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Topological Hausdorff dimension and Poincaré inequality

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Abstract: A relationship between Poincaré inequalities and the topological Hausdorff dimension is exposed—a lower bound on the dimension of Ahlfors regular spaces satisfying a weak $(1, p)$-Poincaré inequality is given.

Key words: Poincaré inequality, metric space, Cantor sets, topological dimension, Hausdorff dimension, bi-Lipschitz map, Ahlfors regular.

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1. Introduction

Let (X, d) be a separable metric space. The subscript of dim indicates the type of dimension, and we set $\dim \varnothing = -1$ for every dimension.

Poincaré inequalities are the forms of the Fundamental Theorem of Calculus that work in general metric spaces. Indeed, a one-dimensional Poincaré inequality is a direct consequence of the Fundamental Theorem of Calculus:

Remark 1.1. Let $f : [a, b] \to \mathbb{R}$ be differentiable. The Intermediate Value Theorem gives a point $c \in [a, b]$ with $f(c) = \fint_a^b f$, the average of f on $[a, b]$. The Fundamental Theorem of Calculus then yields

$$\fint_a^b \left| f(x) - \fint_a^b f \right| \, dx \leq (b - a) \fint_a^b |f'| ,$$

which is inequality (1.1) found below, with $p = \lambda = K = 1$.

There is an inherent connection between Poincaré inequalities and topological Hausdorff dimension because both concepts take connectivity into account. In order to discuss Poincaré inequalities, we include the following definition, which can be found in [4, p. 55].

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Definition 1.2. Given a real valued function u in a metric space X, a Borel function $\rho : X \to [0,\infty]$ is an *upper gradient* of u if

$$|u(x) - u(y)| \leq \int_\gamma \rho \, ds$$

for each rectifiable curve γ joining x and y in X.

To prove the main result, we will use the *upper pointwise dilation* as a suitable upper gradient (see [2, p. 342]).

Fact 1.3. If $f : X \to \mathbb{R}$ is a locally Lipschitz function, the *upper pointwise dilation*

$$\operatorname{Lip} f(x) = \limsup_{r\to 0} \sup_{y\in B(x,r)} \frac{|f(x) - f(y)|}{r}$$

is an upper gradient of f.

The following definition of a *weak Poincaré inequality* is from [4, p. 68], and a broader definition can be found in [2, p. 84].

Definition 1.4. Let (X,μ) be a metric measure space and let $1 \leq p < \infty$. Say that X admits a *weak* $(1,p)$*-Poincaré inequality* if there are constants $0 < \lambda \leq 1$ and $K \geq 1$ so that

$$\fint_{\lambda B} |u - u_{\lambda B}| \, d\mu \leq K(\operatorname{diam} B) \left(\fint_B \rho^p \, d\mu \right)^{1/p} \tag{1.1}$$

for all balls $B \subset X$, for all bounded continuous functions u on B, and for all upper gradients ρ of u, where $u_{\lambda B}$ is the average value of u on the set λB. Also assume $\mu(B(x,r)) > 0$ whenever $r > 0$.

It is not difficult to show that if a space supports a weak Poincaré inequality, then it is connected, and $\partial B(x,r) \neq \varnothing$ whenever $r < \frac{1}{2} \operatorname{diam} X$ [5, Proposition 8.1.6]. Such spaces are also quasiconvex, i.e., any two points can be connected by a curve of controlled length [5, Theorem 8.2.3]. Like the Hausdorff dimension, Poincaré inequalities are preserved by bi-Lipschitz maps, but the constants λ and K may change after application of a Lipschitz map. For a precise statement, see [2, Proposition 4.16].

Recently, results have surfaced that explain the relationship between Poincaré inequalities and some particular fractals. Mackay, Tyson, and Wildrick investigated the potential presence of Poincaré inequalities on various *carpets*—metric measure spaces that are homemorphic to the standard Sierpinskí carpet. In short, a carpet of this kind is constructed in the same manner as the Sierpinskí carpet, except at each step the scaling factor need not be $1/3$. Requiring that the sequence of scaling factors $\mathbf{a} = (a_1, a_2, \dots)$ contain only reciprocals of odd integers that decrease to zero, one obtains a carpet $(S_{\mathbf{a}}, |\cdot|, \mu)$ with Euclidean metric $|\cdot|$ and measure μ, where μ arises as the weak limit of normalized Lebesgue measure on the precarpets. For the construction, see [8]. They provided a complete characterization of these carpets in terms of $(1,p)$-Poincaré inequalities as follows.

Theorem 1.5. (Mackay, Tyson, Wildrick [8])

(i) *The carpet* $(S_{\mathbf{a}}, |\cdot|, \mu)$ *supports a* $(1,1)$*-Poincaré inequality if and only if* $\mathbf{a} \in \ell^1$.

(ii) *The following are equivalent:*

  (a) $(S_{\mathbf{a}}, |\cdot|, \mu)$ *supports a* $(1,p)$*-Poincaré inequality for each* $p > 1$.
  (b) $(S_{\mathbf{a}}, |\cdot|, \mu)$ *supports a* $(1,p)$*-Poincaré inequality for some* $p > 1$.
  (c) $\mathbf{a} \in \ell^2$.

To see how topological Hausdorff dimension is related to connectivity, one need only consider Theorem 3.6 in [1]. That theorem gives an equivalent definition of topological Hausdorff dimension for separable metric spaces:

$$\dim_{tH} X = \min \big\{ d \,:\, \exists\, A \subset X \text{ such that } \dim_H A \leq d-1 \text{ and } \dim_t(X \setminus A) \leq 0 \big\}.$$

A significant advantage of imposing a Poincaré inequality like (1.1) is the flexibility that exists in choosing the function u and one of its upper gradients ρ. To apply (1.1) to the topological Hausdorff dimension of a given space X, one can apply the inequality to the boundary of an arbitrary open set U of X to determine a lower bound on $\dim_H \partial U$. If a non-trivial lower bound on $\dim_H \partial U$ is achieved, then so is a lower bound on $\dim_{tH} X$. In the next section we apply this technique and exploit the Poincaré inequality to accomplish exactly that goal.

A closely related concept was recently investigated by Lotfi in [7], which generalized the topological Hausdorff dimension by combining the definitions of topological dimension and μ-Hausdorff dimension. They presented upper and lower bounds for the so-called μ-topological Hausdorff dimension of the Sierpinskí carpet, and gave a large class of measures μ, where the associated μ-topological Hausdorff dimension of the Sierpinskí carpet coincides with these lower and upper bounds.

The main result requires that a space X satisfies a weak $(1,p)$-Poincaré inequality, and that it is Ahlfors regular. The following definition can be found in [4, p. 62].

Definition 1.6. If X is a metric space admitting a Borel regular measure μ such that

$$C^{-1}R^b \leq \mu(B_R) \leq CR^b$$

for some constant $C \geq 1$, for some exponent $b > 0$, and for all closed balls B_R of radius $0 < R < \operatorname{diam} X$, then X is called *Ahlfors b-regular*.

An Ahlfors b-regular space has Hausdorff dimension b [4, p. 62], and is *doubling*:

Definition 1.7. A metric measure space (X, d, μ) is *doubling* if there is $C > 0$ such that $0 < \mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all $x \in X$ and for all $r > 0$.

There is much interplay between Ahlfors regularity and weak$(1,p)$-Poincaré inequalities in metric spaces. For example, in [6], Lohvansuu and Rajala recently studied the duality of moduli in this context, where the Ahlfors regularity constant is assumed to be greater than one. They proved that there is something of a dual relationship, with exponents p and $p^* = \frac{p}{p-1}$, between the path modulus and the modulus of separating surfaces.

It can be challenging to obtain nontrivial lower bounds on the topological Hausdorff dimension. In the presence of Ahlfors regularity, however, this problem becomes more tractable. We now state the main result, which provides a lower bound in terms of the regularity and Poincaré constants.

Theorem. *Let (X, μ, d) be a complete, Ahlfors b-regular, $(1,p)$-Poincaré metric measure space. Then $\dim_{tH} X \geq b - p + 1$.*

Due to Ahlfors regularity, equality is achieved if $p = 1$ because $\dim_{tH} X \leq \dim_H X = b$. On the other hand, it is not clear whether a space exists that yields equality for any $p > 1$.

2. Preliminaries

The symbol $B(x, \varepsilon)$ denotes the open ball centered at x of radius ε. For $x \in \mathbb{R}^n$, the Euclidean modulus of x is denoted $|x|$. Unless otherwise stated, distance in the metric space Y is denoted d_Y or simply d. We use the notation

$$f_E = \fint_E f \, d\mu = \frac{1}{\mu(E)} \int f \, d\mu$$

for the average value of an integrable function f on $E \subset X$, where (X, d, μ) is a metric measure space. For any $A \subset (X, d)$, the set A_δ is the δ-neighborhood of A in X. The symbol χ_U represents the characteristic function of any $U \subset X$.

In order to define topological Hausdorff dimension, we include the definition of Hausdorff dimension:

DEFINITION 2.1. The *p-dimensional Hausdorff measure* of X is

$$\mathcal{H}^p(X) = \lim_{\delta \to 0} \inf \left\{ \sum_{j=1}^{\infty} (\operatorname{diam} E_j)^p \,:\, X \subset \bigcup_{j=1}^{\infty} E_j \text{ and } \operatorname{diam} E_j \leq \delta \text{ for all } j \right\};$$

the *Hausdorff dimension* of X is $\dim_H X = \inf\{p : \mathcal{H}^p(X) = 0\}$.

An interesting combination of the Hausdorff and topological dimensions, called *topological Hausdorff dimension*, was introduced in [1]:

$$\dim_{tH} X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \leq d - 1 \ \forall U \in \mathcal{U}\}.$$

By Theorem 4.4 in [1], the topological Hausdorff dimension always falls between the topological dimension ($\dim_t X$) and the Hausdorff dimension ($\dim_H X$):

THEOREM 2.2. (BALKA, BUCZOLICH, ELEKES[1]) *For any metric space* X,

$$\dim_t X \leq \dim_{tH} X \leq \dim_H X. \tag{2.1}$$

In certain favorable circumstances, the Hausdorff and topological Hausdorff dimensions are additive under products. For any product space $X \times Y$, we use the metric

$$d((x_1, y_1), (x_2, y_2)) = \max(d_X(x_1, x_2), d_Y(y_1, y_2)).$$

For sake of completeness, we include Theorem 4.21 from [1] and several product formulas for Hausdorff dimension (see e.g. [3, Chapter 7]).

FACT 2.3. If $E \subset \mathbb{R}^n$, $F \subset \mathbb{R}^m$ are Borel sets, then

$$\dim_H (E \times F) \geq \dim_H E + \dim_H F.$$

Let $\overline{\dim}_H X$ be the upper box-counting dimension of X (see e.g. [3]).

FACT 2.4. For any sets $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$

$$\dim_H(E \times F) \leq \dim_H E + \overline{\dim}_B F.$$

We call a Cantor set in $[0,1]$ *uniform* if it is constructed in the same way as the usual middle-thirds example, allowing for any scaling factor $0 < r < 1/2$. Since uniform Cantor sets have equal Hausdorff and upper box dimensions, Facts 2.3 and 2.4 yield the following formula.

FACT 2.5. If $F \subset \mathbb{R}$ is a uniform Cantor set, then for any $E \subset \mathbb{R}^n$

$$\dim_H(E \times F) = \dim_H E + \dim_H F. \tag{2.2}$$

In light of Facts 2.3 and 2.4, we observe the following convenient additivity property.

FACT 2.6. If $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are Borel sets with $\dim_H X = \overline{\dim}_B X$,

$$\dim_H(X \times Y) = \dim_H X + \dim_H Y. \tag{2.3}$$

The condition $\dim_H X = \overline{\dim}_B X$ holds for a wide variety of spaces.

THEOREM 2.7. *If X is a nonempty separable metric space, then*

$$\dim_{tH}(X \times [0,1]) = \dim_H(X \times [0,1]) = \dim_H X + 1\,. \tag{2.4}$$

In particular, for any value $c > 2$, $R = X \times [0,1]$ can be chosen such that $\dim_{tH} R = c$.

The first equality in (2.4) is due to Balka, Buczolich, and Elekes [1]. Because $\dim_H[0,1] = \overline{\dim}_B[0,1] = 1$, the second equality in (2.7) is readily obtained considering Fact 2.6.

Recall that the Hausdorff dimension is invariant under *bi-Lipschitz maps.*

DEFINITION 2.8. An embedding f is *L-bi-Lipschitz* if both f and f^{-1} are L-Lipschitz, and we say f is *bi-Lipschitz* if it is L-bi-Lipschitz for some L.

3. A lower bound on topological Hausdorff dimension for Poincaré Ahlfors regular spaces

To provide a nontrivial lower bound on $\dim_{tH} X$, it suffices to consider an arbitrary bounded basis element U for the topology on X, and show that $\dim_H \partial U \geq b - p$, where b and p are the regularity and Poincaré constants of X, respectively.

Theorem 3.1. *Let (X, μ, d) be a complete, Ahlfors b-regular, $(1, p)$-Poincaré metric measure space. Then* $\dim_{tH} X \geq b - p + 1$.

Proof. Let $\mathcal{U}$ be basis for the topology on X, and consider a bounded element $U \in \mathcal{U}$, $U \neq X$. Choose $\delta > 0$ small enough that $\delta < \frac{1}{2}\operatorname{diam}(U)$, and both $U \setminus \overline{(\partial U)_\delta}$ and $\overline{U_\delta}^c$ are nonempty. Let $0 < \lambda \leq 1$ and $K \geq 1$ be as in Definition 1.4, and choose $z_0 \in U \setminus \overline{(\partial U)_\delta}$. Choose $R > 0$ large enough that $B(z_0, R) \supset \overline{U_\delta}$ and $B(z_0, R) \setminus \overline{U_\delta} \neq \varnothing$, and put $B = B(z_0, R/\lambda)$. Then R is large enough that $\overline{U_\delta} \subset \lambda B = B(z_0, R)$.

Fix an arbitrary finite covering $\mathcal{D}$ of ∂U by open balls as follows:

$$\mathcal{D} = \{D_i = B(x_i, 2r_i) : x_i \in \partial U\}, \qquad 2r_i \leq \delta \text{ for all } i\,. \tag{3.1}$$

We will show that there is a constant $C > 0$ such that $\sum_i (\operatorname{diam} D_i)^{b-p} \geq C$. Note that X is doubling because it is Ahlfors regular, and X is proper because it is complete and doubling [5, Lemma 4.1.14]. Therefore ∂U is compact because it is closed and bounded. Given a finite covering $\mathcal{D}$ of ∂U satisfying (3.1), define the functions

$$u_i(x) = \min\left\{\frac{d(x, D_i^c)}{r_i}, 1\right\} \text{ and } u = \max\left(\max_i u_i, \chi_U\right).$$

Notice that u_i is $\frac{1}{r_i}$-Lipschitz, u is bounded, and u is continuous because $\mathcal{D}$ is a finite covering.

Considering that $0 \leq u \leq 1$, we have $0 \leq u_{\lambda B} \leq 1$, and hence

$$\begin{aligned}
\fint_{\lambda B} |u - u_{\lambda B}|\, d\mu &\geq \frac{1}{\mu(\lambda B)}\left(\int_{\{x \in \lambda B : u(x) = 1\}} |u - u_{\lambda B}|\, d\mu\right) \\
&\quad + \frac{1}{\mu(\lambda B)}\left(\int_{\{x \in \lambda B : u(x) = 0\}} |u - u_{\lambda B}|\, d\mu\right) \\
&= \frac{1}{\mu(\lambda B)}\big[(1 - u_{\lambda B})\mu\big(\{u(x) = 1\}\big) + u_{\lambda B}\mu\big(\{u(x) = 0\}\big)\big]
\end{aligned}$$

$$\geq \frac{1}{\mu(\lambda B)} \min\big\{\mu\big(\{u(x)=1\}\big), \mu\big(\{u(x)=0\}\big)\big\}$$
$$\geq \frac{1}{\mu(\lambda B)} \min\big\{\mu(\lambda B \cap U), \mu\big(\lambda B \cap (U_\delta)^c\big)\big\} \tag{3.2}$$
$$\geq \frac{1}{\mu(\lambda B)} \min\big\{\mu(U), \mu\big(\lambda B \setminus \overline{U_\delta}\big)\big\}.$$

The fact that X is b-regular provides a constant $M \geq 1$ with $M^{-1}r^b \leq \mu(B_r) \leq Mr^b$ for any ball of radius r. In particular $\mu(\lambda B) \leq MR^b$, and $\mu(U) > 0$ because U is open and non-empty. Also, recall that δ and R were chosen so that $\lambda B \setminus \overline{U_\delta} = B(z_0, R) \setminus \overline{U_\delta}$ is open and nonempty. So there is a point z_1 and an integer $N > 0$ such that

$$B(z_1, 1/N) \subset \lambda B \setminus \overline{U_\delta}\,.$$

Applying regularity gives

$$\mu(\lambda B \setminus \overline{U_\delta}) \geq \mu(B(z_1, 1/N)) \geq \frac{1}{MN^b}\,. \tag{3.3}$$

In light of (3.2) and (3.3), we see that

$$\begin{aligned} \fint_{\lambda B} |u - u_{\lambda B}|\, d\mu &\geq \frac{1}{\mu(\lambda B)} \min\{\mu(U), \mu\left(\lambda B \setminus \overline{U_\delta}\right)\} \\ &\geq \frac{1}{MR^b} \min\left\{\mu(U), \frac{1}{MN^b}\right\} = C', \end{aligned} \tag{3.4}$$

where the constant $C' > 0$ is independent of the covering $\mathcal{D}$.

Next, we show that $\fint_{\lambda B} |u - u_{\lambda B}| d\mu \leq C'' \sum_i r_i^{b-p}$ for some $C'' > 0$. To this end, recall that the upper pointwise dilation of any locally Lipschitz function f is denoted $\operatorname{Lip} f$, and note that

$$\begin{aligned} \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x,y)} &= \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(y) - f(x)|}{d(y,x)} \\ &\geq \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(y) - f(x)|}{r} \\ &= \operatorname{Lip} f(x)\,. \end{aligned} \tag{3.5}$$

The fact that u_i is $\frac{1}{r_i}$-Lipschitz, along with equation (3.5), show $\operatorname{Lip}\ u_i(x) \le \frac{1}{r_i}$ for all x. Also $\operatorname{Lip}\ u \le \max_i \operatorname{Lip}\ u_i$, and $\operatorname{Lip} u_i(x) = 0$ for $x \notin D_i$. Ahlfors regularity implies $\mu(D_i) \le M(2r_i)^b$ for all i, and therefore

$$\begin{aligned}\int_B |\operatorname{Lip}\ u|^p\ d\mu &= \int_B (\operatorname{Lip}\ u)^p\ d\mu \le \int_B \left[\max_i(\operatorname{Lip}\ u_i)\right]^p\ d\mu \\ &\le \int_B \sum_i (\operatorname{Lip}\ u_i)^p\ d\mu \le \sum_i \int_X (\operatorname{Lip}\ u_i)^p\ d\mu \\ &\le \sum_i \mu(D_i) r_i^{-p} \le 2^b M \sum_i r_i^{b-p}.\end{aligned} \tag{3.6}$$

Finally, with the Poincaré inequality (1.1), (3.4), and (3.6), the regularity lower bound $\mu(B) \ge M^{-1}\,({}^R\!/\!_\lambda)^b$ gives

$$\begin{aligned}C' \le \fint_{\lambda B} |u - u_{\lambda B}|\ d\mu &\le K(\operatorname{diam} B)\left(\fint_B |\operatorname{Lip}\ u|^p\ d\mu\right)^{1/p} \\ &\le \frac{K\,(2R/\lambda)}{\mu(B)^{1/p}}\left(\int_B |\operatorname{Lip}\ u|^p\ d\mu\right)^{1/p} \\ &\le \frac{K\,(2R/\lambda)}{M^{-1/p}\,(R/\lambda)^{b/p}}\left(2^b M \sum_i r_i^{b-p}\right)^{1/p} \\ &\le \frac{K\,(2R/\lambda)}{M^{-1/p}\,(R/\lambda)^{b/p}}(2^b M)^{1/p}\left(\sum_i r_i^{b-p}\right)^{1/p} \\ &= C''\left(\sum_i r_i^{b-p}\right)^{1/p}.\end{aligned} \tag{3.7}$$

Therefore $0 < C \le \sum_i r_i^{b-p}$, where $C = (C'/C'')^p$ is independent of the covering $\mathcal{D}$.

Suppose $\mu(X) < \infty$. We will show that for any $D_i \in \mathcal{D}$, the radius r_i is bounded above by a constant multiple of $\operatorname{diam} D_i$, where the constant depends only on X. To this end, consider the ball $s_i D_i$, where $s_i = (\operatorname{diam} D_i)^{-1}$. Then $s_i D_i$ has radius $\frac{r_i}{\operatorname{diam} D_i}$, and Ahlfors regularity provides

$$\begin{aligned}\frac{1}{M}\left(\frac{r_i}{\operatorname{diam} D_i}\right)^b &\le \mu(s_i D_i) \le \mu(X) < \infty\,, \\ r_i &\le M^{1/b}\mu(X)^{1/b} \operatorname{diam} D_i\,.\end{aligned} \tag{3.8}$$

In light of (3.8) it is evident that

$$0 < C \leq \sum_i r_i^{b-p} \leq \sum_i \left(M^{1/b} \mu(X)^{1/b} \right)^{b-p} (\operatorname{diam} D_i)^{b-p} ,$$

and hence $0 < \sum_i (\operatorname{diam} D_i)^{b-p}$. Therefore $\dim_H \partial U \geq b - p$ for any such U, from which it follows that $\dim_{tH} X \geq b - p + 1$.

If $\mu(X) = \infty$, put $E = \overline{B(z_0, a)}$, $0 < a < \operatorname{diam} X$, and notice that E is complete and inherits both the Ahlfors b-regularity and $(1, p)$-Poincaré properties from X (with the same constants M, b, p, and λ). By Ahlfors regularity $\mu(E) \leq Ma^b < \infty$, so E satisfies the assumptions of the theorem in the case that has already been proven. Finally, monotonicity of tH-dimension shows that

$$\dim_{tH} X \geq \dim_{tH} E \geq b - p + 1 .$$

∎

If $p = 1$, then equality holds in Theorem 3.1 because (2.1) guarantees that $\dim_{tH} X \leq \dim_H X = b$, but whether equality can be achieved for some $(1, p)$-Poincaré space (X, μ) with $p > 1$ is a mystery.

Question 3.2. Is there a number $p > 1$ with a space (X, μ) for which equality holds in Theorem 3.1?

In order to answer Question 3.2, one needs a supply of spaces that support weak $(1, p)$-Poincaré inequalities for $p > 1$. Theorem 1.5 provides one source of potential examples.

It is tempting to try to answer Question 3.2 with a carpet $S_{\mathbf{a}} = (S_{\mathbf{a}}, |\cdot|, \mu)$ that supports a weak $(1, p)$-Poincaré inequality with $p > 1$. A problem arises, however, once one computes the tH-dimension of this space. Indeed, since $S_{\mathbf{a}}$ is Ahlfors 2-regular [8], $\dim_H S_{\mathbf{a}} = 2$, and in order to have equality in Theorem 3.1, we would need $\dim_{tH} S_{\mathbf{a}} = 3 - p$. Let $C_{\mathbf{a}}$ be the Cantor set in $[0, 1]$ obtained from the sequence of scaling factors $\mathbf{a}$. Since $(C_{\mathbf{a}} \times [0, 1]) \subset S_{\mathbf{a}}$ we see that $\dim_{tH} S_{\mathbf{a}} \geq \dim_{tH}(C_{\mathbf{a}} \times [0, 1]) = 2$ by monotonicity and additivity of tH-dimension. Therefore $\dim_{tH} S_{\mathbf{a}} = 2$, and the equation $\dim_{tH} S_{\mathbf{a}} = 3 - p$ is untenable because we assumed $p > 1$.

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