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Extensions, crossed modules and pseudo quadratic Lie type superalgebras

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Abstract: Extensions and crossed modules of Lie type superalgebras are introduced and studied. We construct homology and cohomology theories of Lie-type superalgebras. The notion of left super-invariance for a bilinear form is defined and we consider Lie type superalgebras endowed with nondegenerate, supersymmetric and left super-invariant bilinear form. Such Lie type superalgebras are called pseudo quadratic Lie type superalgebras. We show that any pseudo quadratic Lie type superalgebra induces a Jacobi-Jordan superalgebra. By using the method of double extension, we study pseudo quadratic Lie type superalgebras and theirs associated Jacobi-Jordan superalgebras.

Key words: Lie type superalgebras, Jacobi-Jordan superalgebras, extension, crossed module, homology, cohomology, double extension, pseudo quadratic Lie type superalgebras.

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Introduction

Recently, in order to investigate commutative non-associative algebras, authors in [5] introduce the so-called Jacobi-Jordan algebras that are commutative algebras satisfying the Jacobi identity. Those algebras were first defined in [12] and since then they have been studied in various papers [3, 4, 6, 8] under different name such as Jordan algebras of nil rank 3, Mock-Lie algebras, Lie-Jordan algebras or pathological algebras. It turns out that the commutativity and Jacobi identity satisfied by the product of an algebra $(A, *)$ induce two relations $x*(y*z) = -(x*y)*z - y*(x*z)$ and $x*(y*z) = -(x*y)*z - (x*z)*y$ for all $x, y, z \in A$, called respectivelly left Lie-type identity and right Lie-type identity.

This motivated us to introduce and study in [11] a new type of nonassociative (super)-algebra called left or right Lie-type superalgebra. A left (resp. right) Lie type superalgebra consists of a $\mathbb{Z}_2$-graded vector space $\mathcal{U} := \mathcal{U}_{\bar{0}} \oplus \mathcal{U}_{\bar{1}}$ endowed with an even bilinear map $[\,,\,] : \mathcal{U} \otimes \mathcal{U} \to \mathcal{U}$ such that $[\mathcal{U}_\alpha, \mathcal{U}_\beta] \subseteq \mathcal{U}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}_2$ and $[x, [y, z]] = -[[x, y], z] - (-1)^{|x||y|}[y, [x, z]]$ (resp.

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$[x, [y, z]] = -[[x, y], z] - (-1)^{|y||z|}[[x, z], y])$ for all $x, y, z \in \mathcal{U}$. It is called symmetric Lie type superalgebra if it is simultaneously a left and right Lie type superalgebra. Lie type superalgebras can be seen as generalization of Jacobi-Jordan (super)-algebras introduced in [5] which are sub class of the class of Jordan algebras that plays an important role in physics (see [10]). In fact, unlike Jacobi-Jordan (super)-algebras, Lie-type superalgebras are not necessary (super)-commutative. For more details about Jacobi-Jordan algebras (see [5, 9, 3]).

In this paper, we introduce and study extension and crossed module of Lie type superalgebras. We give a characterization of extension in terms of two bilinear applications and characterize the notion of isomorphism between two extensions in terms of linear applications satisfying some properties. The notion of trivial extension is defined and studied. By following [7] where the authors studied crossed modules of Leibniz algebras, we define crossed module for Lie type superalgebras that we call Lie type crossed module. The notion of normalized, linked and bilateral Lie type crossed module are defined and we also characterize the equivalence of two Lie type crossed modules. A homology and cohomology theory of Lie-type superalgebras is introduced and the first degree cohomology group is given in term of equivalent class of the so-called restricted trivial extensions.

In [11], we studied quadratic Lie-type superalgebras that are Lie-type superalgebras $(\mathcal{U}, [\,,\,])$ endowed with a nondegenerate, supersymmetic and invariant bilinear form B. We notice that the invariant or associative property of B that is $B([x, y], z) = B(x, [y, z])$ for all $x, y, z \in \mathcal{U}$, plays an important role in the study of quadratic structure of Lie-type superalgebras. But the fact that the bracket of Lie-type superalgebras is not necessary supercommutative allows us to define a new type of invariant of B by $B([x, y], z) = (-1)^{|x||y|}B(y, [x, z])$ called left super-invariance that is different from the associative property.

Another purpose of this paper is the study of the so-called pseudo quadratic Lie type superalgebras. A Lie type superalgebra is said pseudo quadratic if it is endowed with a nondegenerate, symmetric and left super-invariant bilinear form. We show that any pseudo quadratic Lie type superalgebra $(\mathcal{U}, [\,,\,])$ induces a Jacobi-Jordan superalgebra $(\mathcal{U}, \wedge)$. By using the double extension extented to Lie type superalgebras, we study pseudo quadratic Lie type superalgebra $(\mathcal{U}, [\,,\,])$ and the associated Jacobi-Jordan superalgebra $(\mathcal{U}, \wedge)$.

This paper is organized as follows. The first section is devoted to the definitions and elementary results. In Section 2, we study homology and cohomology of Lie type superalgebras. In Section 3, we define extension and crossed module of Lie type superalgebras and characterize these notions in terms of linear and bilinear applications. We give a characterization of the notion of isomorphism between two Lie type crossed modules and the notion of equivalence between extensions of Lie type superalgebra. The notion of normalized and bilateral Lie type crossed module are defined and studied. In Section 4, we define left super-invariance for a bilinear form and by using the notion of double extension, we study pseudo quadratic Lie type superalgebras and the induced Jacobi-Jordan superalgebras.

Throughout this paper, all vector spaces and algebras considered are defined over an algebraically closed field $\mathbb{K}$ of characteristic zero.

NOTATIONS: In this paper we shall keep the same notation as in [2].

1. Preliminaries

In this section we give basic definitions and elementary results about Lie-type superalgebras and Jacobi-Jordan superalgebras.

DEFINITION 1.1. Let $\mathcal{U} := \mathcal{U}_{\bar{0}} \oplus \mathcal{U}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space endowed with a bilinear map $[\,,\,] : \mathcal{U} \otimes \mathcal{U} \to \mathcal{U}$ such that $[\mathcal{U}_\alpha, \mathcal{U}_\beta] \subseteq \mathcal{U}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}_2$. Then $(\mathcal{U}, [\,,\,])$ is called left Lie type superalgebra if

$$[x,[y,z]] = -[[x,y],z] - (-1)^{|x||y|}[y,[x,z]] \quad \forall\, x \in \mathcal{U}_{|x|},\, y \in \mathcal{U}_{|y|},\, z \in \mathcal{U}, \tag{1.1}$$

and $(\mathcal{U}, [\,,\,])$ is called right Lie type superalgebra if

$$[x,[y,z]] = -[[x,y],z] - (-1)^{|y||z|}[[x,z],y] \quad \forall\, x \in \mathcal{U}_{|x|},\, y \in \mathcal{U}_{|y|},\, z \in \mathcal{U}_{|z|}. \tag{1.2}$$

The superalgebra $(\mathcal{U}, [\,,\,])$ is called symmetric Lie type superalgebra if it is simultaneously a left and right Lie type superalgebra.

REMARK 1.1. Let $(\mathcal{U}, [\,,\,])$ be a left Lie type superalgebra. Define the bilinear map $\{\,,\,\} : \mathcal{U} \otimes \mathcal{U} \to \mathcal{U}$ by $\{x,y\} = (-1)^{|x||y|}[y,x]$, then $(\mathcal{U}, \{\,,\,\})$ is a right Lie type superalgebra. Therefore the category of left Lie-type superalgebras is isomorphic to the category of right Lie-type superalgebras.

Let $(A, \cdot)$ be a superalgebra. We define the anti-associator of A by the trilinear application $Aasso : A \otimes A \otimes A \to A$ by $Aasso(x,y,z) := x \cdot (y \cdot z) + (x \cdot y) \cdot z$ for all $x, y, z \in A$. The superalgebra $(A, \cdot)$ is said to be anti-associative if $Aasso(x,y,z) = 0$ for all $x, y, z \in A$.

Example 1.1. Let $(\mathcal{U}, \cdot)$ be an anti-associative superalgebra. If we define the bilinear map $[\,,\,] : \mathcal{U} \otimes \mathcal{U} \to \mathcal{U}$ by $[x, y] = x \cdot y + (-1)^{|x||y|} y \cdot x$ for $x \in \mathcal{U}_{|x|}$ and $y \in \mathcal{U}_{|y|}$, then $(\mathcal{U}, [\,,\,])$ is a right Lie type superalgebra.

A homomorphism $f : \mathcal{U} \to \mathcal{W}$ between two $\mathbb{Z}_2$-graded vector spaces is said to be homogeneous of degree $\alpha \in \mathbb{Z}_2$ if $f(\mathcal{U}_\beta) \subseteq \mathcal{W}_{\alpha+\beta}$ for all $\beta \in \mathbb{Z}_2$. Given three $\mathbb{Z}_2$-graded vector spaces $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{H}$, a bilinear map $g : \mathcal{U} \otimes \mathcal{W} \to \mathcal{H}$ is said to be homogeneous of degree $\alpha \in \mathbb{Z}_2$ if $g(\mathcal{U}_\beta, \mathcal{W}_\gamma) \subseteq \mathcal{H}_{\alpha+\beta+\gamma}$ for all $\beta, \gamma \in \mathbb{Z}_2$. The degree of a homogeneous linear or bilinear map f is denoted by $| f |$ and f is said to be an even (resp. odd) map if $| f |= \bar{0}$ (resp. $| f |= \bar{1}$). For any left Lie-type superalgebra $(\mathcal{U}, [\,,\,])$, the left and the right multiplication L and R defined by $L_x(y) := [x, y]$ and $R_x(y) := (-1)^{|x||y|}[y, x]$ satisfy the following relations:

Lemma 1.1. (i) *$L_{[x,y]} = -L_x \circ L_y - (-1)^{|x||y|} L_y \circ L_x$ for all $x \in \mathcal{U}_{|x|}, y \in \mathcal{U}_{|y|}$;*

(ii) *$R_{[x,y]} = -L_x \circ R_y - (-1)^{|x||y|} R_y \circ R_x$ for all $x \in \mathcal{U}_{|x|}, y \in \mathcal{U}_{|y|}$;*

(iii) *$R_{[x,y]} = -L_x \circ R_y - (-1)^{|x||y|} R_y \circ L_x$ for all $x \in \mathcal{U}_{|x|}, y \in \mathcal{U}_{|y|}$;*

(iv) *$R_y \circ R_x = R_y \circ L_x$ for all $x \in \mathcal{U}_{|x|}, y \in \mathcal{U}_{|y|}$.*

Proof. Straightforward computation. ∎

The left centre $Z^l(\mathcal{U})$ and the right centre $Z^r(\mathcal{U})$ are defined by $Z^l(\mathcal{U}) = \{x \in \mathcal{U}, [x, \mathcal{U}] = 0\}$ and $Z^r(\mathcal{U}) = \{x \in \mathcal{U}, [\mathcal{U}, x] = 0\}$. Define by $\mathrm{Ker}(\mathcal{U})$ the subspace generated by elements of the form $[x, y] - (-1)^{|x||y|}[y, x]$ where $x \in \mathcal{U}_{|x|}$ and $y \in \mathcal{U}_{|y|}$. For any left Lie-type superalgebra $(\mathcal{U}, [\,,\,])$, it holds

Lemma 1.2. (i) *$\mathrm{Ker}(\mathcal{U}) \subseteq Z^l(\mathcal{U})$;*

(ii) *$Z^l(\mathcal{U})$ is a two sided ideal and $Z^r(\mathcal{U})$ is a sub-superalgebra.*

Proof. See [11, Lemma 2.6]. ∎

The fact that $\mathrm{Ker}(\mathcal{U}) \subseteq Z^l(\mathcal{U})$ implies that $[[x, y], z] = (-1)^{|x||y|}[[y, x], z]$ for $x, y, z \in \mathcal{U}$. One can sees that an analogous result of Lemma 1.2 holds for right Lie-type superalgebras. In fact, if $(\mathcal{U}, [\,,\,])$ is a right Lie-type superalgebra then $\mathrm{Ker}(\mathcal{U}) \subseteq Z^r(\mathcal{U})$. Therefore

$$[x, [y, z]] = (-1)^{|y||z|}[x, [z, y]] \qquad \forall\, x \in \mathcal{U}_{|x|},\ y \in \mathcal{U}_{|y|},\ z \in \mathcal{U}_{|z|}. \tag{1.3}$$

DEFINITION 1.2. Let $\mathcal{U}$ be a left Lie-type superalgebra and V a $\mathbb{Z}_2$-graded vector space. A representation of $\mathcal{U}$ over V is a couple of even linear maps (φ, λ) where $\varphi, \lambda : \mathcal{U} \to \mathrm{End}(V)$ such that

$$\begin{aligned}
\varphi_{[x,y]} &= -\varphi_x \cdot \phi_y - (-1)^{|x||y|}\varphi_y \cdot \phi_x\,,\\
\lambda_{[x,y]} &= -\varphi_x \cdot \lambda_y - (-1)^{|x||y|}\lambda_y \cdot \lambda_x\,,\\
\lambda_{[x,y]} &= -\varphi_x \cdot \lambda_y - (-1)^{|x||y|}\lambda_y \cdot \varphi_x\,,
\end{aligned}$$

for all homogeneous elements $x, y \in \mathcal{U}$. If $\varphi = \lambda = 0$, the representation is called trivial representation. We denote $\mathrm{Rep}_V^{\mathcal{U}}$ the set of all representations of $\mathcal{U}$ over a given $\mathbb{Z}_2$-graded vector space V.

EXAMPLE 1.2. Let $\mathcal{U}$ be a left Lie-type superalgebra. Then according to Lemma 1.1, $(L, R) \in \mathrm{Rep}_{\mathcal{U}}^{\mathcal{U}}$ and is called the adjoint representation or the regular representation of $\mathcal{U}$.

Let $(\mathcal{U}, [\ ,\ ])$ be a left (resp. right) Lie type superalgebra, $\mathcal{V} := \mathcal{V}_{\bar{0}} \oplus \mathcal{V}_{\bar{1}}$ a $\mathbb{Z}_2$-graded vector space and (φ, λ) a representation of $\mathcal{U}$ in $\mathcal{V}$. Then the even bilinear application $\psi : \mathcal{U} \otimes \mathcal{U} \to \mathcal{V}$ is said to be an even bi-cocycle of left (resp. right) Lie type superalgebra with respect to (φ, λ) if for all $x, y, z \in \mathcal{U}$ we have

$$\begin{aligned}
&\psi(x,[y,z]) + \psi([x,y],z) + (-1)^{|x||y|}\psi(y,[x,z]) + \varphi_x(\psi(y,z))\\
&\qquad + (-1)^{|x||y|}\varphi_y(\psi(x,z)) + (-1)^{|z|(|x|+|y|)}\lambda_z(\psi(x,y)) = 0
\end{aligned}$$

(resp.

$$\begin{aligned}
&\psi(x,[y,z]) + \psi([x,y],z) + (-1)^{|y||z|}\psi([x,z],y) + \varphi_x(\psi(y,z))\\
&\qquad + (-1)^{|z|(|x|+|y|)}\lambda_z(\psi(x,y)) + (-1)^{|x||y|}\lambda_y(\psi(x,z)) = 0\,).
\end{aligned}$$

Let $(\mathcal{U}, [\ ,\ ])$ be a left(resp. right) Lie type superalgebra, $\mathcal{V}$ a $\mathbb{Z}_2$-graded vector space and $\psi : \mathcal{U} \otimes \mathcal{U} \to \mathcal{V}$ an even bilinear map. Then, the $\mathbb{Z}_2$-graded space $\overline{\mathcal{U}} := \mathcal{U} \oplus \mathcal{V}$ endowed with the product

$$[x+u, y+v]_\psi = [x,y] + \psi(x,y) \qquad \forall\, x, y \in \mathcal{U},\ u, v \in \mathcal{V}$$

is a left (resp. right) Lie type superalgebra if and only if

$$\psi(x,[y,z]) + \psi([x,y],z) + (-1)^{|x||y|}\psi(y,[x,z]) = 0$$

(resp.
$$\psi(x,[y,z])+\psi([x,y],z)+(-1)^{|y||z|}\psi([x,z],y)=0)\,.$$

Moreover, $(\overline{U},[\,,\,]_\psi)$ is a symmetric Lie type superalgebra if and only if $(\mathcal{U},[\,,\,])$ is symmetric and ψ is an even bi-cocycle of $\mathcal{U}$ with respect the trivial representation such that

$$\psi(x,[y,z])=(-1)^{|x|(|y|+|z|)}\psi([y,z],x)\qquad \forall\, x,y,z\in\mathcal{U}\,.$$

In this case ψ is called an even Lie-type bi-cocycle of $\mathcal{U}$ on the trivial $\mathcal{U}$-module $\mathcal{V}$. We denote by $(Z_{Ltype}(\mathcal{U},\mathcal{V}))_{\bar 0}$ the set of even Lie-type bi-cocycles of $\mathcal{U}$ on the trivial $\mathcal{U}$-module $\mathcal{V}$.

Lemma 1.3. *Let $\mathcal{U}$ be a left Lie-type superalgebra. Then $\mathcal{U}$ is a right Lie-type superalgebra if and only if*

$$[x,[y,z]]=(-1)^{|x|(|y|+|z|)}[[y,z],x]\qquad \forall\, x,y,z\in\mathcal{U}\,. \tag{1.4}$$

Proof. See the proof of [11, Lemma 3.1]. ∎

According to the above lemma, a Lie-type superalgebra is symmetric if and only if relation (1.4) holds.

Definition 1.3. A Jacobi-Jordan superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathcal{J}:=\mathcal{J}_{\bar 0}\oplus\mathcal{J}_{\bar 1}$ endowed with an even bilinear map $[\,,\,]:\mathcal{J}\otimes\mathcal{J}\to\mathcal{J}$ such that $[\mathcal{J}_\alpha,\mathcal{J}_\beta]\subseteq\mathcal{J}_{\alpha+\beta}$ for all $\alpha,\beta\in\mathbb{Z}_2$ and

1. $[x,y]=(-1)^{|x||y|}[y,x]$ for all $x\in\mathcal{J}_{|x|}$, $y\in\mathcal{J}_{|y|}$;
2. $(-1)^{|x||z|}[x,[y,z]]+(-1)^{|x||y|}[y,[z,x]]+(-1)^{|y||z|}[z,[x,y]]=0$ for all $x\in\mathcal{J}_{|x|}$, $y\in\mathcal{J}_{|y|}$, $z\in\mathcal{J}_{|z|}$.

Example 1.3. ([1]) The $(2n+1)$-dimensional Heisenberg Jacobi-Jordan superalgebra $\mathfrak{h}(2n+1,\mathbb{K})=(\mathfrak{h}_{\bar 0}\oplus\mathfrak{h}_{\bar 1},\cdot)$ where $\mathfrak{h}_{\bar 0}\oplus\mathfrak{h}_{\bar 1}=\{e_1,\ldots,e_n\}\oplus\{f_1,\ldots,f_n,z\}$ and

$$e_i\cdot f_i=f_i\cdot e_i:=z\qquad \forall\, i=1,\ldots,n\,.$$

Every Jacobi-Jordan superalgebra is a Lie-type superalgebra. A Lie-type superalgebra $(\mathcal{U},[\,,\,])$ is a Jacobi-Jordan superalgebra if and only if $\mathrm{Ker}(\mathcal{U})=\{0\}$.

2. Homology and cohomology of Lie-type superalgebras

In this section we study homology and cohomology of right Lie-type superalgebras.

Definition 2.1. A $\mathbb{Z}_2$-graded vector space $V := V_{\bar{0}} \oplus V_{\bar{1}}$ is called right $\mathcal{U}$-module if it endowed with an action $[\, ,\,] : V \otimes \mathcal{U} \to V$ such that

$$[v,[x,y]] = -[[v,x],y] - (-1)^{|x||y|}[[v,y],x] \quad \forall\, x \in \mathcal{U}_{|x|},\ y \in \mathcal{U}_{|y|},\ v \in V. \quad (2.1)$$

Let us consider the canonical surjection $\varphi : \mathcal{U} \to \mathcal{U}^{ab} := \mathcal{U}/[\mathcal{U},\mathcal{U}]$ and V a right $\mathcal{U}$-module. We define $C_n(\mathcal{U},V) := V \otimes (\varphi(\mathcal{U}))^{\otimes n}$ for all $n \in \mathbb{N}$. Then one can easily see that $C_n(\mathcal{U},V)$ is a $\mathcal{U}$-module through the following action

$$[v \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_n, x] = (-1)^{|x|\sum_{1\leqslant k\leqslant n}|x_k|}[v,x] \otimes x_1 \otimes \cdots \otimes x_n$$

for all $v \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_n \in C_n(\mathcal{U},V)$ and $x \in \mathcal{U}_{|x|}$.

In the sequel for simplicity, we denote by $x_0 \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_n$ an element of $C_n(\mathcal{U},V)$ with $x_0 \in V$.

Let $\delta : C_n(\mathcal{U},V) \to C_{n-1}(\mathcal{U},V)$ be the application defined by

$$\delta(x_0,x_1,\ldots,x_n) = \sum_{j=1}^{n}(-1)^{|x_j|\sum_{0<l<j}|x_l|}[x_0,x_j] \otimes \cdots \otimes \hat{x_j} \otimes \cdots \otimes \cdots$$

where the sign $\hat{}$ over a variable x means that x has to disappear. Let $\underline{x} = x_0 \otimes x_1 \otimes \cdots \otimes x_n \in C_n(\mathcal{U},V)$ for any integer $n > 0$ and $y,z \in \mathcal{U}^{ab}$. Then we have the following relations:

Proposition 2.1. (i) $\delta(\underline{x} \otimes y) = \delta\underline{x} \otimes y + [\underline{x},y]$;

(ii) $[\underline{x} \otimes y, z] = (-1)^{|y||z|}[\underline{x},z] \otimes y$;

(iii) $\delta[\underline{x},y] = -[\delta\underline{x},y]$;

(iv) $\delta^2 = 0$.

Proof. For relation (i), let us set $x_{n+1} = y$. We have

$$\begin{aligned}
\delta(\underline{x} \otimes y) &= \delta(x_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) \\
&= \sum_{j=1}^{n+1} (-1)^{|x_j| \sum_{0<l<j} |x_l|} [x_0, x_j] \otimes \cdots \otimes \hat{x_j} \otimes \cdots \\
&= \sum_{j=1}^{n} (-1)^{|x_j| \sum_{0<l<j} |x_l|} [x_0, x_j] \otimes \cdots \otimes \hat{x_j} \otimes \cdots \otimes x_{n+1} \\
&\quad + (-1)^{|x_{n+1}| \sum_{l=1}^{n} |x_l|} [x_0, x_{n+1}] \otimes x_1 \otimes \cdots \otimes x_n \\
&= \left(\sum_{j=1}^{n} (-1)^{|x_j| \sum_{0<l<j} |x_l|} [x_0, x_j] \otimes \cdots \otimes \hat{x_j} \otimes \cdots \right) \otimes y \\
&\quad + (-1)^{|y| \sum_{l=1}^{n} |x_l|} [x_0, x] \otimes x_1 \otimes \cdots \otimes x_n \\
&= \delta \underline{x} \otimes y + [\underline{x}, y] \,.
\end{aligned}$$

For the second relation, we set $x_{n+1} = y$ and with a simple calculation we obtain

$$\begin{aligned}
[\underline{x} \otimes y, z] &= [x_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}, z] \\
&= (-1)^{|z| \sum_{i=1}^{n+1} |x_i|} [x_0, z] \otimes \cdots \otimes x_n \otimes x_{n+1} \\
&= (-1)^{|z||x_{n+1}| + |z| \sum_{i=1}^{n} |x_i|} [x_0, z] \otimes \cdots \otimes x_n \otimes x_{n+1} \\
&= (-1)^{|y||z| + |z| \sum_{i=1}^{n} |x_i|} [x_0, z] \otimes \cdots \otimes x_n \otimes y \;=\; (-1)^{|y||z|} [\underline{x}, z] \otimes y \,.
\end{aligned}$$

For relation (iii), we shall proceed by induction over n. For that let us recall that since $\mathcal{U}^{ab} := \mathcal{U}/[\mathcal{U},\mathcal{U}]$ is abelian, then for all $x, y \in \mathcal{U}^{ab}$, the relation (2.1) becomes

$$[[v, x], y] = -(-1)^{|x||y|} [[v, y], x] \,. \tag{2.2}$$

If $n = 1$ then $\underline{x} = x_0 \otimes x_1$. Hence according to (2.2) and relation 2, we have

$$\begin{aligned}
\delta([\underline{x}, y]) &= \delta([x_0 \otimes x_1, y]) \\
&= (-1)^{|x_1||y|} \delta([x_0, y] \otimes x_1) \\
&= -[[x_0, x_1], y] \;=\; -[\delta \underline{x}, y] \,.
\end{aligned}$$

Let us assume now that the relation is true up to some integer n. Let $\underline{x'} := \underline{x} \otimes z$ an element of $C_{n+1}(\mathcal{U}, V)$. Hence according to (i), (ii), relation (2.2) and the induction hypothesis, we have

$$\begin{aligned}
\delta[\underline{x}', x] &= \delta([\underline{x} \otimes z, y]) \;=\; (-1)^{|y||z|}\delta([\underline{x}, y] \otimes z) \\
&= (-1)^{|y||z|}(\delta[\underline{x}, y] \otimes z + [[\underline{x}, y], z]) \\
&= -(-1)^{|y||z|}[\delta\underline{x}, y] \otimes z + (-1)^{|y||z|}[[\underline{x}, y], z] \\
&= -[\delta\underline{x} \otimes z, y] - [[\underline{x}, z], y] \\
&= -[\delta\underline{x} \otimes z + [\underline{x}, z], y] \;=\; -[\delta(\underline{x} \otimes z), y] \;=\; -[\delta\underline{x}', y]\,;
\end{aligned}$$

then the relation yields for $n+1$, and this proves the relation 3). Finally in order to prove that $\delta^2 = 0$, we proceed by induction. For $n = 2$, by using the Lie-type identity (1.2) and the fact that $\mathcal{U}^{ab}$ is abelian, we have

$$\begin{aligned}
\delta^2(x_0 \otimes x_1 \otimes x_2) &= \delta([x_0, x_1 \otimes x_2] + (-1)^{|x_1||x_2|}[x_0, x_2] \otimes x_1) \\
&= [[x_0, x_1], x_2] + (-1)^{|x_1||x_2|}[[x_0, x_2], x_1] \\
&= -[x_0, [x_1, x_2]] \;=\; 0\,.
\end{aligned}$$

Let us assume that the result is true up to some integer n. Let $\underline{x}' = \underline{x} \otimes y \in C_{n+1}(\mathcal{U}, V)$ where $\underline{x} \in C_n(\mathcal{U}, V)$ and $y \in \mathcal{U}^{ab}$. With the help of relation 1), 3) and the induction hypothesis, we have

$$\begin{aligned}
\delta^2(\underline{x}') &= \delta^2(\underline{x} \otimes y) \;=\; \delta(\delta(\underline{x} \otimes y)) \;=\; \delta(\delta\underline{x} \otimes y + [\underline{x}, y]) \\
&= \delta(\delta\underline{x} \otimes y) + \delta[\underline{x}, y] = \delta^2\underline{x} \otimes y + [\delta\underline{x}, y] + \delta[\underline{x}, y] \\
&= [\delta\underline{x}, y] + \delta[\underline{x}, y] \;=\; [\delta\underline{x}, y] - [\delta\underline{x}, y] \;=\; 0
\end{aligned}$$

and this proves that $\delta^2 = 0$. ∎

The above result shows that δ defines a differential of degree -1 over $C_n(\mathcal{U}, V)$. Then we have the complex $(C_*(\mathcal{U}, V), \delta)$ which the homological groups are called Lie type homological groups of $\mathcal{U}$ with coefficients in V and are denoted by $H_*(\mathcal{U}, V)$. We obtain a similar result to the case of Leibniz homology concerning the homological groups of degree 0 and 1.

Lemma 2.1. *$H_0(\mathcal{U}, V) = V/[V, \mathcal{U}^{ab}]$ and if V is a trivial module then $H_1(\mathcal{U}, V) = V \otimes \mathcal{U}^{ab}$.*

Proof. The proof is straightforward. ∎

Let $\mathcal{U}$ be a Lie type superalgebra and $V := V_{\bar{0}} \oplus V_{\bar{1}}$ a $\mathbb{Z}_2$-graded vector space. For $n \in \mathbb{N}$, define by $C^n(\mathcal{U}, V) := \mathrm{Hom}_{\mathbb{K}}(\mathcal{U} \otimes (\mathcal{U}^{ab})^{\otimes n}, V)$. Then $C^n(\mathcal{U}, V)$ admits a structure of $\mathcal{U}$-module through the following action

$$(f \cdot x)(y \otimes y_1 \otimes \cdots \otimes y_n) = (-1)^{|x||y|} f([y, x] \otimes y_1 \otimes \cdots \otimes y_n)$$

for all $x \in \mathcal{U}, y \otimes y_1 \otimes \cdots \otimes y_n \in \mathcal{U} \otimes (\mathcal{U}^{ab})^{\otimes n}$. Let $\mathcal{D}_n : C^n(\mathcal{U}, V) \to C^{n+1}(\mathcal{U}, V)$ defined by

$$\mathcal{D}_n f(x_0 \otimes x_1 \otimes \cdots \otimes x_{n+1}) = \sum_{j=1}^{n+1} (-1)^{|x_j| \sum_{0<l<j} |x_l|} f([x_0, x_j] \otimes \cdots \otimes \hat{x_j} \otimes \cdots).$$

The application $\mathcal{D}$ defines a differential over the graded $\mathcal{U}$-module $C^*(\mathcal{U}, V)$. In fact we have the following result

Proposition 2.2. $\mathcal{D}^2 = 0$.

Proof. It's sufficient to notice that for all integer n, we have $\mathcal{D}_n f = f \delta_{n+1}$ and according to 4) of Proposition 2.1 we obtain $\mathcal{D}^2 = 0$. ∎

Indeed, we obtain the cochain complex $(C^*(\mathcal{U}, V), \mathcal{D})$ and the cohomological groups denoted by $H^*(\mathcal{U}, V)$.

3. Extensions and crossed modules of Lie type superalgebras

In this section, we introduce and study the theory of extension and crossed module for Lie type superalgebras. In particular we characterize these notions with more explicit objects such as bilinear applications satisfying some conditions. The cohomological group $H^1(\mathcal{U}, \mathcal{V})$ is investigated through a particular type of extension.

3.1. Extensions of Lie type superalgebras.

Let $(\mathcal{U}, [\,,\,])$ and $(\mathcal{U}_1, [\,,\,])$ be two Lie type superalgebras. Let $\pi : \mathcal{U}_1 \to \mathcal{U}$ an epimorphism and $\mathcal{V} := \mathrm{Ker}(\pi)$. Then the ideal $\mathcal{V}$ admits a structure of $\mathcal{U}_1$-module via the action $(v, x) \in \mathcal{V} \otimes \mathcal{U}_1 \mapsto [v, x] \in \mathcal{V}$. Moreover if $\mathcal{V}$ is an abelian Lie type superalgebra, then $\mathcal{V}$ admits a structure of $\mathcal{U}$-module through the following action

$$v \cdot x = [y, v] \qquad \forall\, x \in \mathcal{U},\ y \in \mathcal{U}_1 \text{ such that } x = \pi y\,. \tag{3.1}$$

Conversely if $\mathcal{V}$ admits a structure of $\mathcal{U}$-module through the action defined in (3.1), then $\mathcal{V}$ is abelian.

DEFINITION 3.1.1. Let $\mathcal{U}$ be a Lie type superalgebra and $\mathcal{V}$ a right $\mathcal{U}$-module. An extension of $\mathcal{U}$ by $\mathcal{V}$ is an exact sequence

$$0 \longrightarrow \mathcal{V} \xrightarrow{i} \mathcal{U}_1 \xrightarrow{\pi} \mathcal{U} \longrightarrow 0\,, \tag{3.2}$$

where $\mathcal{U}_1$ is a Lie type superalgebra and $\mathcal{V}$ becomes a $\mathcal{U}_1$-module that can be considered as an abelian Lie type superalgebra.

EXAMPLE 3.1.1. Let $(\mathcal{L}, [\,,\,])$ be a Lie type superalgebra. Then $\mathcal{U}$ is an extension of $\mathcal{U}/_{\mathrm{Ker}(\mathcal{U})}$ by $\mathrm{Ker}(\mathcal{U})$. In fact we have the following exact sequence

$$0 \longrightarrow \mathrm{Ker}(\mathcal{U}) \xrightarrow{i} \mathcal{U} \xrightarrow{\pi} \mathcal{U}/_{\mathrm{Ker}(\mathcal{U})} \longrightarrow 0\,.$$

The $\mathcal{U}$-module $\mathcal{V}$ is called the kernel of the extension (3.2) and the Lie type superalgebra $\mathcal{U}_1$ can be identified by $\mathcal{V} \oplus \mathcal{U}$. In what follows, we give an explicit description of the notion of an extension of $\mathcal{U}$ with kernel $\mathcal{V}$.

THEOREM 3.1.1. *An extension of $\mathcal{U}$ by $\mathcal{V}$ is equivalent to the set of two even bilinear applications $\alpha : \mathcal{U} \otimes \mathcal{U} \to \mathcal{V}$ and $\phi : \mathcal{U} \otimes \mathcal{V} \to \mathcal{V}$ such that*

(i) $\alpha(x, [x', x'']) + \phi(x, \alpha(x', x'')) = -\alpha([x, x'], x'') - [\alpha(x, x'), x''] - (-1)^{|x'||x''|}\alpha([x, x''], x') - (-1)^{|x'||x''|}[\alpha(x, x''), x']\,,$

(ii) $\phi(x, \phi(x', v)) = -\phi([x, x'], v) - [\phi(x, v), x']$

for all $x \in \mathcal{U}_{|x|}$, $x' \in \mathcal{U}_{|x'|}$, $x'' \in \mathcal{U}_{|x''|}$ and $v \in \mathcal{V}$.

Proof. Let $0 \to \mathcal{V} \xrightarrow{i} \mathcal{U}_1 \xrightarrow{\pi} \mathcal{U} \to 0$ be an extension of $\mathcal{U}$ by $\mathcal{V}$. The Lie type superalgebra $\mathcal{U}_1$ is identified by $\mathcal{V} \oplus \mathcal{U}$. Since $\mathcal{V}$ is a right $\mathcal{U}$-module and an abelian superalgebra, then there exists two even bilinear applications $\alpha : \mathcal{U} \otimes \mathcal{U} \to \mathcal{V}$ and $\phi : \mathcal{U} \otimes \mathcal{V} \to \mathcal{V}$ such that the bracket of $\mathcal{U}_1$ is given by

$$[(x, v), (x', v')] = ([x, x'], \alpha(x, x') + \phi(x, v') + [v, x']) \qquad \forall\, (x, v),\ (x', v') \in \mathcal{U}_1.$$

By applying the Lie type identity (1.2) to $((x, 0), (x', 0), (x'', 0))$ we obtain the relation 1). For the relation 2), we apply the Lie type identity (1.2) to $((x, 0), (x', 0), (0, v))$. ∎

DEFINITION 3.1.2. Two extensions $0 \to \mathcal{V} \xrightarrow{i} \mathcal{U}_1 \xrightarrow{\pi} \mathcal{U} \to 0$ and $0 \to \mathcal{V} \xrightarrow{i} \mathcal{U}_2 \xrightarrow{\pi} \mathcal{U} \to 0$ of $\mathcal{U}$ by $\mathcal{V}$ are said to be equivalent or isomorphic if there exists a homomorphism $\psi : \mathcal{U}_1 \to \mathcal{U}_2$ such that the following diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{V} \arrow[r, "i"] \arrow[d, equal] & \mathcal{U}_1 \arrow[r, "\pi"] \arrow[d] & \mathcal{U} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{V} \arrow[r, "i'"] & \mathcal{U}_2 \arrow[r, "\pi'"] & \mathcal{U} \arrow[r] & 0
\end{tikzcd}
\tag{3.3}
$$

is commutative. In other words $\pi' \circ \psi = \pi$ and $\psi \circ i = i'$.

One can easily sees that the application ψ of the definition above is bijective. In the sequel, the extension $0 \to \mathcal{V} \xrightarrow{i} \mathcal{U}_1 \xrightarrow{\pi} \mathcal{U} \to 0$ will be identified by the couple (α, ϕ) of Theorem 3.1.1.

PROPOSITION 3.1.1. *Two extensions (α, ϕ) and (α', ϕ') are equivalent if and only if, there exists an even linear application $\lambda : \mathcal{U} \to \mathcal{V}$ such that*

$$\lambda([x, x']) + \alpha(x, x') = \alpha'(x, x') + \phi'(x, \lambda x') + [\lambda x, x'] \qquad \forall\, x, x' \in \mathcal{U}\,.$$

Proof. Let $\psi : \mathcal{U}_1 \cong \mathcal{U} \oplus \mathcal{V} \to \mathcal{U}_2 \cong \mathcal{U} \oplus \mathcal{V}$ be a bijective application between the two extensions. The commutativity of the diagram (3.3) and the fact that i and i' are injective imply that $\psi(\mathcal{V}) \subseteq \mathcal{V}$ and $\psi(\mathcal{U}) \subseteq \mathcal{U} + \mathcal{V}$. Hence there exists a linear application $\lambda : \mathcal{U} \to \mathcal{V}$ such that for all $(x, v) \in \mathcal{U}_1$ we have $\psi((x, v)) = (x, \lambda x + v)$. Since ψ is compatible with the brackets of $\mathcal{U}_1$ and $\mathcal{U}_2$, then we have

$$
\begin{aligned}
&\psi([(x,0),(x',0)]) = [\psi(x,0), \psi(x',0)] \qquad \forall\,(x,0),(x',0) \in \mathcal{U}_1 \\
&\quad\Leftrightarrow \psi\big(([x,x'], \alpha(x,x'))\big) = [(x, \lambda x), (x', \lambda x')] \\
&\quad\Leftrightarrow \big([x,x'], \lambda([x,x']) + \alpha(x,x')\big) = \big([x,x'], \alpha'(x,x') + \phi'(x, \lambda x') + [\lambda x, x']\big)\,.
\end{aligned}
$$

Hence $\lambda([x, x']) + \alpha(x, x') = \alpha'(x, x') + \phi'(x, \lambda x') + [\lambda x, x']$ for all $x, x' \in \mathcal{U}$. ∎

An extension (α, ϕ) of $\mathcal{U}$ by $\mathcal{V}$ is said to be trivial if $\mathcal{V}$ is a trivial $\mathcal{U}$-module and $\phi = 0$. Let $(\alpha, 0)$ be a trivial extension of $\mathcal{U}$ and denote α_R the restriction of the bilinear map α over $\mathcal{U} \otimes \mathcal{U}^{ab}$. We call $(\alpha_R, 0)$ the restricted trivial extension of $\mathcal{U}$ by $\mathcal{V}$.

PROPOSITION 3.1.2. *Let $(\alpha, 0)$ be a trivial extension of $\mathcal{U}$ by $\mathcal{V}$. Then $\alpha_R \in \mathrm{Ker}(\mathcal{D}_1)$.*

Proof. Let $x \in \mathcal{U}$ and $x', x'' \in \mathcal{U}^{ab}$. According to the first relation of Theorem 3.1.1 and the fact that $\mathcal{U}^{ab}$ is abelian we have

$$\begin{aligned}\mathcal{D}\alpha_R(x, x', x'') &= \alpha_R([x, x'], x'') + (-1)^{|x'||x''|}\alpha_R([x, x''], x') \\ &= -\alpha_R(x, [x', x'']) \; = \; 0\end{aligned}$$

which implies that $\alpha_R \in \mathrm{Ker}(\mathcal{D}_1)$. ∎

LEMMA 3.1.1. *Two restricted trivial extensions* $(\alpha_R, 0)$ *and* $(\alpha'_R, 0)$ *of* $\mathcal{U}$ *by* $\mathcal{V}$ *are equivalent if and only if* $\alpha'_R - \alpha_R \in \mathrm{Im}(\mathcal{D}_0)$.

Proof. According to Proposition (3.1.1), we have $(\alpha, 0)$ and $(\alpha', 0)$ are equivalent if and only if there exists an even linear map $\lambda : \mathcal{U} \to \mathcal{V}$ such that $\lambda[x, x'] = \alpha'(x, x') - \alpha(x, x')$ for all $x, x' \in \mathcal{U}$, that is $(\alpha' - \alpha)(x, x') = \lambda([x, x'])$. Hence $(\alpha'_R - \alpha_R)(x, x') = \lambda([x, x'])$. Therefore $\alpha'_R - \alpha_R \in \mathrm{Im}(\mathcal{D}_0)$ ∎

The following result gives the cohomological group $H^1(\mathcal{U}, \mathcal{V})$.

THEOREM 3.1.2. *Let* $(\mathcal{U}, [\,,\,])$ *be a Lie-type superalgebra and* $\mathcal{V}$ *a* $\mathcal{U}$-*module. The isomorphic class of restricted trivial extensions of* $\mathcal{U}$ *by* $\mathcal{V}$ *is isomorphic to* $H^1(\mathcal{U}, \mathcal{V})$.

Proof. The result follows from a combination of Proposition 3.1.2 and Lemma 3.1.1. ∎

3.2. CROSSED MODULES OF LIE TYPE SUPERALGEBRAS. Let $(\mathcal{U}, [\,,\,])$ be a Lie type superalgebra. A homogeneous endomorphism f of $\mathcal{U}$ is called $(-1,-1)$-superderivation if

$$f([x, y]) = -[f(x), y] - (-1)^{|f||y|}[x, f(y)] \qquad \forall\, x \in \mathcal{U},\ y \in \mathcal{U}_{|y|}\,.$$

The set of all $(-1,-1)$-superderivations of $\mathcal{U}$ will be denoted by $(-1,-1)\_\mathrm{SDer}(\mathcal{U})$.

DEFINITION 3.2.1. A crossed module of $(\mathcal{U}, [\,,\,])$ is a triplet $(\mathcal{W}, d, \eta)$ where $\mathcal{W}$ is a Lie type superalgebra, $d : \mathcal{U} \to \mathcal{W}$ an even morphism and $\eta : \mathcal{W} \to (-1,-1)\_\mathrm{SDer}(\mathcal{U})$ an even linear map such that

(a) $\eta([w, w']) = -\eta(w') \circ \eta(w) - (-1)^{|w||w'|}\eta(w) \circ \eta(w')$ for all $w \in \mathcal{W}_{|w|}$, $w' \in \mathcal{W}_{|w'|}$;

(b) $d(\eta(w)(u)) = [du, w]$ for all $u \in \mathcal{U}$, $w \in \mathcal{W}$;

(c) $\eta(du)(u') = [u', u]$ for all $u, u' \in \mathcal{U}$.

In what follows, we characterize crossed modules of Lie type superalgebras. For that, let us set

$$\mathcal{P} = coker(d)\,, \qquad \mathcal{L} = \mathrm{Ker}(d) \qquad \text{and} \qquad \mathcal{V} = \mathrm{Im}(d)\,.$$

We have the following properties:

Lemma 3.2.1. (i) *$\mathcal{L} \subseteq Z^r(\mathcal{U})$ and $d \in \mathrm{Aut}(\mathcal{V})$;*

(ii) *$\mathcal{V}$ is a left ideal of $\mathcal{W}$ and $\mathcal{P}$ is a complement of $\mathcal{V}$ in $\mathcal{W}$;*

(iii) *$\mathcal{U}$ and $\mathcal{L}$ are $\mathcal{P}$-modules through η.*

Proof. For relation (i), it's clear that $d \in \mathrm{Aut}(\mathcal{V})$. Moreover from relation (c) of Definition 3.2.1, we have $[\mathcal{U}, \mathcal{L}] \subseteq \eta(d\mathcal{L})(\mathcal{U}) = 0$ because $d\mathcal{L} = 0$. Then $\mathcal{L} \subseteq Z^r(\mathcal{U})$. For the second relation, according to relation (b) of Definition 3.2.1 we have

$$[\mathcal{V}, \mathcal{W}] = [d(\mathcal{U}), \mathcal{W}] \subseteq d(\eta(\mathcal{W})(\mathcal{U})) \subseteq \mathrm{Im}(d) = \mathcal{V}$$

then $\mathcal{V}$ is a left ideal of $\mathcal{W}$. One can easily see that $\mathcal{P}$ is a complement of $\mathcal{V}$ in $\mathcal{W}$. Finally, relation (a) of Definition 3.2.1 and the fact that $d(\eta(\mathcal{P})(\mathcal{L})) \subseteq [d\mathcal{L}, \mathcal{W}] = 0$ imply that $\mathcal{U}$ and $\mathcal{L}$ are $\mathcal{P}$-modules through η. ∎

Hence a crossed module $(\mathcal{W}, d, \eta)$ of $\mathcal{U}$ leads to the following exact sequence

$$0 \longrightarrow \mathcal{L} \xrightarrow{i} \mathcal{U} \xrightarrow{d} \mathcal{W} \xrightarrow{\pi} \mathcal{P} \longrightarrow 0 \tag{3.4}$$

where $\mathcal{U}$ can be identified by $\mathcal{L} \oplus \mathcal{V}$ and $\mathcal{W}$ by $\mathcal{P} \oplus \mathcal{V}$. The crossed module $(\mathcal{W}, d, \eta)$ is called Lie-type crossed module of kernel $\mathcal{L}$ and cokernel $\mathcal{P}$ or for simplicity, a $(\mathcal{L}, \mathcal{P})$-Lie-type crossed module.

Let $(\mathcal{W}, d, \eta)$ be a $(\mathcal{L}, \mathcal{P})$-Lie-type crossed module. We have the following result:

Theorem 3.2.1. *The $(\mathcal{L}, \mathcal{P})$-Lie-type crossed module $(\mathcal{W}, d, \eta)$ is equivalent to a set composed by a structure of $\mathcal{W}$-module over $\mathcal{L}$, an automorphism d of $\mathcal{V}$, two even bilinear maps $\alpha : \mathcal{V} \otimes \mathcal{V} \to \mathcal{L}$ and $\eta_0 : \mathcal{P} \otimes \mathcal{V} \to \mathcal{L}$ such that*

(i) $$\alpha(v, [v', v'']) = -[\alpha(v, v'), v''] - \alpha([v, v'], v'') - (-1)^{|v'||v''|}\alpha([v, v''], v') \\ -(-1)^{|v'||v''|}[\alpha(v, v''), v']\,,$$

$$\text{(ii)}\quad \eta_0(p\otimes[v,v'])+[\alpha(v,v'),p] = -\alpha(d^{-1}[dv,p],v') - [\eta_0(p\otimes v),v'] \\ -(-1)^{|p||v'|}\alpha(v,d^{-1}[dv,p])\,,$$

$$\text{(iii)}\quad \eta_0([p,p']_{\mathcal{P}}) = -\eta_0(p')\circ\eta_0(p) - (-1)^{|p||p'|}\eta_0(p)\circ\eta_0(p')$$

for all $v\in\mathcal{V}_{|v|}$, $v'\in\mathcal{V}_{|v'|}$, $v''\in\mathcal{V}_{|v''|}$ *and* $p\in\mathcal{P}_{|p|}$, $p'\in\mathcal{P}_{|p'|}$.

Proof. Since $\mathcal{U}$ can be identified by $\mathcal{L}\oplus\mathcal{V}$, $[\mathcal{L},\mathcal{V}]\subseteq\mathcal{L}$ and $\mathcal{L}\subseteq Z^r(\mathcal{U})$, then there exist bilinear applications $[\,,\,]_{\mathcal{V}}:\mathcal{V}\otimes\mathcal{V}\to\mathcal{V}$ and $\alpha:\mathcal{V}\otimes\mathcal{V}\to\mathcal{L}$ such that the bracket of $\mathcal{U}$ is given by

$$[(x,v),(x',v')] = ([x,v']+\alpha(v,v'),[v,v']_{\mathcal{V}}) \qquad \forall\,(x,v),(x',v')\in\mathcal{L}\oplus\mathcal{V}.$$

By applying the Lie type identity (1.2) to the triplet $((0,v),(0,v'),(0,v''))$ we obtain the relation (i).

For (ii), since $\mathcal{W}=\mathcal{P}\oplus\mathcal{V}$ and $\eta(w)\in\mathrm{End}(\mathcal{L})$ for all $w\in\mathcal{W}$ then $\eta(\mathcal{W})(\mathcal{V})=\eta(\mathcal{P})(\mathcal{V})+\eta(\mathcal{V})(\mathcal{V})$. Moreover, according to relation (b) of Definition 3.2.1, we have

$$\eta(\mathcal{V})(\mathcal{V}) = \eta(d\mathcal{U})(\mathcal{V}) = [\mathcal{V},\mathcal{U}] = [\mathcal{V},\mathcal{V}] = [\mathcal{V},\mathcal{V}]_{\mathcal{V}}+\alpha(\mathcal{V},\mathcal{V})\,.$$

Since $\eta(\mathcal{P})(\mathcal{V})\subseteq\mathcal{L}\oplus\mathcal{V}$, then there exists a bilinear application $\eta_0:\mathcal{P}\otimes\mathcal{V}\to\mathcal{L}$ such that

$$\eta(p)(v) = \eta_{\mathcal{V}}(p)(v)+\eta_0(p)(v)$$

where $\eta_{\mathcal{V}}(p)(v)$ is the component in $\mathcal{V}$ of $\eta(p)(v)$ for all $p\in\mathcal{P}$ and $v\in\mathcal{V}$. According to the relation (b) of Definition 3.2.1, we have

$$d(\eta(p)(v)) = d(\eta_{\mathcal{V}}(p)(v))+d(\eta_0(p\otimes v)) = d(\eta_{\mathcal{V}}(p)(v)) = [dv,p]$$

and since $d\in\mathrm{Aut}(\mathcal{V})$ then $\eta_{\mathcal{V}}(p)(v)=d^{-1}[dv,p]$. This implies that $\eta(p)(v)=d^{-1}[dv,p]+\eta_0(p\otimes v)$. From the fact $\eta:\mathcal{W}\to(-1,-1)_{-}\mathrm{SDer}(\mathcal{U})$ we have

$$\begin{aligned}
&\eta(p)([v,v']) = -[\eta(p)(v),v'] - (-1)^{|p||v'|}[v,\eta(p)(v')]\\
\iff\ &\eta(p)([v,v']_{\mathcal{V}}+\alpha(v,v')) = -[d^{-1}[dv,p]+\eta_0(p\otimes v),v']\\
&\quad - (-1)^{|p||v'|}[v,d^{-1}[dv',p]+\eta_0(p\otimes v')]\\
\iff\ &d^{-1}[d[v,v'],p]+\eta_0(p\otimes[v,v'])+[\alpha(v,v'),p] = -[d^{-1}[dv,p],v']\\
&\quad - [\eta_0(p\otimes v),v'] - (-1)^{|p||v'|}[v,d^{-1}[dv',p]] - (-1)^{|p||v'|}[v,\eta_0(p\otimes v')]\\
\iff\ &d^{-1}[d[v,v'],p]+\eta_0(p\otimes[v,v'])+[\alpha(v,v'),p] = -[d^{-1}[dv,p],v']\\
&\quad - \alpha(d^{-1}[dv,p],v') - [\eta_0(p\otimes v),v'] - (-1)^{|p||v'|}[v,d^{-1}[dv',p]]\\
&\quad - (-1)^{|p||v'|}\alpha(v,d^{-1}[dv',p]) - (-1)^{|p||v'|}[v,\eta_0(p\otimes v')]
\end{aligned}$$

a projection over $\mathcal{U}$ of the later relation gives us the relation (ii). The relation (iii) is obtained from relation (a) of the Definition 3.2.1. ∎

In the sequel, a Lie type crossed module $(\mathcal{W}, d, \eta)$ will be denoted by the exact sequence (3.4) or by the triplet (d, α, η_0) of the Theorem 3.2.1.

Definition 3.2.2. Two Lie type crossed modules $0 \to \mathcal{L} \xrightarrow{i} \mathcal{U} \xrightarrow{d} \mathcal{W} \xrightarrow{\pi} \mathcal{P} \to 0$ and $0 \to \mathcal{L} \xrightarrow{i'} \mathcal{U}' \xrightarrow{d} \mathcal{W} \xrightarrow{\pi'} \mathcal{P} \to 0$ are isomorphic or equivalent if there exists an isomorphism $\psi : \mathcal{U} \to \mathcal{U}'$ such that the following diagrams:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L} \arrow[r, "i"] \arrow[d, equal] & \mathcal{U} \arrow[r, "d"] \arrow[d] & \mathcal{W} \arrow[r, "\pi"] \arrow[d, equal] & \mathcal{U} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{L} \arrow[r, "i'"] & \mathcal{U}' \arrow[r, "d'"] & \mathcal{W} \arrow[r, "\pi'"] & \mathcal{U} \arrow[r] & 0 \end{tikzcd} \tag{3.5}$$

and

$$\begin{tikzcd} \mathcal{W} \otimes \mathcal{U} \arrow[r, "\eta"] \arrow[d, "Id_{\mathcal{W}} \otimes \psi"'] & \mathcal{U} \arrow[d, "\psi"] \\ \mathcal{W} \otimes \mathcal{U}' \arrow[r, "\eta'"] & \mathcal{U}' \end{tikzcd} \tag{3.6}$$

are commutative.

Proposition 3.2.1. *Two Lie type crossed modules (d, α, η_0) and (d', α', η'_0) are equivalent if there exists an even linear map $\theta : \mathcal{V} \to \mathcal{L}$ such that*

(1) $\alpha(v, v') + \theta([v, v']) = [\theta v, d'^{-1}dv'] + \alpha'(d'^{-1}dv, d'^{-1}dv')$,

(2) $\eta_0(p \otimes v) + \theta(d^{-1}[dv, p]) = [\theta v, p] + \eta'_0(p \otimes d'^{-1}v)$,

(3) $[l, d^{-1}v] = [l, d'^{-1}v]$.

Proof. Let $\psi : \mathcal{U} \cong \mathcal{L} \oplus \mathcal{V} \to \mathcal{U}' \cong \mathcal{L} \oplus \mathcal{V}'$ be an isomorphism between $\mathcal{U}$ and $\mathcal{U}'$ such that the diagrams (3.5) and (3.6) be commutative. According to the commutativity of the diagram (3.5), we have $\psi \circ i = i' \circ Id_{\mathcal{L}}$ then for all $x \in \mathcal{L}$, $\psi(x) = x$. Moreover, since $\psi(v) \in \mathcal{L} \oplus \mathcal{V}'$ for all $v \in \mathcal{V}$ then there exists a linear application $\theta : \mathcal{V} \to \mathcal{L}$ such that

$$\psi(l, v) = (l + \theta v, \psi_{\mathcal{V}}(v)) \qquad \forall\, (l, v) \in \mathcal{L} \oplus \mathcal{V}$$

where $\psi_{\mathcal{V}}(v)$ is the component of $\psi(v)$ in $\mathcal{V}'$. We have $\psi_{\mathcal{V}}(v) = d'^{-1}dv$. In fact, according to the diagram (3.5) we have

$$\begin{aligned} Id_{\mathcal{W}} \circ d(0,v) = d' \circ \psi(0,v) &\iff (0,dv) = d'(\theta v, \psi_{\mathcal{V}}(v)) \\ &\iff (0,dv) = (d'\theta v, d'\psi_{\mathcal{V}}(v)) \\ &\Rightarrow dv = d'\psi_{\mathcal{V}}(v)\,, \end{aligned}$$

hence $\psi_{\mathcal{V}}(v) = d'^{-1}dv$. Therefore, we have $\psi(l,v) = (l+\theta v, d'^{-1}dv)$. The compatibility of ψ with the brackets implies that

$$\begin{aligned} &\psi[(0,v),(0,v')] = [\psi(0,v),\psi(0,v')] \\ \iff &\psi\left(\alpha(v,v'),[v,v']_{\mathcal{V}}\right) = [(\theta v, d'^{-1}dv),(\theta v', d'^{-1}dv')] \\ \iff &\left(\alpha(v,v') + \theta([v,v']_{\mathcal{V}}), d'^{-1}d([v,v']_{\mathcal{V}})\right) \\ &\qquad = \left([\theta v, d'^{-1}dv'] + \alpha'(d'^{-1}dv, d'^{-1}dv'), [d'^{-1}dv, d'^{-1}dv']_{\mathcal{V}}\right) \end{aligned}$$

which proves the first relation.

For relation 2), the commutativity of the diagram (3.6) implies that for all $p \otimes v \in \mathcal{P} \otimes \mathcal{V}$ we have

$$\begin{aligned} &\psi \circ \eta(p\otimes v) = \eta' \circ (Id_{\mathcal{W}} \otimes \psi)(p \otimes v) \\ \iff &\psi(d^{-1}[dv,p] + \eta_0(p\otimes v)) = \eta'(p \otimes \theta v) + \eta'(p \otimes d'^{-1}dv) \\ \iff &\eta_0(p\otimes v) + \theta(d^{-1}[dv,p]) + d'^{-1}[dv,p] \\ &\qquad = [\theta v, p] + d'^{-1}[d'd'^{-1}dv, p] + \eta'_0(p \otimes d'^{-1}dv) \\ &\eta_0(p\otimes v) + \theta(d^{-1}[dv,p]) + d'^{-1}[dv,p] \\ &\qquad = [\theta v, p] + d'^{-1}[dv, p] + \eta'_0(p \otimes d'^{-1}dv) \end{aligned}$$

a projection over $\mathcal{L}$ of the later relation gives us $\eta_0(p\otimes v) + \theta(d^{-1}[dv,p]) = [\theta v, p] + \eta'_0(p \otimes d'^{-1}v)$.

Let $v \in \mathcal{V}$ and $l \in \mathcal{L}$. According to the commutativity of the diagram (3.6) we have $\psi \circ \eta(v \otimes l) = \eta' \circ (Id_{\mathcal{W}} \otimes \psi)(v \otimes l)$ hence $\psi\left(\eta(v)(l)\right) = \eta'(v\otimes l)$, therefore $\psi([l, d^{-1}v]) = [l, d'^{-1}]$. Since $[l,d^{-1}v] \in \mathcal{L}$ then $[l,d^{-1}v] = [l,d'^{-1}v]$. This proves relation 3). ∎

Definition 3.2.3. A Lie type crossed module $(\mathcal{W}, d, \eta)$ of $(\mathcal{U}, [\,,\,])$ is said to be:

- normalized if $d = Id_{\mathcal{U}}$,
- bilateral if $\mathcal{L} \subseteq Z(\mathcal{U})$.

By adapting the characterization of isomorphism between two Lie type crossed modules (see Proposition 3.2.1) to the normalized crossed modules, we obtain the following result:

COROLLARY 3.2.1. *Two normalized Lie type crossed modules* (α, η_0) *and* (α', η_0') *are isomorphic if and only if there exists an even linear map* $\theta : \mathcal{V} \to \mathcal{L}$ *such that*

(1) $\alpha(v, v') + \theta([v, v']) = [\theta v, v'] + \alpha'(v, v')$,

(2) $\eta_0'(p \otimes v) = \eta_0(p \otimes v) + \theta[v, p] - [\theta v, p]$.

Define by $\beta(p, p') = [p, p']_{\mathcal{P}} - [p, p']$ where $[p, p']_{\mathcal{P}}$ is the component in $\mathcal{P}$ of the product in $\mathcal{W}$ of p and p'. The following result gives a characterization of bilateral Lie-type crossed module.

PROPOSITION 3.2.2. *A bilateral Lie type crossed module* $(\mathcal{W}, d, \eta)$ *of kernel* $\mathcal{L}$ *and co-kernel* $\mathcal{P}$ *is equivalent the set composed by bilinear applications* $\alpha : \mathcal{V} \otimes \mathcal{V} \to \mathcal{L}$ *and* $\eta_0 : \mathcal{P} \otimes \mathcal{V} \to \mathcal{L}$ *such that*

(1) $\alpha(v, [v', v'']) = -\alpha([v, v'], v'') - (-1)^{|v'||v''|}\alpha([v, v''], v')$;

(2) $\alpha(v, [p, v']) = (-1)^{|p||v'|}\alpha(v, [v', p])$;

(3) $\eta_0([p, p'] \otimes v) - \alpha(v, d^{-1}\beta(p, p')) = -\eta_0(p' \otimes d^{-1}[dv, p]) - [\eta_0(p \otimes v), p'] - (-1)^{|p||p'|}\eta_0(p \otimes d^{-1}[dv, p']) - (-1)^{|p||p'|}[\eta_0(p' \otimes v), p]$;

(4) $\eta_0(p\otimes[v, v'])+[\alpha(v, v'), p] = -\alpha(d^{-1}[dv, p], v')-(-1)^{|p||v'|}\alpha(v, d^{-1}[dv', p])$.

Proof. The relations (1) and (4) can be obtained respectively from relations (i) and (iii) of Theorem 3.2.1 by using the fact that $\alpha(v, v') \in \mathcal{L} \subseteq Z(\mathcal{U})$ for all $v, v' \in \mathcal{V}$.

For (2), let $v \in \mathcal{V}_{|v|}$, $v' \in \mathcal{V}_{|v'|}$ and $p \in \mathcal{P}_{|p|}$. Since $\mathcal{W}$ is a right Lie-type superalgebra then according to relation (1.3) we have

$$[v, [p, v']] = (-1)^{|p||v'|}[v, [v', p]],$$

hence

$$\alpha(v, [p, v']) + [v, [p, v']]_{\mathcal{V}} = (-1)^{|p||v'|}\alpha(v, [v', p]) + (-1)^{|p||v'|}[v, [v', p]]_{\mathcal{V}}$$

which implies that $\alpha(v, [p, v']) = (-1)^{|p||v'|}\alpha(v, [v', p])$.

Let $p \in \mathcal{P}_{|p|}$, $p' \in \mathcal{P}_{|p'|}$ and $l + v \in \mathcal{L} \oplus \mathcal{V}$, we have then

$$\begin{aligned}
\eta([p,p'])(l+v) &= \eta(p')\circ\eta(p)(l) - (-1)^{|p||p'|}\eta(p)\circ\eta(p')(l) - \eta(p')\circ\eta(p)(v) \\
&\quad - (-1)^{|p||p'|}\eta(p)\circ\eta(p')(v) \\
&= -\eta(p')([l,p]) - (-1)^{|p||p'|}\eta(p)([l,p']) - \eta(p)(d^{-1}[dv,p]) \\
&\quad - \eta(p')(\eta_0(p\otimes v)) - (-1)^{|p||p'|}\eta(p)(d^{-1}[dv,p']) \\
&\quad - (-1)^{|p||p'|}\eta(p)\eta_0(p'\otimes v)) \\
&= -[[l,p],p'] - (-1)^{|p||p'|}[[l,p'],p] - d^{-1}[[dv,p],p'] \\
&\quad - \eta_0(p'\otimes d^{-1}[dv,p]) \\
&\quad - [\eta_0(p\otimes v),p'] - (-1)^{|p||p'|}d^{-1}[[dv,p'],p] \\
&\quad - (-1)^{|p||p'|}\eta_0(p\otimes d^{-1}[dv,p']) - (-1)^{|p||p'|}[\eta_0(p'\otimes v),p] \\
&= [l,[p,p']] + d^{-1}[dv,[p,p']] - \eta_0(p'\otimes d^{-1}[dv,p]) \\
&\quad - [\eta_0(p\otimes v),p'] - (-1)^{|p||p'|}\eta(p\otimes d^{-1}[dv,p']) \\
&\quad - (-1)^{|p||p'|}[\eta_0(p'\otimes v),p]\,.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\eta([p,p'])(l+v) &= \eta([p,p']_{\mathcal{P}})(l+v) - \eta(\beta(p,p'))(l+v) \\
&= \eta([p,p']_{\mathcal{P}})(l) + \eta([p,p']_{\mathcal{P}})(v) \\
&\quad - \eta(\beta(p,p'))(l) - \eta(\beta(p,p'))(v) \\
&= [l,[p,p']_{\mathcal{P}}] + d^{-1}[dv,[p,p']_{\mathcal{P}}] + \eta_0([p,p']_{\mathcal{P}}\otimes v) \\
&\quad - [l,d^{-1}\beta(p,p')] - [v,d^{-1}\beta(p,p')] \\
&= [l,[p,p']_{\mathcal{P}}] + d^{-1}[dv,[p,p']_{\mathcal{P}}] + \eta_0([p,p']_{\mathcal{P}}\otimes v) \\
&\quad - [v,d^{-1}\beta(p,p')]_{\mathcal{V}} - \alpha(v,d^{-1}\beta(p,p'))
\end{aligned}$$

which implies relation (3). ∎

In what follows, we define and study another relationship between two Lie-type crossed modules of a Lie-type superalgebra. In fact we study the notion of linked bilateral Lie-type crossed modules and characterize this relation.

DEFINITION 3.2.4. Let $(MC_1) : 0 \to \mathcal{L} \xrightarrow{i} \mathcal{U} \xrightarrow{d} \mathcal{W} \xrightarrow{\pi} \mathcal{P} \to 0$ and $(MC_2) : 0 \to \mathcal{L} \xrightarrow{i} \mathcal{U}' \xrightarrow{d'} \mathcal{W}' \xrightarrow{\pi} \mathcal{P} \to 0$ be two bilateral Lie type crossed modules. (MC_1) and (MC_2) are said to be linked if there exist Lie type superalgebras homomorphisms $\psi : \mathcal{U} \to \mathcal{U}'$ and $\phi : \mathcal{W} \to \mathcal{W}'$ such that the following diagrams

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{L} & \xrightarrow{i} & \mathcal{U} & \xrightarrow{d} & \mathcal{W} & \xrightarrow{\pi} & \mathcal{U} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{L} & \xrightarrow{i'} & \mathcal{U}' & \xrightarrow{d'} & \mathcal{W}' & \xrightarrow{\pi'} & \mathcal{U} & \longrightarrow & 0 \end{array} \tag{3.7}$$

and

$$\begin{array}{ccc} \mathcal{W} \otimes \mathcal{U} & \xrightarrow{\eta} & \mathcal{U} \\ {\scriptstyle \phi\otimes\psi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathcal{W}' \otimes \mathcal{U}' & \xrightarrow{\eta'} & \mathcal{U}' \end{array} \tag{3.8}$$

are commutative.

The following result gives a characterization of two linked bilateral Lie type crossed modules.

PROPOSITION 3.2.3. *Two Lie type crossed modules (MC_1) and (MC_2) are linked if and only if, there exist even linear maps $f : \mathcal{V} \to \mathcal{V}'$ and $g : \mathcal{V} \to \mathcal{L}$ such that*

(1) $\alpha(v, v') + g([v, v']) = \alpha'(f(v), f(v'))$,

(2) $\eta'_0(p \otimes f(v)) + [g(v), p] = \eta_0(p \otimes v) + g(d^{-1}[dv, p])$,

(3) $f(\beta(p, p')) = \beta'(p, p')$.

Proof. Recall that $\mathcal{U} \cong \mathcal{L} \oplus \mathcal{V}$, $\mathcal{U}' \cong \mathcal{L} \oplus \mathcal{V}'$, $\mathcal{W} \cong \mathcal{P} \oplus \mathcal{V}$ and $\mathcal{W}' \cong \mathcal{P} \oplus \mathcal{V}'$. According to the commutativity of the diagram (3.7), we have $\psi(x) = x$ and $\psi(v) \in \mathcal{L} \oplus \mathcal{V}'$ for all $x \in \mathcal{L}$ and $v \in \mathcal{V}$. Therefore, there exist even linear maps $f : \mathcal{V} \to \mathcal{V}'$ and $g : \mathcal{V} \to \mathcal{L}$ such that $\psi(x + v) = x + f(v) + g(v)$ for all $x + v \in \mathcal{U}$.

Let $x + v \in \mathcal{U}$ and $x' + v' \in \mathcal{U}$. Then by using the compatibility of ψ with the brackets and the fact that $\mathcal{L} \subseteq Z(\mathcal{U})$, we obtain

$$\begin{aligned} & \psi([x + v, x' + v']) = [\psi(x + v), \psi(x' + v')] \\ \Longleftrightarrow\ & \psi([v, v']) = [f(v) + g(v), f(v') + g(v')] \\ \Longleftrightarrow\ & \psi(\alpha(v, v') + [v, v']_{\mathcal{V}}) = [f(v), f(v')] \\ \Longleftrightarrow\ & \alpha(v, v') + f([v, v']_{\mathcal{V}}) + g([v, v']) = \alpha'(f(v), f(v')) + [f(v), f(v')]_{\mathcal{V}} \end{aligned}$$

which implies that $\alpha'(f(v), f(v')) = \alpha(v, v') + g([v, v'])$. Hence we have relation (1).

For relation (2), let $p \in \mathcal{P}$ and $v \in \mathcal{V}$. According to the commutativity of the diagram (3.8), we have

$$\begin{aligned}
& \eta' \circ (\phi \otimes \psi)(p \otimes v) = \psi \circ \eta(p \otimes v) \\
\Longleftrightarrow\ & \eta'(\phi(p))(\psi(v)) = \psi(\eta(p)(v)) \\
\Longleftrightarrow\ & \eta'(p)(f(v) + g(v)) = \psi\left(d^{-1}[dv, p] + \eta_0(p \otimes v)\right) \\
\Longleftrightarrow\ & \eta'(p)(f(v)) + \eta'(p)(g(v)) = \eta_0(p \otimes v) + f(d^{-1}[dv, p]) + g(d^{-1}[dv, p]) \\
\Longleftrightarrow\ & d'^{-1}[d' f(v), p] + \eta'_0(p \otimes f(v)) + [g(v), p] \\
& \qquad = \eta_0(p \otimes v) + f(d^{-1}[dv, p]) + g(d^{-1}[dv, p])\,,
\end{aligned}$$

hence $\eta'_0(p \otimes f(v)) + [g(v), p] = \eta_0(p \otimes v) + g(d^{-1}[dv, p])$.

For the proof of relation 3), let $p, p' \in \mathcal{P}$. The compatibility of ϕ with the brackets of $\mathcal{W}$ and $\mathcal{W}'$ gives us

$$\begin{aligned}
\phi([p, p']) = [\phi(p), \phi(p')] \quad & \Longleftrightarrow\ \phi([p, p']_{\mathcal{P}} - \beta(p, p')) = [p, p'] \\
& \Longleftrightarrow\ [p, p']_{\mathcal{P}} - f(\beta(p, p')) = [p, p']_{\mathcal{P}} - \beta'(p, p') \\
& \Longleftrightarrow\ f(\beta(p, p')) = \beta'(p, p')\,.
\end{aligned}$$

∎

4. Pseudo quadratic Lie type superalgebras and theirs associated Jacobi-Jordan superalgebras

In [11], we studied quadratic Lie-type superalgebras that are Lie-type superalgebras $(\mathcal{U}, [\ ,\ ])$ endowed with a nondegenerate, supersymmetric and invariant bilinear form B. We notice that the invariant property of B that is

$$B([x, y], z) = B(x, [y, z]) \qquad \forall\, x, y, z \in \mathcal{U} \tag{4.1}$$

plays an important role in the study of quadratic structure of Lie-type superalgebras. But the fact that the bracket of Lie-type superalgebras is not necessary supercommutative allows us to define a new type of invariance of B called left super-invariance that is different from the one in relation (4.1).

In this section, we aim to investigate Lie-type superalgebras endowed with a nondegenerate, supersymmetric and left super-invariant bilinear form B^{ig}.

Definition 4.1. Let $(\mathcal{U},[\,,\,])$ be a Lie-type superalgebra and $B^{ig}:\mathcal{U}\otimes\mathcal{U}\to\mathbb{K}$ a bilinear form. Then B^{ig} is said to be

- supersymmetric if $B^{ig}(x,y)=(-1)^{|x||y|}B^{ig}(y,x)$ for all $x\in\mathcal{U}_{|x|}$, $y\in\mathcal{U}_{|y|}$;
- nondegenerate if $B^{ig}(x,y)=0$ for all $y\in\mathcal{U}$ implies $x=0$.

Definition 4.2. Let $(\mathcal{U},[\,,\,])$ be a Lie-type superalgebra. A bilinear form B^{ig} over $\mathcal{U}$ is left super-invariant if

$$B^{ig}\left([x,y],z\right)=(-1)^{|x||y|}B^{ig}\left(y,[x,z]\right)\qquad\forall\, x\in\mathcal{U}_{|x|},\ y\in\mathcal{U}_{|y|},\ z\in\mathcal{U}_{|z|}\,.$$

If $(\mathcal{U},[\,,\,])$ is endowed with nondegenerate, supersymmetric and left super-invariant bilinear form B^{ig}, then $(\mathcal{U},[\,,\,],B^{ig})$ is called pseudo-quadratic Lie type superalgebra.

We can notice that the definition of left super-invariance is equivalent to say that for all $x\in\mathcal{U}_{|x|}$, we have L_x is B^{ig}-supersymmetric that is $B^{ig}\left(L_x(y),z\right)=(-1)^{|x||y|}B^{ig}\left(y,L_x(z)\right)$ for all $x\in\mathcal{U}_{|x|}$, $y\in\mathcal{U}_{|y|}$ and $z\in\mathcal{U}_{|z|}$.

Lemma 4.1. *Let $(\mathcal{U},[\,,\,],B^{ig})$ be a pseudo-quadratic Lie type superalgebra. If moreover B^{ig} is invariant then $(\mathcal{U},[\,,\,])$ is a Jacobi-Jordan superalgebra.*

Proof. Let $x\in\mathcal{U}_{|x|},y\in\mathcal{U}_{|y|}$ and $z\in\mathcal{U}_{|z|}$. Then

$$\begin{aligned}B^{ig}\Big([x,y]-(-1)^{|x||y|}[y,x],z\Big)&=B^{ig}\left([x,y],z\right)-(-1)^{|x||y|}B^{ig}\left([y,x],z\right)\\&=(-1)^{|x||y|}B^{ig}\left(y,[x,z]\right)-(-1)^{|x||y|}B^{ig}\left(y,[x,z]\right)=0\,,\end{aligned}$$

and since B^{ig} is nondegenerate then $[x,y]=(-1)^{|x||y|}[y,x]$. Hence $(\mathcal{U},[\,,\,])$ is a Jacobi-Jordan superalgebra. ∎

Let $(\mathcal{U},[\,,\,],B^{ig})$ be a pseudo-quadratic Lie type superalgebra. Define the even bilinear application $\wedge:\mathcal{U}\otimes\mathcal{U}\to\mathcal{U}$ by

$$B^{ig}\left([x,y],z\right)=B^{ig}\left(x,y\wedge z\right)\qquad\forall\, x\in\mathcal{U}_{|x|},\ y\in\mathcal{U}_{|y|},\ z\in\mathcal{U}_{|z|}\,;$$

$(\mathcal{U},\wedge)$ is a superalgebra and will be called the induced superalgebra of $(\mathcal{U},[\,,\,],B^{ig})$.

In what follows, we shall establish some properties of the induced superalgebra and show that the induced superalgebra of any pseudo-quadratic Lie type superalgebra is a Jacobi-Jordan superalgebra.

PROPOSITION 4.1. *Let $(\mathcal{U}, [\ ,\ ], B^{ig})$ be a pseudo-quadratic Lie type superalgebra and $(\mathcal{U}, \wedge)$ the induced superalgebra. Then we have:*

(1) *$(\mathcal{U}, \wedge)$ is a Jacobi-Jordan superalgebra;*

(2) *$x \wedge (y \wedge z) = [x, y \wedge z]$ for all $x \in \mathcal{U}_{|x|}$, $y \in \mathcal{U}_{|y|}$, $z \in \mathcal{U}_{|z|}$;*

(3) *L_x is a $(-1,-1)$-superderivation of $(\mathcal{U}, \wedge)$ that is*

$$[x, y \wedge z] = -[x, y] \wedge z - (-1)^{|x||y|} y \wedge [x, z] \quad \forall\, x \in \mathcal{U}_{|x|},\ y \in \mathcal{U}_{|y|},\ z \in \mathcal{U}_{|z|};$$

(4) *for all $n \in \mathbb{N} \setminus \{0\}$ we have $L^n_{\wedge x} = L^{n-1}_x \cdot L_{\wedge x}$, where $L_{\wedge x}(y) = x \wedge y$.*

Proof. Let $x \in \mathcal{U}_{|x|}, y \in \mathcal{U}_{|y|}, z \in \mathcal{U}_{|z|}$ and $t \in \mathcal{U}_{|t|}$. Let us show first the super-commutativity of $(\mathcal{U}, \wedge)$

$$\begin{aligned} B^{ig}\Big(z, x \wedge y - (-1)^{|x||y|} y \wedge x\Big) &= B^{ig}(z, x \wedge y) - (-1)^{|x||y|} B^{ig}(z, y \wedge x) \\ &= B^{ig}([z, x], y) - (-1)^{|x||y|} B^{ig}([z, y], x) \\ &= (-1)^{|x||z|} B^{ig}(x, [z, y]) - (-1)^{|x||y|} B^{ig}([z, y], x) \\ &= (-1)^{|x||z|} \big(B^{ig}(x, [z, y]) - B^{ig}(x, [z, y])\big) \;=\; 0\,, \end{aligned}$$

and since B^{ig} is nondegenerate then $x \wedge y = (-1)^{|x||y|} y \wedge x$. Hence the product $\wedge$ is supercommutative. Let us show that $x \wedge (y \wedge z) = -(x \wedge y) \wedge z - (-1)^{|x||y|} y \wedge (x \wedge z)$.

$$\begin{aligned} &B^{ig}\big(t, x \wedge (y \wedge z) + (x \wedge y) \wedge z + (-1)^{|x||y|} y \wedge (x \wedge z)\big) \\ &\quad= B^{ig}\big([[t, x], y], z\big) + B^{ig}\big([t, x \wedge y], z\big) + (-1)^{|x||y|} B^{ig}\big([[t, y], x], z\big) \\ &\quad= (-1)^{|y|(|x|+|t|)} \Big(B^{ig}\Big(y, [[t, x], z] + [t, [x, z]] + (-1)^{|x||z|}[[t, z], x]\Big)\Big) \\ &\quad= (-1)^{|y|(|x|+|t|)} \Big(B^{ig}\Big(y, -(-1)^{|x||t|}[x, [t, z]] + (-1)^{|x||z|}[[t, z], x]\Big)\Big) \\ &\quad= (-1)^{|y|(|x|+|t|)} \Big(B^{ig}\Big(y, -(-1)^{|x||z|}[[t, z], x] + (-1)^{|x||z|}[[t, z], x]\Big)\Big) = 0\,, \end{aligned}$$

which implies that $x \wedge (y \wedge z) = -(x \wedge y) \wedge z - (-1)^{|x||y|} y \wedge (x \wedge z)$. Therefore $(\mathcal{U}, \wedge)$ is a Jacobi-Jordan superalgebra. Let us show that $x \wedge (y \wedge z) = [x, y \wedge z]$

$$\begin{aligned} B^{ig}(t, x \wedge (y \wedge z) - [x, y \wedge z]) &= B^{ig}([t, x], y \wedge z) - (-1)^{|x||t|} B^{ig}([x, t], y \wedge z) \\ &= B^{ig}([[t, x], y], z) - (-1)^{|x||t|} B^{ig}([[x, t], y], z) \\ &= B^{ig}\Big(\Big[[t, x] - (-1)^{|x||t|}[x, t], y\Big], z\Big) \;=\; 0\,, \end{aligned}$$

and the fact that B^{ig} is nondegenerate gives us $x\wedge(y\wedge z) = [x, y\wedge z]$. A similar computation of (2) gives us relation (3). For relation (4), we will proceed by induction over n. If $n = 2$ then according to (2) we have

$$L^2_{\wedge x}(y) = x \wedge (x \wedge y) = [x, x \wedge y] = L_x \cdot L_{\wedge x}(y)$$

therefore $L^2_{\wedge x} = L_x \cdot L_{\wedge x}$.

Let us assume that $L^n_{\wedge x} = L^{n-1}_x \cdot L_{\wedge x}$ and show that $L^{n+1}_{\wedge x} = L^n_x \cdot L_{\wedge x}$. By using the relation (2) above and the induction hypothesis, we obtain

$$\begin{aligned} L^{n+1}_{\wedge x}(y) \;&=\; L^n_{\wedge x}(x \wedge y) \;=\; L^{n-1}_x \cdot L_{\wedge x}(x \wedge y) \\ &=\; L^{n-1}_x(x \wedge (x \wedge y)) \;=\; L^{n-1}_x([x, x \wedge y]) \;=\; L^n_x \circ L_{\wedge x}(y)\,. \end{aligned}$$

Therefore, we have $L^{n+1}_{\wedge x} = L^n_x \cdot L_{\wedge x}$ for all $n \in \mathbb{N} \setminus \{0\}$. ∎

The above proposition shows that the superalgebras $(\mathcal{U}\wedge\mathcal{U}, \wedge)$ and $(\mathcal{U}\wedge\mathcal{U}, [\,,\,])$ are the same and the existence of a nondegenerate, supersymmetric and left super-invariant bilinear form over Lie-type superalgebra $(\mathcal{U}, [\,,\,])$ induces over the underlying $\mathbb{Z}_2$-graded vector space $\mathcal{U}$ a structure of Jacobi-Jordan superalgebra. The following result gives the expression of the product $\wedge$ of the induced Jacobi-Jordan superalgebra.

Proposition 4.2. *Let $(\mathcal{U}, [\,,\,], B^{ig})$ be a pseudo quadratic symmetric Lie type superalgebra. Then the induced product $\wedge : \mathcal{U}\otimes\mathcal{U} \to \mathcal{U}$ is given as follow*

$$x \wedge y = x * y - \psi(x, y) \qquad \forall\, x, y \in \mathcal{U}$$

where $x * y = \dfrac{1}{2}\big([x, y] + (-1)^{|x||y|}[y, x]\big)$ *and* $\psi \in (Z_{tLie}(\mathcal{U},\mathcal{U}))_{\bar 0}\,.$

Proof. Set $\mu(x, y) = [x, y] - x \wedge y$. Then we have $\mu(x, y) \in Z^r(\mathcal{U})$ for all $x, y \in \mathcal{U}$. Indeed, let $t, z \in \mathcal{U}$. We have

$$\begin{aligned} &B^{ig}([z, \mu(x, y)], t) \\ &\quad= B^{ig}([z, [x, y]], t) - B^{ig}([z, x \wedge y], t) \\ &\quad= B^{ig}([z, [x, y]], t) - (-1)^{|z|(|x|+|y|)}B^{ig}(x \wedge y, [z, t]) \\ &\quad= B^{ig}([z, [x, y]], t) - (-1)^{|t|(|x|+|y|)}B^{ig}([[z, t], x], y) \\ &\quad= (-1)^{|z|(|x|+|y|)}B^{ig}([x, y], [z, t]) - (-1)^{|t|(|x|+|y|)}B^{ig}([[z, t], x], y) \\ &\quad= (-1)^{|z|(|x|+|y|)+|x||y|}B^{ig}(y, [x, [z, t]]) - (-1)^{|t|(|x|+|y|)}B^{ig}([[z, t], x], y) \\ &\quad= (-1)^{|z|(|x|+|y|)+|y|(|z|+|t|)}B^{ig}([x, [z, t]], y) - (-1)^{|t|(|x|+|y|)}B^{ig}([[z, t], x], y) \end{aligned}$$

$$= (-1)^{|x||z|+|y||t|} B^{ig}([x,[z,t]],y) - (-1)^{|t|(|x|+|y|)} B^{ig}([[z,t],x],y)$$
$$= (-1)^{|t|(|x|+|y|)} B^{ig}([[z,t],x],y) - (-1)^{|t|(|x|+|y|)} B^{ig}([[z,t],x],y) \;=\; 0$$

and the fact that B^{ig} is nondegenerate implies that $\mu(x,y) \in Z^r(\mathcal{U})$ for all $x, y \in \mathcal{U}$. According to Proposition 4.1, the product $\wedge$ is supercommutative. Therefore, for all $x \in \mathcal{U}_{|x|}$ and $y \in \mathcal{U}_{|y|}$ we have

$$\begin{aligned}
x \wedge y &= \frac{1}{2}\Big(x \wedge y + (-1)^{|x||y|} y \wedge x\Big) \\
&= \frac{1}{2}\Big([x,y] - \mu(x,y) + (-1)^{|x||y|}[y,x] - (-1)^{|x||y|}\mu(y,x)\Big) \\
&= \frac{1}{2}\Big([x,y] + (-1)^{|x||y|}[y,x]\Big) - \frac{1}{2}\Big(\mu(x,y) + (-1)^{|x||y|}\mu(y,x)\Big) \\
&= x * y - \psi(x,y)\,,
\end{aligned}$$

where

$$x * y := \frac{1}{2}\Big([x,y] + (-1)^{|x||y|}[y,x]\Big),\;\; \psi(x,y) := \frac{1}{2}\Big(\mu(x,y) + (-1)^{|x||y|}\mu(y,x)\Big).$$

Since $\wedge$ and $*$ are supercommutative and satisfy the Lie-type identity then ψ is a bi-cocycle of $(\mathcal{U}, \wedge)$ with value in $Z^r(\mathcal{U})$. ∎

LEMMA 4.2. *Let $(\mathcal{U}, [\ ,\ ], B^{ig})$ be a pseudo quadratic symmetric Lie type superalgebra. Then the application $\wedge : \mathcal{U} \otimes \mathcal{U} \to \mathcal{U}$ is a bi-cocycle of $(\mathcal{U}, [\ ,\ ])$.*

Proof. Let $x \in \mathcal{U}_{|x|}, y \in \mathcal{U}_{|y|}, z \in \mathcal{U}_{|z|}$ and $t \in \mathcal{U}_{|t|}$. We want to show that $x \wedge [y,z] + [x,y] \wedge z + (-1)^{|x||y|} y \wedge [x,z] = 0$ and $[y,z] \wedge x + y \wedge [z,x] + (-1)^{|x||z|}[y,x] \wedge z = 0$. For the first relation we have

$$\begin{aligned}
&B^{ig}\Big(t, x \wedge [y,z] + [x,y] \wedge z + (-1)^{|x||y|} y \wedge [x,z]\Big) \\
&\quad = B^{ig}\left(t, x \wedge [y,z]\right) + B^{ig}\left(t, [x,y] \wedge z\right) + (-1)^{|x||y|} B^{ig}\left(t, y \wedge [x,z]\right) \\
&\quad = B^{ig}\left([t,x], [y,z]\right) + B^{ig}\left([t,[x,y]], z\right) + (-1)^{|x||y|} B^{ig}\left([t,y], [x,z]\right) \\
&\quad = B^{ig}\left([[t,x],y], z\right) + B^{ig}\left([t,[x,y]], z\right) + (-1)^{|x||t|} B^{ig}\left([x,[t,y]], z\right) \\
&\quad = B^{ig}\Big([[t,x],y] + [t,[x,y]] + (-1)^{|x||t|}[x,[t,y]], z\Big) \;=\; 0\,,
\end{aligned}$$

the fact that B^{ig} is nondegenerate gives us

$$x \wedge [y,z] + [x,y] \wedge z + (-1)^{|x||y|} y \wedge [x,z] = 0$$

and by proceeding in the same way we show the second relation. ∎

4.1. Inductive description of pseudo-quadratic Lie type superalgebras. In this subsection, we continue our study of pseudo-quadratic Lie type superalgebras. We extend the notion of double extension to pseudo-quadratic Lie type superalgebras by using central extensions and representations. we give an inductive description of pseudo-quadratic Lie type superalgebras.

Lemma 4.1.1. *Let $(\mathcal{U}, [\ ,\ ], B^{ig})$ be a symmetric pseudo-quadratic Lie type superalgebra and $\gamma \in \mathrm{End}(\mathcal{U})_{\bar{0}}$. Then the bilinear application $\psi : \mathcal{U} \otimes \mathcal{U} \to \mathbb{K}$ defined by $\psi(x, y) = B^{ig}(\gamma(x), y)$ is a bi-cocycle if and only if*

$$\gamma([x, y]) = -[x, \gamma(y)] - (-1)^{|x||y|}[y, \gamma(x)] \qquad \forall\, x \in \mathcal{U}_{|x|},\ y \in \mathcal{U}_{|y|}$$

and

$$(\bar{\gamma} - \gamma)([\mathcal{U}, \mathcal{U}]) = 0\,,$$

where $\bar{\gamma}$ is the adjoint of γ with respect to B^{ig}.

Proof. Let $x \in \mathcal{U}_{|x|}, y \in \mathcal{U}_{|y|}$ and $z \in \mathcal{U}_{|z|}$. By using the fact that B^{ig} is supersymmetric and left super-invariant, we have

$$\begin{aligned}
& \psi(x, [y, z]) + \psi([x, y], z) + (-1)^{|x||y|}\psi(y, [x, z]) = 0 \\
\Longleftrightarrow\ & B^{ig}(\gamma(x), [y, z]) + B^{ig}(\gamma([x, y]), z) + (-1)^{|x||y|}B^{ig}(\gamma(y), [x, z]) = 0 \\
\Longleftrightarrow\ & (-1)^{|x||y|}B^{ig}([y, \gamma(x)], z) + B^{ig}(\gamma([x, y]), z) + B^{ig}([x, \gamma(y)], z) = 0 \\
\Longleftrightarrow\ & B^{ig}\Big(\gamma([x, y]) + [x, \gamma(y)] + (-1)^{|x||y|}[y, \gamma(x)]\,, z\Big) = 0\,,
\end{aligned}$$

and since B^{ig} is nondegenerate this equivalent to $\gamma([x, y]) = -[x, \gamma(y)] - (-1)^{|x||y|}[y, \gamma(x)]$. On the other hand, we have

$$\begin{aligned}
& \psi(x, [y, z]) = (-1)^{|x|(|y|+|z|)}\psi([y, z], x) \\
\Longleftrightarrow\quad & B^{ig}(\gamma(x), [y, z]) = (-1)^{|x|(|y|+|z|)}B^{ig}(\gamma([y, z]), x) \\
\Longleftrightarrow\quad & B^{ig}(x, \bar{\gamma}([y, z])) = B^{ig}(x, \gamma([y, z])) \Leftrightarrow \bar{\gamma}([y, z]) = \gamma([y, z])\,.
\end{aligned}$$

This proves the lemma. ∎

Lemma 4.1.2. *Let $(\mathcal{U}, [\ ,\ ])$ be a Lie type superalgebra, V a $\mathbb{Z}_2$-graded vector space and $\psi : \mathcal{U} \otimes \mathcal{U} \to V$ a bi-cocycle. Then the space $\overline{\mathcal{U}} := \mathcal{U} \oplus V$ endowed with the product*

$$[x + u, y + w] = [x, y] + \psi(x, y) \qquad \forall\, x + u, y + w \in \overline{\mathcal{U}}$$

is a Lie type superalgebra.

Proof. Straightforward calculation. ∎

The Lie-type superalgebra $\overline{\mathcal{U}}$ constructed in the Lemma 4.1.2 is called central extension of $\mathcal{U}$ by means of ψ.

Lemma 4.1.3. *Let $(\mathcal{U},[\,,\,])$ be a Lie type superalgebra, H a $\mathbb{Z}_2$-graded vector space and $(\varphi,\lambda)\in \mathrm{Rep}_H^{\mathcal{U}}$. Then the space $\mathcal{U}_1 := \mathcal{U}\oplus H$ endowed with the product*

$$[x+h,y+h'] = [x,y]+\varphi_x(h')+(-1)^{|x||y|}\lambda_y(h') \quad \forall\, x+h\in(\mathcal{U}_1)_{|x|}\,,\ y+h\in(\mathcal{U}_1)_{|y|}$$

is a Lie type superalgebra.

Proof. Straightforward calculation. ∎

Given a vector space $\mathcal{H}$, we denote $\mathcal{H}^*$ the dual space. The ground field $\mathbb{K}$ admits a $\mathbb{Z}_2$-graduation as follows

$$\mathbb{K}=\mathbb{K}_{\bar 0}\oplus\mathbb{K}_{\bar 1} \qquad \text{with}\ \ \mathbb{K}_{\bar 0}=\mathbb{K}\ \ \text{and}\ \ \mathbb{K}_{\bar 1}=\{0\}\,.$$

The pseudo-quadratic Lie type superalgebra $(\mathcal{U},[\,,\,],B^{ig})$ is called even (resp. odd) pseudo-quadratic Lie type superalgebra if $\mid B^{ig}\mid=\bar 0$ (resp. $\mid B^{ig}\mid=\bar 1$). The following result gives the double extension of pseudo-quadratic Lie type superalgebra.

Theorem 4.1.1. *Let $(\mathcal{U},[\,,\,],B^{ig})$ be a symmetric pseudo-quadratic Lie type superalgebra, $\mathcal{H}=\mathbb{K}e$ an one dimensional $\mathbb{Z}_2$-graded vector space, $\gamma\in\mathrm{End}(\mathcal{U})_{\bar 0}$ and $(F,G)\in\mathrm{Rep}_{\mathcal{H}}^{\mathcal{U}}$ such that*

(1) $\gamma([x,y]) = -[x,\gamma(y)]-(-1)^{|x||y|}[y,\gamma(x)]$ *for all* $x\in\mathcal{U}_{|x|}$, $y\in\mathcal{U}_{|y|}$;

(2) $(\bar\gamma-\gamma)([\mathcal{U},\mathcal{U}])=0\,$;

(3) $F_x\cdot F_y = G_x\cdot F_y,\ \ F_{[x,y]}=G_{[x,y]}$ *and* $F_x\cdot G_y=G_x\cdot G_y\,.$

Then the space $\tilde{\mathcal{U}}=\mathcal{H}\oplus\mathcal{U}\oplus\mathcal{H}^*$ endowed with the product

$$[h+x+f,h'+y+g] = [x,y]_{\mathcal{U}}+B^{ig}(\gamma(x),y)\,e^*+F_x(h')+(-1)^{|x||y|}G_y(h)$$

is a symmetric Lie type superalgebra.

Proof. Set $\psi : \mathcal{U}\otimes\mathcal{U} \to H^*$ defined by $\psi(x,y) = B^{ig}(\gamma(x),y)e^*$. Then according to Lemma 4.1.1 and the relations (1) and (2) we have ψ is a bi-cocycle. Hence by Lemma 4.1.2, the space $\overline{\mathcal{U}} := \mathcal{U}\oplus H^*$ endowed with the bracket

$$[x+f,y+g] = [x,y]+\psi(x,y) = [x,y]+B^{ig}(\gamma(x),y)e^*$$

is a Lie type superalgebra. Now let us define $\overline{F},\overline{G} : \overline{\mathcal{U}} \to \mathrm{End}(H)$ by $\overline{F}_{x+f} = F_x$ and $\overline{G}_{x+f} = G_x$. Since $(F,G) \in \mathrm{Rep}_H^{\mathcal{U}}$ then it is clear that $(\overline{F},\overline{G}) \in \mathrm{Rep}_H^{\overline{U}}$. Therefore, according to Lemma 4.1.3, the space $\tilde{\mathcal{U}} := \overline{\mathcal{U}}\oplus H$ endowed with the bracket

$$\begin{aligned}[(x+f)+h,(y+g)+h'] &= [x+f,y+g]_{\overline{\mathcal{U}}} + \overline{F}_{x+f}(h') + (-1)^{|x||y|}\overline{G}_{y+g}(h)\\ &= [x,y]+\psi(x,y)+F_x(h')+(-1)^{|x||y|}G_y(h)\\ &= [x,y]+B^{ig}(\gamma(x),y)e^* + F_x(h')+(-1)^{|x||y|}G_y(h)\end{aligned}$$

is a Lie type superalgebra. And by using (3), one can easily sees that $\tilde{\mathcal{U}}$ is symmetric. ∎

The triplet (F,G,γ) in the above theorem is called context of double extension and the Lie-type superalgebra $\tilde{\mathcal{U}}$ is the double extension of $\mathcal{U}$ by $\mathcal{H}$ by means of (F,G,γ).

In what follows, we will show that odd pseudo-quadratic Lie type superalgebra can be constructed from a finite number of odd pseudo-quadratic Jacobi-Jordan superalgebras via the notion of double extension.

Theorem 4.1.2. *Let $(\mathcal{U},[\,,\,],B^{ig})$ be a symmetric odd pseudo-quadratic Lie type superalgebra such that $\mathrm{Ker}(\mathcal{U})_{\bar 0} \neq \{0\}$. Then $\mathcal{U}$ is isomorphic to a double extension of a Lie type superalgebra.*

Proof. Since $\mathrm{Ker}(\mathcal{U})_{\bar 0} \neq \{0\}$ then there exists $0 \neq e \in \mathrm{Ker}(\mathcal{U})_{\bar 0}$ and the fact that B^{ig} is a nondegenerate odd bilinear form implies that there exists $0 \neq d \in \mathcal{U}_{\bar 1}$ such that $B^{ig}(e,d) = 1$. Set $\mathcal{H} = \mathbb{K}e, \mathcal{V} = \mathbb{K}d$ and $\mathcal{E} = (\mathcal{H}\oplus\mathcal{V})^{\perp}$. Following the same way as in the proof of [11, Theorem 6.3], one can easily see that $\mathcal{U} = \mathcal{H}\oplus\mathcal{E}\oplus\mathcal{V}$ and $\mathcal{H}^{\perp} = \mathcal{H}\oplus\mathcal{E}$ is an ideal of $\mathcal{U}$. Therefore $[\mathcal{E},\mathcal{E}] \subseteq \mathcal{H}\oplus\mathcal{E}$. Hence

$$[x,y]_{\mathcal{E}} = [x,y]-\psi(x,y)e \qquad \forall\, x,y\in\mathcal{E}\,, \tag{4.2}$$

where $\psi : \mathcal{E}\otimes\mathcal{E} \to \mathbb{K}$ and $[\,,\,]_{\mathcal{E}} : \mathcal{E}\otimes\mathcal{E} \to \mathcal{E}$. By using the Lie-type identity (1.1) of $(\mathcal{U},[\,,\,])$ and relation (4.2) we show that $(\mathcal{E},[\,,\,]_{\mathcal{E}})$ is a symmetric Lie-type superalgebra and ψ is a bi-cocycle.

Since $\mathcal{U} = \mathcal{H} \oplus \mathcal{E} \oplus \mathcal{V}$ and $\mathcal{H}^\perp = \mathcal{H} \oplus \mathcal{E}$ is an ideal of $\mathcal{U}$ then the bracket of the Lie-type superalgebra $\mathcal{U}$ is given by

$$\begin{aligned}
[x,d] &= \alpha_x^\varphi e + \xi(x) && \text{with } \alpha_x^\varphi \in \mathbb{K}, \xi \in \mathrm{End}(\mathcal{E})\,,\\
[d,x] &= \alpha_x^\phi e + \delta(x) && \text{with } \alpha_x^\phi \in \mathbb{K}, \delta \in \mathrm{End}(\mathcal{E})\,,\\
[d,d] &= \alpha e + x_0 + \lambda d && \text{where } \alpha, \lambda \in \mathbb{K}, x_0 \in \mathcal{E}\,.
\end{aligned}$$

Define the applications $\varphi, \phi : \mathcal{E} \to \mathrm{End}(\mathcal{H})$ by $\varphi_x(e) = \alpha_x^\varphi e$ and $\phi_x(e) = \alpha_x^\phi e$ for all $x \in \mathcal{E}_{|x|}$. By proceeding in a same way as in the proof of [11, Theorem 6.3], we show that (φ, ϕ, ξ) is a context of double extension of $\mathcal{E}$ by $\mathcal{H}$. Hence $\tilde{\mathcal{E}} = \mathcal{H} \oplus \mathcal{E} \oplus \mathcal{H}^*$ is a double extension of $(\mathcal{E}, [\,,\,]_{\mathcal{E}}, B_{\mathcal{E}}^{ig})$ by $\mathcal{H}$ by means of (φ, ϕ, ξ) where $B_{\mathcal{E}}^{ig}$ is the restriction of B^{ig} over $\mathcal{E} \otimes \mathcal{E}$. It is easy to see that $\tilde{\mathcal{E}}$ is isomorphic to $\mathcal{U} = \mathcal{H} \oplus \mathcal{E} \oplus \mathcal{V}$. ∎

Corollary 4.1.1. *Let $(\mathcal{U}, [\,,\,]B^{ig})$ be a symmetric odd pseudo-quadratic Lie type superalgebra which is not a Jacobi-Jordan superalgebra. Then $\mathcal{U}$ can be obtained from a finite number of Jacobi-Jordan superalgebras via the double extension.*

Proof. The proof is analogous to the one of [11, Corollary 6.4]. ∎

4.2. The Jacobi-Jordan superalgebra $(\mathcal{U}, \wedge)$. In the previous subsection, we have proved through an inductive description that the study of odd pseudo-quadratic Lie type superalgebras can be reduced to the study of pseudo-quadratic Jacobi-Jordan superalgebras. Then it is necessary to study pseudo-quadratic Jacobi-Jordan superalgebras. For that study, we will reconsider the double extension defined above with some modifications because unlike Lie type superalgebras, Jacobi-Jordan superalgebra $(\mathcal{U}, \wedge)$ are supercommutative . Moreover, since all pseudo-quadratic Lie type superalgebras $(\mathcal{U}, [\,,\,], B^{ig})$ is isomorphic to a double extension of a pseudo-quadratic Lie type superalgebra $(\mathcal{E}, [\,,\,]_{\mathcal{E}}, B_{\mathcal{E}}^{ig})$ (see Theorem 4.1.2), we shall investigate the relationships between the Jacobi-Jordan superalgebra $(\mathcal{U}, \wedge)$ induced by $(\mathcal{U}, [\,,\,], B^{ig})$ and the Jacobi-Jordan superalgebra $(\mathcal{E}, \wedge_{\mathcal{E}})$ induced by $(\mathcal{E}, [\,,\,]_{\mathcal{E}}, B_{\mathcal{E}}^{ig})$.

Definition 4.2.1. Let $(\mathcal{U}, \wedge)$ be a Jacobi-Jordan superalgebra and $V = V_{\bar 0} \oplus V_{\bar 1}$ a $\mathbb{Z}_2$-graded vector space. A representation of $\mathcal{U}$ in V is given by an even linear map $\varphi : \mathcal{U} \to (\mathrm{End}(V))_{\bar 0}$ such that

$$\varphi_{x \wedge y} = -\varphi_x \cdot \varphi_y - (-1)^{|x||y|} \varphi_y \cdot \varphi_x \qquad \forall\, x \in \mathcal{U}_{|x|}\,,\, y \in \mathcal{U}_{|y|}\,.$$

LEMMA 4.2.1. *Let $(\mathcal{U}, \wedge)$ be a Jacobi-Jordan superalgebra and $\varphi \in \mathrm{Rep}_V^{\mathcal{U}}$. Then the space $\mathcal{U} \oplus V$ endowed with the product*

$$(x+u) \,\bar{\wedge}\, (y+v) = x \wedge y + \varphi_x(v) + (-1)^{|x||y|}\varphi_y(u)$$

for all $x+u \in (\mathcal{U} \oplus V)_{|x|}$ and $y+v \in (\mathcal{U} \oplus V)_{|y|}$, is a Jacobi-Jordan superalgebra.

Proof. Straightforward computation. ∎

The following result gives the double extension of a Jacobi-Jordan superalgebra.

THEOREM 4.2.1. *Let $(\mathcal{U}, \wedge)$ be a Jacobi-Jordan superalgebra, $\mathcal{H}$ a $\mathbb{Z}_2$-graded vector space, $\varphi \in \mathrm{Rep}_{\mathcal{H}}^{\mathcal{U}}$ and $\psi : \mathcal{U} \otimes \mathcal{U} \to \mathcal{H}^*$ a bi-cocycle of $(\mathcal{U}, \wedge)$. Then the space $\tilde{\mathcal{U}} := \mathcal{H} \oplus \mathcal{U} \oplus \mathcal{H}^*$ with the product*

$$(h+x+f) \,\bar{\bar{\wedge}}\, (h'+y+g) = x \wedge y + \psi(x,y) + \varphi_x(h') + (-1)^{|x||y|}\varphi_y(h)$$

for all $h+x+f \in \tilde{\mathcal{U}}_{|x|}$ and $h'+y+g \in \tilde{\mathcal{U}}_{|y|}$, is a Jacobi-Jordan superalgebra.

Proof. The proof is analogous to the one of Theorem 4.1.1. ∎

The couple (φ, ψ) is called JJ-context of double extension and the Jacobi-Jordan superalgebra $\tilde{\mathcal{U}}$ obtained in the Theorem 4.2.1 is called the JJ-double extension of $(\mathcal{U}, \wedge)$ by $\mathcal{H}$ by means of (φ, ψ).

Given a symmetric odd pseudo-quadratic Lie type superalgebra $(\mathcal{U}, [\,,\,], B^{ig})$ such that $\mathrm{Ker}(\mathcal{U})_{\bar{0}} \neq \{0\}$, let $(\mathcal{U}, \wedge)$ be the Jacobi-Jordan superalgebra induced by $(\mathcal{U}, [\,,\,])$. According to Theorem 4.1.2, $\mathcal{U}$ is isomorphic to a double extension $\mathcal{H} \oplus \mathcal{E} \oplus \mathcal{H}^*$. Consider the Jacobi-Jordan superalgebra $(\mathcal{E}, \wedge_{\mathcal{E}})$ induced by $(\mathcal{E}, [\,,\,]_{\mathcal{E}})$. Then we have the following result:

THEOREM 4.2.2. *The Jacobi-Jordan superalgebra $(\mathcal{U}, \wedge)$ is the JJ-double extension of $(\mathcal{E}, \wedge_{\mathcal{E}})$.*

Proof. Since $\mathrm{Ker}(\mathcal{U})_{\bar{0}} \neq \{0\}$ and B^{ig} is odd and nondegenerate then, there exists $0 \neq e \in \mathrm{Ker}(\mathcal{U})_{\bar{0}}$ and $d \in \mathcal{U}_{\bar{1}}$ such that $B^{ig}(e,d) = 1$. Put $\mathcal{H} = \mathbb{K}e$, $\mathcal{V} = \mathbb{K}d$ and $\mathcal{E} = (\mathcal{H} \oplus \mathcal{V})^{\perp}$.

By using the fact that B^{ig} is odd and nondegenerate we obtain $\mathcal{U} = \mathcal{H} \oplus \mathcal{E} \oplus \mathcal{V}$. Let us show that $\mathcal{H}^{\perp}$ is an ideal of $(\mathcal{U}, \wedge)$. Let $a \in \mathrm{Ker}(\mathcal{U})_{|a|}, x \in \mathcal{U}_{|x|}$ and $y \in \mathcal{U}_{|y|}$ then we have $B^{ig}(y, a \wedge x) = B^{ig}([y,a],x) = 0$, therefore $a \wedge x = 0$ because B^{ig} is nondegenerate. Which implies that $\mathrm{Ker}(\mathcal{U}) \subseteq Z_{\wedge}(\mathcal{U})$, and since

$\mathcal{H} \subseteq \mathrm{Ker}(\mathcal{U})$ then $\mathcal{H}$ is an ideal of $(\mathcal{U}, \wedge)$. Therefore one can easily show that $\mathcal{H}^{\perp} = \mathcal{H} \oplus \mathcal{E}$ is an ideal of $(\mathcal{U}, \wedge)$.

Hence the product in $(\mathcal{U}, \wedge)$ is given as follow, let $x \in \mathcal{E}_{|x|}$ and $y \in \mathcal{E}_{|y|}$ then

$$\begin{aligned}
&x \wedge y = (-1)^{|x||y|} y \wedge x = x \wedge_{\mathcal{E}} y + \psi_{\mathcal{E}}(x, y)e\,,\\
&\qquad\qquad \psi_{\mathcal{E}} : \mathcal{E} \otimes \mathcal{E} \to \mathbb{K}\,, \quad \wedge_{\mathcal{E}} : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E},\\
&x \wedge d = (-1)^{|x|} d \wedge x = \pi(x) + \lambda(x)e\,, \qquad \pi \in \mathrm{End}(\mathcal{E})\,, \quad \lambda \in \mathcal{E}^*,\\
&d \wedge d = 0 \qquad \text{because} \ \ d \wedge d = (-1)^{|d|} d \wedge d = -d \wedge d\,.
\end{aligned}$$

Let us show that $(\mathcal{E}, \wedge_{\mathcal{E}})$ is a Jacobi-Jordan superalgebra induced by $(\mathcal{E}, [\ ,\ ]_{\mathcal{E}})$. Let $x \in \mathcal{E}_{|x|}, y \in \mathcal{E}_{|y|}$ and $z \in \mathcal{E}_{|z|}$ by using relation (4.2) in the proof of Theorem 4.1.2, we have

$$\begin{aligned}
B^{ig}_{\mathcal{E}}\left([x, y]_{\mathcal{E}}, z\right) &= B^{ig}\left([x, y]_{\mathcal{U}} - \psi(x, y)e, z\right) = B^{ig}\left([x, y], z\right)\\
&= B^{ig}\left(x, y \wedge z\right) = B^{ig}\left(x, y \wedge_{\mathcal{E}} z + \psi_{\mathcal{E}}(y, z)e\right) = B^{ig}_{\mathcal{E}}\left(x, y \wedge_{\mathcal{E}} z\right)
\end{aligned}$$

therefore $(\mathcal{E}, \wedge_{\mathcal{E}})$ is a Jacobi-Jordan superalgebra induced by $(\mathcal{E}, [\ ,\ ]_{\mathcal{E}})$. Let us consider now the application $\varphi : (\mathcal{E}, \wedge_{\mathcal{E}}) \to \mathrm{End}(\mathcal{H})_{\bar{0}}$ defined by $\varphi_x(e) = \lambda(x)e$. It is clear that if $x \in \mathcal{E}_{\bar{1}}$ then $\lambda(x) = 0$ because φ_x changes the degree of e. If $x \in \mathcal{E}_{\bar{0}}$ then since B^{ig} is odd and $d \in \mathcal{U}_{\bar{1}}$, we have

$$0 = B^{ig}\left(x \wedge d, d\right) = B^{ig}\left(\pi(x) + \lambda(x)e, d\right) = \lambda(x)\,,$$

hence $\lambda(x) = 0$. Which implies that φ is a trivial representation of $(\mathcal{E}, \wedge_{\mathcal{E}})$ in $\mathcal{H}$. Moreover, from the relation

$$x \wedge y = (-1)^{|x||y|} y \wedge x = x \wedge_{\mathcal{E}} y + \psi_{\mathcal{E}}(x, y)e$$

we can deduce that $\psi_{\mathcal{E}}$ is a bi-cocycle of $(\mathcal{E}, \wedge_{\mathcal{E}})$. Therefore, $(\varphi, \psi_{\mathcal{E}})$ is a JJ-context of double extension of $(\mathcal{E}, \wedge_{\mathcal{E}})$. and according to Theorem 4.2.1 we obtain that the JJ-double extension $(\mathcal{H} \oplus \mathcal{E} \oplus \mathcal{H}^*, \wedge)$ of $(\mathcal{E}, \wedge_{\mathcal{E}})$ by $\mathcal{H}$ by means of $(\varphi, \psi_{\mathcal{E}})$. One can easily sees that $(\mathcal{U} = \mathcal{H} \oplus \mathcal{E} \oplus \mathcal{V}, \wedge)$ is isomorphic to $(\mathcal{H} \oplus \mathcal{E} \oplus \mathcal{H}^*, \wedge)$. This implies that $(\mathcal{U}, \wedge)$ is the JJ-double extension of $(\mathcal{E}, \wedge_{\mathcal{E}})$. ∎

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