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Prolongations of *G*-structures related to Weil bundles and some applications

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Abstract: Let M be a smooth manifold of dimension $m \geq 1$ and P be a G-structure on M, where G is a Lie subgroup of linear group $GL(m)$. In [8], it has been defined the prolongations of G-structures related to tangent functor of higher order and some properties have been established. The aim of this paper is to generalize these prolongations to a Weil bundles. More precisely, we study the prolongations of G-structures on a manifold M, to its Weil bundle $T^A M$ (A is a Weil algebra) and we establish some properties. In particular, we characterize the canonical tensor fields induced by the A-prolongation of some classical G-structures.

Key words: G-structures, Weil-Frobenius algebras, Weil functors, gauge functors and natural transformations.

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Introduction

We recall that, a Weil algebra A is a real commutative algebra with unit which is of the form $A = \mathbb{R} \cdot 1_A \oplus N_A$, where N_A is a finite dimensional ideal of nilpotent elements of A (see [4] or [8]). It exists several examples of Weil algebra, for instance the algebra generated by 1 and ε with $\varepsilon^2 = 0$ denoted by $\mathbb{D}$ (sometimes it is called the algebra of dual numbers, it is also the truncated polynomial algebra of degree 1). Another Weil algebra is given by the spaces of all r-jets of $\mathbb{R}^k$ into $\mathbb{R}$ with source $0 \in \mathbb{R}^k$ and denoted by $J_0^r(\mathbb{R}^k, \mathbb{R})$. The ideal of nilpotent elements is the finite vector space $J_0^r(\mathbb{R}^k, \mathbb{R})_0$. Let $A = \mathbb{R} \cdot 1_A \oplus N_A$ be a Weil algebra, we adopt the covariant approach of Weil functor described by I. Kolàr in [6]. We denote by N_A^k the ideal generated by the product of k elements of N_A, there is one and only one natural number h such that $N_A^h \neq 0$ and $N_A^{h+1} = 0$. The integer h is called the order of A and the dimension k of

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the vector space N_A/N_A^2 is said to the width of A. In this case, the Weil algebra A is called (k,h)-algebra. If $\varrho, \varrho_1 : J_0^h(\mathbb{R}^k, \mathbb{R}) \to A$ are two surjective algebra homomomorphisms, then there is an isomorphism $\sigma : J_0^h(\mathbb{R}^k, \mathbb{R}) \to J_0^h(\mathbb{R}^k, \mathbb{R})$ such that: $\varrho_1 \circ \sigma = \varrho$. We say that, two maps $\varphi, \psi : \mathbb{R}^k \to M$ determine the same A-velocity if for every smooth map $f : M \to \mathbb{R}$

$$\varrho\left(j_0^h(f \circ \varphi)\right) = \varrho\left(j_0^h(f \circ \psi)\right).$$

The equivalence class of the map $\varphi : \mathbb{R}^k \to M$ is denoted by $j^A\varphi$ and will called A-velocity at 0 (see [6], [7] or [8]). We denote by T^AM the space of all A-velocities on M. More precisely,

$$T^AM = \left\{ j^A\varphi,\ \varphi : \mathbb{R}^k \to M \right\}.$$

T^AM is a smooth manifold of dimension $m \times \dim A$. For a local chart $\left(U, u^1, \dots, u^m\right)$ of M, the local chart of T^AM is $\left(T^AU, u_0^i, \dots, u_K^i\right)$ such that:

$$\begin{cases} u_0^i\left(j^A\varphi\right) = u^i(\varphi(0)) \\ u_\alpha^i\left(j^A\varphi\right) = a_\alpha^*\left(j^A(u^i \circ \varphi)\right) \end{cases} \quad 1 \leq \alpha \leq K$$

where $(a_0, \dots, a_K)$ is basis of A and $(a_0^*, \dots, a_K^*)$ is a dual basis. We denote by $\pi_M^A : T^AM \to M$ the natural projection such that $\pi_M^A(j^A\varphi) = \varphi(0)$, so $\left(T^AM, M, \pi_M^A\right)$ is a fibered manifold. For every smooth map $f : M \to \overline{M}$, induces a smooth map $T^Af : T^AM \to T^A\overline{M}$ such that: for any $j^A\varphi \in T^AM$,

$$T^Af(j^A\varphi) = j^A(f \circ \varphi).$$

In particular we have that $\left(f, T^Af\right)$ is a fibered morphism from $\left(T^AM, M, \pi_M^A\right)$ to $\left(T^A\overline{M}, \overline{M}, \pi_{\overline{M}}^A\right)$. This defines a bundle functor $T^A : \mathcal{M}f \to \mathcal{FM}$ called Weil functor induced by A. The bundle functor T^A preserves product in the sense, that for any manifolds M and $\overline{M}$, the map

$$\left(T^A(\mathrm{pr}_M), T^A(\mathrm{pr}_{\overline{M}})\right) : T^A(M \times \overline{M}) \longrightarrow T^AM \times T^A\overline{M}$$

where $\mathrm{pr}_M : M \times \overline{M} \to M$ and $\mathrm{pr}_{\overline{M}} : M \times \overline{M} \to \overline{M}$ are the projections, is an $\mathcal{FM}-$isomorphism. Hence we can identify $T^A(M \times \overline{M})$ with $T^AM \times T^A\overline{M}$.

Let B be another (s,r) Weil algebra and $\mu : A \to B$ be an algebra homomorphism, $\varrho' : J_0^r(\mathbb{R}^s, \mathbb{R}) \to B$ the surjective algebra homomorphism. Then there is an algebra homomorphism $\widetilde{\mu} : J_0^h(\mathbb{R}^k, \mathbb{R}) \to J_0^r(\mathbb{R}^s, \mathbb{R})$ such that the following diagram

$$\begin{array}{ccc} J_0^h(\mathbb{R}^k,\mathbb{R}) & \xrightarrow{\widetilde{\mu}} & J_0^r(\mathbb{R}^s,\mathbb{R}) \\ \varrho \downarrow & & \downarrow \varrho' \\ A & \xrightarrow[\mu]{} & B \end{array}$$

commutes. In particular, there is map $f_\mu : \mathbb{R}^s \to \mathbb{R}^k$ such that, $\widetilde{\mu}(j_0^h g) = j_0^r(g \circ f_\mu)$, where $g \in C^\infty(\mathbb{R}^k)$. For any manifold M of dimension $m \geq 1$, it is proved in [7] that there is smooth map $\mu_M : T^A M \to T^B M$ defined by:

$$\mu_M(j^A \varphi) = j^B(\varphi \circ f_\mu).$$

More precisely, $\mu_M : T^A M \to T^B M$ is a natural transformations and denoted by $\overline{\mu} : T^A \to T^B$. The fundamental result, which reads that every product preserving bundle functor on $\mathcal{M}f$ is a Weil functor. More precisely, if F is a product preserving bundle functor on $\mathcal{M}f$, $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the addition and the multiplication of reals, then $Fa : F\mathbb{R} \times F\mathbb{R} \to F\mathbb{R}$ and $F\lambda : F\mathbb{R} \times F\mathbb{R} \to F\mathbb{R}$ is the vector addition and the algebra multiplication in the Weil algebra $F\mathbb{R}$ and F coincides with the Weil functor $T^{F\mathbb{R}}$. Every natural transformation $\mu : T^A \to T^B$ are in bijection with the algebra homomorphism $\mu_{\mathbb{R}} : A \to B$ (see [8]). Since $\varrho : J_0^h(\mathbb{R}^k, \mathbb{R}) \to A$ is determined up to an isomorphism $J_0^h(\mathbb{R}^k, \mathbb{R}) \to J_0^h(\mathbb{R}^k, \mathbb{R})$ it follows that this construction is independent of the choice of ϱ. The Weil functor generalizes the tangent functor, more precisely, when A is the space of all r-jets of $\mathbb{R}^k$ into $\mathbb{R}$ with source $0 \in \mathbb{R}^k$ denoted by $J_0^r(\mathbb{R}^k, \mathbb{R})$, the corresponding Weil functor is the functor of k-dimensional velocities of order r and denoted by T_k^r. For $k = 1$, it is called tangent functor of order r and denoted by T^r, this functor plays an essential role in the reduction of some hamiltonian systems of higher order. It has been clarified that, the theory of Weil functor represents a unified technique for studying a large class of geometric problems related with product preserving functor.

Let $A = \mathbb{R} \cdot 1_A \oplus N_A$ be a Weil algebra. For any multiindex $0 < |\alpha| \leq h$, we put $e_\alpha = j^A(x^\alpha)$ is an element of N_A. For any $\varphi \in C^\infty(\mathbb{R}^k, \mathbb{R})$, we have:

$$j^A \varphi = \varphi(0) \cdot 1_A + \sum_{1 \leq |\alpha| \leq h} \tfrac{1}{\alpha!} D_\alpha \varphi(0) e_\alpha.$$

In particular the family $\{e_\alpha\}$ generates the ideal N_A. We denote by B_A the set of all multiindex such that $(e_\alpha)_{\alpha \in B_A}$ is a basis of N_A and $\overline{B}_A$ her complementary with respect to the set of all multiindex $\gamma \in \mathbb{N}^n$ such that $1 \leq| \gamma |\leq h$. For $\beta \in \overline{B}_A$, we have $e_\beta = \lambda_\beta^\alpha e_\alpha$. In particular,

$$e_\alpha \cdot e_\beta = \begin{cases} e_{\alpha+\beta} & \text{if } \alpha+\beta \in B_A \,, \\ \lambda_{\alpha+\beta}^{\gamma} e_\gamma & \text{if } \alpha+\beta \in \overline{B}_A \,. \end{cases}$$

It follows that, for any $\varphi \in C^\infty(\mathbb{R}^k, \mathbb{R})$, we have:

$$j^A\varphi = \varphi(0)\cdot 1_A + \sum_{\alpha\in B_A}\left(\tfrac{1}{\alpha!}D_\alpha\varphi(0) + \sum_{\beta\in\overline{B_A}} \tfrac{\lambda_\beta^\alpha}{\beta!}D_\beta\varphi(0)\right)e_\alpha \,.$$

Let $\left(U, x^i\right)$ be a local coordinate system of M, a coordinate system induced by $\left(U, x^i\right)$ over the open T^AU of T^AM denoted by $\left(x^i, x_\alpha^i\right)$ is given by

$$\begin{cases} x^i \;=\; x^i\circ\pi_M^A \;=\; x_0^i \,, \\ x_\alpha^i \;=\; \overline{x}_\alpha^i + \sum_{\beta\in\overline{B}_A}\lambda_\beta^\alpha\overline{x}_\beta^i \,, \end{cases}$$

where $\overline{x}_\beta^i(j^Ag) = \frac{1}{\beta!}\cdot D_\beta(x^i\circ g)(0)$ and $j^Ag \in T^AU$. In the particular case where $A = \mathbb{D}$, the local coordinate system of TM induced by $\left(U, x^i\right)$ is denoted by $\left(x^i, \dot{x}^i\right)$.

Let M be a smooth manifold of dimension $m \geq 1$, with (TM, M, π_M) we denote its tangent bundle, and with $(F(M), M, p_M)$ we denote the frame bundles of M. Let G be a Lie subgroup of $GL(m)$, a G-structure on a manifold M is a G-subbundle (P, M, π) of the frame bundle $F(M)$ of M. For the general theory of G-structures see, for instance [1]. The prolongations of G-structures from a manifold M to its tangent bundles of higher order T^rM has been studied by A. Morimoto in [12]. In particular, it proves that if a manifold M has an integrable structure (resp. almost complex structure, symplectic structure, pseudo-Riemannian structure), then T^rM has canonically the same kind of structure. Since the tangent functor of higher order T^r on the manifolds, considers all derivatives of higher order (up to order r), all the proofs are obtained by calculation in local coordinate. The situation should be much complicated for the Weil functor T^A. Thus, the aim of this paper is to define the prolongations of G-structures from a manifold M to its Weil bundle T^AM. In particular, we construct a canonical embedding $j_{A,E}$ of $T^A(FE)$ into $F(T^AE)$, where $F(E)$ denote the frame bundle of the vector bundle $(E \to M)$. Using the natural isomorphism $\kappa_{A,M} : T^A(TM) \to T(T^AM)$ (see [5]) and the embedding $j_{A,TM}$, we define this A-prolongation $\mathcal{T}^AP$ of a G-structure P of a manifold M, to its Weil bundle T^AM. In particular, we prove that $\mathcal{T}^AP$ is integrable if and only if P is integrable. In the last section, we use the theory of lifting of tensor fields defined in [3] and [6], to characterized the canonical tensor fields induced by the A-prolongation of some classical G-structures.

In this paper, all manifolds and mappings are assumed to be differentiable of class C^∞. In the sequel A will be a Weil algebra of order $h \geq 2$ and of width $k \geq 1$.

1. Preliminaries

1.1. Lifts of functions and vector fields. Let $\ell : A \to \mathbb{R}$ be a smooth function, for any smooth function $f : M \to \mathbb{R}$, we define the ℓ-lift of f to T^AM by:

$$f^{(\ell)} = \ell \circ T^A(f)\,;$$

$f^{(\ell)}$ is a smooth function on T^AM.

Remark 1. Let $(e_\beta)_{\beta\in B_A}$ a basis of N_A, we denote by $\left(e^0, e^\beta\right)_{\beta\in B_A}$ the dual basis of A. For $\ell = e^\alpha$, the smooth function $f^{(\ell)}$ is denoted by $f^{(\alpha)}$. In particular, for any $j^A\varphi \in T^AM$,

$$f^{(\alpha)}(j^A\varphi) = \tfrac{1}{\alpha!}D_\alpha(f\circ\varphi)(z)\big|_{z=0} + \sum_{\beta\in\overline{B_A}} \tfrac{\lambda_\beta^\alpha}{\beta!}D_\beta(f\circ\varphi)(z)\Big|_{z=0}$$

and $f^{(0)} = f\circ\pi_M^A$. For a coordinate system $\left(U, x^1, \ldots, x^m\right)$ in M, the induced coordinate system $\left\{x_0^i, x_\alpha^i\right\}$ on T^AM is such that, $x_\alpha^i = \left(x^i\right)^{(\alpha)}$.

Remark 2. For any smooth map $\ell : A \to \mathbb{R}$, the map

$$\begin{array}{ccc} C^\infty(M) & \longrightarrow & C^\infty(T^AM) \\ f & \longmapsto & f^{(\ell)} \end{array}$$

is $\mathbb{R}$-linear.

For all multiindex α such that $|\alpha| \leq h$, we denote by $\chi^{(\alpha)} : T^A \to T^A$ the natural transformation defined for any vector bundle $(E \to M)$ and $\varphi \in C^\infty(\mathbb{R}^k, E)$ by:

$$\chi_E^{(\alpha)}(j^A\varphi) = j^A(z^\alpha\varphi)$$

where $z^\alpha\varphi$ is a smooth map defined for any $z \in \mathbb{R}^k$ by $(z^\alpha\varphi)(z) = z^\alpha\varphi(z)$.

Proposition 1. *Let A be a Weil algebra. There exists one and only one family $\kappa_{A,M} : T^A(TM) \to T(T^AM)$ of vector bundle isomorphisms such that $\pi_{T^AM} \circ \kappa_{A,M} = T^A(\pi_M)$ and the following conditions hold:*

1. *For every smooth mapping $f : M \to N$ the following diagram*

$$\begin{array}{ccc} T^A(TM) & \xrightarrow{T^A(Tf)} & T^A(TN) \\ \kappa_{A,M}\Big\downarrow & & \Big\downarrow\kappa_{A,N} \\ T(T^AM) & \xrightarrow[T(T^Af)]{} & T(T^AN) \end{array}$$

   commutes.

2. *For two manifolds M, N we have $\kappa_{A,M\times N} = \kappa_{A,M} \times \kappa_{A,N}$.*

Proof. See [5]. ∎

Let $X : M \to TM$ be a vector field on a manifold M, then we put

$$X^{(\alpha)} = \kappa_{A,M} \circ \chi^{(\alpha)}_{TM} \circ T^A(X)\,.$$

It is a vector bundle field on $T^A(M)$ called α-lift of X to T^AM. In the particular case where $\alpha = 0$, the vector field $X^{(0)}$ is denoted by $X^{(c)}$ and it is called complete lift of X to T^AM. We put $X^{(\alpha)} = 0$, for $|\alpha| > h$ or $\alpha \notin \mathbb{N}^k$.

Remark 3. For any $|\alpha| \leq h$, the map

$$\begin{array}{ccc} \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(T^AM) \\ X & \longmapsto & X^{(\alpha)} \end{array}$$

is $\mathbb{R}$-linear and for any smooth map $\varphi : M \to N$ and any φ-related vector fields $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$, the vector fields $X^{(\alpha)} \in \mathfrak{X}(T^AM)$, $Y^{(\alpha)} \in \mathfrak{X}(T^AN)$ are $T^A(\varphi)$ related.

Proposition 2. *For $X, Y \in \mathfrak{X}(M)$, we have:*

$$\left[X^{(\alpha)}, Y^{(\beta)}\right] = [X,Y]^{(\alpha+\beta)}$$

for all $0 \leq |\alpha, \beta| \leq h$.

Proof. See [5]. ∎

Remark 4. The family of α-lift of vector fields is very important, because, if S and S' are two tensor fields of type $(1,p)$ or $(0,p)$ on $T^A(M)$ such that, for all $X_1,\dots,X_p \in \mathfrak{X}(M)$, and multiindex $\alpha_1,\dots,\alpha_p$, the equality

$$S\Big(X_1^{(\alpha_1)},\dots,X_p^{(\alpha_p)}\Big) = S'\Big(X_1^{(\alpha_1)},\dots,X_p^{(\alpha_p)}\Big)$$

holds, then $S = S'$ (see [2]).

1.2. LIFTS OF TENSOR FIELDS OF TYPE $(1,q)$. Let S be a tensor field of type $(1,q)$, we interpret the tensor S as a q-linear mapping

$$S : TM \times_M \cdots \times_M TM \longrightarrow TM$$

of the bundle product over M of q copies of the tangent bundle TM. For all $0 \le |\alpha| \le h$, we put:

$$S^{(\alpha)} : T(T^AM) \times_{T^AM} \cdots \times_{T^AM} T(T^AM) \longrightarrow T(T^AM)$$

with $S^{(\alpha)} = \kappa_{A,M} \circ \chi_{TM}^{(\alpha)} \circ T^A(S) \circ \Big(\kappa_{A,M}^{-1} \times \cdots \times \kappa_{A,M}^{-1}\Big)$. It is a tensor field of type $(1,q)$ on $T^A(M)$ called α-prolongation of the tensor field S from M to $T^A(M)$. In the particular case where $\alpha = 0$, it is denoted by $S^{(c)}$ and is called complete lift of S from M to $T^A(M)$.

PROPOSITION 3. *The tensor $S^{(\alpha)}$ is the only tensor field of type $(1,q)$ on $T^A(M)$ satisfying*

$$S^{(\alpha)}\Big(X_1^{(\alpha_1)},\dots,X_q^{(\alpha_q)}\Big) = \big(S(X_1,\dots,X_q)\big)^{(\alpha+\alpha_1+\cdots+\alpha_q)}$$

for all $X_1,\dots,X_q \in \mathfrak{X}(M)$ and multiindex $\alpha_1,\dots,\alpha_q$.

Proof. See [2]. ∎

For some properties of these lifts, see [2] and [3].

1.3. LIFTS OF TENSOR FIELDS OF TYPE $(0,s)$. We fix the linear map $p : A \to \mathbb{R}$, for any vector bundle (E,M,π), we consider the natural vector bundle morphism $\tau_{A,E}^p : T^AE^* \to \big(T^AE\big)^*$ (see [10]) defined for any $j^A\varphi \in T^AE^*$ and $j^A\psi \in T^AE$ by:

$$\tau_{A,E}^p(j^A\varphi)(j^A\psi) = p\big(j^A(\langle\psi,\varphi\rangle_E)\big)$$

where $\langle\psi,\varphi\rangle_E:\mathbb{R}^k\to\mathbb{R},\ z\mapsto\langle\psi(z),\varphi(z)\rangle_E$ and $\langle\cdot,\cdot\rangle_E$ the canonical pairing.

For any manifold M of dimension m, we consider the vector bundle morphism

$$\varepsilon^p_{A,M}=\left[\kappa^{-1}_{A,M}\right]^*\circ\tau^p_{A,TM}:T^AT^*M\longrightarrow T^*T^AM.$$

It is clear that the family of maps $\left(\varepsilon^p_{A,M}\right)$ defines a natural transformation between the functors $T^A\circ T^*$ and $T^*\circ T^A$ on the category $\mathcal{M}f_m$ of m-dimensional manifolds and local diffeomorphisms, denoted by

$$\varepsilon^p_{A,*}:T^A\circ T^*\longrightarrow T^*\circ T^A.$$

When (A,p) is a Weil-Frobenius algebra (see [4]), the mapping $\varepsilon^p_{A,M}$ is an isomorphism of vector bundles over id_{T^AM}. Being $\left\{x^1,\dots,x^m\right\}$ a local coordinate system of M, we introduce the coordinates $\left(x^i,\dot{x}^i\right)$ in TM, $\left(x^i,\pi_i\right)$ in T^*M, $(x^i,\dot{x}^i,\overline{x}^i_\beta,\overline{\dot{x}^i_\beta})$ in T^ATM, $(x^i,\pi_j,\overline{x}^i_\beta,\overline{\pi}^\beta_j)$ in T^AT^*M, $(x^i,\overline{x}^i_\beta,\dot{x}^i,\dot{\overline{x}}^i_\beta)$ in TT^AM and $(x^i,\overline{x}^i_\beta,\overline{\xi}_j,\overline{\xi}^\beta_j)$ in T^*T^AM. We have

$$\varepsilon^p_{A,M}\left(x^i,\pi_j,\overline{x}^i_\beta,\overline{\pi}^\beta_j\right)=\left(x^i,\overline{x}^i_\beta,\overline{\xi}_j,\overline{\xi}^\beta_j\right)\qquad\text{with}\quad\begin{cases}\overline{\xi}_j=\pi_jp_0+\sum\limits_{\mu\in B_A}\overline{\pi}^\mu_jp_\mu\,,\\ \overline{\xi}^\beta_j=\sum\limits_{\mu\in B_A}\overline{\pi}^{\mu-\beta}_jp_\mu\,,\end{cases}$$

and $p_\alpha=p\left(e_\alpha\right)$.

Let G be a tensor fields of type $(0,s)$ on a manifold M. It induces the vector bundle morphism $G^\sharp:TM\times_M\cdots\times_MTM\to T^*M$ of the bundle product over M of $s-1$ copies of TM. We define,

$$G^{(p)}:T(T^AM)\times_{T^AM}\cdots\times_{T^AM}T(T^AM)\longrightarrow T^*(T^AM)$$

as $G^{(p)}=\varepsilon^p_{A,M}\circ T^A(G^\sharp)\circ\left(\kappa^{-1}_{A,M}\times\cdots\times\kappa^{-1}_{A,M}\right)$. It is a T^AM-morphism of vector bundles, so $G^{(p)}$ is tensor field of type $(0,s)$ on T^AM called p-prolongation of G from M to T^AM.

Example 1. In a particular case, where $s=2$ and locally $G=G_{ij}dx^i\otimes dx^j$ then

$$G^{(p)}=G_{ij}p_0dx^i\otimes dx^j+\sum_{\alpha\in B_A}p_\alpha\left(\sum_{\beta\in B_A}G^{(\alpha-\beta)}_{ij}\right)dx^i\otimes dx^j_\beta$$

$$+\sum_{\mu,\beta\in B_A}\left(\sum_{\alpha\in B_A}p_\alpha G^{(\alpha-\beta-\mu)}_{ij}\right)dx^i_\mu\otimes dx^j_\beta.$$

In the particular case where $A = J_0^r(\mathbb{R}^k, \mathbb{R})$ and $p(j_0^r\varphi) = \frac{1}{\alpha!} D_\alpha (\varphi(z))|_{z=0}$, then $G^{(p)}$ coincides with the α-prolongation of G from M to $T_k^r M$ defined in [13].

Example 2. If Ω_M is a Liouville 2-form on T^*M defined in local coordinate system $\left(x^i, \xi_j\right)$ by:

$$\Omega_M = dx^i \wedge d\xi_i,$$

then we have:

$$\Omega_M^{(p)} = p_0 dx^i \wedge d\xi_i + \sum_{\alpha \in B_A} p_\alpha dx^i \wedge d\overline{\xi}_i^\alpha + \sum_{\alpha,\beta \in B_A} p_\alpha d\overline{x}_\beta^i \wedge d\overline{\xi}_i^{\alpha-\beta}.$$

Proposition 4. *The tensor field $G^{(p)}$ is the only tensor field of type $(0, s)$ on $T^A(M)$ satisfying, for all $X_1, \dots, X_s \in \mathfrak{X}(M)$ and multiindex $\alpha_1, \dots, \alpha_s$*

$$G^{(p)}\left(X_1^{(\alpha_1)}, \dots, X_s^{(\alpha_s)}\right) = (G(X_1, \dots, X_s))^{\left(p \circ l_{\alpha_1 + \dots + \alpha_s}\right)}$$

where $l_a : A \to A$ is given by $l_a(x) = ax$.

Proof. See [5]. ∎

2. The natural transformations $j_{A,E} : T^A(FE) \to F(T^AE)$

Let V be a real vector space of dimension n, we denote by $GL(V)$ the Lie group of automorphisms of V.

2.1. The embedding $j_{A,V} : T^A(GL(V)) \to GL(T^AV)$. Let G be a Lie group and M be a $m-$dimensional manifold, $m \geq 1$. We consider the differential action $\rho : G \times M \to M$, then the Lie group T^AG acts to T^AM by the differential action $T^A\rho : T^AG \times T^AM \to T^AM$.

Lemma 1. *If the Lie group G operates on M effectively, then T^AG operates on T^AM effectively by the differential action $T^A(\rho)$.*

Proof. See [5]. ∎

Let $\rho_V : GL(V) \times V \to V$ be the canonical action of $GL(V)$, then the Lie group $T^A(GL(V))$ operates effectively on the vector space T^AV by the induced action

$$\begin{aligned} T^A(\rho_V) : T^A(GL(V)) \times T^AV &\longrightarrow T^AV \\ \left(j^A\varphi, j^Au\right) &\longmapsto j^A(\varphi * u) \end{aligned}$$

where $\varphi * u : \mathbb{R}^k \to V$ is defined for any $z \in \mathbb{R}^k$ by:

$$\varphi * u(z) = \varphi(z)(u(z)).$$

We deduce an injective map $j_{A,V} : T^A(GL(V)) \to GL(T^AV)$ such that,

$$\begin{array}{rcl} j_{A,V}(j^A g) : T^AV & \longrightarrow & T^AV \\ j^A\xi & \longmapsto & j^A(g * \xi)\,. \end{array}$$

Proposition 5. *The map $j_{A,V} : T^A(GL(V)) \to GL(T^AV)$ is an embedding of Lie groups.*

Proof. By calculation, it is clear that $j_{A,V}$ is a homomorphism of Lie groups. ∎

Remark 5. Let $\{e_1, \ldots, e_n\}$ be a basis of V ($\dim V = n$), we consider the global coordinate system of V, $\left(e^1, \ldots, e^n\right)$, we denote by $\left(y^i_j\right)$ the global coordinate of $GL(V)$, for any $f \in GL(V)$,

$$y^i_j(f) = \left\langle e^i, f(e_j)\right\rangle$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket $V^* \times V \to \mathbb{R}$. We deduce that, the coordinate system of $T^A(GL(V))$ is denoted by $\left(y^i_j, y^i_{j,\alpha}\right)_{\alpha \in B_A}$. On the other hand, the global coordinate system of T^AV is $\left(e^i, e^i_\alpha\right)$, such that:

$$\left\{\begin{array}{l} e^i\left(j^A u\right) = e^i(u(0))\,, \\ e^i_\alpha\left(j^A u\right) = \frac{1}{\alpha!}\, D_\alpha(e^i \circ u)(z)\big|_{z=0} + \sum\limits_{\beta \in \overline{B_A}} \frac{\lambda^\alpha_\beta}{\beta!}\, D_\beta(e^i \circ u)(z)\big|_{z=0}\,, \end{array}\right. \quad j^A u \in T^AV,$$

the global coordinate of $GL(T^AV)$ denoted $\left(z^i_j, z^{i,\beta}_{j,\alpha}\right)_{\alpha,\beta \in B_A}$ is such that:

$$\left\{\begin{array}{l} z^i_j(\xi) = \left\langle e^i, \pi_{A,V}(\xi)(e_j)\right\rangle\,, \\ z^{i,\beta}_{j,\alpha}(\xi) = \left\langle e^i_\beta, \xi\left(e^\alpha_j\right)\right\rangle\,, \end{array}\right. \qquad \xi \in GL(T^AV)\,,$$

we deduce that the local coordinate of the map $j_{A,V}$ is given by:

$$j_{A,V}\left(y^i_j, y^i_{j,\alpha}\right) = \begin{pmatrix} y^i_j & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ \cdot & \cdots & y^i_{j,\alpha} & \cdots & y^i_j \end{pmatrix}$$

In fact,

$$\begin{aligned} z_{j,\alpha}^{i,\beta}\left(j_{A,V}\left(j^{A}g\right)\right) &= \left\langle e_{\beta}^{i}, j_{A,V}\left(j^{A}g\right)\left(e_{j}^{\alpha}\right)\right\rangle \\ &= \frac{1}{\beta!}\, D_{\beta}\big(t^{\alpha}\left\langle e^{i}, g(t)(e_{j})\right\rangle\big)\big|_{t=0} \\ &\quad + \sum_{\mu\in\overline{B_A}} \frac{\lambda_{\beta}^{\mu}}{\mu!}\, D_{\mu}\left(t^{\alpha}\left\langle e^{i}, g(t)(e_{j})\right\rangle\right)\big|_{t=0} \end{aligned}$$

for any $j^{A}g \in T^{A}(GL(V))$.

2.2. Frame gauge functor on the vector bundles. We denote by $\mathcal{VB}_m$ the category of vector bundles with m-dimensional base together with local isomorphism. Let $\mathcal{B}_{\mathcal{VB}_m} : \mathcal{VB}_m \to \mathcal{M}f$ and $\mathcal{B}_{\mathcal{FM}} : \mathcal{FM} \to \mathcal{M}f$ be the respective base functors.

Definition 1. (See [11]) A gauge bundle functor on $\mathcal{VB}_m$ is a covariant functor $\mathbb{F} : \mathcal{VB}_m \to \mathcal{FM}$ satisfying:

1. (Base preservation) $\mathcal{B}_{\mathcal{FM}} \circ \mathbb{F} = \mathcal{B}_{\mathcal{VB}_m}$;
2. (Locality) for any inclusion of an open vector bundle $\imath_{E|_U} : E|_U \to E$, $\mathbb{F}\left(E|_U\right)$ is the restriction $p_E^{-1}(U)$ of $p_E : E \to \mathcal{VB}_m(E)$ over U and $\mathbb{F}\left(\imath_{E|_U}\right)$ is the inclusion $p_E^{-1}(U) \to \mathbb{F}E$.

Definition 2. Let G be a Lie group. A principal fiber bundle is a fiber bundle (P, M, π) of standard fiber G such that: there is a fiber bundle atlas $\left(U_\alpha, \varphi_\alpha : \pi^{-1}\left(U_\alpha\right) \to U_\alpha \times G\right)_{\alpha\in A}$, the family of smooth maps $\theta_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ which satisfies the cocycle condition ($\theta_{\alpha\beta}(x) \cdot \theta_{\beta\gamma}(x) = \theta_{\alpha\gamma}(x)$ for $x \in U_\alpha \cap U_\beta \cap U_\gamma$ and $\theta_{\alpha\alpha}(x) = e$) and

$$\text{for each } x \in U_\alpha \cap U_\beta\,,\ \text{ for each } g \in G\,, \qquad \varphi_\alpha \circ \varphi_\beta^{-1}(x, g) = (x,\, \theta_{\alpha\beta}(x) \cdot g)\,.$$

Example 3. Let (E, M, π) be a vector bundle of standard fiber the real vector space V of dimension $n \geq 1$. For any $x \in M$, we denote by F_xE the set of all linear isomorphisms of V on E_x and we set $FE = \bigcup_{x\in M} F_xE$, it is clear that FE is an open set of the manifold $\hom(M \times V, E)$. We denote by $p_E : FE \to M$ the canonical projection. Let $(U_\alpha, \psi_\alpha)_{\alpha\in\Lambda}$ the fiber bundle atlas of (E, M, p), so for all $x \in U_\alpha \cap U_\beta$ and $v \in V$, $\psi_\alpha \circ \psi_\beta^{-1}(x, v) = (x, \theta_{\alpha\beta}(x)(v))$, where $\theta_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$ satisfies the cocycle condition. We consider the smooth map $\varphi_\alpha : p_E^{-1}(U_\alpha) \to U_\alpha \times GL(V)$ such that, for any $x \in U_\alpha$ and $f_x \in p_M^{-1}(U_\alpha)$,

$$\varphi_\alpha(f_x) = (x, \psi_\alpha|_{E_x} \circ f_x) .$$

It is clear that, $(U_\alpha, \varphi_\alpha)_{\alpha\in\Lambda}$ is the fiber bundle atlas of (FE, M, p_E). As $\varphi_\beta \circ \varphi_\alpha^{-1}(x, f) = (x, \theta_{\alpha\beta}(x) \circ f)$, it follows that (FE, M, p_E) is a principal bundle of standard fiber, the linear Lie group $GL(V)$. It is called the frame bundle of the vector bundle (E, M, π).

Remark 6. Let $\left(U, x^i\right)$ be a local coordinate system of M, we denote by $\left(x^i, x^i_j\right)$ the local coordinate of FM induced by $\left(U, x^i\right)$, it is such that:

$$\begin{cases} x^i\,(\xi) \;=\; x^i(p_E(\xi))\,, \\ x^i_j\,(\xi) \;=\; \left\langle dx^i, (\xi(e_j))\right\rangle , \end{cases}$$

for $\xi \in FM$ and $(e_1, \ldots, e_n)$ is a basis of V.

Definition 3. $\Phi : (P, M, p, G) \to (P', M', p', G')$ is a homomorphism of principal bundles over the homomorphism of Lie groups $\phi : G \to G'$ if $\Phi : P \to P'$ is smooth and satisfies

$$\text{for each } u \in P\,, \ \text{ for each } g \in G\,, \qquad \Phi(u \cdot g) = \Phi(u) \cdot \phi(g)\,.$$

The collection of principal bundles and their homomorphisms form a category, it is called the category of principal bundles and denoted by $\mathcal{PB}$. In particular, it is subcategory of the category $\mathcal{FM}$.

Example 4. Let $f : E_1 \to E_2$ an isomorphism of vector bundles over the diffeomorphism $\overline{f} : M_1 \to M_2$. The smooth map $F(f) : FE_1 \to FE_2$ defined for any $\varphi_x \in F_xE_1$ by:

$$F(f)(\varphi_x) = f_x \circ \varphi_x \in F_{\overline{f}(x)}E_1$$

is such that $\left(\overline{f}, F(f)\right) : (FE_1, M_1, p_{E_1}) \to (FE_2, M_2, p_{E_2})$ is an isomorphism of principal bundles. We obtain in particular a functor $F : \mathcal{VB}_n \to \mathcal{PB}$, it is a covariant functor.

Proposition 6. *The functor $F : \mathcal{VB}_n \to \mathcal{FM}$ is a gauge bundle functor on $\mathcal{VB}_n$ which do not preserves the fiber product. It is called the frame gauge functor on $\mathcal{VB}_n$.*

Proof. The properties of gauge functor $F : \mathcal{VB}_n \to \mathcal{FM}$ are easily verified by calculation. Since do not exists an isomorphism between the Lie groups $GL(V_1) \times GL(V_2)$ and $GL(V_1 \oplus V_2)$, it follows that the gauge functor F do not preserves the fiber product. ∎

Remark 7. Let (P, M, π) be a principal fiber bundle with total space P, base space M, projection π and structure group G. If $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open covering of M, for each $\alpha \in \Lambda$, P giving a trivial bundle over U_α, and if $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ are the transition functions of P, we express this fiber bundle by $P = \{U_\alpha, g_{\alpha\beta}\}$. When G is a Lie subgroup of a Lie group G' and $j : G \to G'$ is the injection map, then there is a fiber bundle $P' = \{U_\alpha, j \circ g_{\alpha\beta}\}$ and an injection $\overline{j} : P \to P'$ which is a bundle homomorphism i.e. $\overline{j}(p \cdot a) = \overline{j}(p) \cdot a$, for any $p \in P$ and $a \in G$.

2.3. THE NATURAL EMBEDDING $j_{A,E} : T^A(FE) \to F(T^AE)$. We denote with (E, M, π) a vector bundle of standard fiber the real vector space V of dimension $n \geq 1$. Then, $\left(T^AE, T^AM, T^A\pi\right)$ is a real vector bundle of standard fiber T^AV, in particular the frame bundle of this vector bundle is a $GL(T^AV)$-principal $\left(F(T^AE), T^AM, p_{T^AE}\right)$. On the other hand, (FE, M, p_E) is a $GL(V)$-principal bundle, so $\left(T^A(FE), T^AM, T^A(p_E)\right)$ is a $T^A(GL(V))$-principal bundle. Let $(U_\alpha, \psi_\alpha)_{\alpha\in\Lambda}$ a fiber bundle atlas of (E, M, π), so that $\left(T^AU_\alpha, T^A\psi_\alpha\right)_{\alpha\in\Lambda}$ is a fiber bundle atlas of $\left(T^AE, T^AM, T^A\pi\right)$. The bundle atlas of the principal bundle (FE, M, p_E) is denoted by $(U_\alpha, \varphi_\alpha)_{\alpha\in\Lambda}$ where

$$\begin{array}{rcl} \varphi_\alpha : p_E^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times GL(V) \\ g & \longmapsto & \Big(p_E(g), (\psi_\alpha)_{p_E(g)} \circ g\Big), \end{array}$$

we deduce that $\left(T^AU_\alpha, T^A(\varphi_\alpha)\right)_{\alpha\in\Lambda}$ is the following fiber bundle atlas of $\left(T^A(FE), T^AM, T^A(p_E)\right)$,

$$\begin{array}{rcl} T^A(\varphi_\alpha) : \left(T^Ap_E\right)^{-1}\left(T^AU_\alpha\right) & \longrightarrow & T^AU_\alpha \times T^A(GL(V)) \\ j^Ag & \longmapsto & \left(T^Ap_E(j^Ag), j^A(\psi_\alpha \cdot g)\right), \end{array}$$

where $(\psi_\alpha \cdot g)(z) = (\psi_\alpha)_{p_E(g(z))} \circ g(z) : V \to V$ is a linear isomorphism, for all $z \in \mathbb{R}^k$.

As $\left(T^AU_\alpha, T^A\psi_\alpha\right)_{\alpha\in\Lambda}$ is a fiber bundle atlas of $\left(T^AE, T^AM, T^A\pi\right)$, it follows that the fiber bundle atlas of the principal bundle $\left(F(T^AE), T^AM, p_{T^AE}\right)$

is denoted by $\left(T^AU_\alpha,\varphi_{\alpha,A}\right)_{\alpha\in\Lambda}$ where

$$\begin{array}{rcl}\varphi_{\alpha,A}p^{-1}_{T^AE}(T^AU_\alpha) & \longrightarrow & T^AU_\alpha\times GL(T^AV)\\ \xi & \longmapsto & \left(p_{T^AE}(\xi),\left(T^A(\psi_\alpha)\right)_{p_{T^AE}(\xi)}\circ\xi\right)\end{array}$$

and $\varphi^{-1}_{\alpha,A}\left(\widetilde{x},\widetilde{\xi}\right)=\left(T^A\psi_\alpha\right)^{-1}(\widetilde{x},\cdot)\circ\widetilde{\xi}$, for any $\left(\widetilde{x},\widetilde{\xi}\right)\in T^AU_\alpha\times GL(T^AV)$. For any $\alpha\in\Lambda$, we put

$$j_{A,U_\alpha}=\varphi^{-1}_{\alpha,A}\circ(\mathrm{id}_{T^AU_\alpha},j_{A,V})\circ T^A(\varphi_\alpha):\left(T^Ap_E\right)^{-1}(T^AU_\alpha)\longrightarrow p^{-1}_{T^AE}(T^AU_\alpha)$$

and for any $j^Ag\in\left(T^Ap_E\right)^{-1}(T^AU_\alpha)$, we have:

$$\begin{aligned}j_{A,U_\alpha}(j^Ag)&=\varphi^{-1}_{\alpha,A}\left(j^A(p_E\circ g),j_{A,V}\left(j^A(\psi_\alpha\cdot g)\right)\right)\\&=\left(T^A\psi_\alpha\right)^{-1}\left(j^A(p_E\circ g),\cdot\right)\circ j_{A,V}\left(j^A(\psi_\alpha\cdot g)\right).\end{aligned}$$

For $\beta\in\Lambda$ such that $U_\alpha\cap U_\beta\neq\emptyset$, we have $j_{A,U_\alpha}\big|_{(T^Ap_E)^{-1}\left(T^AU_\alpha\cap T^AU_\beta\right)}=j_{A,U_\beta}\big|_{(T^Ap_E)^{-1}\left(T^AU_\alpha\cap T^AU_\beta\right)}$, it follows that, it exists one and only one principal fiber bundle homomorphism $j_{A,E}:T^A(FE)\to F(T^AE)$ such that, for any $\alpha\in A$, $j_{A,E}\big|_{(T^Ap_E)^{-1}(T^AU_\alpha)}=j_{A,U_\alpha}$. In particular, for any $\widetilde{\xi}\in T^A(FE)$ and $\widetilde{u}\in T^A(GL(V))$,

$$j_{A,E}\left(\widetilde{\xi}\cdot\widetilde{u}\right)=j_{A,E}\left(\widetilde{\xi}\right)\cdot j_{A,V}\left(\widetilde{u}\right).$$

THEOREM 1. *The map $j_{A,E}:T^A(FE)\to F(T^AE)$ is a principal fiber bundle homomorphism over the homomorphism of Lie groups $j_{A,V}:T^A(GL(V))\to GL(T^AV)$. In particular, $j_{A,E}$ is an embedding.*

Proof. It is clear that, $j_{A,E}:T^A(FE)\to F(T^AE)$ is a principal fiber bundle homomorphism over $j_{A,V}$, because for any $\widetilde{\xi}\in T^A(FE)$ and $\widetilde{u}\in T^A(GL(V))$,

$$j_{A,E}\left(\widetilde{\xi}\cdot\widetilde{u}\right)=j_{A,E}\left(\widetilde{\xi}\right)\cdot j_{A,V}\left(\widetilde{u}\right).$$

On the other hand, for any $\alpha\in A$, $j_{A,E}\big|_{(T^Ap_E)^{-1}(T^AU_\alpha)}=j_{A,U_\alpha}$, it follows that $j_{A,E}$ is an embedding. ∎

Remark 8. Let $\left(\pi^{-1}(U),x^i,y^j\right)$ be a fiber chart of E, then the local coordinate of FE and T^AE are $\left(p_E^{-1}(U_i),x^i,y^j_k\right)$ and $\left(\left(T^A\pi\right)^{-1}(T^AU),x^i_\alpha,y^j_\alpha\right)$.

We deduce that, the local coordinate of $T^A(FE)$ and $F(T^AE)$ are given by $\left(T^A(p_E^{-1}(U_i)), x_\alpha^i, y_k^j, y_{k,\alpha}^j\right)$ and $\left(p_{T^AE}^{-1}\left(T^AU\right), x_\alpha^i, y_{k,\beta}^{j,\alpha}\right)$, so the local expression of $j_{A,E}$ is given by:

$$j_{A,E}\big|_{(T^Ap_E)^{-1}(T^AU)}\left(x_\alpha^i, y_k^j, y_{k,\alpha}^j\right) = \left(x_\alpha^i, \begin{pmatrix} y_k^j & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ \cdots & y_{k,\alpha}^j & \cdots & y_k^j \end{pmatrix}\right).$$

PROPOSITION 7. *Let $f : E \to E'$ is an isomorphism of vector bundles over the diffeomorphism $\overline{f} : M \to M'$. The following diagram*

$$\begin{array}{ccc} T^A(FE) & \xrightarrow{T^A(Ff))} & T^A(FE') \\ {\scriptstyle j_{A,E}}\downarrow & & \downarrow{\scriptstyle j_{A,E'}} \\ F(T^AE) & \xrightarrow[F(T^Af)]{} & F(T^AE') \end{array}$$

commutes.

Proof. Let $(U_\alpha, \psi_\alpha)_{\alpha\in\Lambda}$ and $(U'_\alpha, \psi'_\alpha)_{\alpha\in\Lambda}$ the bundle atlas of (E, M, π) and (E', M', π') such that $\overline{f}(U_\alpha) = U'_\alpha$, $\alpha \in \Lambda$. As $f : E \to E'$ is an isomorphism of vector bundles over $\overline{f}$, it follows that it exists a smooth map $f_\alpha : U_\alpha \times V \to V$ such that $\psi'_\alpha \circ f\big|_{\pi^{-1}(U_\alpha)} \circ \psi_\alpha^{-1}(x, v) = \left(\overline{f}(x),\ f_\alpha(x, v)\right)$, for any $(x, v) \in U_\alpha \times V$ and $f_\alpha(x, \cdot)$ is a linear isomorphism. It follows that, the diagram

$$\begin{array}{ccc} p_E^{-1}(U_\alpha) & \xrightarrow{Ff\big|_{p_E^{-1}(U_\alpha)}} & p_{E'}^{-1}(U'_\alpha) \\ {\scriptstyle \varphi_\alpha}\downarrow & & \downarrow{\scriptstyle \varphi'_\alpha} \\ U_\alpha \times GL(V) & \xrightarrow[\widetilde{f_\alpha}]{} & U'_\alpha \times GL(V) \end{array}$$

commutes, and $\widetilde{f_\alpha}(x, g) = \left(\overline{f}(x),\ f_\alpha(x, \cdot) \circ g\right)$, for each $(x, g) \in U_\alpha \times GL(V)$. It is clear that the following diagram

$$\begin{array}{ccc} \left(T^Ap_E\right)^{-1}\left(T^AU_\alpha\right) & \xrightarrow{T^A(Ff)\big|_{(T^Ap_E)^{-1}(T^AU_\alpha)}} & \left(T^Ap_{E'}\right)^{-1}\left(T^AU'_\alpha\right) \\ {\scriptstyle T^A\varphi_\alpha}\downarrow & & \downarrow{\scriptstyle T^A\varphi'_\alpha} \\ T^AU_\alpha \times T^A(GL(V)) & \xrightarrow[T^A\left(\widetilde{f_\alpha}\right)]{} & T^AU'_\alpha \times T^A(GL(V)) \end{array}$$

commutes. On the other hand, as the diagram following commutes

$$\begin{array}{ccc} T^AU_\alpha \times T^A(GL(V)) & \xrightarrow{T^A\left(\widetilde{f_\alpha}\right)} & T^AU'_\alpha \times T^A(GL(V)) \\ {\scriptstyle \left(id_{U_\alpha}, j_{A,V}\right)}\downarrow & & \downarrow{\scriptstyle \left(id_{U'_\alpha}, j_{A,V}\right)} \\ T^AU_\alpha \times GL\left(T^AV\right) & \xrightarrow[\widetilde{f_{\alpha,A}}]{} & T^AU'_\alpha \times GL\left(T^AV\right) \end{array}$$

with $\widetilde{f_{\alpha,A}}\left(\widetilde{x}, \widetilde{\xi}\right) = \left(T^A\overline{f}\left(\widetilde{x}\right),\, f_{\alpha,A}(\widetilde{x}, \cdot) \circ \widetilde{\xi}\right)$ where

$$T^A(\psi'_\alpha) \circ T^Af\big|_{(T^A\pi)^{-1}(T^AU_\alpha)} \circ \left(T^A\psi_\alpha\right)^{-1}(\widetilde{x}, v) = \left(T^A\overline{f}\left(\widetilde{x}\right),\, f_{\alpha,A}(\widetilde{x}, \cdot)\right),$$

it follows that

$$\begin{aligned} &\left(id_{U'_\alpha}, j_{A,V}\right) \circ T^A\left(\widetilde{f_\alpha}\right)\left(j^Au, j^A\xi\right) \\ &\qquad = \left(id_{U'_\alpha}, j_{A,V}\right)\left(T^A\overline{f}\left(j^Au\right), j^A\left(\widetilde{f}\left(u, \cdot\right) \circ \xi\right)\right) \\ &\qquad = \left(T^A\overline{f}(j^Au),\, j_{A,V}\left(j^A\left(\widetilde{f}\left(u, \cdot\right) \circ \xi\right)\right)\right). \end{aligned}$$

As $j_{A,V}\left(j^A\left(\widetilde{f}\left(u, \cdot\right) \circ \xi\right)\right)\left(j^Av\right) = j^A\left(\left(\widetilde{f}\left(u, \cdot\right) \circ \xi\right) \cdot v\right)$ and

$$\left(\widetilde{f}(u, \cdot) \circ \xi\right) \cdot v\left(z\right) = \widetilde{f}\left(u(z), \xi(z)(v(z))\right),$$

for any $z \in \mathbb{R}^k$, thus,

$$\begin{aligned} F(T^Af) \circ j_{A,U_\alpha}\left(j^Au, j^A\xi\right) &= F(T^Af)\left(j^Au,\, j_{A,V}\left(j^A\xi\right)\right) \\ &= \left(T^A\overline{f}\left(j^Au\right), T^A\widetilde{f}\left(j^Au, \circ\right) \circ j_{A,V}\left(j^A\xi\right)\right). \end{aligned}$$

For any $j^Av \in T^AV$, as $j_{A,V}\left(j^A\xi\right)\left(j^Av\right) = j^A\left(\xi * v\right)$ with $\xi * v(z) = \xi(z)(v(z))$, for all $z \in \mathbb{R}^k$, we deduce that

$$\begin{aligned} T^A\widetilde{f}\left(j^Au, \circ\right) \circ j_{A,V}\left(j^A\xi\right)\left(j^Av\right) &= T^A\widetilde{f}\left(j^Au, j^A\left(\xi * v\right)\right) \\ &= j^A\left(\widetilde{f}\left(u, \xi * v\right)\right), \end{aligned}$$

so $T^A\widetilde{f}\left(j^Au,\circ\right)\circ j_{A,V}\left(j^A\xi\right)\left(j^Av\right) = j_{A,V}\left(j^A\left(\widetilde{f}\left(u,\cdot\right)\circ\xi\right)\right)\left(j^Av\right)$ for any $j^Av\in T^AV$. More precisely, $j_{A,U'_\alpha}\circ T^A\left(\widetilde{f_\alpha}\right)=\widetilde{f_{\alpha,A}}\circ j_{A,U_\alpha}$,

$$\begin{aligned}
j_{A,E'}\big|_{(T^Ap_{E'})^{-1}(T^AU'_\alpha)}\circ T^A(Ff) &= \varphi'^{-1}_{\alpha,A}\circ j_{A,U'_\alpha}\circ T^A\left(\varphi'_\alpha\right)\circ T^A(Ff)\\
&= \varphi'^{-1}_{\alpha,A}\circ j_{A,U'_\alpha}\circ T^A\left(\varphi'_\alpha\circ Ff\circ\varphi_\alpha^{-1}\right)\circ T^A(\varphi_\alpha^{-1})\\
&= \varphi'^{-1}_{\alpha,A}\circ j_{A,U'_\alpha}\circ T^A\left(\widetilde{f_\alpha}\right)\circ T^A(\varphi_\alpha^{-1})\\
&= \varphi'^{-1}_{\alpha,A}\circ \widetilde{f_{\alpha,A}}\circ j_{A,U_\alpha}\circ T^A(\varphi_\alpha^{-1})\\
&= \left(\varphi'^{-1}_{\alpha,A}\circ \widetilde{f_{\alpha,A}}\circ\varphi_{\alpha,A}\right)\circ\varphi_{\alpha,A}^{-1}\circ j_{A,U_\alpha}\circ T^A(\varphi_\alpha^{-1})\\
&= F(T^Af)\circ j_{A,E}\big|_{(T^Ap_E)^{-1}(T^AU_\alpha)},
\end{aligned}$$

thus, $j_{A,E'}\circ T^A(Ff)=F(T^Af)\circ j_{A,E}$. ∎

Let (E,M,π) be a vector bundle of standard fiber V, for any $t\in\mathbb{R}$, we consider the linear automorphism of E, $g_t:E\to E$ defined by: $g_t(u)=\exp(t)u$, for any $u\in E$. We consider the principal bundle isomorphism over id_M, $\varphi_t\doteqdot F(g_t):FE\to FE$ such that, for any $x\in M$,

$$\begin{aligned}
\varphi_t\big|_{F_xE}:F_xE &\longrightarrow F_xE\\
h_x &\longmapsto h_x\circ g_t\,.
\end{aligned}$$

In particular, we deduce a smooth map $\varphi:\mathbb{R}\times FE\to FE$, $(t,\xi)\mapsto\varphi_t(\xi)$. For any multi index α, we consider the smooth map

$$\begin{aligned}
\varphi_{\alpha,E}:T^A(FE) &\longrightarrow T^A(FE)\\
\xi &\longmapsto T^A\varphi(e_\alpha,\xi).
\end{aligned}$$

Then $T^A(p_E)\circ\varphi_{\alpha,E}=T^A(p_E)$. In particular, it is a homomorphism of principal bundle of $T^A(FE)$ in to $T^A(FE)$.

PROPOSITION 8. *Let $f:E\to E'$ be an isomorphism of vector bundles over the diffeomorphism $\overline{f}:M\to M'$. Then the following diagram*

$$\begin{array}{ccc}
T^A(FE) & \xrightarrow{T^A(Ff)} & T^A(FE')\\
{\scriptstyle\varphi_{\alpha,E}}\downarrow & & \downarrow{\scriptstyle\varphi_{\alpha,E'}}\\
T^A(FE) & \xrightarrow[T^A(Ff)]{} & T^A(FE')
\end{array}$$

commutes.

Proof. Let $j^A\xi \in T^A(FE)$, we have:

$$\begin{aligned}
\varphi_{\alpha,E'} \circ T^A(Ff)\left(j^A\xi\right) &= \varphi_{\alpha,E'}\left(j^A(F(f)\circ\xi)\right)\\
&= T^A\varphi\left(j^A(t^\alpha), j^A(F(f)\circ\xi)\right)\\
&= j^A\left(\varphi(t^\alpha, F(f)\circ\xi)\right)\\
&= j^A\left(F(f)\circ\varphi(t^\alpha,\xi)\right)\\
&= T^A(F(f))\circ\varphi_{\alpha,E}\left(j^A\xi\right).
\end{aligned}$$

Therefore, $\varphi_{\alpha,E'} \circ T^A(Ff) = T^A(F(f))\circ\varphi_{\alpha,E}$. ∎

3. Prolongations of G-structures to Weil bundles

3.1. The natural embedding $j_{A,M} : T^A(FM) \to F(T^AM)$. Let M be a smooth manifold of dimension $n \geq 1$, we denote by $GL(n)$ the Lie group $GL(\mathbb{R}^n)$ and $(F(M), M, p_M)$ the frame bundle of the tangent vector bundle (TM, M, π_M), so that $\left(T^A(FM), T^AM, T^A(p_M)\right)$ is a principal fiber bundle over the Lie group $T^A(GL(n))$. By the same way $\left(F(T^AM), T^AM, p_{T^AM}\right)$ is a frame bundle of the vector bundle $\left(T(T^AM), T^AM, \pi_{T^AM}\right)$. If $f : M \to N$ is a local diffeomorphism, we denote with $F(f)$ the principal bundle homomorphism $F(Tf) : FM \to FN$.

Let M be a smooth n-dimensional manifold,

$$F(\kappa_{A,M}) : F(T^ATM) \longrightarrow F(T^AM)$$

is an isomorphism of principal bundles over id_{T^AM} and $p_{T^AM}\circ F(\kappa_{A,M}) = p_{T^ATM}$, where $\kappa_{A,M} : T^A(TM) \to T(T^AM)$ is the canonical isomorphism defined in [7]. We put

$$j_{A,M} = F(\kappa_{A,M})\circ j_{A,TM} : T^A(FM) \longrightarrow F(T^AM)$$

such that $p_{T^AM}\circ j_{A,M} = T^A(p_M)$ and $j_{A,M}(\tilde{x}\cdot g) = j_{A,M}(\widetilde{x})\cdot j_{A,\mathbb{R}^n}(g)$. In particular $j_{A,M}$ is a homomorphism of principal bundles over $j_{A,\mathbb{R}^n}$. We identify $T^A\mathbb{R}^n$ with the euclidian vector space $\mathbb{R}^{n\times \dim A}$, it follows that $T^A(GL(n))$ is a Lie subgroup of $GL(n\times\dim A)$.

Proposition 9. *Let M and N be two manifolds and $f : M \to N$ be a diffeomorphism between them. Then the following diagram*

$$\begin{array}{ccc} T^A(FM) & \xrightarrow{T^A(Ff)} & T^A(FN) \\ {\scriptstyle j_{A,M}}\big\downarrow & & \big\downarrow{\scriptstyle j_{A,N}} \\ F(T^AM) & \xrightarrow[F(T^Af)]{} & F(T^AN) \end{array}$$

commutes.

Proof. Let $f : M \to N$ a diffeomorphism,

$$\begin{aligned} j_{A,N} \circ T^A(Ff) &= F(\kappa_{A,N}) \circ j_{A,TN} \circ T^A(Ff) \\ &= F(\kappa_{A,N}) \circ F(T^ATf) \circ j_{A,TM} \\ &= F\left(\kappa_{A,N} \circ T^A(Tf)\right) \circ j_{A,TM} \\ &= F(T(T^Af) \circ \kappa_{A,M}) \circ j_{A,TM} \\ &= F(T(T^Af)) \circ F(\kappa_{A,M}) \circ j_{A,TM} \\ &= F(T^Af) \circ F(\kappa_{A,M}) \circ j_{A,TM}. \end{aligned}$$

We deduce that $j_{A,N} \circ T^A(Ff) = F\left(T^Af\right) \circ j_{A,M}$. ∎

Remark 9. Let $\left(U, x^i\right)$ be a local coordinate on a manifold M, the local coordinate of FM is denoted by $\left(p_M^{-1}(U), x^i, x^i_j\right)$, $\left(T^AU, x^i, x^i_\alpha\right)$ the local coordinate of T^AM , $\left(\left(T^Ap_M\right)^{-1}\left(T^AU\right), x^i, x^i_j, x^i_\alpha, x^i_{j,\alpha}\right)$ the local coordinate of $T^A(FM)$ and $\left(p^{-1}_{T^AM}\left(T^AU\right), x^i, x^i_\alpha, x^i_j, x^{i,\beta}_{j,\alpha}\right)$ local coordinate of $F(T^AM)$. The formula

$$j_{A,M}(x^i, x^i_j, x^i_\alpha, x^i_{j,\alpha}) = \left(x^i,\, x^i_\alpha, \begin{pmatrix} x^i_j & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ \cdots & x^i_{j,\alpha} & \cdots & x^i_j \end{pmatrix} \right)$$

is a local expression of the natural embedding $j_{A,M}$.

3.2. Prolongations of G-structures. Let G be a Lie subgroup of $GL(n)$, we denote by $G_{A,n}$ the image of T^AG by the homomorphism $j_{A,\mathbb{R}^n}$, i.e. $G_{A,n} = j_{A,\mathbb{R}^n}(T^AG)$. Clearly $G_{A,n}$ is a Lie subgroup of $GL(n \times \dim A)$.

Let (P, M, π) be a G-structure on M, we denote by π^A the restriction of the projection $p_{T^AM} : F(T^AM) \to T^AM$ to the subbundle $\mathcal{T}^AP = j_{A,M}(T^AP)$. Then we obtain a $G_{A,n}$-structure $\left(\mathcal{T}^AP, T^AM, \pi^A\right)$ on the Weil bundle T^AM of M related to A. It is called the A-prolongation of the G-structure P to the Weil bundle T^AM to M.

Proposition 10. *Let P (resp. P') be a G-structure on M (resp. M') and $f : M \to M'$ be a diffeomorphism. Then f is an isomorphism of P on P' if and only if $T^Af : T^AM \to T^AM'$ is an isomorphism of $\mathcal{T}^AP$ on $\mathcal{T}^AP'$.*

Proof. The diffeomorphism $f : M \to M'$ is an isomorphism of P on P', if and only if $F(f)(P) = P'$. By the equality $j_{A,M'} \circ T^A(Ff) = F\left(T^Af\right) \circ j_{A,M}$ it follows that, if f is an isomorphism of P on P', then

$$\begin{aligned}\mathcal{T}^AP' &= j_{A,M'}\left(T^AP'\right) = j_{A,M'} \circ T^A(Ff)\left(T^AP\right) \\ &= F\left(T^Af\right) \circ j_{A,M}\left(T^AP\right) = F\left(T^Af\right)\left(\mathcal{T}^AP\right).\end{aligned}$$

Inversely, if $T^Af : T^AM \to T^AM'$ is an isomorphism of $\mathcal{T}^AP$ on $\mathcal{T}^AP'$, then

$$\begin{aligned}j_{A,M'}(T^AP') &= F(T^Af)(T^AP) \\ &= F(T^Af) \circ j_{A,M}(T^AP) = j_{A,M'} \circ T^A(Ff)(T^AP).\end{aligned}$$

Therefore, $T^AP' = T^A(Ff)(T^AP)$. In particular, $P' = \pi_{A,P'}(T^AP') = \pi_{A,P'} \circ T^A(Ff)(T^AP) = F(f) \circ \pi_{A,P}(T^AP) = F(f)(P)$. So f is an isomorphism of P on P'. ∎

Corollary 1. *Let f be a diffeomorphism of M into itself, and P be a G-structure on M. Then f is an automorphism of P if and only if T^Af is an automorphism of the A-prolongation $\mathcal{T}^AP$.*

Let $\phi : M \to FM$ be a smooth section, the we define $\widetilde{\phi}_A = j_{A,M} \circ T^A(\phi)$, where $j_{A,M} : T^A(FM) \to F(T^AM)$ is the natural embedding from Subsection 3.1. It is a smooth section of the frame bundle $F(T^AM)$ called complete lift of ϕ to $F(T^AM)$.

Remark 10. Let $\left(U, x^1, \ldots, x^n\right)$ be a local coordinate of M, we introduce the coordinate $\left(T^AU, x^i_\alpha\right)$ of T^AM. Let $\phi : M \to FM$ be a smooth section such that

$$\phi\big|_U = \phi^i_j\left(\tfrac{\partial}{\partial x^i}\right) \otimes e^j,$$

then

$$\widetilde{\phi}_A\big|_{T^AU} = \left(\phi_j^i\right)^{(\alpha-\beta)} \left(\frac{\partial}{\partial x_\alpha^i}\right) \otimes e_\beta^j\,,$$

where $\left\{e^i\right\}_{i=1,\ldots,n}$ and $\left\{e^i, e_\alpha^i\right\}_{(i,\alpha)\in\{1,\ldots,n\}\times B_A}$ are the dual basis of the canonical basis of $\mathbb{R}^n$ and $T^A(\mathbb{R}^n)$.

DEFINITION 4. Let (P, M, π) be a G-structure on M. The G-structure P is called integrable (or flat) if for each point $x \in M$, there is a coordinate neighborhood U with local coordinate system $\left(x^1, \ldots, x^n\right)$ such that the frame

$$\left(\left(\tfrac{\partial}{\partial x^1}\right)_y, \ldots, \left(\tfrac{\partial}{\partial x^n}\right)_y\right) \in P_y$$

for any $y \in U$.

PROPOSITION 11. *Let P be a G-structure on a manifold M. Then, P is integrable if and only if the A-prolongation $\mathcal{T}^AP$ of P is integrable.*

Proof. We suppose that P is integrable, then there is a cross section $\phi : U \to P$ of P over $U \subset M$ of FM such that

$$\phi = \sum_{i=1}^{n} \left(\tfrac{\partial}{\partial x^i}\right) \otimes e^i.$$

Then $\widetilde{\phi}_A = j_{A,M} \circ T^A(\phi)$ is a cross section of $\mathcal{T}^AP$ over T^AU and,

$$\widetilde{\phi}_A = \sum_{\alpha\in B_A} \left(\frac{\partial}{\partial x_\alpha^i}\right) \otimes e_\alpha^i$$

so, the A-prolongation $\mathcal{T}^AP$ of P is integrable.

Inversely, taking $(a_1, \ldots, a_K)$ be a basis of N_A over $\mathbb{R}$. We consider the basis $\mathcal{B} = (1_A, a_1, \ldots, a_K)$ as a linear isomorphism $A \to \mathbb{R}^{K+1}$ and let $\pi_{\mathcal{B}}^\alpha : A \to \mathbb{R}$ be the composition of $\mathcal{B}$ with the projection $\mathbb{R}^{K+1} \to \mathbb{R}$ on α-factor, $\alpha = 1, \ldots, K+1$. For a coordinate system $\left(U, x^i\right)$ in M we define the induced coordinate system $\left\{x_0^i, x_\alpha^i\right\}$ on T^AM by:

$$\begin{cases} x_0^i = x^i \circ \pi_M^A\,, \\ x_\alpha^i = \left(x^i\right)^{\left(\pi_{\mathcal{B}}^\alpha\right)}, \end{cases} \qquad \alpha = 1, \ldots, K\,.$$

Using these arguments, the proof is similar as for the case of tangent bundle of higher order establish in [12]. ∎

4. Prolongations of some classical G-structures

4.1. Complex structures. We take $J_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ a linear automorphism such that $J_0 \circ J_0 = -\mathrm{id}_{\mathbb{R}^n}$ and denote by $G(n, J_0)$ the group of all $a \in GL(2n)$ such that $a \circ J_0 = J_0 \circ a$. We consider $\{1_A, e_\alpha,\ \alpha \in B_A\}$ be a basis of A over $\mathbb{R}$. We consider this basis as a linear isomorphism $T^A(\mathbb{R}^{2n}) \to \mathbb{R}^{2n \dim A}$. The map $T^A(J_0)$ is a linear automorphism of $T^A(\mathbb{R}^{2n})$ such that $T^A(J_0) \circ T^A(J_0) = -\mathrm{id}_{T^A(\mathbb{R}^n)}$. We put,

$$\widetilde{G} = j_{A,\mathbb{R}^{2n}}\left(T^A(G(n, J_0))\right).$$

Proposition 12. *The Lie group $\widetilde{G}$ is a Lie subgroup of $G\big(n \times \dim A, T^A(J_0)\big)$.*

Proof. Let $\widetilde{a} \in \widetilde{G}$, then there is an element $X \in T^A(G(n, J_0))$, such that $\widetilde{a} = j_{A,\mathbb{R}^n}(X)$. We put $X = j^A\varphi$, with $\varphi : \mathbb{R}^k \to G(n, J_0)$ smooth map. For any $j^A\xi \in T^A\mathbb{R}^n$, we have:

$$T^A(J_0) \circ \widetilde{a}\left(j^A\xi\right) = T^A(J_0)\left(j^A(\varphi * \xi)\right) = j^A(J_0 \circ (\varphi * \xi)).$$

As, for any $z \in \mathbb{R}^k$,

$$\begin{aligned} J_0 \circ (\varphi * \xi)(z) &= J_0 \circ \varphi(z)(\xi(z)) \\ &= \varphi(z) \circ J_0(\xi(z)) = \varphi * (J_0 \circ \xi)(z), \end{aligned}$$

we deduce that

$$\begin{aligned} T^A(J_0) \circ \widetilde{a}\left(j^A\xi\right) &= j^A(\varphi * (J_0 \circ \xi)) = j_{A,\mathbb{R}^n}\left(j^A\varphi\right)\left(j^A(J_0 \circ \xi)\right) \\ &= j_{A,\mathbb{R}^n}(X) \circ T^A(J_0)\left(j^A\xi\right). \end{aligned}$$

So, $T^A(J_0) \circ \widetilde{a}\left(j^A\xi\right) = \widetilde{a} \circ T^A(J_0)\left(j^A\xi\right)$, for all $j^A\xi \in T^A\mathbb{R}^n$. ∎

Remark 11. Let M be a smooth manifold of dimension $2n$, M has an almost complex structure if and only if M has a $G(n, J_0)$-structure P. Applying Subsection 2.2, we see that T^AM has canonically a $\widetilde{G}$-structure $\mathcal{T}^AP$. By Proposition 9, $\mathcal{T}^AP$ induces canonically a $G(n \dim A, T^A(J_0))$-structure $\widetilde{P}^A$. Which means that T^AM has a canonical almost complex structure.

Theorem 2. *The canonical almost complex structure $\widetilde{J}^A$ on T^AM induced by a $G(n \dim A, T^A(J_0))$-structure $\widetilde{P}^A$ is just the complete lift $J^{(c)}$ of the associated almost complex structure J with P.*

Proof. Let $\phi : M \to P$ be a smooth section, then $J(x) = \phi(x) \circ J_0 \circ \phi(x)^{-1}$, for any $x \in M$. Consider the vector $e_{i,\alpha} = j^A(z^\alpha e_i)$, with $\alpha \in B_A$ and $i \in \{1, \dots, 2n\}$. The family $(e_{i,\alpha})$ is a basis of the real vector space $T^A(\mathbb{R}^n)$. If $\phi\big|_U = \phi_i^j \left(\frac{\partial}{\partial x^j}\right) \otimes e^i$ then $\widetilde{\phi}_A\big|_{T^A U} = \left(\phi_i^j\right)^{(\alpha-\beta)} \left(\frac{\partial}{\partial x_\alpha^j}\right) \otimes e_\beta^i$. In particular

$$\widetilde{\phi}_A(e_{i,\alpha}) = \left(\phi_i^j\right)^{(\alpha-\beta)} \left(\frac{\partial}{\partial x_\beta^j}\right) = (\phi(e_i))^{(\alpha)},$$

so $\widetilde{J}^A \circ \widetilde{\phi}_A(e_{i,\alpha}) = \widetilde{J}^A\left((\phi(e_i))^{(\alpha)}\right)$. For any $j^A\xi \in T^A M$, we have

$$\begin{aligned}
\widetilde{\phi}_A \circ T^A(J_0)(e_{i,\alpha})\left(j^A\xi\right) &= \kappa_{A,M} \circ j_{A,TM}\left(T^A(\phi) \circ T^A(J_0)\left(j^A(z^\alpha e_i)\right)\right)\left(j^A\xi\right) \\
&= \kappa_{A,M} \circ j_{A,TM}\left(j^A(\phi \circ \xi)\right)\left(j^A(z^\alpha J_0(e_i))\right) \\
&= \kappa_{A,M}\left(j^A\left((\phi \circ \xi) * (z^\alpha J_0(e_i))\right)\right).
\end{aligned}$$

For any $z \in \mathbb{R}^k$,

$$\begin{aligned}
(\phi \circ \xi) * (z^\alpha J_0(e_i))(z) &= \phi(\xi(z))(z^\alpha J_0(e_i)) \\
&= z^\alpha \phi(\xi(z)) \circ J_0(e_i) \\
&= z^\alpha J(\xi(z)) \circ \phi(\xi(z))(e_i) \\
&= J(\xi(z)) \circ \phi(z^\alpha e_i)(\xi(z)),
\end{aligned}$$

we deduce that

$$\begin{aligned}
\widetilde{\phi}_A \circ T^A(J_0)(e_{i,\alpha})\left(j^A\xi\right) &= \kappa_{A,M} \circ T^A J\left(\chi_{TM}^{(\alpha)} \circ T^A(\phi(e_i))\right)\left(j^A\xi\right) \\
&= \left(\kappa_{A,M} \circ T^A(J) \circ k_{A,M}^{-1}\right) \circ \left(\kappa_{A,M} \circ \chi_{TM}^{(\alpha)} \circ T^A(\phi(e_i))\right)\left(j^A\xi\right) \\
&= J^{(c)}\left((\phi(e_i))^{(\alpha)}\right)\left(j^A\xi\right).
\end{aligned}$$

As $\widetilde{\phi}_A \circ T^A(J_0)(e_{i,\alpha}) = \widetilde{J}^A \circ \widetilde{\phi}_A(e_{i,\alpha})$, we deduce that, $\widetilde{J}^A\left((\phi(e_i))^{(\alpha)}\right) = J^{(c)}\left((\phi(e_i))^{(\alpha)}\right)$, for any $\alpha \in B_A$. So $\widetilde{J}^A$ is the complete lift of J. ∎

4.2. Almost symplectic structure. Let $f : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ be a skew-symmetric non degenerate bilinear form on $\mathbb{R}^{2n}$. We denote by $G(f)$ the group of all $a \in GL(2n)$ such that $f(a(x), a(y)) = f(x, y)$, for all $x, y \in \mathbb{R}^{2n}$. We consider the basis of A over $\mathbb{R}$, $\{1_A, e_\alpha,\ \alpha \in B_A\}$ as a linear isomorphism $T^A(\mathbb{R}^{2n}) \to \mathbb{R}^{2n \dim A}$. We suppose that, A is a Weil-Frobenius algebra, so there is a linear form $p : A \to \mathbb{R}$ such that the bilinear form $q : A \times A \to \mathbb{R}$, $(a,b) \mapsto p\,(ab)$ is non degenerate. The map $p \circ T^A(f) : T^A(\mathbb{R}^{2n}) \times T^A(\mathbb{R}^{2n}) \to \mathbb{R}$ is a skew-symmetric non degenerate bilinear form on $T^A\left(\mathbb{R}^{2n}\right)$. We put, $f^{(A)} = p \circ T^A(f)$ and

$$\widetilde{G} = j_{A,\mathbb{R}^{2n}}\left(T^A(G(f))\right).$$

Proposition 13. *The Lie group $\widetilde{G}$ is a Lie subgroup of $G(f^{(A)})$.*

Proof. Let $u = j^A\xi \in T^A(G(f))$, then $j_{A,\mathbb{R}^{2n}}(u) = \widetilde{u}$ is the linear automorphism of $T^A(\mathbb{R}^{2n})$ defined for any $j^A\varphi \in T^A(\mathbb{R}^{2n})$ by:

$$\widetilde{u}\left(j^A\varphi\right) = j^A(\xi * \varphi)$$

where $(\xi * \varphi)(z) = \xi(z)(\varphi(z))$, for any $z \in \mathbb{R}^k$.

For any $j^A\varphi, j^A\psi \in T^A(\mathbb{R}^{2n})$, we have:

$$\begin{aligned} f^{(A)}\left(\widetilde{u}\left(j^A\varphi\right), \widetilde{u}\left(j^A\psi\right)\right) &= f^{(A)}\left(j^A(\xi * \varphi), j^A(\xi * \psi)\right) \\ &= p \circ T^A(f)\left(j^A(\xi * \varphi), j^A(\xi * \psi)\right) \\ &= p\left(j^A(f(\xi * \varphi, \xi * \psi))\right). \end{aligned}$$

On the other hand, for any $z \in \mathbb{R}^k$,

$$f(\xi * \varphi, \xi * \psi)(z) = f(\xi(z)(\varphi(z)), \xi(z)(\psi(z))) = f(\varphi(z), \psi(z)).$$

Therefore,

$$f^{(A)}\left(\widetilde{u}\left(j^A\varphi\right), \widetilde{u}\left(j^A\psi\right)\right) = p \circ T^A(f)\left(j^A\varphi, j^A\psi\right) = f^{(A)}\left(j^A\varphi, j^A\psi\right).$$

∎

Theorem 3. *The almost symplectic form on T^AM associated with the A-prolongation of a $G(f)$ structure P on a manifold M is the p-prolongation of the almost symplectic form associated with the G-structure P.*

Proof. Let $\phi : M \to P$ be a smooth section, consider the vector $e_{i,\alpha} = j^A(z^\alpha e_i)$, with $\alpha \in B_A$ and $i \in \{1, \dots, 2n\}$. The family $(e_i, e_{i,\alpha})$ is a basis of the real vector space $T^A(\mathbb{R}^n)$. If $\phi\big|_U = \phi_i^j\left(\frac{\partial}{\partial x^j}\right) \otimes e^i$ then $\widetilde{\phi}_A\big|_{T^AU} = \left(\phi_i^j\right)^{(\alpha-\beta)}\left(\frac{\partial}{\partial x_\alpha^j}\right) \otimes e_\beta^i$. In particular,

$$\widetilde{\phi}_A(e_{i,\alpha}) = \left(\phi_i^j\right)^{(\alpha-\beta)}\left(\frac{\partial}{\partial x_\beta^j}\right) = (\phi(e_i))^{(\alpha)}.$$

We denote by ω the almost symplectic form induced by P and ω_A the almost symplectic form induced by $\mathcal{T}^A P$. For all $i,j \in \{1,\dots,2n\}$ and $\alpha,\beta \in B_A$, we have:

$$\begin{aligned}
&\omega_A\Big((\phi(e_i))^{(\alpha)},(\phi(e_j))^{(\beta)}\Big)\\
&\quad = f^{(A)}\left(\left(\widetilde{\phi}_A\right)^{-1}\left((\phi(e_i))^{(\alpha)}\right),\left(\widetilde{\phi}_A\right)^{-1}\left((\phi(e_j))^{(\beta)}\right)\right)\\
&\quad = p\circ T^A(f)(e_{i,\alpha},e_{j,\beta}) = p\circ T^A(f)\left(j^A(z^\alpha e_i),j^A(z^\beta e_j)\right)\\
&\quad = p\left(j^A(f(z^\alpha e_i,z^\beta e_j))\right) = p\left(j^A(z^{\alpha+\beta}f(e_i,e_j))\right)\\
&\quad = (\omega(\phi(e_i),\phi(e_j)))^{(\alpha+\beta)} = \omega^{(p)}\left((\phi(e_i))^{(\alpha)},(\phi(e_j))^{(\beta)}\right).
\end{aligned}$$

It follows that, $\omega_A = \omega^{(p)}$, where $\omega^{(p)}$ is the complete p-lift of ω described in [9] and [10]. ∎

Remark 12. When $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is a bilinear symmetric non degenerate form and G the Lie subgroup generated by all elements of linear group invariant with respect to f, then, the pseudo riemannian metric on T^AM associated with the A-prolongation of a G-structure P on a manifold M is the p-prolongation of the pseudo riemannian metric associated with the structure P.

4.3. Regular foliations induced by A-prolongations of $G(V)$-structures. Let V be a vector subspace of $\mathbb{R}^n$ ($\dim V = p$). We denote by $G(V)$ the group of all $a\in GL(n)$ such that $a(V)=V$. We consider the basis $\{1_A, e_\alpha,\ \alpha\in B_A\}$ of A over $\mathbb{R}$ and the linear isomorphism induced $T^A(\mathbb{R}^n)\to\mathbb{R}^{n\dim A}$. Therefore $GL(T^A(\mathbb{R}^{2n}))$ is identified to $GL(n\dim A)$.

Proposition 14. *The Lie group $\widetilde{G} = j_{A,\mathbb{R}^n}(T^A(G(V)))$ is a Lie subgroup of $G(T^A(V))$.*

Proof. Let $X = j_{A,\mathbb{R}^n}\left(j^A\varphi\right)$ where $\varphi:\mathbb{R}^k\to G(V)$ is a smooth map. So that, $X: T^A(\mathbb{R}^n)\to T^A(\mathbb{R}^n)$ is a linear isomorphism and for any $j^A\xi\in T^A(\mathbb{R}^n)$,

$$X\left(j^A\xi\right) = j^A(\varphi*\xi).$$

For any $j^A\xi\in T^A(V)$, we have $X\left(j^A\xi\right) = j^A(\varphi*\xi)$, as for any $z\in\mathbb{R}^k$, $(\varphi*\xi)(z) = \varphi(z)(\xi(z))\in V$, it follows that $X\left(j^A\xi\right)\in T^A(V)$. Thus, $X\left(T^A(V)\right)\subset T^A(V)$. ∎

Let D be a smooth regular distribution on M of rank p, we denote by $\mathfrak{X}_D$ the set of all local vector fields X such that: for all $x \in M$, $X(x) \in D_x$. Let us notice that for a completely integrable distribution D, the family $\mathfrak{X}_D$ is a Lie subalgebra of the Lie algebra of vector fields on M. We denote by $D^{(A)}$ the distribution generated by the family $\left\{X^{(\alpha)}, 0 \leq \alpha \leq h\right\}$. As $\left[X^{(\alpha)}, X^{(\beta)}\right] = [X, Y]^{(\alpha+\beta)}$ and by the Frobenius theorem, it follows that $D^{(A)}$ is a smooth regular and completely integrable distribution on $T^A M$. It is called A-complete lift of D from M to $T^A M$. In particular $D^{(A)} = \kappa_{A,M}\left(T^A(D)\right) \subset T\left(T^A M\right)$.

PROPOSITION 15. *If $S \subset M$ is a leaf of regular completely integrable distribution D, then $T^A S$ is a leaf of the regular distribution $D^{(A)}$.*

Proof. As S is connected, then $T^A S$ is also connected. In fact, let $\xi_1, \xi_2 \in T^A S$, we put $\pi_S^A(\xi_i) = s_i$, $i = 1, 2$. We consider $X_0 : M \to T^A M$ the smooth section defined by for any $x \in M$ by:

$$X_0(x) = j^A(z \mapsto x).$$

In particular $\pi_S^A \circ X_0(s_i) = s_i$, for $i = 1, 2$. There is a continuous curve $\alpha_1 : [0,1] \to T_{s_1}^A M$ such that $\alpha_1(0) = \xi_1$ and $\alpha_1(1) = X_0(s_1)$. By the same way, there is a continuous curve $\alpha_2 : [0,1] \to T_{s_2}^A M$ such that $\alpha_2(0) = X_0(s_2)$ and $\alpha_2(1) = \xi_2$. Let $\alpha_0 : [0,1] \to S$ be a continuous curve such that $\alpha_0(0) = s_1$ and $\alpha_0(1) = s_2$. Consider the following curve $\alpha : [0,1] \to T^A S$ defined by:

$$\alpha(t) = \begin{cases} \alpha_1(3t) & \text{if } 0 \leq t \leq \frac{1}{3} \\ X_0 \circ \alpha_0(3t-1) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}\,, \\ \alpha_2(3t-2) & \text{if } \frac{2}{3} \leq t \leq 1\,. \end{cases}$$

The curve α is continuous and $\alpha(0) = \xi_1$, $\alpha(1) = \xi_2$. So, $T^A S$ is connected.

For any $\xi \in T^A S$, we have,

$$\begin{aligned} T_\xi(T^A S) &= T_\xi\left(\left(\pi_M^A\right)^{-1}(S)\right) \\ &= \left(T_\xi \pi_M^A\right)^{-1}\left(T_{\pi_M^A(\xi)} S\right) \\ &= \left(T_\xi \pi_M^A\right)^{-1}\left(D_{\pi_M^A(\xi)}\right) = D_\xi^{(A)}. \end{aligned}$$

Thus, $T^A S$ is a leaf of $D^{(A)}$. ∎

THEOREM 4. *The regular foliation on T^AM associated with the A-prolongation of a $G(V)$-structure P on a manifold M is the A-complete lift of the regular foliation associated with the structure P.*

Proof. Let $\phi : M \to P$ be a smooth section. If locally $\phi\big|_U = \phi_i^j \left(\frac{\partial}{\partial x^j}\right) \otimes e^i$ then $\widetilde{\phi}_A\big|_{T^AU} = \left(\phi_i^j\right)^{(\alpha-\beta)} \left(\frac{\partial}{\partial x^j_\alpha}\right) \otimes e^i_\beta$. In particular,

$$\widetilde{\phi}_A(e_{i,\alpha}) = \left(\phi_i^j\right)^{(\alpha-\beta)} \left(\frac{\partial}{\partial x^j_\beta}\right) = (\phi(e_i))^{(\alpha)}.$$

Let D the regular smooth distribution induced by the $G(V)$-structure P and $\widetilde{D}$ the smooth distribution induced by $\mathcal{T}^AP$, for any $\xi \in T^AM$,

$$\begin{aligned}\widetilde{D}_\xi = \widetilde{\phi}_A(\xi)\left(T^AV\right) &= \left\langle \widetilde{\phi}_A(\xi)(e_{i,\alpha}),\, i \in \{1,\dots,p\},\, 0 \le |\alpha| \le h \right\rangle \\ &= \left\langle (\phi(e_i))^{(\alpha)}(\xi),\, i \in \{1,\dots,p\},\, 0 \le |\alpha| \le h \right\rangle.\end{aligned}$$

It follows that, $\widetilde{D}_\xi = D_\xi^{(A)}$. ∎

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