Order isomorphisms between bases of topologies [*]

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Abstract: In this paper we will study the representations of isomorphisms between bases of topological spaces. It turns out that the perfect setting for this study is that of regular open subsets of complete metric spaces, but we have been able to show some results about arbitrary bases in complete metric spaces and also about regular open subsets in Hausdorff regular topological spaces.

Key words: Lattices; complete metric spaces; locally compact spaces; open regular sets; partial ordered sets.

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1. Introduction

Back in the 1930's, Stefan Banach and Marshall Stone proved one of the most celebrated results in Functional Analysis. The usual statement the reader can find of the Banach-Stone Theorem is, give or take, the following:

Theorem. *Let X and Y be compact Hausdorff spaces and let $T : C(X) \to C(Y)$ be a surjective linear isometry. Then there exist a homeomorphism $\tau : X \to Y$ and $g \in C(Y)$ such that $|g(y)| = 1$ for all $y \in Y$ and $(Tf)(y) = g(y)f(\tau(y))$ for all $y \in Y, f \in C(X)$.*

This result is, however, much deeper. It allows to determine X by means of the structure of $C(X)$, in the sense that X turns out to be homeomorphic to the set of extreme points of the unit sphere of $(C(X))^*$ (after quotienting by the sign).

Since then, similar results began to appear, as Gel'fand-Kolmogorov Theorem [13] or the subsequent works by Milgram, Kaplansky or Shirota, [18, 19, 22, 24]. In spite of this rapid development, after Shirota's 1952 work

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–which we will discuss later– a standstill lasts until the last few years of the XXth century. Then the topic forks in two different ways. On the one hand, there are mathematicians that begin to suspect that the proof of [24, Theorem 6] did not work, so they began to study lattice isomorphisms between spaces of uniformly continuous functions (see, e.g., [10]). On the other hand, it begins to appear a significant amount of papers that deal with representation of isomorphisms between other spaces of functions or, in general, between subsets of $C(X)$ and $C(Y)$, see [2, 11, 12, 14, 17, 23].

Anyhow, the papers where we find some of the most accurate results about isomorphisms of spaces of uniformly continuous functions [5, 6], Lipschitz functions [4] and smooth functions [3] have something in common: the result labelled in the present paper as Lemma 2.1; the interested reader should take a close look at [21], where the authors were able to unify all these results and find new ones. This lemma has been key in these works, and has recently lead to similar results, see [7, 9]. In the present paper we study Lemma 2.1, generalizing it in two ways and providing a more accurate description of the isomorphisms between lattices of regular open sets in complete metric spaces.

In the first part of the second section, we shall restrict ourselves to the study of complete metric spaces and order preserving bijections between arbitrary bases of their topologies. Namely, we will show that given a couple of complete metric spaces, say X and Y, every order preserving bijection between bases of their topologies induces a homeomorphism between dense G_δ subspaces $X_0 \subset X$ and $Y_0 \subset Y$ –subspaces that can be endowed with (equivalent) metrics that turns them into complete spaces. Later, we restrict ourselves to the bases of regular open sets on the wider class of Hausdorff, regular topological spaces and show that whenever $X_0 \subset X$ is dense, the lattices $R(X)$ and $R(X_0)$ of regular open subsets are naturally isomorphic and analyse some consequences of this. Joining both parts we get an explicit representation of every isomorphism between lattices of regular open sets in complete metric spaces that may be considered as the main result in this paper.

Remark 1.1. Apart from this Introduction, the present paper contains Section 2, where we prove the main results of the paper, and Section 3, that contains some remarks and applications.

Remark 1.2. In this paper, X and Y will always be topological spaces.

We will denote the interior of $A \subset X$ as $\text{int}_X A$, unless the space X is clear by the context, in which case we will just write $\text{int}\, A$. The same way, $\overline{A}^X$ or $\overline{A}$ will denote the closure of A in X.

We will denote by $R(X)$ the lattice of regular open subsets of X and $\mathcal{B}_X$ will be a basis of the topology of X, please recall that an open subset U of some topological space X is regular if and only if $U = \operatorname{int} \overline{U}$.

We say that $\mathcal{T} : \mathcal{B}_X \to \mathcal{B}_Y$ is an isomorphism when it is a bijection that preserves inclusion, i.e., when $\mathcal{T}(U) \subset \mathcal{T}(V)$ is equivalent to $U \subset V$.

2. The main result

In this Section we will prove our main result, Theorem 2.14. Actually, it is just a consequence of Theorem 2.8 and Proposition 2.13, but as both results are more general than Theorem 2.14 we have decided to separate them. We have split the proof in several intermediate minor results.

LEMMA 2.1. *Let (X, d_X) and (Y, d_Y) be completely metrizable metric spaces and $\mathcal{B}_X, \mathcal{B}_Y$, bases of their topologies. Suppose there is an isomorphism $\mathcal{T} : \mathcal{B}_X \to \mathcal{B}_Y$. Then, there exist dense subspaces $X_0 \subset X$ and $Y_0 \subset Y$ and a homeomorphism $\tau : X_0 \to Y_0$ such that $\tau(x) \in \mathcal{T}(U)$ if and only if $x \in U \cap X_0$.*

Proof. The proof is the same as in [5, Lemma 2], after endowing X and Y with equivalent complete distances. ∎

Remark 2.2. In the conditions of Lemma 2.1, we will denote

$$\mathcal{R}_X(x) = \bigcap_{x \in U \in \mathcal{B}_X} \mathcal{T}(U), \qquad \mathcal{R}_Y(y) = \bigcap_{y \in V \in \mathcal{B}_Y} \mathcal{T}^{-1}(V).$$

What the proof of [5, Lemma 2] shows is that the subset X_0 is dense in X, where X_0 consists of the points $x \in X$ for which there exists $y \in Y$ such that $\mathcal{R}_X(x) = \{y\}$ and $\mathcal{R}_Y(y) = \{x\}$. Once we have that X_0 is dense, it is clear that the map sending each $x \in X_0$ to the only point in $\mathcal{R}_X(x)$ is a homeomorphism.

The following Theorem is just a translation of the *Théorème fondamental* in [20].

THEOREM 2.3. *If there exists a bicontinuous, univocal and reciprocal correspondence between two given sets (inside an m-dimensional space), it is possible to determine another correspondence with the same nature between the points of two G_δ sets containing the given sets, the second correspondence agreeing with the first in the points of the two given sets.*

A more general statement of Lavrentieff's Theorem is the following, that can be found in [27, Theorem 24.9]:

THEOREM 2.4. (LAVRENTIEFF) *If X and Y are complete metric spaces and h is a homeomorphism of $A \subset X$ onto $B \subset Y$, then h can be extended to a homeomorphism h^* of A^* onto B^*, where A^* and B^* are G_δ-sets in X and Y, respectively, and $A \subset A^* \subset \overline{A}$, $B \subset B^* \subset \overline{B}$.*

As for the following Theorem, the author has been unable to find Alexandroff's work [1], but Hausdorff references the result in [15] as follows:

THEOREM 2.5. (ALEXANDROFF–HAUSDORFF, [1, 15]) *Every G_δ subset in a complete space is homeomorphic to a complete space.*

Remark 2.6. As can be seen in [25], every locally compact metric space is open in its completion, so the class of completely metrizable spaces includes that of locally compact metrizable spaces.

Combining Theorems 2.4 and 2.5 with Lemma 2.1 we obtain:

PROPOSITION 2.7. *Let X and Y be complete metric spaces and $\mathcal{T} : B_X \to B_Y$ an inclusion preserving bijection. Then, there exist a complete metric space Z and dense G_δ subspaces $X_1 \subset X$, $Y_1 \subset Y$ such that Z, X_1 and Y_1 are mutually homeomorphic.*

Of course, if Z is an in Proposition 2.7 then every dense G_δ subset $Z' \subset Z$ fulfils the same, so it is clear that there is no *minimal Z* whatsoever. In spite of this, it is very easy to determine some *maximal Z*. Consider (X_0, d_Z), where X_0 is the subset given in Lemma 2.1 and

$$d_Z(x, x') = \max \left\{ d_X(x, x'), d_Y\big(\tau(x), \tau(x')\big) \right\}. \qquad (2.1)$$

THEOREM 2.8. *The metric d_Z makes X_0 complete and, moreover, if Z' embeds into both X and Y respectively via ϕ'_X and ϕ'_Y in such a way that $\phi'_X(z) \in U$ if and only if $\phi'_Y(z) \in \mathcal{T}(U)$, then ϕ'_X embeds Z' into X_0.*

Proof. For the first part, take a d_Z-Cauchy sequence (x_n) in X_0 and let $y_n = \tau(x_n)$ for every n. It is clear that both (x_n) and (y_n) are d_X-Cauchy and d_Y-Cauchy, respectively, so let $x = \lim(x_n) \in X, y = \lim(y_n) \in Y$, these limits exist because X and Y are complete. It is clear that any sequence

$(\tilde{x}_n) \subset X_0$ converges to x if and only if $y = \lim(\tau(\tilde{x}_n))$. This readily implies that $\mathcal{R}_X(x) = \{y\}$, so $x \in X_0$ and this means that (X_0, d_Z) is complete.

Now we must see that every metric space Z' that embeds in both X and Y is embeddable in X_0, whenever the embeddings respect the isomorphism between the bases. For this, as X_0 is endowed with the restriction of the topology of X and Z' is homeomorphic to $\phi'_X(Z') \subset X$, the only we need is $\phi'_X(Z') \subset X_0$. So, suppose $x \in \phi'_X(Z') \setminus X_0$ and let $z \in (\phi'_X)^{-1}(x)$. As Z' also embeds in Y, there exists $y = \phi'_Y(z) \in \phi'_Y(Z') \setminus Y_0$, too, with the property that $x \in U$ if and only if $y \in \mathcal{T}(U)$. By the very definition of X_0 and Y_0 this means that $x \in X_0, y \in Y_0$ and $\tau(x) = y$. ∎

Now, we approach Proposition 2.13, the main result about regular topological spaces. For this, the following three elementary results will come in handy; all their proofs are clear.

LEMMA 2.9. *Let Z be a topological space and $A \subset Z$. A is a regular open subset of Z if and only if for every open $V \subset Z$, $V \subset \overline{A}$ implies $V \subset A$.*

LEMMA 2.10. *Let X be a topological space. Whenever $Y \subset X$ is dense and $U \subset X$ is open, one has $\overline{U}^X = \overline{U \cap Y}^X$.*

LEMMA 2.11. *Let X be a topological space and $U, V \in R(X)$ such that $U \subset V, U \neq V$. Then, there is $\emptyset \neq W \in R(X)$ such that $W \cap U = \emptyset$ and $W \subset V$.*

Remark 2.12. If in Lemma 2.11 X is regular and Hausdorff, then V can be taken as any open subset that contains U strictly.

PROPOSITION 2.13. *Let X be a topological space and $Y \subset X$ a dense subset. Then $\mathcal{T} : R(X) \to R(Y)$, defined as $\mathcal{T}(U) = U \cap Y$, is a lattice isomorphism with inverse $\mathcal{S}(V) = \mathrm{int}\,\overline{V}$.*

Proof. We need to show that $\mathcal{T}$ and $\mathcal{S}$ are mutually inverse.

Let $U \in R(X)$, the first we need to show is that $\mathcal{T}$ is well-defined, i.e., that $\mathcal{T}(U) = U \cap Y$ is regular in Y.

Let $V \subset X$ an open subset such that $V \cap Y \subset \overline{U \cap Y}^Y$. Then, as the closure in X preserves inclusions, we have

$$\overline{V}^X = \overline{V \cap Y}^X \subset \overline{\overline{U \cap Y}^Y}^X \subset \overline{U}^X,$$

where the first equality holds because of Lemma 2.10. Taking interiors in X also preserves inclusions, so we obtain

$$V \subset \operatorname{int}_X\left(\overline{V}^X\right) \subset \operatorname{int}_X\left(\overline{U}^X\right) = U,$$

which readily implies that $V \cap Y \subset U \cap Y$ and we obtain $V \cap Y \in R(Y)$ from Lemma 2.9. It is clear that $\mathcal{S}(V) \in R(X)$ for every $V \in R(Y)$, so both maps are well-defined.

Furthermore, Lemma 2.10 implies that, for any regular $U \subset X$,

$$\mathcal{S} \circ \mathcal{T}(U) = \mathcal{S}(U \cap Y) = \operatorname{int}_X\left(\overline{U \cap Y}^X\right) = \operatorname{int}_X\left(\overline{U}^X\right) = U.$$

As for the composition $\mathcal{T} \circ \mathcal{S}$, we have

$$\mathcal{T} \circ \mathcal{S}(V) = \mathcal{T}\left(\operatorname{int}_X\left(\overline{V}^X\right)\right) = \operatorname{int}_X\left(\overline{V}^X\right) \cap Y$$

for any $V \in R(Y)$. Let $W \subset X$ be an open subset for which $V = W \cap Y$, the very definition of inherited topology implies that there exists such W. The previous equalities can be rewitten as

$$\mathcal{T} \circ \mathcal{S}(W \cap Y) = \mathcal{T}\left(\operatorname{int}_X\left(\overline{W \cap Y}^X\right)\right) = \mathcal{T}\left(\operatorname{int}_X\left(\overline{W}^X\right)\right) = \operatorname{int}_X\left(\overline{W}^X\right) \cap Y,$$

so we need $W \cap Y = \operatorname{int}_X\left(\overline{W}^X\right) \cap Y$. It is clear that $W \cap Y \subseteq \operatorname{int}_X\left(\overline{W}^X\right) \cap Y$, so what we need is $\operatorname{int}_X\left(\overline{W}^X\right) \cap Y \subseteq W \cap Y$. Both subsets are regular in Y, so if this inclusion does not hold, there would exist an open $H \subset Y$

$$H \neq \emptyset, \qquad H \subset \operatorname{int}_X\left(\overline{W}^X\right) \cap Y, \qquad H \cap (W \cap Y) = \emptyset,$$

so, by Lemma 2.11 there is an open $G \subset X$ such that $H = G \cap Y$ and so

$$G \cap Y \neq \emptyset, \quad G \cap Y \overset{(*)}{\subset} \operatorname{int}_X\left(\overline{W}^X\right) \cap Y, \quad (G \cap Y) \cap (W \cap Y) = \emptyset, \quad (2.2)$$

and this is absurd. Indeed, the inclusion marked with $(*)$ implies that we may substitute G by $G \cap \operatorname{int}_X\left(\overline{W}^X\right)$, so both inequalities in (2.2) hold for some open $G \subset \operatorname{int}_X\left(\overline{W}^X\right)$. As Y is dense and G and W are open, the last equality implies that $G \cap W = \emptyset$. Of course, this implies $G \cap \operatorname{int}_X\left(\overline{W}^X\right) = \emptyset$, which means $G = \emptyset$ and we are done. ∎

Now we are in conditions to state our main result:

THEOREM 2.14. *Let X, Y and Z be complete metric spaces, $\phi_X : Z \hookrightarrow X$ and $\phi_Y : Z \hookrightarrow Y$ be continuous, dense, embeddings and $X_0 = \phi_X(Z)$. Then, $\mathcal{T} : R(X) \to R(Y)$ given by the composition*

$$U \mapsto U \cap X_0 \mapsto \phi_X^{-1}(U \cap X_0) \mapsto \phi_Y(\phi_X^{-1}(U \cap X_0)) \mapsto \mathrm{int}\left(\overline{\phi_Y(\phi_X^{-1}(U \cap X_0))}\right)$$

is an isomorphism between the lattices of open regular subsets of X and Y and every isomorphism arises this way.

The "every isomorphism arises this way" part is due to Theorem 2.8, while the "the composition is an isomorphism" part is consequence of Proposition 2.13.

3. APPLICATIONS AND REMARKS

In this Section, we are going to show how Proposition 2.13 leads to some properties of $\beta\mathbb{N}$ and conclude with a couple of examples that show that the hypotheses imposed in the main results are necessary. But first, we need to deal with an error in some outstanding work. In [4] F. Cabello and the author of the present paper showed that some results in [24] were not properly proved. Later in [5] the same authors proved that, even when the proof of [24, Theorem 6] was incorrect, the result was true. Now, we are going to explain what the error was. The following Definitions and Theorems can be found in [24]:

DEFINITION 3.1. (DEFINITION 2) A distributive lattice with smallest element 0 satisfying Wallman's disjunction property is an *R-lattice* if there exists a binary relation $\gg$ in L which satisfies:

- If $h \geq f$ and $f \gg g$, then $h \gg g$.

- If $f_1 \gg g_1$ and $f_2 \gg g_2$, then $f_1 \wedge f_2 \gg g_1 \wedge g_2$.

- If $f \gg g$, then there exists h such that $f \gg h \gg g$.

- For every $f \neq 0$ there exist g_1 and $g_2 \neq 0$ such that $g_1 \gg f \gg g_2$.

- If $g_1 \gg f \gg g_2$, then there exists h such that $h \vee f = g_1$ and $h \wedge g_2 = 0$.

Immediately after Definition 2 we find this:

THEOREM 3.2. (THEOREM 1) *A distributive lattice with smallest element* 0 *is an R-lattice if and only if it is isomorphic to a sublattice of the lattice of all regular open sets on a locally compact space* X. *This sublattice is an open base and its elements have compact closures.*

The open regular set in X associated to $f \in L$ is denoted by $U(f)$. With this notation, the next statement is:

THEOREM 3.3. (THEOREM 2) *Let* L *be an R-lattice. Then there exists uniquely a locally compact space* X *which satisfies the property in Theorem 1 and where* $U(f) \supset \overline{U(g)}$ *if and only if* $f \gg g$.

Our Proposition 2.13 contradicts the uniqueness of X in the statement of Theorem 2 and we may actually explicit a lattice isomorphism between $R(X)$ and $R(Y)$ for different compact metric spaces X and Y. Namely, we just need to take the simplest compactifications of $\mathbb{R}$ and the composition of the lattice isomorphisms predicted by Proposition 2.13:

EXAMPLE 3.4. Let $X = \mathbb{R} \cup \{-\infty, \infty\}$ and $Y = \mathbb{R} \cup \{N\}$. Then, $\mathcal{T} : R(X) \to R(Y)$, defined by

$$\mathcal{T}(U) = \begin{cases} U & \text{if } U \cap \{-\infty, \infty\} = \emptyset, \\ U \cap \mathbb{R} & \text{if } U \cap \{-\infty, \infty\} = \{\infty\}, \\ U \cap \mathbb{R} & \text{if } U \cap \{-\infty, \infty\} = \{-\infty\}, \\ (U \cap \mathbb{R}) \cup \{N\} & \text{if } \{-\infty, \infty\} \subset U, \end{cases}$$

is a lattice isomorphism whose inverse is given by

$$\mathcal{S}(V) = \begin{cases} V & \text{if } N \notin V, \\ (V \cap \mathbb{R}) \cup \{-\infty, \infty\} & \text{if } N \in V. \end{cases}$$

It seems that the problem here is that the definition of R-lattice, Definition 2, does not include the relation $\gg$, but in Theorem 2 and its consequences the author considers $\gg$ as a unique, fixed, relation given by $(\mathcal{L}, \leq)$. It is clear that the above spaces generate, say, different $\gg_X$ and $\gg_Y$ in the isomorphic lattices $R(X)$ and $R(Y)$. This leads to the error already noted in [4, Section 5].

Actually, with the definition of R-lattice given in [24], in seems that the original purpose of the definition is lost. Indeed, the relation $\gg$ may be taken as $\geq$ in quite a few lattices. This leads to a topology where every regular open set is clopen, in Section 3.1 we will see an example of a far from

trivial topological space where this is true. Given a lattice $(\mathcal{L}, \geq)$, the relation between each possible $\gg$ and the unique locally compact topological space given by Theorem 2 probably deserves a closer look.

Anyway, if we include $\gg$ in the definition, then [24, Theorem 2] is true. So let us put everything in order.

DEFINITION 3.5. (SHIROTA) Let $(\mathcal{L}, \leq)$ be a distributive lattice with $\min \mathcal{L} = 0$ and $\gg$ be a relation in $\mathcal{L}$. The triple $(\mathcal{L}, \leq, \gg)$ is an R-lattice if the following hold:

1. For every $a \neq b \in \mathcal{L}$, there exists $h \in \mathcal{L}$ such that either $a \wedge h = 0$ and $b \wedge h \neq 0$ or the other way round.

2. If $h \geq f$ and $f \gg g$, then $h \gg g$.

3. If $f_1 \gg g_1$ and $f_2 \gg g_2$, then $f_1 \wedge f_2 \gg g_1 \wedge g_2$.

4. If $f \gg g$, then there exists h such that $f \gg h \gg g$.

5. For every $f \neq 0$ there exist g_1 and $g_2 \neq 0$ such that $g_1 \gg f \gg g_2$.

6. If $g_1 \gg f \gg g_2$, then there exists h such that $h \vee f = g_1$ and $h \wedge g_2 = 0$.

With Definition 3.5 everything works and this result remains valid, but it does not lead to the consequences stated there as Theorems 3 to 6.

THEOREM 3.6. (THEOREM 2) *Let $(\mathcal{L}, \leq, \gg)$ be an R-lattice. Then there exists uniquely a locally compact space X which satisfies the property in Theorem 1 and where $U(f) \supset \overline{U(g)}$ if and only if $f \gg g$.*

Remark 3.7. It is worth noting that the main result in [16] states that, in the more general setting of uniform spaces, every lattice isomorphism $\mathcal{T} : U(X) \to U(Y)$ induces another lattice isomorphism $\mathcal{T} : U^*(X) \to U^*(Y)$, thus leading to another proof of Shirota's Theorem.

3.1. THE STONE-ČECH COMPACTIFICATION OF $\mathbb{N}$. We will analyse the isomorphism given in Proposition 2.13 when $Y = \mathbb{N}$ and $X = \beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$. This is not going to lead to new results, but it seems to be interesting in spite of this. These are very different spaces, but they share the same lattice of regular open subsets. In any case, as $\mathbb{N}$ is discrete, every $V \subset \mathbb{N}$ is regular and this, along with Proposition 2.13, implies that

$$R(\beta\mathbb{N}) = \big\{ \mathrm{int}(\overline{U}) \,:\, U \subset \mathbb{N} \big\}.$$

As $\beta\mathbb{N}$ is regular, every open $W \subset \beta\mathbb{N}$ is the union of its regular open subsets, and these regular open subsets are determined by the integers they contain, so W is determined by a collection $\mathcal{A}_W$ of subsets of $\mathbb{N}$. Of course, if W contains $\mathcal{S}(J)$ for some $J \subset \mathbb{N}$ and $I \subset J$ then $\mathcal{S}(I) \subset W$, too. This means that $\mathcal{A}_W$ is closed for inclusions. Furthermore, as $\overline{J}$ is open in $\beta\mathbb{N}$, see [26, p. 144, Subsection 3.9], if $\mathcal{S}(I) \subset W$ and $\mathcal{S}(J) \subset W$ then $\mathcal{S}(I) \cup \mathcal{S}(J) = \mathcal{S}(I \cup J)$, so $\mathcal{S}(I \cup J) \subset W$ and $\mathcal{A}_W$ is closed for pairwise supremum. Summing up, $\mathcal{A}_W$ is an ideal of the lattice $\mathcal{P}(\mathbb{N})$ for every proper open subset $\emptyset \subsetneq W \subsetneq \beta\mathbb{N}$ –and every $\mathcal{S}(J) \in R(\beta\mathbb{N})$ is closed, so "clopen" and "regular open" are equivalent in $\beta\mathbb{N}$.

It is also clear that every ideal $\mathcal{A}$ in $\mathcal{P}(\mathbb{N})$ defines an open $W_\mathcal{A} \in \beta\mathbb{N}$ as $\cup\{\mathcal{S}(I) : I \in \mathcal{A}\}$, that these identifications are mutually inverse and that $W \subset V$ if and only if $\mathcal{A}_W \subset \mathcal{A}_V$, so each maximal ideal in $\mathcal{P}(\mathbb{N})$ defines a maximal open proper subset of $\beta\mathbb{N}$. As $\beta\mathbb{N}$ is Hausdorff, these maximal open subsets are exactly $\beta\mathbb{N} \setminus \{x\}$ for some x, so each point is dually defined by a maximal ideal. In other words, every point in $\beta\mathbb{N}$ is associated to an ultrafilter in $\mathcal{P}(\mathbb{N})$.

As our final comment in this *Just for fun* Remark, we have that $\beta\mathbb{N}$ is the only Hausdorff compactification of $\mathbb{N}$ that fulfils:

♠ If $J, I \subset \mathbb{N}$ are disjoint, then their closures in the compactification are disjoint, too,

although this is just a particular case of a result by Čech, see [26, pp. 25-26].

3.2. THE HYPOTHESES ARE MINIMAL. In some sense, Theorem 2.14 is optimal. Here we see that there is no way to generalise it if we omit any of the hypotheses.

Remark 3.8. Consider any infinite set Z endowed with the cofinite topology τ_{cof}. It is clear that every pair of nonempty open subsets of Z meet, so every nonempty open subset is dense in Z and this implies that the only regular open subsets of Z are Z and $\emptyset$. Of course the same applies to any uncountable set endowed with the cocountable topology τ_{con}, so $(\mathbb{R}, \tau_{cof})$ and $(\mathbb{R}, \tau_{con})$ have the same regular open subsets. Nevertheless, there is no way to identify homeomorphically any couple of dense subsets of $\mathbb{R}$ with each topology. In order to avoid this pathological behaviour we needed to consider only regular Hausdorff spaces since these spaces are the only reasonable spaces for which the regular subsets comprise a base of the topology. In other words, Theorem 2.14 will not extend to general topological spaces.

Remark 3.9. Consider $X = [0,1]$ endowed with its usual topology and let Y be its Gleason cover. The lattices $R(X)$ and $R(Y)$ are canonically isomorphic, but it is well known that no point in Y has a countable basis of neighbourhoods, so

$$\bigcap_{x \in U \in \mathcal{B}_X} \mathcal{T}(U) = \bigcap_{n=1}^{\infty} \mathcal{T}\big(B(x,1/n)\big)$$

is never a singleton.

It is remarkable that [8, Example 1.7.16] is the *only* place where the author has been able to find a statement that explicitly confirms that the Gleason cover of some compact space K is the same topological space as the Stone space associated to $R(K)$, i.e., $G_K = \mathrm{St}(R(K))$.

Remark 3.10. Consider $A = \mathbb{Q} \cap [0,1]$, $B = \mathbb{I} \cap [0,1]$ and their Stone-Čech compactifications $X = \beta A$, $Y = \beta B$. There is a lattice isomorphism between $R(X)$ and $R(Y)$, say $\mathcal{T}$, given by the composition of the following isomorphisms:

$$U \in R(X) \mapsto U \cap A \in R(A), \qquad U \in R(A) \mapsto \mathrm{int}\,\overline{U} \in R\big([0,1]\big),$$
$$U \in R\big([0,1]\big) \mapsto U \cap B \in R(B), \qquad U \in R(B) \mapsto \mathrm{int}\,\overline{U} \in R(Y).$$

In spite of this, it is intuitively evident that

$$\mathcal{R}_X(x) = \bigcap_{x \in U \in \mathcal{B}_X} \mathcal{T}(U) = \bigcap_{n=1}^{\infty} \mathcal{T}\Big(\mathrm{int}\Big(\overline{B(x,1/n)}\Big)\Big)$$

is never a singleton when $x \in A$, and

$$\mathcal{R}_Y(y) = \bigcap_{y \in V \in \mathcal{B}_Y} \mathcal{T}^{-1}(V) = \bigcap_{n=1}^{\infty} \mathcal{T}^{-1}\Big(\mathrm{int}\Big(\overline{B(y,1/n)}\Big)\Big)$$

is neither a singleton for $y \in B$. It seems clear that $X = A \sqcup \{\mathcal{R}_Y(y) : y \in B\}$ and $Y = B \sqcup \{\mathcal{R}_X(x) : x \in A\}$, so there is no point in X_0.

Remark 3.11. There is no "non-complete metric spaces" result. Indeed, $\mathbb{I}$ and $\mathbb{Q}$ have the obvious isomorphism between their bases of regular open subsets and they are, nevertheless, disjoint subsets in $\mathbb{R}$. This means that when trying to generalise Theorem 2.14 the problem may come not only from

the lack of separation of the topologies as in Remark 3.8, from the excess or points as in Remark 3.9 or from the points in X not squaring with those in Y as in Remark 3.10 but also from the, so to say, lack of points in the spaces even when they are metric.

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