_{(r,s)} v_s \text{ with } \left|\lambda_{(r,s)}\right| = 1 \text{ for all } 1 \le r \le N\right\}. \quad (4.1)$$

It is possible that $\kappa = 0$, in which case, the limiting vector is the zero vector (as you may observe by the end of this proof). Recall from Equation (2.2) that

$$\begin{aligned}\mathcal{LC}(E) \;=\; \big\{\alpha_1 v_1 \;+\; \cdots \;+\; \alpha_d v_d \;:\; & \alpha_j \in \mathbb{C} \text{ satisfying} \\ & \alpha_{s_1} = \overline{\alpha_{s_2}} \text{ for all } v_{s_1} = \overline{v_{s_2}} \\ & \text{and } \alpha_j \in \mathbb{R} \text{ otherwise}\big\}.\end{aligned}$$

Owing to the hypotheses on the spectral radius in the statement of the theorem, we have that for every $x \in \mathcal{LC}(E)$, the sequence $\big\{\|A_\omega^k x\|\big\}_{k\geq 1}$ is bounded. In fact,

$$\big\|A_\omega^k x\big\| \;=\; \big\|\alpha_1 A_\omega^k v_1 + \cdots + \alpha_d A_\omega^k v_d\big\| \;\leq\; \big|\alpha_1\big|\big\|v_1\big\| + \cdots + \big|\alpha_d\big|\big\|v_d\big\|\,.$$

Let $q_1, \ldots, q_N$ be positive integers that satisfies the outcome of Theorem 1.1, for the matrices $A_1, \ldots, A_N$ respectively. For some $p > N$, let ω be a p-lettered word in Σ_N^p such that for all $1 \leq r \leq N$, there exists $1 \leq j \leq p$ such that $\omega_j = r$. Define q to be the least common multiple of the numbers $\{q_1, \ldots, q_N\}$.

For every $s \in \big\{1, \ldots, d\big\}$ and $r \in \big\{1, \ldots, N\big\}$, we enumerate the following possibilities that can occur for the values of $\lambda_{(r,s)}$:

Case 1. $\big(\lambda_{(r,s)}\big)^q = 1$ for every r and for some s with $\lambda_{(r,s)} \in \mathbb{R}$. This implies that the corresponding eigenvector v_s lies in $\mathcal{LC}(E)$.

Case 2. $\big(\lambda_{(r,s)}\big)^q = 1$ for every r and for some s with $\lambda_{(r,s)} \in \mathbb{C}$. This implies that there exists eigenvectors v_s and $\overline{v_s}$ with corresponding eigenvalues conjugate to each other such that $\alpha_s v_s + \overline{\alpha_s v_s}$ lies in $\mathcal{LC}(E)$.

Case 3. $\big|\lambda_{(r,s)}\big| < 1$ for some s and for some r. In this case, the iterates of v_s under the map A_ω goes to 0; that is, $\lim\limits_{k\to\infty} A_\omega^k v_s = 0$.

For any $x \in \mathcal{LC}(E)$ that can be written as $x \;=\; \alpha_1 v_1 + \cdots + \alpha_d v_d$, we have

$$\begin{aligned}\lim_{k\to\infty} A_\omega^{kq} x \;&=\; \alpha_1 \lim_{k\to\infty} \big(\lambda_{(\omega,1)}\big)^{kq} v_1 + \cdots + \alpha_d \lim_{k\to\infty} \big(\lambda_{(\omega,d)}\big)^{kq} v_d \\ &=\; \alpha_1 v_1 + \cdots + \alpha_\kappa v_\kappa \\ &=:\; \xi_{(x,\,\omega)}.\end{aligned}$$

Observe that $\xi_{(x,\omega)}$ and q are independent of the length of the word and in fact, the word ω itself. We denote $\xi_{(x,\omega)} \;=\; \xi_x$. Moreover, $\xi_x \;\in\; \mathcal{LC}(E)$.

Further, for $\xi_x = \alpha_1 v_1 + \cdots + \alpha_\kappa v_\kappa$ and $1 \leq r \leq N$, we have

$$\begin{aligned} A_r^{q_r}\xi_x &= \alpha_1 A_r^{q_r} v_1 + \cdots + \alpha_\kappa A_r^{q_r} v_\kappa \\ &= \alpha_1 \left(\lambda_{(r,1)}\right)^{q_r} v_1 + \cdots + \alpha_\kappa \left(\lambda_{(r,\kappa)}\right)^{q_r} v_\kappa \\ &= \xi_x\,. \end{aligned}$$

Since ξ_x is a periodic point of $A_1, \ldots, A_N$ with periods $q_1, \ldots, q_N$ respectively, we have ξ_x to be a periodic point of A_ω with period q. ∎

We now state a corollary to Theorem 2.1, where we include conditions in the hypotheses that ensures $\mathcal{LC}(E) = \mathbb{R}^n$. The corollary can be proved analogously to the above theorem. However, we present a simpler proof in this case.

Corollary 4.2. *In addition to the hypothesis of Theorem* 2.1*, assume that the considered collection of matrices is pairwise commuting with each matrix being diagonalizable over* $\mathbb{C}$*. Then, for any* $x \in \mathbb{R}^n$*, the same conclusion, as in Theorem* 2.1 *holds.*

Proof. We first observe that the extra hypotheses in the statement of the corollary ensures that the matrices $A_1, \ldots, A_N$ are simultaneously diagonalizable, i.e., there exists a nonsingular matrix Q such that the matrix $A_r = Q^{-1} D_r Q$, for $D_r = \operatorname{diag}\left(\lambda_{(r,1)}, \lambda_{(r,2)}, \ldots, \lambda_{(r,n)}\right)$, where $\lambda_{(r,s)}$ are the eigenvalues of the matrix A_r, arranged in non-increasing modulus.

Let $\omega \in \Sigma_N^p$ with corresponding matrix product A_ω. Then, $A_\omega = Q^{-1} D_\omega Q$. By Theorem 4.1, we have $\lambda_{(r,s)}^q = 1$ for every eigenvalue of modulus 1. Hence, for every $x \in \mathbb{R}^n$, we have

$$\lim_{k\to\infty} A_\omega^{kq} x = \lim_{k\to\infty} \left(Q^{-1} D_\omega^{kq} Q\right) x = \xi_x\,,$$

where ξ_x is a periodic point of A_r for every $1 \leq r \leq N$ and is given by $\xi_x = \alpha_1 v_1 + \ldots + \alpha_\kappa v_\kappa$, the definition of κ, as in the proof of Theorem 2.1. ∎

We now show that the diagonalizability condition can not be weakened in the hypothesis of Corollary 4.2. However, since the matrices in the collection $\{A_1, \ldots, A_N\}$ commute pairwise, we still obtain a collection of common eigenvectors, $E = \{v_1, \ldots, v_d\}$. For example, consider the following pair of non-diagonalizable, commuting matrices:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Observe that $e_1 = (1,0)^t$ is a common eigenvector for A_1 and A_2, whereas $e_2 = (0,1)^t$ is a common generalized eigenvector for A_1 and A_2. We further note that for any word ω that contains both the letters, the orbit of e_2 under A_ω is unbounded while that of e_1 is bounded.

Remarks 4.3. A few remarks are in order.

(a) If all the matrices $A_1, \ldots, A_N$ are pairwise commuting nonnegative symmetric matrices, subject to the spectral radius assumption in Corollary 4.2, then the periods of all the periodic points corresponding to the eigenvalues 1 and -1 for all A_r's is at most 2. Hence, for a matrix product A_ω corresponding to a word ω, we have $q = 2$.

(b) We have proved Theorem 2.1 for a special choice of ω that contains all the N letters. Suppose ω' is any arbitrary p-lettered word. Then we can take the appropriate subset of $\{1, \ldots, N\}$, whose members have been used for the writing of the word ω' and the same result as above follows for ω'.

Suppose the p-lettered word $\omega = (r\, r \cdots r)$ for some $1 \le r \le N$. Then the above theorem reduces to a particular case, as one may find in [7, 9]. We now state the same as a corollary.

Corollary 4.4. *Let A be an $n \times n$ matrix with nonnegative entries that is diagonalizable over $\mathbb{C}$ and of spectral radius* 1. *Then there exists an integer $q \ge 1$ such that for every $x \in \mathbb{R}^n$, we have $\lim\limits_{k\to\infty} A^{qk}x = \xi_x$, where ξ_x is a periodic point of A with its period dividing q.*

5. Examples

In this section, we provide several examples that illustrate the various results, that have been proved until now. We first fix a few notations. We denote the standard basis vectors of $\mathbb{R}^n$ by $e_1, \ldots, e_n$, while I_n denotes the identity matrix of order n. We write the permutation matrices of order n in column partitioned form denoted by P_n; for instance, we denote the 2×2 permutation matrix $\left[e_2 \,|\, e_1\right]$ by P_2. The matrix of 1's (of order n) is denoted by J_n. The diagonal matrix of order n with diagonal entries $d_1, \ldots, d_n$ is denoted by $\mathrm{diag}(d_1, \ldots, d_n)$. Our first example is a fairly simple one and illustrates the scenario in Corollary 4.4.

Example 5.1. Consider the diagonalizable matrix $A = P_2$ with spectral radius 1. If $x = e_2 \in \mathbb{R}^2$, then observe that

$$A^k x = \begin{cases} e_2\,, & \text{if } k \text{ is even}\,, \\ e_1\,, & \text{if } k \text{ is odd}\,. \end{cases}$$

In this example, we obtain $q = 2$.

The next two examples illustrate Corollary 4.2. The first one involves a pair of 6×6 commuting nonnegative matrices.

EXAMPLE 5.2. Consider

$$A_1 = \begin{bmatrix} P_4 & 0 \\ 0 & A_1^{(22)} \end{bmatrix} \qquad \text{where} \quad P_4 = \big[e_4 \,|\, e_1 \,|\, e_2 \,|\, e_3\big]$$

$$\text{and} \quad A_1^{(22)} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} A_2^{(11)} & 0 \\ 0 & A_2^{(22)} \end{bmatrix} \qquad \text{where} \quad A_2^{(11)} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\text{and} \quad A_2^{(22)} = \frac{1}{10}\begin{bmatrix} 3 & \sqrt{7} \\ \sqrt{7} & 3 \end{bmatrix}.$$

It can be easily seen that the matrices A_1 and A_2 commute and are diagonalizable over $\mathbb{C}$ and therefore, are simultaneously diagonalizable. The following table gives the common eigenvectors of A_1 and A_2 and the corresponding eigenvalues of the matrices A_1 and A_2.

Eigenvectors	v_1	v_2	v_3	v_4	v_5	v_6
Eigenvalues of A_1	1	-1	$-i$	i	1	$-1/3$
Eigenvalues of A_2	1	-1	0	0	λ_1	λ_2

where $\lambda_1 = \dfrac{3+\sqrt{7}}{10}$ and $\lambda_2 = \dfrac{3-\sqrt{7}}{10}$. The common eigenvectors are given by

$$\begin{aligned} v_1 &= (1,1,1,1,0,0)^t\,, & v_2 &= (1,-1,1,-1,0,0)^t\,, \\ v_3 &= (1,i,-1,-i,0,0)^t, & v_4 &= (1,-i,-1,i,0,0)^t\,, \\ v_5 &= (0,0,0,0,1,1)^t\,, & v_6 &= (0,0,0,0,1,-1)^t\,. \end{aligned}$$

Following the lines of the proof of Corollary 4.2, we consider the nonsingular matrix (written in column partitioned form) $Q = \big[v_1|\cdots|v_6\big]$. Then, for $r = 1, 2$ we have $A_r = QD_rQ^{-1}$, where D_r is the diagonal matrix consisting of the eigenvalues of A_r. Looking at the table of eigenvalues, one can see that $q_1 = 4$ and $q_2 = 2$. Consider any $x \in \mathbb{R}^6$ given by $x = \sum_{i=1}^{6} \alpha_i v_i$ where $\alpha_1, \alpha_2, \alpha_5, \alpha_6 \in \mathbb{R}$ and $\alpha_3, \alpha_4 \in \mathbb{C}$ with $\alpha_3 = \overline{\alpha_4}$. For any word A_ω that contains both A_1 and A_2 , we have

$$\lim_{k\to\infty} A_\omega^{4k} x = \left(Q \lim_{k\to\infty} D_\omega^{4k} Q^{-1}\right) x = \alpha_1 v_1 + \alpha_2 v_2 \,,$$

a periodic point of A_ω with period at most 2, that divides the least common multiple of q_1 and q_2.

We now present another pair of commuting and diagonalizable matrices, this time in $\mathbb{R}^7$, where we exhibit a periodic point of A_ω, whose period is equal to the least common multiple of the relevant q_i's.

Example 5.3. Let

$$A_1 = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & D_1 \end{bmatrix} \qquad \text{where } D_1 = \operatorname{diag}\left(\frac{1}{2}, \frac{1}{3}\right);$$

$$A_2 = \begin{bmatrix} P_3 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & D_2 \end{bmatrix} \qquad \text{where } D_2 = \operatorname{diag}\left(\frac{1}{5}, \frac{1}{6}\right)$$

$$\text{and } P_3 = \big[e_3 \,|\, e_1 \,|\, e_2\big].$$

As earlier, we write a table with the common eigenvectors and the corresponding eigenvalues for the matrices A_1 and A_2.

Eigenvectors	v_1	v_2	v_3	v_4	v_5	v_6	v_7
Eigenvalues of A_1	1	1	1	1	-1	$1/2$	$1/3$
Eigenvalues of A_2	1	ω	ω^2	1	1	$1/5$	$1/6$

where ω is the cubic root of unity and the v_i's are

$$v_1 = (1,1,1,0,0,0,0)^t\,, \qquad v_2 = (1,\omega,\omega^2,0,0,0,0)^t\,,$$
$$v_3 = (1,\omega^2,\omega,0,0,0,0)^t\,, \qquad v_4 = (0,0,0,1,1,0,0)^t\,,$$
$$v_5 = (0,0,0,1,-1,0,0)^t\,, \qquad v_6 = e_6\,, \qquad v_7 = e_7\,.$$

In this example, we have $q_1 = 2$ and $q_2 = 3$. Consider $x \in \mathbb{R}^7$ given by $x = \sum_{i=1}^{7} \alpha_i v_i$ where $\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \in \mathbb{R}$, $\alpha_2, \alpha_3 \in \mathbb{C}$ with $\alpha_2 = \overline{\alpha_3}$ and α_2, α_5 being non-zero. Then, for the word $A_\omega = A_1^{p_1} A_2^{p_2}$ with $p_1 \neq 0 \pmod 2$ and $p_2 \neq 0 \pmod 3$, we have $\lim\limits_{k\to\infty} A_\omega^{6k} x = \xi_x$, a periodic point of A_ω of period 6, the least common multiple of q_1 and q_2. If p_1 violates the above condition, then the period of ξ_x is 3; if p_2 violates the above condition, then the period of ξ_x is 2 and if both p_1 and p_2 violate the above conditions, then the period of ξ_x is 1, all three numbers being factors of the least common multiple of q_1 and q_2.

We now write two examples in the non-commuting set up that illustrates Theorem 2.1.

EXAMPLE 5.4. Let

$$A_1 = \begin{bmatrix} P_4 & 0 \\ 0 & A_1' \end{bmatrix} \qquad \text{where } A_1' = \begin{bmatrix} 1/5 & 1/6 \\ 1/6 & 1/5 \end{bmatrix}$$

$$\text{and } P_4 = \big[e_4 \,|\, e_1 \,|\, e_2 \,|\, e_3\big];$$

$$A_2 = \begin{bmatrix} P_2 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & A_2' \end{bmatrix} \qquad \text{where } A_2' = \begin{bmatrix} 1/7 & 1/8 \\ 1/7 & 1/8 \end{bmatrix}.$$

Observe that the matrices A_1 and A_2 do not commute, but partially commute giving rise to the existence of a set of common eigenvectors that are given by

$$v_1 = (1,1,1,1,0,0)^t\,, \quad v_2 = (1,-1,1,-1,0,0)^t\,, \quad v_3 = (0,0,0,0,1,1)^t\,.$$

The corresponding eigenvalues of the respective matrices are given in the following table.

Eigenvectors	v_1	v_2	v_3
Eigenvalues of A_1	1	-1	$1/5+1/6$
Eigenvalues of A_2	1	-1	$1/7+1/8$

In this case, we obtain $q_1 = 4$ and $q_2 = 2$. Here,

$$\mathcal{LC}(E) \subsetneq \left\{ x \in \mathbb{R}^6 \,:\, \sup \left\| A_\omega^k x \right\| < \infty \right\},$$

where A_ω contains both A_1 and A_2. Suppose $x \in \mathcal{LC}(E)$ given by $x = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ for $\alpha_i \in \mathbb{R}$. Then, we have $\lim\limits_{k\to\infty} A_\omega^{4k} x = \xi_x$, a periodic point of A_ω, however with period at most 2, that divides the least common multiple of q_1 and q_2.

As in the commuting case, we now present an example in the non-commuting setup and exhibit a periodic point for a particular choice of A_ω whose period is equal to the least common multiple of the appropriate q_i's.

EXAMPLE 5.5. Let

$$A_1 = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & \frac{1}{2}J_2 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} P_3 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & A_2' \end{bmatrix}, \quad \text{where } A_2' = \begin{bmatrix} 1/3 & 1/4 \\ 1/3 & 1/4 \end{bmatrix}$$

and P_3 is the permutation matrix as defined in Example 5.3.

In this example, A_1 and A_2 form a Laffey pair. They have the following six common eigenvectors:

$$\begin{aligned} v_1 &= \left(1, \omega, \omega^2, 0, 0, 0, 0\right)^t, & v_2 &= \left(1, \omega^2, \omega, 0, 0, 0, 0\right)^t, \\ v_3 &= (1, 1, 1, 0, 0, 0, 0)^t, & v_4 &= (0, 0, 0, 1, 1, 0, 0)^t, \\ v_5 &= (0, 0, 0, 1, -1, 0, 0)^t, & v_6 &= (0, 0, 0, 0, 0, 1, 1)^t. \end{aligned}$$

As earlier, we write the corresponding the eigenvalues of the matrices in the following table:

Eigenvectors	v_1	v_2	v_3	v_4	v_5	v_6
Eigenvalues of A_1	1	1	1	1	-1	1
Eigenvalues of A_2	ω	ω^2	1	1	1	$1/3 + 1/4$

Here, $q_1 = 2$ and $q_2 = 3$. Let A_ω be a matrix product such the matrix A_r occurs p_r times in A_ω and satisfies $p_1 \neq 0 \pmod 2$ and $p_2 \neq 0 \pmod 3$. For any vector $x = \alpha_1 v_1 + \cdots + \alpha_6 v_6$ with $\alpha_1, \alpha_2 \in \mathbb{C}$ satisfying $\alpha_1 = \overline{\alpha_2}$, $\alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \mathbb{R}$ and α_1, α_5 being non-zero, we obtain $\lim\limits_{k\to\infty} A_\omega^{6k} x = \xi_x$, a periodic point of A_ω of period 6, the least common multiple of q_1 and q_2. If p_1 violates the above condition, then the period of ξ_x is 3; if p_2 violates the above condition, then the period of ξ_x is 2 and if both p_1 and p_2 violate the above conditions, then the period of ξ_x is 1, all three numbers being factors of the least common multiple of q_1 and q_2.

At this juncture, we write one more example that showcases the dependence of the limiting periodic point on the word ω, in the non-commuting set-up, even when the non-common eigenvectors have a bounded orbit.

EXAMPLE 5.6. Let

$$A_1 = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \frac{1}{5}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

It can be easily seen that $A_1 A_2 \neq A_2 A_1$. The eigenvalues of A_1 and A_2 are $1, -\frac{1}{3}$ and $1, -\frac{1}{5}$ respectively. The vector $(1,1)^t$ is a common eigenvector for A_1 and A_2 corresponding to the eigenvalue 1. Moreover, the eigenvalues of $A_1 A_2$ are $1, \frac{1}{15}$ (and so the same is true for $A_2 A_1$). It easily follows from this that any $x \in \mathbb{R}^2$ has a bounded orbit. The eigenvector corresponding to the eigenvalue $-\frac{1}{3}$ for A_1 is $(1,-1)^t$ and the eigenvector corresponding to the eigenvalue $-\frac{1}{5}$ for A_2 is $(2,-1)^t$. Note that

$$(A_1A_2)^k - (A_2A_1)^k = \begin{bmatrix} -\alpha(k) & \alpha(k) \\ -\alpha(k) & \alpha(k) \end{bmatrix} \qquad \text{for } k \geq 1,$$

and therefore the commutator has rank 1, making this a Laffey pair. It is now obvious that

$$\lim_{k\to\infty} (A_1A_2)^k (1,1)^t = \lim_{k\to\infty} (A_2A_1)^k (1,1)^t$$

since $\Big((A_1A_2)^k - (A_2A_1)^k\Big)(1,1)^t = (0,0)^t$. Nevertheless,

$$\Big((A_1A_2)^k - (A_2A_1)^k\Big)(2,-1)^t = -3\alpha(k)(1,1)^t \qquad \text{whereas}$$
$$\Big((A_1A_2)^k - (A_2A_1)^k\Big)(1,-1)^t = -2\alpha(k)(1,1)^t.$$

Therefore, if x is one of the points $(2,-1)^t$ or $(1,-1)^t$, then, $\lim_{k\to\infty} (A_1A_2)^k x \neq \lim_{k\to\infty} (A_2A_1)^k x$, since $\lim_{k\to\infty} \alpha(k) \neq 0$.

It is possible to study these examples under the action of the appropriate non-homogeneous map, described in Corollary 2.2.

We conclude this section by describing another way of writing Theorem 2.1. Recall that Σ_N^∞ denotes the set of all infinite-lettered words on the set of symbols $\{1,\ldots,N\}$. Considering the Cartesian product of the symbolic space Σ_N^∞ and $\mathbb{R}^n$, one may describe the dynamical system discussed in this paper thus: Given a collection $\{A_1, \ldots, A_N\}$ of $n \times n$ matrices, let $T : X = \Sigma_N^\infty \times \mathbb{R}^n \to X$ be defined by $T(\tau, x) = (\sigma\tau, A_{\tau_1} x)$ where $\tau = (\tau_1\tau_2\tau_3\cdots)$ and σ is the shift map defined on Σ_N^∞ by $(\sigma\tau)_n = \tau_{n+1}$ for $n \geq 1$. We equip X with the corresponding product topology and study T as a non-invertible map.

Theorem 5.7. *Let $\{A_1, \ldots, A_N\}$, $N < \infty$, be a collection of $n \times n$ matrices that satisfy the hypotheses of Theorem 2.1. Suppose E denotes the set of all common eigenvectors of the collection of matrices. Let $\tau \in \Sigma_N^\infty$ be any arbitrary infinite lettered word that encounters all the N letters within a finite time, say m. Let $\left\{\overline{\tau^{[p]}}\right\}_{p \geq m}$ be a sequence of infinite-lettered words that converges to τ. Let $A_{\tau^{[p]}}$ be the matrix associated to the p-lettered word $\tau^{[p]} \in \Sigma_N^p$. Then, for every $p \geq m$ and any vector $x \in \mathcal{LC}(E)$, there exists an integer $q \geq 1$ such that*

$$\lim_{k \to \infty} T^{kpq}\left(\overline{\tau^{[p]}}, x\right) = \left(\overline{\tau^{[p]}}, \xi_x\right),$$

where $\left(\overline{\tau^{[p]}}, \xi_x\right)$ is a periodic point of T, whose period divides the least common multiple of p and q.

6. Words of infinite length

We conclude this paper with this final section where we write the proof of Theorem 2.3. Recall that the hypotheses of Theorem 2.3 and Corollary 4.2 are one and the same.

Proof of Theorem 2.3. Recall from Equation (2.5) that whenever $x \in \mathcal{LC}(E) = \mathbb{R}^n$ (in this case), we have

$$\widetilde{A_{\tau^{[p]}}} x = \left(\xi_x\,, A_{\tau^{[p]}}\xi_x\,, \ldots, A_{\tau^{[p]}}^{q-1}\xi_x\right), \qquad \text{for } p \geq m\,,$$

where m is the finite stage by when the word $\tau^{[p]}$ encounters all the letters in $\{1, 2, \ldots, N\}$. In general, it is not necessary that $A_{\tau^{[m]}}\xi_x = A_{\tau^{[m+1]}}\xi_x$. However, owing to ξ_x being a periodic point of $A_{\tau^{[p]}}$ for $p \geq m$, whose period divides q (> 1, say), a simple application of the pigeonhole principle ensures $A_{\tau^{[m]}}\xi_x = A_{\tau^{[m']}}\xi_x$, for some $m' > m$. We choose m' that guarantees

$$\widetilde{A_{\tau^{[m]}}} x = \widetilde{A_{\tau^{[m']}}} x\,, \qquad \text{as vectors in } (\mathbb{R}^n)^q\,.$$

Proceeding along similar lines, one obtains an increasing sequence, say $\{p_\gamma\}$ such that $\left\{\widetilde{A_{\tau^{[p_\gamma]}}x}\right\}_{\gamma\geq 1}$ is a constant sequence of vectors in $(\mathbb{R}^n)^q$ for every $x \in \mathbb{R}^n$. Thus, for any two integers p_{γ_k} and $p_{\gamma_{k'}}$ from the sequence $\{p_\gamma\}$, we have $A^j_{\tau^{[p_{\gamma_k}]}}\xi_x = A^j_{\tau^{[p_{\gamma_{k'}}]}}\xi_x$ for every $0 \leq j \leq q-1$. Since $\xi_x = \sum_{s=1}^{\kappa} \alpha_s v_s$, we obtain

$$\lambda_{\left(\tau^{[p_{\gamma_k}]},s\right)} = \lambda_{\left(\tau^{[p_{\gamma_{k'}}]},s\right)} \qquad \text{for every } \ s \in I(x) \cap \{1,\ldots,\kappa\}\,.$$

This implies that for every $s \in I(x) \cap \{1,\ldots,\kappa\}$, we have

$$\prod_{r=1}^{N} \lambda_{(r,s)}^{\Phi_{(\tau,r)}(p_{\gamma_k})} = \prod_{r=1}^{N} \lambda_{(r,s)}^{\Phi_{(\tau,r)}(p_{\gamma_{k'}})} \quad \Longleftrightarrow \quad \prod_{r=1}^{N} \lambda_{(r,s)}^{\Phi_{(\tau,r)}(p_{\gamma_k})-\Phi_{(\tau,r)}(p_{\gamma_{k'}})} = 1\,.$$

Since the numbers $\lambda_{(r,s)}$'s are q-th roots of unity, we obtain positive integers $\Lambda_{(r,s)}$ that satisfies

$$\sum_{r=1}^{N} \Lambda_{(r,s)} \left[\Phi_{(\tau,r)}(p_{\gamma_k}) - \Phi_{(\tau,r)}(p_{\gamma_{k'}})\right] \equiv 0 \pmod q$$

for all $s \in I(x) \cap \big\{1,\ldots,\kappa\big\}$. ∎

As pointed out after the statement of Theorem 2.3 in Section 2 and as one may observe from the proof above, $\left\{\widetilde{A_{\tau^{[p_\gamma]}}x}\right\}_{\gamma\geq 1}$ is constructed to be a constant sequence, thus proving Corollary 2.4.

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