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Perturbation ideals and Fredholm theory in Banach algebras

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Abstract: In this paper we characterize perturbation ideals of sets that generate the familiar spectra in Fredholm theory.

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1. Introduction and preliminaries

In 1971, Lebow and Schechter in [10] introduced and studied the notion of perturbation classes. They proved results that will be useful in establishing new results in this paper. The sets which will be of our interest are those in Fredholm theory such as *Fredholm elements*, *Weyl elements, Browder elements and almost invertible Fredholm elements*. Another goal in this paper is to identify whether the particular set of interest is a regularity or a semiregularity. The concepts of a regularity and a semiregularity were first identified by V. Kordula and V. Müller ([9] and [15]). This was a new axiomatic framework to spectral theory which was an improvement to the approach by W. Żelazko ([18]) in the sense that there exist spectra for a single element in a Banach algebra which could not be covered by the axiomatic theory of Żelazko.

Let $\mathcal{A}$ be a complex Banach algebra with a unit element $1_{\mathcal{A}}$ and for any $\lambda \in \mathbb{C} \setminus \{0\}$, simply write λ for $\lambda \cdot 1_{\mathcal{A}}$. We will denote by $\mathcal{A}^{-1}$ the group of all invertible elements in $\mathcal{A}$ while $\mathcal{A}_l^{-1}$ ($\mathcal{A}_r^{-1}$) represents the set of all left (right) invertible elements in $\mathcal{A}$. We say $p \in \mathcal{A}$ is an *idempotent* if it satisfies $p = p^2$. The set

$$\sigma(x) = \sigma_{\mathcal{A}}(x) = \{\lambda \in \mathbb{C} : \lambda - x \notin \mathcal{A}^{-1}\},$$

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is the usual spectrum of $x \in \mathcal{A}$ and the corresponding *spectral radius of x in $\mathcal{A}$* is denoted by

$$r(x) = r_{\mathcal{A}}(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

It is well known that $\sigma(x)$ is non-empty and a compact subset of the complex plane $\mathbb{C}$. The element $a \in \mathcal{A}$ is called a *quasinilpotent element* if $\sigma(a) = \{0\}$ and the set of all quasinilpotent elements in a Banach algebra $\mathcal{A}$ is denoted by $QN(\mathcal{A})$. If $K \subseteq \mathbb{C}$ or $K \subseteq \mathcal{A}$, then we will denote by $\mathtt{acc}\, K$, the set of accumulation points of K. The topological boundary of K is denoted by ∂K and its closure is denoted by $\overline{K}$. Let M be a subset of a Banach algebra $\mathcal{A}$. The *commutant* of M is defined by $\mathtt{comm}(M) = \{a \in \mathcal{A} : am = ma,\ m \in M\}$. By an *ideal* in $\mathcal{A}$ we mean a two-sided ideal. An ideal J is *proper* if $J \subsetneq \mathcal{A}$. A *maximal left (right) ideal* is a proper left (right) ideal which is not contained in any proper left (right) ideal. A *minimal left (right) ideal* of $\mathcal{A}$ is a left (right) ideal $J \neq \{0\}$, such that $\{0\}$ and J are the only left (right) ideals contained in J. The *radical* of $\mathcal{A}$, denoted by $\mathrm{Rad}(\mathcal{A})$, is the intersection of all maximal ideals of $\mathcal{A}$. Hence $\mathrm{Rad}(\mathcal{A})$ is a two-sided ideal. If $\mathrm{Rad}(\mathcal{A}) = \{0\}$, then we say that $\mathcal{A}$ is *semisimple*. If $\mathcal{A}$ has minimal left (right) ideals, then the smallest left (right) ideal containing all the minimal left (right) ideals is called the *left (right) socle*. If $\mathcal{A}$ has both minimal left and right ideals, and if the left and right socles of $\mathcal{A}$ are equal, we say that the *socle* of $\mathcal{A}$ exists and it is denoted by $\mathrm{Soc}(\mathcal{A})$. Let $\mathcal{A}$ be a Banach algebra and I be an ideal in $\mathcal{A}$. We say that I is an *inessential ideal* if, for every $x \in I$, $\sigma(x)$, the spectrum of x has at most 0 as a limit point, i.e.,

$$x \in I \Rightarrow \mathtt{acc}\, \sigma(x) \subseteq \{0\}.$$

Every inessential ideal in a Banach algebra determines a Fredholm Theory. For a comprehensive account of the abstract Fredholm Theory in Banach algebras, see [1, Chapter 5].

Next we define the concepts of a regularity and a semiregularity since they are key in this paper.

Definition 1.1. ([16, Definition I.6.1]) Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called *regularity* if it satisfies the following conditions:

(i) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$, then $a \in R \Leftrightarrow a^n \in R$;

(ii) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ and $ac + bd = 1_{\mathcal{A}}$, then

$$ab \in R \qquad \Longleftrightarrow \qquad a \in R \text{ and } b \in R\,.$$

The following well known sets are examples of regularities (see [16, I.6, Example 5]):

$$\mathcal{A}\,, \quad \mathcal{A}^{-1}\,, \quad \mathcal{A}_l^{-1} \quad \text{and} \quad \mathcal{A}_r^{-1}\,.$$

In many cases it is possible to verify the axioms of a regularity by using the following criterion:

THEOREM 1.2. ([16, Theorem I.6.4]) *Let R be a non-empty subset of a Banach algebra $\mathcal{A}$ satisfying*

$$ab \in R \qquad \Longleftrightarrow \qquad a \in R \text{ and } b \in R \tag{P1}$$

for all commuting elements $a, b \in \mathcal{A}$. Then R is a regularity.

One can divide the definition of a regularity into two parts:

DEFINITION 1.3. ([16, Definition III.23.1]) Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called *lower semiregularity* if

(i) $a \in \mathcal{A}$, $n \in \mathbb{N}$, $a^n \in R \;\Rightarrow\; a \in R$;

(ii) a, b, c, d are mutually commuting elements of $\mathcal{A}$ satisfying $ac+bd = 1_{\mathcal{A}}$, and $ab \in R$, then $a, b \in R$.

Remark 1.4. ([16, Remark III.23.3]) Let R be a non-empty subset of a Banach algebra $\mathcal{A}$ satisfying

$$a, b \in \mathcal{A}\,,\ ab = ba\,,\ ab \in R \qquad \Rightarrow \qquad a \in R \text{ and } b \in R\,. \tag{P1$'$}$$

Then clearly R is a lower semiregularity.

DEFINITION 1.5. ([16, Definition III.23.10]) Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called *upper semiregularity* if

(i) $a \in R$, $n \in \mathbb{N} \;\Rightarrow\; a^n \in R$;

(ii) a, b, c, d are mutually commuting elements of $\mathcal{A}$ satisfying $ac+bd = 1_{\mathcal{A}}$, and $a, b \in R$, then $ab \in R$;

(iii) R contains a neighbourhood of the unit element $1_{\mathcal{A}}$.

Remark 1.6. A semigroup containing a neighborhood of the unit element of $\mathcal{A}$ is an upper semiregularity because it already satisfies conditions (i) and (ii) of Definition 1.5.

We can deduce that R is regularity if and only if it is both a lower semiregularity and an upper semiregularity.

If $\mathcal{A}$ is a Banach algebra and $\mathcal{S} \subseteq \mathcal{A}$, then one can define in a natural way a spectrum relative to $\mathcal{S}$ for any $a \in \mathcal{A}$ by

$$\sigma_{\mathcal{S}}(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin \mathcal{S}\}.$$

If $\mathcal{S}$ is a regularity or a semiregularity, then $\sigma_{\mathcal{S}}(a)$ has interesting properties, see ([16, Theorem I.6.7, Theorem III.23.4 and Theorem III.23.18]). Suppose $\mathcal{A}$ is a Banach algebra and I is a closed ideal in $\mathcal{A}$. We denote the canonical homomorphism from $\mathcal{A}$ to $\mathcal{A}/I$ by $\pi : \mathcal{A} \to \mathcal{A}/I$ and it is defined by $\pi(x) = x + I$ $(x \in \mathcal{A})$. Let

$$\Phi(I) = \{x \in \mathcal{A} : x + I \in (\mathcal{A}/I)^{-1}\} = \pi^{-1}((\mathcal{A}/I)^{-1}), \tag{1.1}$$

$$\Phi_l(I) = \{x \in \mathcal{A} : x + I \in (\mathcal{A}/I)_l^{-1}\} = \pi^{-1}((\mathcal{A}/I)_l^{-1}), \tag{1.2}$$

$$\Phi_r(I) = \{x \in \mathcal{A} : x + I \in (\mathcal{A}/I)_r^{-1}\} = \pi^{-1}((\mathcal{A}/I)_r^{-1}), \tag{1.3}$$

$$\mathcal{W}(I) = \{x \in \mathcal{A} : x = a + b \text{ with } a \in \mathcal{A}^{-1} \text{ and } b \in I\}, \tag{1.4}$$

$$\mathcal{B}(I) = \{x \in \mathcal{A} : x = a + b \text{ with } a \in \mathcal{A}^{-1},\ b \in I \text{ and } ab = ba\}. \tag{1.5}$$

The sets $\Phi(I)$, $\Phi_l(I)$ and $\Phi_r(I)$ are called *Fredholm elements relative to I*, *left Fredholm elements relative to I* and *right Fredholm elements relative to I* respectively. The elements in $\mathcal{W}(I)$ are called *Weyl elements relative to I* and the elements in $\mathcal{B}(I)$ are called *Browder elements relative to I*. It is clear that

$$\mathcal{A}_l^{-1} \subseteq \Phi_l(I)\,, \qquad \mathcal{A}_r^{-1} \subseteq \Phi_r(I)\,, \tag{1.6}$$

$$\mathcal{A}^{-1} \subseteq \mathcal{B}(I) \subseteq \mathcal{W}(I) \subseteq \Phi(I) \subseteq \Phi_l(I) \cup \Phi_r(I)\,. \tag{1.7}$$

Since the sets $(A/I)_l^{-1}$, $(A/I)_r^{-1}$ and $(A/I)^{-1}$ are open in the Banach algebra $\mathcal{A}/I$ and since the natural homomorphism $\pi : \mathcal{A} \to \mathcal{A}/I$ is continuous, it follows that the sets $\Phi_l(I)$, $\Phi_r(I)$ and $\Phi(I)$ are open in $\mathcal{A}$. Also, since $\mathcal{A}^{-1}$ is an open set in $\mathcal{A}$, it is easy to see that the sets $\mathcal{W}(I)$ and $\mathcal{B}(I)$ are open subsets of $\mathcal{A}$. One can clearly see that if $\mathcal{A}$ is commutative, then $\mathcal{W}(I) = \mathcal{B}(I)$.

The sets defined above generate in a natural way the spectra that are relevant in Fredholm theory, i.e., if $x \in \mathcal{A}$ define

$$\sigma_{\Phi(I)}(x) = \{\lambda \in \mathbb{C} : \lambda - x \notin \Phi(I)\} \qquad \text{(Fredholm spectrum)}, \tag{1.8}$$

$$\sigma_{\Phi_l(I)}(x) = \{\lambda \in \mathbb{C} : \lambda - x \notin \Phi_l(I)\} \qquad \text{(left Fredholm spectrum)}, \tag{1.9}$$

$$\sigma_{\Phi_r(I)}(x) = \{\lambda \in \mathbb{C} : \lambda - x \notin \Phi_r(I)\} \qquad \text{(right Fredholm spectrum)}, \tag{1.10}$$

$$\sigma_{\mathcal{B}(I)}(x) = \{\lambda \in \mathbb{C} : \lambda - x \notin \mathcal{B}(I)\} \qquad \text{(Browder spectrum)}, \tag{1.11}$$

$$\sigma_{\mathcal{W}(I)}(x) = \{\lambda \in \mathbb{C} : \lambda - x \notin \mathcal{W}(I)\} \qquad \text{(Weyl spectrum)}. \tag{1.12}$$

This together with equation (1.6) and equation (1.7) gives

$$\sigma_{\Phi_l(I)}(x) \subseteq \sigma_l(x), \qquad \sigma_{\Phi_r(I)}(x) \subseteq \sigma_r(x),$$

$$\sigma_{\Phi_l(I)} \cup \sigma_{\Phi_r(I)}(x) \subseteq \sigma_{\Phi(I)}(x) \subseteq \sigma_{\mathcal{W}(I)}(x) \subseteq \sigma_{\mathcal{B}(I)}(x) \subseteq \sigma(x),$$

where $\sigma_l(x)$ and $\sigma_r(x)$ are the left and right spectra, denoting the spectrum of x with respect to $\mathcal{A}_l^{-1}$ and $\mathcal{A}_r^{-1}$ respectively.

If I is a closed ideal of a Banach algebra $\mathcal{A}$, then we define the set $kh(I)$ by

$$kh(I) = \{b \in \mathcal{A} : b + I \in \mathrm{Rad}(\mathcal{A}/I)\} = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I)).$$

2. Perturbation classes

In 1971, Lebow and Schechter in [10] introduced and studied the notion of perturbation classes. In this section we highlight some of the results which will play a role in this paper.

Definition 2.1. Let X be a complex Banach space, and let $\mathcal{S}$ be a subset of X. The *perturbation* of $\mathcal{S}$, denoted by $\mathcal{P}(\mathcal{S})$, is a set of all $x \in X$ such $x + s \in \mathcal{S}$ for all $s \in \mathcal{S}$, i.e.,

$$\mathcal{P}(\mathcal{S}) = \{x \in X : x + s \in \mathcal{S} \text{ for all } s \in \mathcal{S}\}.$$

We say $\mathcal{P}(\mathcal{S})$ is the set of elements of X that *perturb* $\mathcal{S}$ into itself.

Remark 2.2. Let $\mathcal{S}$ be a subset of a Banach space X closed under multiplication by nonzero scalars. If $0 \in \mathcal{S}$, then it is easy to see that $\mathcal{P}(\mathcal{S}) \subseteq \mathcal{S}$. We refer the reader to [10, Section 2] for the basic properties of the set $\mathcal{P}(\mathcal{S})$. If $\mathcal{A}$ is a Banach algebra and $\mathcal{S} \subseteq \mathcal{A}$ is closed under scalar multiplication, the set $\mathcal{P}(\mathcal{S})$ is in general not an ideal. However, we will choose to call the set $\mathcal{P}(\mathcal{S})$ the perturbation ideal of $\mathcal{S}$.

Proposition 2.3. *Let $\mathcal{A}$ be a Banach space and suppose $R \subseteq \mathcal{A}$ is closed under multiplication by nonzero scalars. Then*

$$\mathcal{P}(R) = \mathcal{P}(\mathcal{A} \setminus R)\,.$$

Proof. Since R is closed under multiplication by nonzero scalars, it also follows that $\mathcal{A} \setminus R$ is closed under multiplication by nonzero scalars. Let $a \in \mathcal{P}(R)$ and let $y \in \mathcal{A} \setminus R$. Now assume that $a + y \in R$, then in view of [10, Lemma 2.1], $-a + (a + y) = y \in R$ which leads to a contradiction, hence $a+y \in \mathcal{A}\setminus R$ and it then follows that $a \in \mathcal{P}(\mathcal{A}\setminus R)$. Hence, $\mathcal{P}(R) \subseteq \mathcal{P}(\mathcal{A}\setminus R)$. Similarly we can prove the inclusion $\mathcal{P}(\mathcal{A} \setminus R) \subseteq \mathcal{P}(R)$. ∎

The connection between the spectrum that a set in a Banach algebra generate and the perturbation ideal of the set is the following: Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{S} \subseteq A$ have the property that it is closed under multiplication by nonzero scalars. If $a \in \mathcal{A}$, then

$$x \in \mathcal{P}(\mathcal{S}) \text{ if and only if } \sigma_{\mathcal{S}}(a + x) = \sigma_{\mathcal{S}}(a)\,.$$

3. Fredholm elements

In this section we investigate perturbation ideals for Fredholm elements relative to some ideal I in a Banach algebra. For more account on Fredholm elements, we refer the reader to [1, Chapter 5, Section 5].

Let $\mathcal{A}$ be a Banach algebra and let I be a closed ideal in $\mathcal{A}$. Since $\mathcal{A}_l^{-1}$ and $\mathcal{A}_r^{-1}$ are regularities in $\mathcal{A}$, similarly, $(\mathcal{A}/I)_l^{-1}$ and $(\mathcal{A}/I)_r^{-1}$ are regularities in the quotient algebra $\mathcal{A}/I$. It then follows by [16, Theorem I.6.3 (iii)] that $\Phi_l(I) = \pi^{-1}((\mathcal{A}/I)_l^{-1})$ and $\Phi_r(I) = \pi^{-1}((\mathcal{A}/I)_r^{-1})$ are regularities in $\mathcal{A}$, hence $\Phi(I) = \Phi_l(I) \cap \Phi_r(I)$ is also a regularity in $\mathcal{A}$.

Remark 3.1. Let $\mathcal{A}$ be a Banach algebra and let I be a closed ideal in $\mathcal{A}$. Then $\mathcal{P}(\Phi(I))$, $\mathcal{P}(\Phi_l(I))$ and $\mathcal{P}(\Phi_r(I))$ are closed ideals: Let $ab \in \Phi(I)\mathcal{A}^{-1}$ where $a \in \Phi(I)$ and $b \in \mathcal{A}^{-1}$. Now since $a + I, b + I \in (\mathcal{A}/I)^{-1}$, then $(a + I)(b + I) = ab + I \in (\mathcal{A}/I)^{-1}$. Hence $ab \in \Phi(I)$ and $\Phi(I)\mathcal{A}^{-1} \subseteq \Phi(I)$. In the same way, we can also show that $\mathcal{A}^{-1}\Phi(I) \subseteq \Phi(I)$. From this and the fact that $\Phi(I)$ is open in $\mathcal{A}$ and closed under multiplication by nonzero scalars, it then follows by [10, Lemma 2.1 and Theorem 2.4] that $\mathcal{P}(\Phi(I))$ is a closed ideal. Similarly, it can be shown that $\mathcal{P}(\Phi_l(I))$ and $\mathcal{P}(\Phi_r(I))$ are also closed ideals in $\mathcal{A}$.

THEOREM 3.2. ([10, Theorem 2.7]) *Let $\mathcal{A}$ be a Banach algebra and let I be a closed ideal in $\mathcal{A}$. Then*

$$\mathcal{P}(\Phi(I)) = \mathcal{P}(\Phi_l(I)) = \mathcal{P}(\Phi_r(I)) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I)).$$

Proof. It follows from [10, Theorem 2.5] in the quotient algebra $\mathcal{A}/I$ that $\mathrm{Rad}(\mathcal{A}/I) = \mathcal{P}((\mathcal{A}/I)^{-1})$. This implies that

$$\begin{aligned}\pi^{-1}(\mathrm{Rad}(\mathcal{A}/I)) &= \pi^{-1}(\mathcal{P}((\mathcal{A}/I)^{-1}))\\ &= \mathcal{P}(\pi^{-1}((\mathcal{A}/I)^{-1})) = \mathcal{P}(\Phi(I))\,.\end{aligned}$$

One can adapt the above proof to conclude that $\mathcal{P}(\Phi_l(I)) = \mathcal{P}(\Phi_r(I)) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))$ if one employs [10, Theorem 2.6] in the quotient algebra $\mathcal{A}/I$. ∎

Set $\widetilde{\Phi} = \Phi_l(I) \cup \Phi_r(I)$. We first show that $\widetilde{\Phi}$ is a lower semiregularity in $\mathcal{A}$ and then we will determine the perturbation ideal $\mathcal{P}(\widetilde{\Phi})$ of $\widetilde{\Phi}$.

PROPOSITION 3.3. *Let $\mathcal{A}$ be a Banach algebra and let I be a closed ideal in $\mathcal{A}$. Then $\widetilde{\Phi} = \Phi_l(I) \cup \Phi_r(I)$ is a lower semiregularity in $\mathcal{A}$.*

Proof. By using that $\Phi_l(I) = \pi^{-1}((\mathcal{A}/I)_l^{-1})$ and $\Phi_r(I) = \pi^{-1}((\mathcal{A}/I)_r^{-1})$, we then have

$$\begin{aligned}\widetilde{\Phi} &= \Phi_l(I) \cup \Phi_r(I)\\ &= \pi^{-1}((\mathcal{A}/I)_l^{-1}) \cup \pi^{-1}((\mathcal{A}/I)_r^{-1})\\ &= \pi^{-1}((\mathcal{A}/I)_l^{-1} \cup (\mathcal{A}/I)_r^{-1}).\end{aligned}$$

Note that $(\mathcal{A}/I)_l^{-1} \cup (\mathcal{A}/I)_r^{-1}$ is a (P1$'$) lower semiregularity in the quotient algebra $\mathcal{A}/I$, see Remark 1.4. Since π is a homomorphism, $\pi^{-1}((\mathcal{A}/I)_l^{-1} \cup (\mathcal{A}/I)_r^{-1})$ is a (P1$'$) lower semiregularity in $\mathcal{A}$. ∎

THEOREM 3.4. *Let $\mathcal{A}$ be a Banach algebra and let I be a closed ideal in $\mathcal{A}$. If $\widetilde{\Phi} = \Phi_l(I) \cup \Phi_r(I)$, then*

$$\mathcal{P}(\widetilde{\Phi}) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))\,.$$

Proof. We first note that $\mathcal{A}^{-1}$ is open in $\mathcal{A}$ and $\mathcal{A}^{-1} \subseteq \mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$. Now we show that $\partial\mathcal{A}^{-1} \cap (\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}) = \emptyset$. If we assume $\partial\mathcal{A}^{-1} \cap (\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}) \neq \emptyset$, then in view of [16, Theorem I.1.14] we arrive at a contradiction. Since both

$\mathcal{A}^{-1}$ and $\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$ are closed under multiplication by nonzero scalars, it follows from [10, Lemma 2.2] that $\mathcal{P}(\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}) \subseteq \mathcal{P}(\mathcal{A}^{-1})$. To prove the inclusion $\mathcal{P}(\mathcal{A}^{-1}) \subseteq \mathcal{P}(\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1})$, let $x \in \mathcal{P}(\mathcal{A}^{-1})$ and $a \in \mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$. If $a \in \mathcal{A}_l^{-1}$, then by [10, Theorem 2.5], $x + a \in \mathcal{A}_l^{-1} \subseteq \mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$. If $a \in \mathcal{A}_r^{-1}$, then similarly $x+a \in \mathcal{A}_r^{-1} \subseteq \mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$. If we combine our arguments, we get $\mathcal{P}(\mathcal{A}^{-1}) \subseteq \mathcal{P}(\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1})$, and hence $\mathcal{P}(\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}) = \mathcal{P}(\mathcal{A}^{-1})=\mathrm{Rad}(\mathcal{A})$, see [10, Theorem 2.5]. Now in the quotient Banach algebra $\mathcal{A}/I$, it then follows that $\mathcal{P}((\mathcal{A}/I)_l^{-1} \cup (\mathcal{A}/I)_r^{-1}) = \mathrm{Rad}(\mathcal{A}/I)$. Hence,

$$\begin{aligned}
\pi^{-1}(\mathrm{Rad}(\mathcal{A}/I)) &= \pi^{-1}(\mathcal{P}((\mathcal{A}/I)_l^{-1} \cup (\mathcal{A}/I)_r^{-1})) \\
&= \mathcal{P}(\pi^{-1}(\mathcal{A}/I)_l^{-1} \cup \pi^{-1}(\mathcal{A}/I)_r^{-1}) \\
&= \mathcal{P}(\Phi_l(I) \cup \Phi_r(I)) \\
&= \mathcal{P}(\widetilde{\Phi}(I))\,.
\end{aligned}$$

∎

Next we calculate the perturbation ideal of $\partial\Phi(I)$, the boundary of the Fredholm elements relative to an ideal I of a Banach algebra $\mathcal{A}$. Since $\mathcal{A}^{-1} \cap \partial\Phi(I) = \emptyset$, $\partial\Phi(I)$ is neither a lower nor an upper semiregularity, see Definition 1.5 and [16, Lemma 23.2].

Theorem 3.5. *Let $\mathcal{A}$ be a Banach algebra and let I be a closed ideal in $\mathcal{A}$. Then*

$$\mathcal{P}(\partial\Phi(I)) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I)).$$

Proof. Let $R = \mathcal{A} \setminus \partial\Phi(I)$. Then $\Phi(I) \subseteq R$ and $\partial\Phi(I) \cap R = \emptyset$. Both R and $\partial\Phi(I)$ are closed under scalar multiplication. Since $\Phi(I)$ is open, it then follows by [10, Lemma 2.2] that $\mathcal{P}(R) \subseteq \mathcal{P}(\Phi(I)) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))$. In view of Proposition 2.3, it follows that $\mathcal{P}(\partial\Phi(I)) \subseteq \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))$. Now let $x \in \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))$ and let $a \in \partial\Phi(I)$. So there exist sequences (a_n) in $\Phi(I)$ and (b_n) in $\mathcal{A}\setminus\Phi(I)$ such that $a_n \to a$ and $b_n \to a$. By Theorem 3.2 and Proposition 2.3, $x+a_n \in \Phi(I)$ with $x+a_n \to x+a$ and $x+b_n \in \mathcal{A}\setminus\Phi(I)$ with $x+b_n \to x+a$. It then follows from this that $x + a \in \partial\Phi(I)$. Hence $x \in \mathcal{P}(\partial\Phi(I))$. If we combine our arguments we get that $\mathcal{P}(\partial\Phi(I)) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))$. ∎

In view of Proposition 2.3 we also have that

$$\mathcal{P}(\mathcal{A} \setminus \partial\Phi(I)) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))\,.$$

Next we investigate the perturbation of $\overline{\Phi(I)}$, the closure of the Fredholm elements relative to some closed ideal I in a Banach algebra $\mathcal{A}$. Since $\overline{\Phi(I)}$

satisfies all the conditions in Definition 1.5, it is an upper semiregularity. In addition, $\mathcal{P}(\overline{\Phi(I)})$ is an ideal: To show $\mathcal{A}^{-1} \cdot \overline{\Phi(I)} \subseteq \overline{\Phi(I)}$, let $ab \in \mathcal{A}^{-1} \cdot \overline{\Phi(I)}$ with $a \in \mathcal{A}^{-1}$ and $b \in \overline{\Phi(I)}$. Hence there is a sequence (b_n) in $\Phi(I)$ with $b_n \to b$. Hence $ab_n \to ab$ with $ab_n \in \Phi(I)$. This means that $ab \in \overline{\Phi(I)}$. It follows likewise that $\overline{\Phi(I)} \cdot \mathcal{A}^{-1} \subseteq \overline{\Phi(I)}$. By [10, Lemma 2.3], $\mathcal{P}(\overline{\Phi(I)})$ is an ideal in $\mathcal{A}$.

Theorem 3.6. *Let $\mathcal{A}$ be a Banach algebra and let I be a closed ideal in $\mathcal{A}$. Then*

$$\pi^{-1}(\mathrm{Rad}(\mathcal{A}/I)) \subseteq \mathcal{P}(\overline{\Phi(I)}) \subseteq \overline{\Phi(I)}\,.$$

Proof. First we note that $\overline{\Phi(I)} = \Phi(I) \cup \partial\Phi(I)$. Let $x \in \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))$ and $a \in \overline{\Phi(I)}$. If $a \in \Phi(I)$, by Theorem 3.2, $x + a \in \Phi(I)$. If $a \in \partial\Phi(I)$, it follows from Theorem 3.5 that $x+a \in \partial\Phi(I)$. Combining the two cases we can conclude that $x + a \in \overline{\Phi(I)}$. It then follows that $\pi^{-1}(\mathrm{Rad}(\mathcal{A}/I)) \subseteq \mathcal{P}(\overline{\Phi(I)})$. Since $0 \in \overline{\mathcal{A}^{-1}} \subseteq \overline{\Phi(I)}$, it follows by Remark 2.2 that $\pi^{-1}(\mathrm{Rad}(\mathcal{A}/I)) \subseteq \mathcal{P}(\overline{\Phi(I)}) \subseteq \overline{\Phi(I)}$. ∎

4. Riesz elements

Our goal in this section is to calculate the perturbation ideal for the set of Riesz elements relative to some closed ideal in a Banach algebra.

Let I be a closed ideal in a Banach algebra $\mathcal{A}$. An element $a \in \mathcal{A}$ is called *Riesz relative* to I if $\sigma_{\mathcal{A}/I}(a + I) = \{0\}$, i.e., $a + I \in QN(\mathcal{A}/I)$. The set of all Riesz elements in $\mathcal{A}$ relative to the ideal I is denoted by $\mathcal{R}(\mathcal{A}, I)$. Whenever the algebra $\mathcal{A}$ is clear from the context, we will simply write $\mathcal{R}(\mathcal{A}, I) = \mathcal{R}(I)$. For properties of Riesz elements we refer the reader to Chapter R of the monograph [3]. It is clear that

$$I \subseteq kh(I) \subseteq \mathcal{R}(I)\,. \tag{4.1}$$

Remark 4.1. It must be noted that $kh(I)$ is the largest ideal in $\mathcal{A}$ consisting of Riesz elements [17, Theorem 4.4].

Since $\mathcal{R}(I)$ is in general not closed under addition or multiplication, it is in general not an ideal. To prove one of our main result, Theorem 5.12, we need the following characterization of Riesz elements.

Proposition 4.2. *Let I be a closed ideal in a Banach algebra $\mathcal{A}$ and $a \in \mathcal{A}$. Then $a \in \mathcal{R}(I)$ if and only $1_{\mathcal{A}} + \lambda a \in \Phi(I)$ for all $\lambda \in \mathbb{C}$.*

Proof. If $a \in \mathcal{R}(I)$, then $\sigma_{\mathcal{A}/I}(a+I) = \{0\}$. Hence for all $\lambda \in \mathbb{C} \setminus \{0\}$

$$\frac{1}{\lambda} + a + I \in (\mathcal{A}/I)^{-1} \qquad \Rightarrow \qquad 1_{\mathcal{A}} + \lambda a \in \Phi(I)\,.$$

Since $1_{\mathcal{A}} \in \Phi(I)$, the statement is also true for $\lambda = 0$.

Conversely, let $1_{\mathcal{A}} + \lambda a \in \Phi(I)$ for $\lambda \in \mathbb{C}$. Then $1_{\mathcal{A}} + \lambda a + I \in (\mathcal{A}/I)^{-1}$. For any $\lambda \neq 0$, we have

$$1_{\mathcal{A}} + \frac{1}{\lambda} a + I \in (\mathcal{A}/I)^{-1} \qquad \Rightarrow \qquad -\lambda - (a+I) \in (\mathcal{A}/I)^{-1}.$$

Now since $\sigma_{\mathcal{A}/I}(a+I)$ is a non-empty set, $\sigma_{\mathcal{A}/I}(a+I) = \{0\}$, and so $a \in \mathcal{R}(I)$. ∎

If I is an inessential ideal in a Banach algebra $\mathcal{A}$ and $a \in \mathcal{R}(I)$, then $\sigma(a)$ the spectrum of a is either a finite set or a sequence converging to zero [2, Corollary 5.7.5]. Let I be a closed ideal in a Banach algebra $\mathcal{A}$. Our next result describes the perturbation ideal of the set of Riesz elements relative to I.

If I be a closed ideal in a Banach algebra $\mathcal{A}$, it follows from the Spectral Mapping Theorem in the quotient algebra $\mathcal{A}/I$ that $\mathcal{R}(I)$ is closed under scalar multiplication.

Proposition 4.3. *Let $\mathcal{A}$ be a Banach algebra and let I be a closed ideal in $\mathcal{A}$. Then*

$$kh(I) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I)) = \mathcal{P}(\mathcal{R}(I))\,.$$

Proof. It follows from the equivalence (i)⇔(iii) in [2, Theorem 5.3.1] in the quotient algebra $\mathcal{A}/I$ that $\mathrm{Rad}(\mathcal{A}/I) = \mathcal{P}(QN(\mathcal{A}/I))$. This implies that

$$\begin{aligned} kh(I) \;=\; \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I)) \;&=\; \pi^{-1}(\mathcal{P}(QN(\mathcal{A}/I))) \\ &=\; \mathcal{P}(\pi^{-1}(QN(\mathcal{A}/I))) \;=\; \mathcal{P}(\mathcal{R}(I))\,. \end{aligned}$$

∎

Remark 4.4. We note that $\mathcal{R}(I)$ is neither a regularity nor a semiregularity for any closed ideal I in a Banach algebra $\mathcal{A}$: Since $1_{\mathcal{A}}$ is the unit element in $\mathcal{A}$, $1_{\mathcal{A}} + I$ is the unit element in the quotient algebra $\mathcal{A}/I$ and so $\sigma_{\mathcal{A}/I}(1_{\mathcal{A}} + I) = \{1\}$. Hence $1_{\mathcal{A}} + I \notin QN(\mathcal{A}/I)$. This means that $1_{\mathcal{A}} \notin \mathcal{R}(I)$. Our claim follows from Definition 1.5 and [16, Lemma III.23.2].

Proposition 4.5. *Let $\mathcal{A}$ be a Banach algebra and let I be a closed ideal in $\mathcal{A}$. Then $\mathcal{A} \setminus \mathcal{R}(I)$ is a lower semiregularity.*

Proof. Let $a, b \in \mathcal{A} \setminus \mathcal{R}(I)$ with $ab = ba$ and suppose $ab \in \mathcal{A} \setminus \mathcal{R}(I)$. Hence, $r_{\mathcal{A}/I}(ab + I) > 0$ and since $ab = ba$,

$$0 < r_{\mathcal{A}/I}(ab + I) \leq r_{\mathcal{A}/I}(a + I) \cdot r_{\mathcal{A}/I}(b + I).$$

This means that $r_{\mathcal{A}/I}(a + I) > 0$ and $r_{\mathcal{A}/I}(b + I) > 0$, and so $a, b \in \mathcal{A} \setminus \mathcal{R}(I)$. In view of Remark 1.4, $\mathcal{A} \setminus \mathcal{R}(I)$ is a lower semiregularity. ∎

5. Index theory

In this section, we consider the set of Fredholm elements relative to some trace ideal I in a Banach algebra $\mathcal{A}$. This enables us to define subsets of $\Phi(I)$ for which we can calculate the perturbation ideal. We refer the reader to [4, 5] for background knowledge on trace ideals in a semisimple Banach algebra. Our main result in this section is Theorem 5.12.

Definition 5.1. (See [4, Section 2.1]) Let I be an ideal in a Banach algebra $\mathcal{A}$. A linear function $\tau : I \to \mathbb{C}$ is called a trace if it satisfies the following two conditions:

(TN) $\tau(p) = 1$ for every rank one idempotent $p \in I$;

(TC) $\tau(ba) = \tau(ab)$ for all $a \in I$ and $b \in \mathcal{A}$.

If I is a trace ideal of a Banach algebra $\mathcal{A}$, then it is possible to define an index function i defined on the set of Fredholm elements relative to I as follows:

Definition 5.2. ([4, Definition 3.3]) Let $\mathcal{A}$ be a Banach algebra and let τ be a trace on an ideal I in $\mathcal{A}$. We define the index function $i : \Phi(I) \to \mathbb{C}$ by

$$i(a) := \tau(aa_0 - a_0a) = \tau([a, a_0]) \qquad \text{for all } a \in \Phi(I),$$

where $a_0 \in \mathcal{A}$ satisfies $aa_0 - 1_{\mathcal{A}} \in I$ and $a_0a - 1_{\mathcal{A}} \in I$.

It has been proven that the index function defined above is well defined on $\Phi(I)$ (see [4, Proposition 3.4]). Next we list some of the important properties of this index function.

Remark 5.3. By remark preceding [4, Proposition 3.5], its should be noted that for any $a \in \mathcal{A}^{-1}$, $i(a) = 0$.

PROPOSITION 5.4. ([4, Proposition 3.5]) *Let $\mathcal{A}$ be a Banach algebra and let I be a trace ideal in $\mathcal{A}$. If $a, b \in \Phi(I)$, then*

$$i(ab) = i(a) + i(b)\,.$$

To exhibit more properties of the index function, we will suppose that the trace ideal satisfies the condition $\mathrm{Soc}(\mathcal{A}) \subseteq I \subseteq kh(\mathrm{Soc}(\mathcal{A}))$. For a motivation of this assumption, see the remarks preceding [4, Section 3].

DEFINITION 5.5. An idempotent p is called a *left Barnes idempotent* for $a \in \mathcal{A}$ if

$$a\mathcal{A} = (1-p)\mathcal{A}\,,$$

while idempotent q is called a *right Barnes idempotent* for $a \in \mathcal{A}$ if

$$\mathcal{A}a = \mathcal{A}(1-q)\,.$$

THEOREM 5.6. ([4, Theorem 3.11]) *Let $\mathcal{A}$ be a semisimple Banach algebra and let the trace ideal I satisfy $Soc(\mathcal{A}) \subseteq I \subseteq kh(Soc(\mathcal{A}))$. Then*

(i) *$a \in \Phi(I)$ if and only if there exist left and right Barnes idempotents p and q in $Soc(\mathcal{A})$ and an element $a_0 \in \mathcal{A}$ such that*

$$aa_0 = 1 - p \qquad \text{and} \qquad a_0 a = 1 - q\,;$$

(ii) *$i(a) = \tau(q) - \tau(p) \in \mathbb{Z}$.*

We are going to use (ii) the above theorem to extend the index function to the set $\widetilde{\Phi} = \Phi_l(I) \cup \Phi_r(I)$: If $a \in \Phi_l(I) \setminus \Phi(I)$ define the index of a by

$$i(a) = \tau(q) - \tau(p) = \tau(q) - \infty = -\infty\,,$$

and if $a \in \Phi_r(I) \setminus \Phi(I)$ define the index of a by

$$i(a) = \tau(q) - \tau(p) = \infty - \tau(p) = \infty\,.$$

Let Z be a subset of the integers $\mathbb{Z}$ and Φ_Z be the set

$$\Phi_Z = \{a \in \widetilde{\Phi} : i(a) \in Z\}\,.$$

LEMMA 5.7. *Let $\mathcal{A}$ be a semisimple Banach algebra and let the trace ideal I satisfy $Soc(\mathcal{A}) \subseteq I \subseteq kh(Soc(\mathcal{A}))$. Then Φ_Z is an open subset of $\mathcal{A}$ and it is closed under multiplication by nonzero scalars.*

Proof. Consider the real numbers $\mathbb{R}$ with the usual topology. If $Z \subset \mathbb{R}$ have the relative topology, then for each $k \in Z$, the singleton $\{k\}$ is an open subset of $\mathbb{Z}$. Since Z is the union of such singletons, it is open. Since the index function is integer valued and continuous, see [4, Proposition 3.7 (vi)] and Theorem 5.6, it follows that $\Phi_Z = i^{-1}(Z)$ is an open subset in $\mathcal{A}$. The second part easily follows since $i(\alpha a) = i(a) \in Z$ for $a \in \Phi_Z$ and $\alpha \in \mathbb{C}$. ∎

Remark 5.8. Let I be a trace ideal in a Banach algebra $\mathcal{A}$. If $Z \subset \mathbb{Z}$, then in general Φ_Z is neither an upper nor a lower semiregularity in $\mathcal{A}$: If $0 \notin Z$, then $\mathcal{A}^{-1} \cap \Phi_Z = \emptyset$, see the remark preceding Proposition 5.4. In view of [16, Lemma 23.2 (ii)], Φ_Z is not a lower semiregularity. If $Z = \{0, 10\}$ and $a \in \Phi_Z$ has the property that $i(a) = 5$, then $i(a^2) = 10$, and hence $a^2 \in \Phi_Z$ and $a \notin \Phi_Z$, showing that Φ_Z is not a lower semiregularity, see Definition 1.3. If we choose $Z = \{0, 10\}$ and if we let $a \in \Phi_Z$ have the property that $i(a) = 10$, it then follows that $i(a^2) = 20$ and so $a^2 \notin \Phi_Z$. By Definition 1.5, Φ_Z is not an upper semiregularity.

However for $\Phi_0(I)$, the set of Fredholm elements of index zero, we have

PROPOSITION 5.9. *Let $\mathcal{A}$ be a Banach algebra and let I be a trace ideal in $\mathcal{A}$. Then $\Phi_0(I)$ is an upper semiregularity.*

Proof. Since $\Phi_0(I)$ is a semigroup with $1_{\mathcal{A}} \in \mathcal{A}^{-1} \subseteq \Phi_0$, it follows that $\Phi_0(I)$ is an upper semiregularity, see Remark 1.6. ∎

If I is a trace ideal in a Banach algebra $\mathcal{A}$, then the perturbation ideal $\mathcal{P}(\Phi_Z)$ of Φ_Z is a closed ideal in $\mathcal{A}$ (see [10, Theorem 2.4] and Lemma 5.7), as well as Remark 5.3 and Proposition 5.4.

Note that Φ_Z can be decomposed into equivalence classes (components) as follows:

$$\Phi_Z = \bigcup_{k \in Z} i^{-1}(\{k\}).$$

LEMMA 5.10. *Let $\mathcal{A}$ be a Banach algebra and let I be a trace ideal in $\mathcal{A}$. If $Z \subset \mathbb{Z}$ and $\widetilde{\Phi} = \Phi_l(I) \cup \Phi_r(I)$, then*

$$\partial \Phi_Z \cap \widetilde{\Phi} = \emptyset, \qquad \text{where } \Phi_Z = \{a \in \widetilde{\Phi} \,:\, i(a) \in Z\}.$$

Proof. By Lemma 5.7, Φ_Z is an open set and it is closed under multiplication by nonzero scalars. Also, $\Phi_Z \subseteq \widetilde{\Phi}$. Suppose there is $x \in \partial\Phi_Z \cap \widetilde{\Phi}$. Since Φ_Z is open, Φ_Z avoids $\partial\Phi_Z$ and so $x \notin \Phi_Z$. Let C_x be the component of $\widetilde{\Phi}$ that contains x. Since $\widetilde{\Phi}$ is an open set, C_x is also an open set. Hence, C_x is a neighbourhood of x that avoids Φ_Z. This is a contradiction because if $x \in \partial\Phi_Z$, then every neighbourhood of x contains points of Φ_Z and points not in Φ_Z. This completes the proof. ∎

Remark 5.11. Let $\mathcal{A}$ be a Banach algebra with $P \subseteq \mathcal{A}$. If $a \in \mathcal{A}$ belongs to some component C_a of P, then a and αa belong to C_a for all $0 \neq \alpha \in \mathbb{C}$: Consider the set $\{a + t(\alpha a - a) : t \in [0,1]\}$. This set is connected and it contains a and αa. Since C_a is the maximal connected subset of P that contains a, it follows that $\{a + t(\alpha a - a) : t \in [0,1]\} \subseteq C_a$.

We are now ready to prove one of our main theorems which is a Banach algebra version of [10, Theorem 2.8] proved by Lebow and Schechter in the Banach algebra $\mathcal{B}(X)$ of bounded operators defined on a Banach space X.

Theorem 5.12. *Let $\mathcal{A}$ be a semisimple Banach algebra and I be a closed trace ideal in $\mathcal{A}$ such that $Soc(\mathcal{A}) \subseteq I \subseteq kh(Soc(\mathcal{A}))$. If $\Phi_Z \neq \emptyset$, then*

$$\mathcal{P}(\Phi_Z) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I)) \,. \tag{5.1}$$

Proof. Set $\widetilde{\Phi} = \Phi_l(I) \cup \Phi_r(I)$. We can deduce from [10, Lemma 2.2], Lemma 5.7, Lemma 5.10 and Theorem 3.4 that

$$\mathcal{P}(\Phi_Z) \supseteq \mathcal{P}(\widetilde{\Phi}) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I)) \,.$$

This proves containment in one direction of equation (5.1). Now we prove the opposite inclusion. Since $\Phi_Z \neq \emptyset$, there exists some $a \in \Phi_Z$. Assume $a \in \Phi_l(I)$ and let $x \in \mathcal{P}(\Phi_Z)$. Since $\mathcal{P}(\Phi_Z)$ is an ideal, then $\lambda ax \in \mathcal{P}(\Phi_Z)$ for every scalar λ. It then follows that $a + \lambda ax \in \Phi_Z$ for all $\lambda \in \mathbb{C}$. Note that $a + \lambda ax = a(1_{\mathcal{A}} + \lambda x) \in \Phi_Z \subseteq \widetilde{\Phi}$ for all $\lambda \in \mathbb{C}$. We claim that a and $a + \lambda ax$ belong to the same component of $\Phi_l(I)$: If $\lambda \to 0$, then $a + \lambda ax \to a$. Since $\Phi_l(I)$ is an open set, it follows that the component of $\Phi_l(I)$ containing a is also an open set. Hence, if $|\lambda|$ is small enough, a and $a + \lambda ax$ belong to the same component of $\Phi_l(I)$. If $|\lambda|$ is large, there exists $\alpha \in \mathbb{C}$ with $|\alpha\lambda|$ small. Hence, $a + \lambda ax = \frac{1}{\alpha}(\alpha a + \alpha\lambda ax)$. By Remark 5.11, $a + \lambda ax$ and $\alpha a + \alpha\lambda ax$ belong to the same component. Again, by applying Remark 5.11, a and αa belong to the same component of $\Phi_l(I)$ hence we can conclude that a and

$a + \lambda ax$ belong to the same component for all $\lambda \in \mathbb{C}$. Hence, for each $\lambda \in \mathbb{C}$ there exists $b_\lambda \in \mathcal{A}$ with $(b_\lambda + I)(a + \lambda ax + I) = 1_{\mathcal{A}} + I$ and so $1_{\mathcal{A}} + \lambda x \in \Phi_l(I)$ for all $\lambda \in \mathbb{C}$.

In the same way as above, $1_{\mathcal{A}}$ and $1_{\mathcal{A}} + \lambda x$ belong to the same component of $\mathcal{A}^{-1}$ for all $\lambda \in \mathbb{C}$. This means that $1_{\mathcal{A}} + \lambda x \in \Phi(I)$ for all $\lambda \in \mathbb{C}$. By Proposition 4.2 we get that $x \in \mathcal{R}(I)$. But $kh(I) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))$ is the largest ideal consisting of Riesz elements relative to I, see Remark 4.1. Hence, $x \in \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))$ and so $\mathcal{P}(\Phi_Z) \subseteq kh(I) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I)$. If we combine our arguments we get $\mathcal{P}(\Phi_Z) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))$. If instead $a \in \Phi_r(I)$, then it can be proved in the same way as above that $\mathcal{P}(\Phi_Z) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))$. ∎

If I is a closed trace ideal in a semisimple Banach algebra $\mathcal{A}$ with $\mathrm{Soc}(\mathcal{A}) \subseteq I \subseteq kh(\mathrm{Soc}(\mathcal{A}))$, denote by $\Phi_n(I)$ the set of Fredholm elements of index n, $n \in \mathbb{Z}$. In view of Lemma 5.7, $\Phi_n(I)$ is an open subset of $\mathcal{A}$ which is closed under multiplication by nonzero scalars. Also, by Theorem 5.12, $\mathcal{P}(\Phi_n(I)) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))$.

6. Weyl and Browder elements

In this section we are going to investigate the perturbation ideals of the collection of Weyl elements and the collection of Browder elements. Let I be a closed ideal in a Banach algebra $\mathcal{A}$. Since $\mathcal{A}^{-1} + I$ is an open set in $\mathcal{A}$, $\mathcal{A}^{-1}\mathcal{W}(I) \subseteq \mathcal{W}(I)$ and $\mathcal{W}(I)\mathcal{A}^{-1} \subseteq \mathcal{W}(I)$, it follows from [10, Theorem 2.4] that $\mathcal{P}(\mathcal{W}(I))$ is a closed ideal in $\mathcal{A}$.

Proposition 6.1. ([14, Corollary 8.1]) *If I is a closed ideal in a Banach algebra $\mathcal{A}$, then the set of Weyl elements relative to I forms an upper regularity in $\mathcal{A}$.*

Proposition 6.2. *Let $\mathcal{A}$ be a Banach algebra and let I be a closed ideal in $\mathcal{A}$. Then*

$$\mathrm{Rad}(\mathcal{A}) + I \subseteq \mathcal{P}(\mathcal{W}(I)) .$$

Proof. Let $x \in \mathrm{Rad}(\mathcal{A}) + I$. Then $x = a + b$ where $a \in \mathrm{Rad}(\mathcal{A})$ and $b \in I$. Suppose $y \in \mathcal{W}(I)$, i.e. $y = c + d$ with $c \in \mathcal{A}^{-1}$ and $d \in I$. It then follows that

$$x + y = (a + b) + (c + d) = (a + c) + (b + d) .$$

It is clear that $a + c \in \mathcal{A}^{-1}$ and $b + d \in I$, hence, $x \in \mathcal{P}(\mathcal{W}(I))$. ∎

If I is a closed trace ideal in $\mathcal{A}$, recall that $\Phi_0(I) = \{a \in \Phi(I) : i(a) = 0\}$. By [4, Proposition 3.7] and Remark 5.3 we get that

$$\mathcal{W}(I) \subseteq \Phi_0(I)\,.$$

Corollary 6.3. *Let $\mathcal{A}$ be a semisimple Banach algebra and let I be a closed trace ideal in $\mathcal{A}$ with $Soc(\mathcal{A}) \subseteq I \subseteq kh(Soc(\mathcal{A}))$. Then*

$$\mathcal{P}(\mathcal{W}(I)) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))\,.$$

Proof. In view of in [5, Corollary 3.5], $\mathcal{W}(I) = \Phi_0(I)$. This together with Theorem 5.12 with $Z = \{0\}$, gives $\mathcal{P}(\mathcal{W}(I)) = \mathcal{P}(\Phi_0(I)) = \pi^{-1}(\mathrm{Rad}(\mathcal{A}/I))$. ∎

We now turn our attention to $\mathcal{B}(I)$, the set of Browder elements relative to I. It is clear from equations (1.4) and (1.5) that Browder elements are Weyl elements.

Remark 6.4. It should however be clear that if $\mathcal{A}$ is a commutative Banach algebra, the set of Weyl elements relative to I and the set of Browder elements relative to I coincide, i.e., $\mathcal{W}(I) = \mathcal{B}(I)$. In this case it then follows that the results in Proposition 6.2 and Corollary 6.3 will also hold if we replace $\mathcal{W}(I)$ by $\mathcal{B}(I)$.

In general, the inclusions $\mathcal{B}(I)\mathcal{A}^{-1} \subseteq \mathcal{B}(I)$ and $\mathcal{A}^{-1}\mathcal{B}(I) \subseteq \mathcal{B}(I)$ are not satisfied if $\mathcal{A}$ is a non-commutative Banach algebra so it is not clear that $\mathcal{P}(B(I))$ is an ideal. But $\mathcal{B}(I)$ is closed under nonzero scalar multiplication.

The next result is a modified version of Proposition 6.2.

Proposition 6.5. *Let $\mathcal{A}$ be a Banach algebra and let I be a closed ideal in $\mathcal{A}$ with $I \subseteq \mathtt{comm}(\mathcal{A}^{-1})$. Then*

$$\mathrm{Rad}(\mathcal{A}) + I \subseteq \mathcal{P}(\mathcal{B}(I))\,.$$

Proof. Since $I \subseteq \mathtt{comm}(\mathcal{A}^{-1})$, the rest of the proof follows from the proof of Proposition 6.2. ∎

If I is a closed inessential ideal in a Banach algebra $\mathcal{A}$, [7, Theorem 7.7.6] tells us that $ab \in \mathcal{B}(I)$ if and only if $a, b \in \mathcal{B}(I)$ whenever $ab = ba$. Hence $\mathcal{B}(I)$ satisfies the (P1) condition of Theorem 1.2. It then follows from this that $\mathcal{B}(I)$ is a regularity in $\mathcal{A}$.

THEOREM 6.6. ([11, Theorem 7.5]) *Suppose I is a closed inessential ideal in a Banach algebra $\mathcal{A}$. Then $\mathcal{B}(I)$ is an open regularity in $\mathcal{A}$.*

The next result describe the perturbation of Browder elements by commuting Riesz elements.

THEOREM 6.7. ([13, Theorem 5.1]) *Suppose I is a closed inessential ideal in a Banach algebra $\mathcal{A}$. If $a \in \mathcal{A}$ and $x \in \mathcal{R}(I)$ satisfy $ax = xa$, then a is Browder if and only if $a + x$ is Browder.*

The next result shows that the Browder spectrum with respect to an inessential ideal is invariant under the addition of commuting Riesz element relative to I.

COROLLARY 6.8. ([13, Corollary 5.2]) *Suppose I is a closed inessential ideal in a Banach algebra $\mathcal{A}$. If $a \in \mathcal{A}$ and $x \in \mathcal{R}(I)$ satisfy $ax = xa$, then*

$$\sigma_{\mathcal{B}(I)}(a) = \sigma_{\mathcal{B}(I)}(a + x) .$$

In view of these observations we have

COROLLARY 6.9. *Suppose I is an closed inessential ideal in a Banach algebra $\mathcal{A}$. If $\mathcal{R}(I) \subseteq \texttt{comm}(\mathcal{B}(I))$, then*

$$\mathcal{R}(I) \subseteq \mathcal{P}(\mathcal{B}(I)) .$$

Proof. This easily follows from Theorem 6.7. ∎

7. ALMOST INVERTIBLE FREDHOLM ELEMENTS

In this section we investigate perturbation ideals of almost invertible elements and perturbation ideals of almost invertible Fredholm elements in a Banach algebra.

An element a in a Banach algebra $\mathcal{A}$ is called *almost invertible* if 0 is not an accumulation point of the spectrum of a. This means, a is almost invertible if a is either invertible or 0 is an isolated point of the spectrum of a. In addition, if I is a closed ideal in $\mathcal{A}$, then a is *almost invertible Fredholm* relative to I if a is almost invertible and Fredholm relative to I. We will denote the collection of all almost invertible elements in $\mathcal{A}$ by $\mathcal{AI}(\mathcal{A})$ and the collection of all almost invertible Fredholm elements relative to closed ideal I by $\mathcal{A}^{\Phi}(I)$. The spectra that these sets generate are

$$\begin{aligned} \sigma_{\mathcal{A}I(\mathcal{A})}(a) &= \{\lambda \in \mathbb{C} : \lambda - a \text{ is not almost invertible}\} \\ &= \operatorname{\mathsf{acc}} \sigma(a)\,, \\ \sigma_{\mathcal{A}^{\Phi}(I)}(a) &= \{\lambda \in \mathbb{C} : \lambda - a \text{ is not almost invertible Fredholm relative to } I\} \\ &= \operatorname{\mathsf{acc}} \sigma(a) \cup \sigma_{\mathcal{A}/I}(a+I)\,, \end{aligned}$$

for all $a \in \mathcal{A}$.

If I is a closed ideal in a Banach algebra $\mathcal{A}$, then every almost invertible Fredholm element is Browder element (see [6, Theorem 1] and [13, Corollary 2.5]).

In light of this, we have the following relationship

$$\begin{aligned} \text{invertible} \quad &\Rightarrow \quad \text{almost invertible Fredholm} \\ &\Rightarrow \quad \text{Browder} \quad \Rightarrow \quad \text{Weyl} \quad \Rightarrow \quad \text{Fredholm}\,. \end{aligned}$$

Remark 7.1. If I is a closed inessential ideal in $\mathcal{A}$, then almost invertible Fredholm elements and Browder elements relative to I coincide (see [13, Corollary 3.6] and [14, Theorem 5.2]).

We note that if I is a closed ideal in Banach algebra $\mathcal{A}$, then almost invertible elements (almost invertible Fredholm elements relative to I) is closed under multiplication by nonzero scalars.

Remark 7.2. An element a in a Banach algebra $\mathcal{A}$ is Koliha-Drazin invertible if and only if 0 is not the accumulation point of the usual spectrum of a (see [8, Theorem 4.2]). This statement shows that the Koliha-Drazin invertible elements and the almost invertible elements are equal.

PROPOSITION 7.3. *In a Banach algebra $\mathcal{A}$ the set $\mathcal{A}I(\mathcal{A})$ is a regularity.*

Proof. The result follows from Remark 7.2 and [12, Theorem 1.2]. ∎

THEOREM 7.4. *Let $\mathcal{A}$ be a Banach algebra. Then*

$$\operatorname{Rad}(\mathcal{A}) \subseteq \mathcal{P}(\mathcal{A}I(\mathcal{A})) \subseteq \mathcal{A}I(\mathcal{A})\,.$$

Proof. Let $x \in \mathrm{Rad}(\mathcal{A})$ and $a \in \mathcal{A}I(\mathcal{A})$. It follows that $0 \notin \mathtt{acc}\,\sigma(a)$. By [2, Theorem 5.3.1], it follows that $\sigma(x+a) = \sigma(a)$, and so $0 \notin \mathtt{acc}\,\sigma(x+a)$. Hence $x + a \in \mathcal{A}I(\mathcal{A})$. From this we can conclude that $x \in \mathcal{P}(\mathcal{A}I(\mathcal{A}))$. Since $\sigma(0) = \{0\}$, it follows that $0 \notin \mathtt{acc}\,\sigma(0)$, hence $0 \in \mathcal{A}I(\mathcal{A})$. Now by Remark 2.2, we get that $\mathcal{P}(\mathcal{A}I(\mathcal{A})) \subseteq \mathcal{A}I(\mathcal{A})$. Combing all our arguments we obtain the required result. ∎

THEOREM 7.5. *Let $\mathcal{A}$ be a Banach algebra and let I be a closed ideal in $\mathcal{A}$. Then*

$$\mathrm{Rad}(\mathcal{A}) \subseteq \mathcal{P}(\mathcal{A}^{\Phi}(I))\,.$$

Proof. The result follows by Theorem 7.4 and Theorem 3.2. ∎

Recall that a closed ideal I in a Banach algebra $\mathcal{A}$ is called an *s-inessential ideal* if

$$a \in \mathcal{A} \Rightarrow \mathtt{acc}\,\sigma(a) \subseteq \sigma_{\mathcal{A}/I}(a+I)\,.$$

For these ideals one can prove

PROPOSITION 7.6. ([11, Proposition 7.1]) *Suppose a closed ideal I in a Banach algebra $\mathcal{A}$ is s-inessential. Then $\mathcal{A}^{\Phi}(I)$ is a regularity in $\mathcal{A}$.*

However one can prove a stronger result.

THEOREM 7.7. ([11, Theorem 7.5]) *Suppose I is a closed inessential ideal in a Banach algebra $\mathcal{A}$. Then $\mathcal{A}^{\Phi}(I)$ is an open regularity in $\mathcal{A}$.*

THEOREM 7.8. *Suppose I is a closed inessential ideal in a Banach algebra $\mathcal{A}$. If $\mathcal{R}(I) \subseteq \mathtt{comm}(\mathcal{A}^{\Phi}(I))$, then*

$$\mathcal{R}(I) \subseteq \mathcal{P}(\mathcal{A}^{\Phi}(I))\,.$$

Proof. This is a consequence of Remark 7.1 and Theorem 6.9. ∎

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