EXTRACTA MATHEMATICAE
Vol. **36**, Num. 2 (2021), 241 – 278

doi:10.17398/2605-5686.36.2.241
Available online October 21, 2021

Support and separation properties of convex sets in finite dimension

V. Soltan

Department of Mathematical Sciences, George Mason University
4400 *University Drive, Fairfax, VA* 22030, *USA*

vsoltan@gmu.edu

Received August 21, 2021
Accepted September 20, 2021

Presented by H. Martini

Abstract: This is a survey on support and separation properties of convex sets in the n-dimensional Euclidean space. It contains a detailed account of existing results, given either chronologically or in related groups, and exhibits them in a uniform way, including terminology and notation. We first discuss classical Minkowski's theorems on support and separation of convex bodies, and next describe various generalizations of these results to the case of arbitrary convex sets, which concern bounding and asymptotic hyperplanes, and various types of separation by hyperplanes, slabs, and complementary convex sets.

Key words: Convex, cone, bound, hyperplane, support, separation.

MSC (2020): 52A20.

Contents

ISSN: 0213-8743 (print), 2605-5686 (online)

1. Introduction

Support and separation properties of convex sets are among the core topics of convexity theory. Introduced and studied by the prominent mathematician Hermann Minkowski (see [63] and [64]), they became useful tools of convex geometry. Further development of Minkowski's ideas occurred at the beginning of 20th century and was summarized in the monograph of Bonnesen and Fenchel [8].

A rapid growth of linear analysis in the first half of 20th century led to numerous generalizations of Minkowski's contribution to the case of infinite-dimensional vector spaces and made this topic an organic part of the discipline. For instance, the survey of Klee [51] from 1969 describes existing results on support and separation properties of convex sets in vector spaces of any dimension, with sporadic divisions into finite- and infinite-dimensional cases. However, besides obvious similarities between these cases, they display nowadays different goals: while finite-dimensional convexity deals, predominantly, with the properties of convex sets, a big part of similar results in linear analysis became a tool for various classifications of topological and normed spaces.

A new wave of interest towards finite-dimensional convexity occurred in the second half of the 20th century due to advancements in linear programming, with polyhedral sets considered as geometric interpretation of solutions sets of systems of linear inequalities (see, e.g., Dantzig [19] and Černikov [18]), and later in convex analysis, optimization theory, and polyhedral geometry. Various books on convex analysis (see, e.g., Güler [37], Panik [69], and Rockafellar [72]) contain separate chapters on support and separation properties of convex sets, illustrating a continuous development of these topics. These books and numerous articles in the field also underline a shift of interest from the study of convex bodies toward arbitrary convex sets, possibly unbounded or nonclosed.

Despite a steady progress in research, no comprehensive survey on support and separation properties of convex sets in finite dimension was published during the last five decades. The present paper aims to fill in (at least partly) this gap and overview new trends and results. It contains an account of existing facts, given either chronologically or in related groups, exhibiting them in a uniform way, including terminology and notation. We do not consider here algorithmic and computational aspects of the theory on support and separation of convex sets, which deserve their own surveys.

Following the necessary preliminaries, given in Section 2, the main text is divided into three parts. Section 3 describes classical Minkowski's theorems on support and separation properties of convex bodies and summarizes immediate contributions of his colleagues. Contemporary approach to the study of these topics is given in Sections 4 and 5. These sections cover existing results on support, bounding and asymptotic planes of arbitrary convex sets, and various types of separation of convex sets by hyperplanes, slabs, and complementary convex halfspaces.

It is interesting to compare methods of research on support and separation of convex sets in classical and contemporary periods. Since the main results of the classical period are related to full-dimensional compact sets, the methods of their proofs are predominantly based on compactness arguments and basic topology of open sets in finite dimension. Contemporary results in this field deal with arbitrary convex sets, possibly unbounded and having intermediate dimension. Consequently, their proofs employ the concept of relative interior and extensively use various types of cones associated with convex sets.

2. Preliminaries

This section describes necessary notation, terminology, and results on convex sets in the n-dimensional Euclidean space $\mathbb{R}^n$ (see, e.g., [76] for details). The elements of $\mathbb{R}^n$ are called vectors (or points), and o stands for the zero vector of $\mathbb{R}^n$. An r-dimensional *plane* L in $\mathbb{R}^n$, where $0 \le r \le n$, is a translate, $L = a + S$, $a \in \mathbb{R}^n$, of a suitable r-dimensional subspace S of $\mathbb{R}^n$, called the *direction space* of L. A *hyperplane* is a plane of dimension $n-1$; it can be described as

$$H = \{x \in \mathbb{R}^n : x{\cdot}e = \gamma\}, \qquad e \ne o,\ \gamma \in \mathbb{R}, \tag{2.1}$$

where $x{\cdot}e$ means the dot product of vectors x and e. Nonzero multiples λe of the vector e in (2.1) are called *normal vectors* of H. The direction space of the hyperplane (2.1) is the $(n-1)$-dimensional subspace given by

$$S = \{x \in \mathbb{R}^n : x{\cdot}e = 0\}, \qquad e \ne o. \tag{2.2}$$

Every hyperplane of the form (2.1) determines the opposite *closed halfspaces*

$$V_1 = \{x \in \mathbb{R}^n : x\cdot e \le \gamma\} \qquad \text{and} \qquad V_2 = \{x \in \mathbb{R}^n : x\cdot e \ge \gamma\}$$

and the pair of opposite *open halfspaces*

$$W_1 = \{x \in \mathbb{R}^n : x\cdot e < \gamma\} \qquad \text{and} \qquad W_2 = \{x \in \mathbb{R}^n : x\cdot e > \gamma\}.$$

The closed and open (line) segments with distinct endpoints u and v in $\mathbb{R}^n$ and the halfline through v with endpoint u are denoted by $[u, v]$, (u, v), and $[u, v\rangle$, respectively. The norm (or the length) of a vector $x \in \mathbb{R}^n$ is denoted $\|x\|$. Given a point $a \in \mathbb{R}^n$ and a scalar $\rho > 0$, the sphere and balls (closed and open) of radius ρ and center a are denoted $S_\rho(a)$, $B_\rho(a)$, and $U_\rho(a)$, respectively.

The topological interior, closure, and boundary of a nonempty set $X \subset \mathbb{R}^n$ are given by $\operatorname{int} X$, $\operatorname{cl} X$, and $\operatorname{bd} X$, respectively. The open ρ-neighborhood of X, denoted $U_\rho(X)$, is the union of all open balls $U_\rho(x)$ of radius $\rho > 0$ centered at $x \in X$. Nonempty sets X_1 and X_2 in $\mathbb{R}^n$ are called *strongly disjoint* provided $U_\rho(X_1) \cap U_\rho(X_2) = \varnothing$ for a suitable $\rho > 0$; the latter occurs if and only if the *inf*-distance $\delta(X_1, X_2)$, defined by

$$\delta(X_1, X_2) = \inf\{\|x_1 - x_2\| : x_1 \in X_1, x_2 \in X_2\},$$

is positive. For a nonempty set $X \subset \mathbb{R}^n$, the notations $\operatorname{span} X$ and $X^\perp$, stand, respectively, for the span and orthogonal complement of X. The set X is called *proper* if $\varnothing \ne X \ne \mathbb{R}^n$.

The *affine span* of X, denoted $\operatorname{aff} X$, is the intersection of all planes containing X, and $\dim X$ is defined as the dimension of the plane $\operatorname{aff} X$. Also, the *direction space* of X is defined by $\operatorname{dir} X = \operatorname{aff} X - \operatorname{aff} X$, and the *orthogonal space* of X by $\operatorname{ort} X = (\operatorname{dir} X)^\perp$ (generally, $\operatorname{ort} X \ne X^\perp$). Given nonempty sets $X, Y \subset \mathbb{R}^n$ and a scalar λ, we let

$$X + Y = \{x + y : x \in X, y \in Y\}, \qquad \lambda X = \{\lambda x : x \in X\}.$$

Nonempty sets X and Y in $\mathbb{R}^n$ are called (directly) *homothetic* provided $X = z + \lambda Y$ for suitable $z \in \mathbb{R}^n$ and $\lambda > 0$.

In what follows, K means a *convex set* in $\mathbb{R}^n$. To avoid trivial cases, we will be assuming that all convex sets involved are *nonempty.* A point x of a convex set $K \subset \mathbb{R}^n$ is said to be relatively interior to K provided there is a scalar $\rho = \rho(x) > 0$ such that $\operatorname{aff} K \cap U_\rho(x) \subset K$. The set of all relatively interior points of K is called the *relative interior* of K and is denoted $\mathrm{rint}\, K$. The set $\mathrm{rint}\, K$ is nonempty and convex; furthermore, $\mathrm{cl}\, K = \mathrm{cl}\,(\mathrm{rint}\, K)$. The difference between topological and relative interiors can be illustrated by the following example: If K is a unit circular disk of the coordinate xy-plane of $\mathbb{R}^3$, then $\mathrm{int}\, K = \varnothing$, while $\mathrm{rint}\, K$ is the interior of this disk. The *relative boundary* of a convex set $K \subset \mathbb{R}^n$, denoted $\mathrm{rbd}\, K$, is defined by $\mathrm{rbd}\, K = \mathrm{cl}\, K \setminus \mathrm{rint}\, K$. It is known that $\mathrm{rbd}\, K \neq \varnothing$ if and only if K is not a plane.

A *convex body* in $\mathbb{R}^n$ is a compact convex set with nonempty interior, a (convex) *polyhedron* is the intersection of finitely many closed halfspaces, and a *polytope* (bounded polyhedron) is the convex hull of finitely many points.

A contemporary approach to the study of support and separation properties of convex sets deals with a family of various associated cones. We recall that a nonempty set C in $\mathbb{R}^n$ is called a *cone* with apex $a \in \mathbb{R}^n$ if $a + \lambda(x - a) \in C$ whenever $\lambda \geq 0$ and $x \in C$. (Obviously, this definition implies that $a \in C$, although a stronger condition $\lambda > 0$ can be beneficial; see, e.g., [57].) The cone C is called convex if it is a convex set. The *apex set* of a convex cone $C \subset \mathbb{R}^n$, denoted $\mathrm{ap}\, C$, is the set of all apices of C. If a is an apex of a convex cone C, then $\mathrm{ap}\, C = C \cap (2a - C)$. Natural generalizations of cones are given by the sets of the form $K = B + C$, where B is a compact convex set and C is a convex cone; these sets are called M-decomposable if C is closed, and M-predecomposable otherwise (see Goberna et al. [35] and Iusemi et al. [41], respectively).

For a convex set $K \subset \mathbb{R}^n$ and a point $a \in \mathbb{R}^n$, the *generated cone* $C_a(K)$ with apex a is defined by

$$C_a(K) = \{a + \lambda(x - a) \,:\, x \in K,\ \lambda \geq 0\}.$$

Both sets $C_a(K)$ and $\mathrm{cl}\, C_a(K)$ are convex cones with apex a (we observe that the cone $C_a(K)$ may be nonclosed even if K is closed). Furthermore, $C_a(K)$ is a plane if and only if $a \in \mathrm{rint}\, K$. In particular, $C_a(K) = \mathbb{R}^n$ if and only if $a \in \mathrm{int}\, K$.

The *recession cone* of a convex set $K \subset \mathbb{R}^n$ is defined by

$$\mathrm{rec}\, K = \{e \in \mathbb{R}^n \,:\, x + \lambda e \in K \text{ whenever } x \in K \text{ and } \lambda \geq 0\}.$$

If K is closed, then $\mathrm{rec}\, K$ is a closed convex cone with apex o, and $\mathrm{rec}\, K \neq \{o\}$ if and only if K is unbounded. The *lineality space* of K is the subspace defined by $\mathrm{lin}\, K = \mathrm{rec}\, K \cap (-\mathrm{rec}\, K)$.

In a standard way, the (negative) *polar cone* of a convex set $K \subset \mathbb{R}^n$ is defined by

$$K^\circ = \{e \in \mathbb{R}^n \,:\, x \cdot e \leq 0 \ \text{ for all } \ x \in K\}.$$

The set K° is a closed convex cone with apex o. Furthermore,

1. $K^\circ = \{o\}$ if and only if $o \in \operatorname{int} K$, and K° is a subspace if and only if $o \in \operatorname{rint} K$.

2. $K^\circ = (C_o(K))^\circ$, $(K^\circ)^\circ = \operatorname{cl} C_o(K)$, $\operatorname{lin} K^\circ = K^\perp$.

3. If $C \subset \mathbb{R}^n$ is a closed convex cone with apex o, then a nonzero vector $e \in \mathbb{R}^n$ belongs to $\operatorname{rint} C^\circ$ if and only if $x \cdot e < 0$ for all $x \in C \setminus \operatorname{lin} C$.

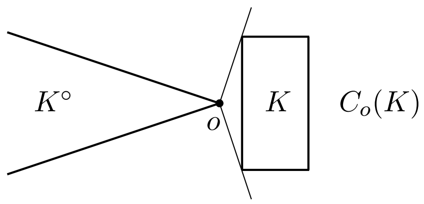


Figure 1: The polar cone of a convex set K.

Given a point z in the closure of a convex set $K \subset \mathbb{R}^n$, the polar cone $(K - z)^\circ$ is often called the *normal cone of* K *at* z and is denoted $N_z(K)$; it consists of all vectors $e \in \mathbb{R}^n$ such that z is the nearest to $e + z$ point in $\operatorname{cl} K$.

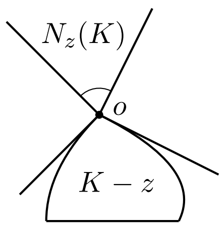


Figure 2: The normal cone $N_z(K)$.

The union of all normal cones $N_z(K)$, where $z \in \operatorname{cl} K$, is denoted $\operatorname{nor} K$ and called the *normal cone* of K. The set $\operatorname{nor} K$ is a cone with apex o, which is not necessarily closed or convex. One more cone associated with K is its *barrier cone*, defined as

$$\operatorname{bar} K = \{e \in \mathbb{R}^n : \exists\, \gamma = \gamma(e) \in \mathbb{R} \text{ such that } x \cdot e \leq \gamma \text{ for all } x \in K\}.$$

It is known that $\operatorname{bar} K$ is a convex (not necessarily closed) cone with apex o. Generally,

$$\operatorname{rint}(\operatorname{rec}(\operatorname{cl} K))^\circ \subset \operatorname{nor} K \subset \operatorname{bar} K \subset (\operatorname{rec}(\operatorname{cl} K))^\circ,$$

which implies the equalities $\operatorname{rec}(\operatorname{cl} K) = (\operatorname{nor} K)^\circ = (\operatorname{bar} K)^\circ$. For instance, if $K = \{(x,y) : y \geq x^2\}$, then $K^\circ = \{(0,y) : y \leq 0\}$ and

$$\operatorname{rec} K = \{(0,y) : y \geq 0\}, \qquad (\operatorname{rec} K)^\circ = \{(x,y) : y \leq 0\},$$
$$\operatorname{nor} K = \operatorname{bar} K = \{o\} \cup \{(x,y) : y < 0\}.$$

A point z of a convex set $K \subset \mathbb{R}^n$ is called an *extreme point* of K provided the set $K \setminus \{z\}$ is convex (equivalently, the equality $z = (1-\lambda)u + \lambda v$, where $u, v \in K$ and $0 < \lambda < 1$, is possible only if $u = v = z$). Similarly, z is an *exposed point* of K provided there is a hyperplane $H \subset \mathbb{R}^n$ satisfying the condition $H \cap K = \{z\}$. The sets $\operatorname{ext} K$ and $\exp K$ of extreme and exposed points of a closed convex set $K \subset \mathbb{R}^n$ have the following properties.

4. $\exp K \neq \varnothing \Leftrightarrow \operatorname{ext} K \neq \varnothing \Leftrightarrow K$ is line-free (that is, K contains no line).
5. $\exp K \subset \operatorname{ext} K \subset \operatorname{cl}(\exp K)$.

3. Minkowski's Theorems

3.1. Support Hyperplanes The concept of *support hyperplane* is attributed to Minkowski [63, § 8]: Given a nonempty set $X \subset \mathbb{R}^n$, a hyperplane $H \subset \mathbb{R}^n$ *supports* X if it contains at least one point of X and *does not cut* X (that is no two points of X belong, respectively, to the opposite open halfspaces determined by H). Clearly, the above condition "H does not cut X" can be equivalently reformulated as "X entirely lies in a closed halfspace determined by H." The following well-known result is due to Minkowski (see [63, § 16] for all $n \geq 3$ and [64, pp. 139–141] for $n = 3$).

Theorem 3.1. ([63, § 16]) *Every boundary point of a convex body $K \subset \mathbb{R}^n$ belongs to a hyperplane supporting K.*

Since general theory of convex sets, with credible geometric arguments, was still in rudimentary stage, Minkowski's proof of Theorem 3.1 used an analytic description of convex bodies in terms of radial distances (see [63, § 1]). The *radial distance* (which later became known as the *Minkowski gauge function*) from a point a to a point b in $\mathbb{R}^n$ is a real-valued function $S(a,b)$ satisfying the following conditions:

1. $S(a,b) > 0$ if $a \neq b$, and $S(a,a) = 0$;
2. if $c = a + t(b-a)$, where $t \geq 0$, then $S(a,c) = tS(a,b)$;
3. $S(a,c) \leq S(a,b) + S(b,c)$ whenever $a,b,c \in \mathbb{R}^n$.

The *standard surface* F and the *standard body* K of the radial distance function $S(o,x)$ are defined, respectively, by

$$F = \{x \in \mathbb{R}^n : S(o,x) = 1\} \quad \text{and} \quad K = \{x \in \mathbb{R}^n : S(o,x) \leq 1\}. \tag{3.1}$$

Minkowski observed (see [63, § 8]) that the standard body K from (3.1) is a compact convex set containing the origin o in its interior, and, conversely, that a certain translate of a convex body can be viewed as the standard body of a suitable radial distance function. Using the properties of $S(a,b)$, and not the convexity of K, Minkowski showed that any point of F belongs to a hyperplane supporting F (and thus supporting K). His proof consists of two steps.

Step 1. Given a point $z \in F$ and a scalar $t > 1$, there is a hyperplane $H(t)$ contained in $\mathbb{R}^n \setminus K$ and meeting the open interval (z, tz) (see Figure 3).

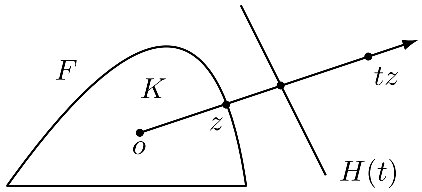


Figure 3: A hyperplane $H(t)$ in $\mathbb{R}^n \setminus K$.

Step 2. Given a sequence of scalars $t_1, t_2, \dots (> 1)$ tending to 1 and a respective sequence of hyperplanes $H(t_1), H(t_2), \dots$ obtained in the above Step 1 and described as

$$\beta_0(t_i) + \beta_1(t_i)x_1 + \cdots + \beta_n(t_i)x_n = 0, \qquad i \geq 1,$$

one can choose $n+1$ infinite subsequences $\beta_j(t_{i_1}), \beta_j(t_{i_2}), \dots, 0 \leq j \leq n$, which converge, respectively, to scalars $\beta_0, \beta_1, \dots, \beta_n$ such that the limit hyperplane $\beta_0 + \beta_1 x_1 + \cdots + \beta_n x_n = 0$ supports F at z.

The analytic nature of Minkowski's proof prompted various mathematicians to consider more geometric approaches. For instance, Carathéodory [16] proved Theorem 3.1 based on the following auxiliary result, afterward widely used in convex geometry (see Figure 4).

THEOREM 3.2. ([16]) *Let $K \subset \mathbb{R}^n$ be a convex body, and u be a point outside K. If z is the nearest to u point in K, then the hyperplane H through z orthogonal to the segment $[u, z]$ supports K such that u and K lie in the opposite closed halfspaces determined by H.*

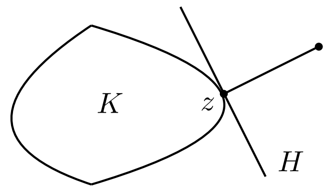


Figure 4: Illustration to Theorem 3.2.

Straszewicz [86, pp. 19 – 21] gave one more method of the proof of Theorem 3.1 which uses an induction argument. Steinitz (see [83, § 11] and [84, § 26]) showed that Theorem 3.1 and Theorem 3.2 hold for the case of any proper n-dimensional convex set $K \subset \mathbb{R}^n$, but his contribution went unnoticed.

Various proofs of Theorem 3.1 for the case of dimensions 2 and 3 were provided at that time. For instance, Brunn [10] uses a similar to Straszewicz [86] method (and later another method in [11]), while Blaschke [7, pp. 53 – 54] finds a suitable orthogonal projection of a convex body $K \subset \mathbb{R}^3$ on a plane and uses the support property of this projection in the plane.

We observe that in all three sources [16, 63, 86], the resulting support hyperplane of a given convex body is obtained as the limit of a suitable converging sequence of hyperplanes. Proofs of Theorem 3.1 which do not employ limit procedures appeared much later. These can be found in the papers of Favard [28, Chapter 2] and Botts [9], as given below.

THEOREM 3.3. ([9]) *If z is a boundary point of a compact convex set $K \subset \mathbb{R}^n$, and v is a point of the unit sphere $S_1(z)$ at a largest distance from K, then the hyperplane through z orthogonal to the segment $[v, z]$ supports K.*

Support properties of convex bodies are used for various classifications of their boundary points. For instance, a point z of a convex body $K \subset \mathbb{R}^n$ is called *regular* if it belongs to a unique support hyperplane of K. It is well-known that the set of regular points of a convex body $K \subset \mathbb{R}^n$ is everywhere dense in $\operatorname{bd} K$. Historical references here are due to Jensen [44] and Bernstein [4, 5], (for the planar case), Kakeya [45], Fujiwara [31], and Reidemeister [71] (for the 3-dimensional case), and Mazur [62] (for the case of linear normed spaces of any dimension).

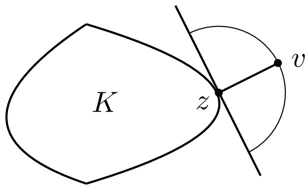


Figure 5: Illustration to Theorem 3.3.

Botts [9] gave the following characteristic property of regular points: If $K \subset \mathbb{R}^n$ is a convex body, $z_0 \in \operatorname{bd} K$, and H is a hyperplane supporting K at z_0, then z_0 is regular if and only if for every sequence of points $z_1, z_2, \ldots \in \operatorname{bd} K \setminus \{z_0\}$ converging to z_0, the sequence of numbers

$$\frac{\delta(z_1, H)}{\|z_1 - z_0\|}, \ \frac{\delta(z_2, H)}{\|z_2 - z_0\|}, \ \ldots$$

tends to 0, where $\delta(z_i, H)$ denotes the distance from z_i to H.

One more classification of boundary points derives from observing contact sets of a convex body $K \subset \mathbb{R}^n$ and its support hyperplanes. For example, the equality $K = \operatorname{cl}(\operatorname{conv}(\exp K))$, proved by Straszewicz [87], implies that K has at least $n + 1$ exposed points, and that the set $\exp K$ is finite if and only if K is a polytope.

3.2. Bounding and Separating Hyperplanes An important class of hyperplanes with respect to a given convex body was considered by Carathéodory [17] and used by him to describe the closed convex hull of a compact set in $\mathbb{R}^n$. Namely, a hyperplane $H \subset \mathbb{R}^n$ is said to *bound* a convex body $K \subset \mathbb{R}^n$ provided $H \cap K = \varnothing$. Equivalently, H bounds K if K lies in one of the open halfspaces determined by H (see Figure 6).

Theorem 3.4. ([17]) *If $z \in \mathbb{R}^n$ is an exterior point of a convex body $K \subset \mathbb{R}^n$, and u is the nearest to z point in K, then the hyperplane through z orthogonal to the segment $[u, z]$ bounds K.*

The following separation theorem of Minkowski [64] originated a variety of related results.

THEOREM 3.5. ([64, p. 141]) *If K_1 and K_2 are convex bodies in $\mathbb{R}^3$ with disjoint interiors, then there is a plane H such that the sets* $\operatorname{int} K_1$ *and* $\operatorname{int} K_2$ *belong to the opposite open halfspaces determined by H.*

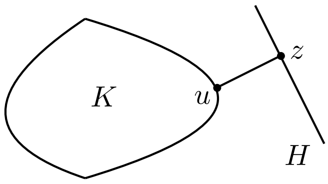


Figure 6: Bounding hyperplane of K through a given point z.

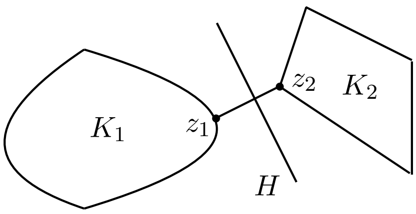


Figure 7: Separating hyperplane of convex bodies K_1 and K_2.

Minkowski's proof of this theorem is divided into two cases.

Case 1: $K_1 \cap K_2 = \varnothing$. If points $z_1 \in K_1$ and $z_2 \in K_2$ are at a minimum possible distance, and if H is a plane orthogonal to the segment $[z_1, z_2]$ and passing through an interior point of this segment, then K_1 and K_2 are contained in the opposite open halfspaces determined by H (see Figure 7).

Case 2: $K_1 \cap K_2 \neq \varnothing$. Assuming that $o \in \operatorname{int} K_1$, any smaller homothetic copy tK_1 of K_1, where $0 < t < 1$, is disjoint from K_2. By the above Case 1, there is a plane $H(t)$ such that tK_1 and K_2 belong to the opposite open halfspaces determined by $H(t)$. Given a sequence of scalars $t_1, t_2, \dots$ in the interval $(0,1)$ which tend to 1, one can chose a sequence of planes $H(t_i)$ separating, respectively, t_iK_1 and K_2 and converging to a plane H such that K_1 and K_2 belong to the opposite closed halfspaces determined by H.

Interestingly, Minkowski did not use the term "separating plane". This term ("Zwischenebene") was used later by Brunn [12]. The assertion of Theorem 3.5 was formulated by Bonnesen and Fenchel [8, p. 5] for the n-dimensional case, with the term "separating hyperplane", without any reference on Minkowski [64]. Theorem 3.5 was generalized in 1936 by Eidelheit [25] for the case of convex bodies in linear normed spaces of any dimension (see the survey [51] for further bibliography).

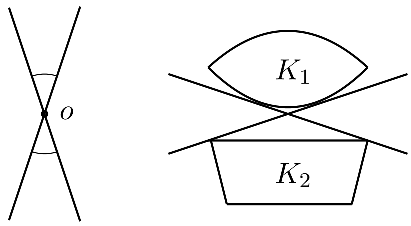


Figure 8: Double cone of normal vectors to separating hyperplanes.

The following result, obtained by Brunn [12], describes the normal vectors of all planes which separate a given pair of convex bodies (compare with Theorem 5.3 below).

Theorem 3.6. ([12]) *Let K_1 and K_2 be disjoint convex bodies in $\mathbb{R}^3$. The set of normal vectors (drawn at the origin of $\mathbb{R}^3$) to all planes separating K_1 and K_2 is a convex double cone, i.e., is the union of two opposite convex cones with the improper common apex o (see Figure 8).*

3.3. Sufficient Conditions for Convexity of Solid Sets The following theorem of Minkowski [63] characterizes bounded convex surfaces in terms of their support property.

Theorem 3.7. ([63, § 17]) *A set $F \subset \mathbb{R}^n$ is the boundary of a convex body in $\mathbb{R}^n$ provided there is a point $z \in \mathbb{R}^n$ such that*

(a) *every open halfline originated at z meets F,*

(b) *every point of F belongs to a hyperplane supporting F.*

Using similar arguments, Brunn [10, p. 293] proved the theorem below.

Theorem 3.8. ([10]) *If a compact set $X \subset \mathbb{R}^n$ has nonempty interior and any boundary point of X belongs to a hyperplane supporting X, then X is a convex body.*

Straszewicz [86, pp. 22 − 24] independently proved Theorem 3.8 under the additional assumption that X is connected and coincides with the closure of its interior. A similar result was obtained by Haalmeijer [38]: If $X \subset \mathbb{R}^n$ is the closure of an open connected set and Y is a dense subset of $\operatorname{bd} X$, then X is convex provided every point $x \in Y$ belongs to a hyperplane supporting X.

Tietze (see [89] and [90]) gave two local versions of Theorem 3.8. Following [89], an open half-ball $Q_\rho(z) \subset \mathbb{R}^n$ with center z and radius $\rho > 0$ means the intersection of the open ball $U_\rho(z)$ and an open halfspace whose boundary hyperplane contains z.

Theorem 3.9. ([89, 90]) *A compact set $X \subset \mathbb{R}^n$ is convex if it satisfies any of the following two conditions:*

(a) *X is the closure of an open connected set, and for every boundary point z of X there is an open half-ball $Q_\rho(z)$, $\rho = \rho(z) > 0$, disjoint from X;*

(b) *X is connected, $\operatorname{int} X \neq \varnothing$, and there is a scalar $\rho > 0$ such that for every boundary point z of X a suitable open half-ball $Q_\rho(z)$ is disjoint from X.*

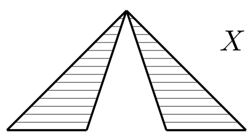


Figure 9: Illustration to Theorem 3.9.

The nonconvex sets $X \subset \mathbb{R}^2$ in Figure 9 illustrate that the assumptions in both conditions (a) and (b) of Theorem 3.9 are essential. Indeed, at any boundary point z, the set X has a support open half-ball $Q_\rho(z)$, where $\rho = \rho(z)$, while $\operatorname{int} X$ is not connected and no constant scalar $\rho > 0$ satisfies condition (b).

A simplified proof of part (b) of Theorem 3.9 was given by Reinhardt [70]. Also, this part was sharpened later in the following ways.

1. Gericke [34] (see also Nöbeling [67]) showed that the centers of disjoint with X half-balls $Q_\rho(z)$ can be chosen in $\operatorname{bd} X \setminus L$, where L is a suitable $(n-3)$-dimensional plane.
2. Süß [88] proved that condition (b) is satisfied if X is connected, has interior points, and there is a scalar $\rho > 0$ such that for every boundary point z of X there is a cylinder $C(z)$ of variable height based on an $(n-1)$-dimensional ball of radius ρ, with $X \cap \operatorname{int} C(z) = \varnothing$.
3. Schmidt (see Bieberbach [6, p. 20]) observed that it is sufficient to require the existence of a hyperplane H through z which does not meet the set $X \cap U_\rho(z)$ (without the assumption that H supports $X \cap U_\rho(z)$).

Burago and Zalgaller [13] (see pp. 395 and 415) observed that the word "compact" can be replaced by "closed" in both Theorem 3.8 and Theorem 3.9. Also, they changed the language of Theorem 3.9, replacing in (b) the requirement $X \cap Q_\rho(z) = \varnothing$ with the following one: $X \cap U_\rho(z)$ is supported at z by a suitable hyperplane.

One more variation of Theorem 3.8 comes from geometric measure theory. Namely, a closed set $X \subset \mathbb{R}^n$ with non-empty interior is convex if and only if it has locally finite perimeter and possesses a support hyperplane at each point of its reduced boundary (see Caraballo [14, 15] for definitions and technical details).

4. Supports and Bounds of Convex Sets

4.1. Support Hyperplanes An extension of Minkowski's definition of support hyperplane says that a hyperplane $H \subset \mathbb{R}^n$ *supports* a nonempty set $X \subset \mathbb{R}^n$ provided H meets its closure, $\operatorname{cl} X$, and does not cut X. Analysis of the proof of Theorem 3.3 shows that its assertion holds for the case of any proper convex set $K \subset \mathbb{R}^n$. Consequently, Theorem 3.1 can be generalized as follows:

Theorem 4.1. *Any boundary point of a convex set $K \subset \mathbb{R}^n$ belongs to a hyperplane supporting K.*

If the dimension of a convex set $K \subset \mathbb{R}^n$ is less than n, then a hyperplane $H \subset \mathbb{R}^n$ supporting K may contain K entirely. Nevertheless, in many instances it is important to know whether H *properly supports* K, that is whether $K \not\subset H$. Equivalent terms used in the literature are *essential support* (see Steinitz [84, § 28]) and *nontrivial support* (see Rockafellar [72, p. 100]).

Theorem 4.2. ([72, Theorem 11.6]) *Let $K \subset \mathbb{R}^n$ be a convex set which is not a plane, and let F be a nonempty convex subset of* $\mathrm{cl}\, K$ *(for instance, F is a singleton). There is a hyperplane containing F and properly supporting K if and only if $F \subset \mathrm{rbd}\, K$.*

It is easy to see that a hyperplane H properly supports a convex set K if and only if it meets $\mathrm{cl}\, K$ such that $H \cap \mathrm{rint}\, K = \varnothing$. The following assertion is a variation of Theorem 4.2.

Theorem 4.3. ([76, Corollary 9.11]) *If a plane $L \subset \mathbb{R}^n$ meets the closure of a convex set $K \subset \mathbb{R}^n$ such that $L \cap \mathrm{rint}\, K = \varnothing$, then there is a hyperplane containing L and properly supporting K.*

The next result shows that the existence of a support hyperplane is a local property.

Theorem 4.4. ([76, Problem 9.3]) *Let $K \subset \mathbb{R}^n$ be a convex set, z be a point in* $\mathrm{cl}\, K$, *and $B_\rho(z) \subset \mathbb{R}^n$ be a closed ball of radius $\rho > 0$ centered at z. A hyperplane $H \subset \mathbb{R}^n$ through z supports (properly supports) K if and only if H supports (respectively, properly supports) the set $K \cap B_\rho(z)$.*

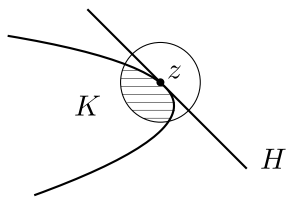


Figure 10: Local support of K at z.

If a support hyperplane $H \subset \mathbb{R}^n$ of a convex set $K \subset \mathbb{R}^n$ is described by the equation (2.1), then its normal vector e and the level scalar γ can be characterized as follows.

1. H supports (properly supports) K at a point $z \in \mathrm{cl}\, K$ if and only if $e \in N_z(K)$ (respectively, $e \in N_z(K) \setminus \mathrm{ort}\, K$) and $\gamma = z \cdot e$.
2. H supports (properly supports) K if and only if $e \in \mathrm{nor}\, K$ (respectively, $e \in \mathrm{nor}\, K \setminus \mathrm{ort}\, K$) and $\gamma = \sup\{x \cdot e \, : \, x \in K\}$.

THEOREM 4.5. ([76, Theorem 12.22]) *If $K \subset \mathbb{R}^n$ is a line-free convex set, then the set of nonzero vectors $e \in \mathbb{R}^n$ for which the hyperplane of the form*

$$H = \{x \in \mathbb{R}^n \, : \, x \cdot e = \gamma\}, \qquad \text{where } \gamma = \sup\{u \cdot e \, : \, u \in K\},$$

supports $\operatorname{cl} K$ *at a single point is dense in* $\operatorname{nor} K$.

One more assertion, independently proved by Durier [24] for the case of convex bodies and by Klee [52] for the case of line-free closed convex sets, complements Theorem 4.5 and the inclusion $\operatorname{ext} K \subset \operatorname{cl}(\exp K)$.

THEOREM 4.6. ([24, 52]) *Let $K \subset \mathbb{R}^n$ be a line-free closed convex set, H be a hyperplane supporting K, and z be an extreme point of K that belongs to H. Then there is a sequence of points $z_i \in \exp K$ and a respective sequence of hyperplanes H_i, $i \geq 1$, satisfying the conditions:*

$$\text{(a)} \;\; H_i \cap K = \{z_i\}, \; i \geq 1, \qquad \text{(b)} \;\; z = \lim_{i \to \infty} z_i \;\; \textit{and} \;\; H = \lim_{i \to \infty} H_i.$$

There are a few results on support properties of special convex sets. For instance:

3. If a hyperplane $H \subset \mathbb{R}^n$ supports a convex cone $C \subset \mathbb{R}^n$, then every apex of C belongs to H. Furthermore, the intersection of all hyperplanes supporting C is precisely the apex set of $\operatorname{cl} C$ (see, e.g., [76], Theorem 9.43 and Theorem 9.46).
4. If a hyperplane $H \subset \mathbb{R}^n$ supports an M-predecomposable set $K \subset \mathbb{R}^n$, expressed as the sum of a compact convex set B and a convex cone C with apex o, then H supports B (see [79]).

4.2. BOUNDING HYPERPLANES AND HALFSPACES Modifying Carathéodory's definition of bounding hyperplane, we say that a hyperplane $H \subset \mathbb{R}^n$ *bounds* a convex set $K \subset \mathbb{R}^n$ provided K is contained in a closed halfspace determined by H. Furthermore: H *nontrivially bounds* K if $K \not\subset H$; *strictly bounds* K if $H \cap K = \varnothing$; and *strongly bounds* K if H bounds a suitable open ρ-neighborhood $U_\rho(K)$ of K.

Analysis of the proof of Theorem 3.4 shows that it can be generalized in the following way.

THEOREM 4.7. *Let K be a proper convex set $\mathbb{R}^n$ and $z \in \mathbb{R}^n \setminus \mathrm{cl}\, K$. If u is the nearest to z point in $\mathrm{cl}\, K$, then the hyperplane $H \subset \mathbb{R}^n$ through z orthogonal to the segment $[u, z]$ strongly bounds K and $\delta(H, K) = \delta(z, K) = \|z - u\|$.*

The results below describe all hyperplanes bounding a convex set $K \subset \mathbb{R}^n$ and having a given direction of normal vectors or containing a given point (or even a plane) of $\mathbb{R}^n$. The following auxiliary statements on a hyperplane $H \subset \mathbb{R}^n$ immediately follow from definitions.

1. H bounds K if and only if H can be expressed in the form (2.1), where $e \in \mathrm{bar}\, K$ and $\sup\{x \cdot e : x \in K\} \le \gamma$.
2. H properly bounds K if and only if H can be expressed in the form (2.1), where $e \in \mathrm{bar}\, K$ and γ satisfies both inequalities

$$\inf\{x \cdot e : x \in K\} < \gamma \qquad \text{and} \qquad \sup\{x \cdot e : x \in K\} \le \gamma.$$

3. H strictly bounds K if and only if H can be expressed in the form (2.1), where $e \in \mathrm{bar}\, K$ and $x \cdot e < \gamma$ for all $x \in K$.
4. H strongly bounds K if and only if H can be expressed in the form (2.1), where $e \in \mathrm{bar}\, K$ and $\sup\{x \cdot e : x \in K\} < \gamma$.

THEOREM 4.8. ([82]) *For a convex set $K \subset \mathbb{R}^n$ and a point $z \in \mathbb{R}^n$, the assertions below hold.*

(a) *There is a hyperplane through z bounding K if and only if $o \notin \mathrm{int}\,(K - z)$, or, equivalently, $(K - z)^\circ \ne \{o\}$.*

(b) *A hyperplane through z bounds K is and only if it can be expressed as*

$$H = \{x \in \mathbb{R}^n : x \cdot e = z \cdot e\}, \tag{4.1}$$

where $e \in (K - z)^\circ \setminus \{o\}$.

THEOREM 4.9. ([82]) *For a convex set $K \subset \mathbb{R}^n$ and a point $z \in \mathbb{R}^n$, the assertions below hold.*

(a) *There is a hyperplane through z properly bounding K if and only if $o \notin \mathrm{rint}\,(K - z)$, or, equivalently, $(K - z)^\circ \setminus \mathrm{lin}\,(K - z)^\circ \ne \varnothing$.*

(b) *A hyperplane through z properly bounds K if and only if it can be expressed in the form (4.1), where $e \in (K - z)^\circ \setminus \mathrm{lin}\,(K - z)^\circ$.*

Theorem 4.10. ([82]) *For a closed convex set* $K \subset \mathbb{R}^n$ *and a point* $z \in \mathbb{R}^n$, *the assertions below hold.*

(a) *There is a hyperplane through* z *strongly bounding* K *if and only if* $o \notin \mathrm{cl}\,(K - z)$.

(b) *If* $o \notin \mathrm{cl}\,(K - z)$ *and* $e \in \mathrm{rint}\,(K - z)^\circ$, *then the hyperplane* (4.1) *strongly bounds* K.

(c) *If a hyperplane through* z *strongly bounds* K, *then it can be expressed in the form* (4.1), *where* $e \in (K - z)^\circ \setminus \mathrm{lin}\,(K - z)^\circ$. *If, additionally,* K *is compact, then* e *can be chosen in* $\mathrm{rint}\,(K - z)^\circ$.

There are various extensions of Theorem 3.4 to the case of hyperplanes bounding a convex set $K \subset \mathbb{R}^n$ and containing a given plane $L \subset \mathbb{R}^n$.

5. If L does not meet the relative interior of K, then there is a hyperplane through L properly bounding K (Rockafellar [72, Theorem 11.2]).
6. If the boundary of K does not contain a halfline and L is disjoint from $\mathrm{cl}\,K$, then there is a hyperplane through L strictly bounding $\mathrm{cl}\,K$ (Klee [47]).
7. If $\delta(K, L) > 0$, then there is a hyperplane H through L strongly bounding K such that $\delta(K, H) = \delta(K, L)$ ([76, Theorem 9.6]).

The families of hyperplanes through a plane $L \subset \mathbb{R}^n$ bounding (properly, strictly, or strongly) a given convex set $K \subset \mathbb{R}^n$ can be described similarly to Theorems 4.8 – 4.10. For instance,

8. There is a hyperplane which bounds K and contains L if and only if $o \notin \mathrm{int}\,(K - L)$, or, equivalently, $(K - L)^\circ \neq \{o\}$.
9. A hyperplane $H \subset \mathbb{R}^n$ bounds K and contains L if and only if H can be expressed in the form (4.1), where $e \in (K - L)^\circ \setminus \{o\}$ and z is any point in L.

We recall (see, e.g., Gale and Klee [33]) that a plane $L \subset \mathbb{R}^n$ is an *asymptote* of a set $X \subset \mathbb{R}^n$ provided $\mathrm{cl}\,X \cap L = \varnothing$ and $\delta(X, L) = 0$. Existence of plane asymptotes is closely related to the properties of various algebraic operations on sets (see, e.g., Auslender and Teboulle [1] and [75], [76] for further references). For instance, the following assertions hold.

10. For a closed set $X \subset \mathbb{R}^n$ and a plane $L \subset \mathbb{R}^n$, the sum $X + L$ is closed if and only if there is a translate of L which is an asymptote of X.

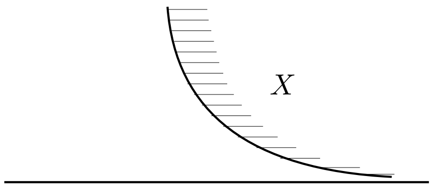


Figure 11: A plane asymptote of a set X.

11. For a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ and a closed set $X \subset \mathbb{R}^n$, the set $f(X)$ is closed if and only if there is a translate of null f which is an asymptote of X.

The family of plane asymptotes of a given convex set is not hereditary. For example, if K is the convex set in $\mathbb{R}^3$ be given by

$$K = \big\{(x, y, z) \,:\, x \geq 0,\ xy \geq 1,\ z \geq (x + y)^2\big\},$$

then the xy- and xz-coordinate planes are the only plane asymptotes of K. Asymptotic properties of convex sets without boundary halflines are studied by Klee [49], and of cones and M-predecomposable sets in [77] and [79]. As proved in [81], every plane asymptote L of a convex set $K \subset \mathbb{R}^n$ contains a line-free closed convex cone which is an asymptote of K.

The following assertions immediately follow the above definitions.

12. Any hyperplane $H \subset \mathbb{R}^n$ disjoint from a closed set $X \subset \mathbb{R}^n$ either strongly bounds X or is an asymptote of X.
13. A hyperplane $H \subset \mathbb{R}^n$ is an asymptote of a closed convex set $K \subset \mathbb{R}^n$ if and only if it can be expressed in the form (2.1), where $e \in \operatorname{bar} K \setminus \operatorname{nor} K$ and $\gamma = \sup\{x \cdot e \,:\, x \in K\}$.

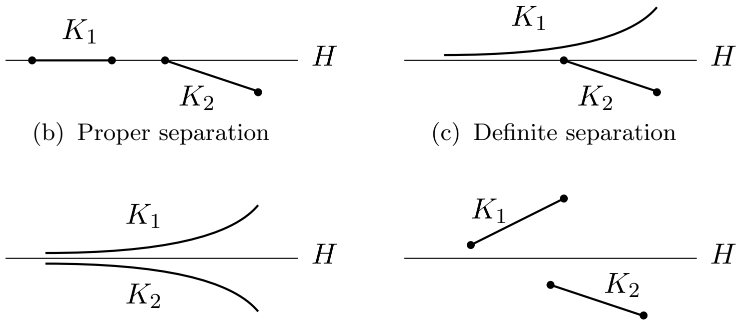
We will say that a closed halfspace $V \subset \mathbb{R}^n$ *bounds* a convex set $K \subset \mathbb{R}^n$ provided $K \subset V$. Furthermore, if H denotes the boundary hyperplane of V, then V *supports* (*properly supports*) K is H supports (properly supports) K. Also, V *strictly* (*strongly*) *bounds* K is H disjoint (strongly disjoint) from K. Various results on the existence or on description of different types of bounding halfspaces can be routinely derived from the above assertions on bounding hyperplanes. The results below deal with various representations of the closure of a given proper convex set $K \subset \mathbb{R}^n$ as intersections of bounding halfspaces (see [76, Chapter 9]).

14. $\operatorname{cl} K$ is the intersection of all closed halfspaces bounding (supporting, strictly, or strongly bounding) K.
15. If K is not a plane, X is a dense subset of $\operatorname{rbd} K$, and $\mathcal{V}(X)$ is the family of all closed halfspaces V each properly supporting K at a point from X, then
$$\operatorname{cl} K = \cap\,(V : V \in \mathcal{V}(X)) \qquad \text{and} \qquad \operatorname{rint} K = \cap\,(\operatorname{int} V : V \in \mathcal{V}(X)).$$
16. If E is a dense subset of the normal cone $\operatorname{nor} K$, then $\operatorname{cl} K$ is the intersection of a countable family $\mathcal{F}$ of closed halfspaces of the form
$$V_e(\gamma) = \{x \in \mathbb{R}^n \,:\, x{\cdot}e \le \gamma\}, \quad \text{where } e \in E,\ \gamma \ge \sup\{u{\cdot}e \,:\, u \in K\}.$$
17. If K is line-free, then $\operatorname{cl} K = \cap\,(V : V \in \mathcal{G})$, where $\mathcal{G}$ denotes the family of all closed halfspaces supporting $\operatorname{cl} K$ at its exposed points.

5. Separation of Convex Sets

5.1. Classification of Separating Hyperplanes Various results on hyperplane separation of convex sets are usually formulated in the following terms. If K_1 and K_2 are convex sets in $\mathbb{R}^n$ and $H \subset \mathbb{R}^n$ is a hyperplane, then we say that

1. H *separates* K_1 and K_2 if K_1 and K_2 lie in the opposite closed halfspaces determined by H (possibly, $K_1 \cup K_2 \subset H$).
2. H *properly separates* K_1 and K_2 if H separates K_1 and K_2 such that $K_1 \cup K_2 \not\subset H$.
3. H *definitely separates* K_1 and K_2 if H separates K_1 and K_2 such that $K_1 \not\subset H$ and $K_2 \not\subset H$.
4. H *strictly separates* K_1 and K_2 if H separates K_1 and K_2 such that both sets are disjoint from H (equivalently, K_1 and K_2 lie in the opposite open halfspaces determined by H).
5. H *strongly separates* K_1 and K_2 if H separates suitable open neighborhoods $U_\rho(K_1)$ and $U_\rho(K_2)$.



(b) Proper separation (c) Definite separation

(d) Strict separation (e) Strong separation

Figure 12: Types of hyperplane separation of convex sets.

The above terminology gradually evolved in time: the term *proper separation* is due to Rockafellar [72, p. 95], *definite separation* is called *real* by Bair and Jongmans [3], *strict separation* and *strong separation* are due to Klee [51].

The following obvious assertions provide analytical equivalences of the above definitions on separation of convex sets K_1 and K_2 in $\mathbb{R}^n$ by a hyperplane $H \subset \mathbb{R}^n$ of the form (2.1):

1.′ H separates K_1 and K_2 if and only if e and γ can be chosen such that

$$\sup\{x\cdot e \,:\, x \in K_1\} \le \gamma \le \inf\{x\cdot e \,:\, x \in K_2\}. \tag{5.1}$$

2.′ H properly separates K_1 and K_2 if and only if e and γ can be chosen to satisfy both inequalities (5.1) and

$$\inf\{x\cdot e \,:\, x \in K_1\} < \sup\{x\cdot e \,:\, x \in K_2\}. \tag{5.2}$$

3.′ H definitely separates K_1 and K_2 if and only if e and γ can be chosen to satisfy both inequalities (5.1) and

$$\inf\{x\cdot e \,:\, x \in K_1\} < \gamma < \sup\{x\cdot e \,:\, x \in K_2\}. \tag{5.3}$$

4.′ H strictly separates K_1 and K_2 if and only if e and γ can be chosen such that both conditions below are satisfied:

$$\begin{aligned} u\cdot e &< \inf\{x\cdot e \,:\, x \in K_2\} \qquad \text{for every } u \in K_1, \\ \sup\{x\cdot e \,:\, x \in K_1\} &< u\cdot e \qquad \text{for every } u \in K_2. \end{aligned} \tag{5.4}$$

5.′ H strongly separates K_1 and K_2 if and only if e and γ can be chosen such that

$$\sup\{x\cdot e \,:\, x \in K_1\} < \gamma < \inf\{x\cdot e \,:\, x \in K_2\}. \tag{5.5}$$

The above types of separation can be refined by using asymmetric conditions (see Klee [51] and [78] for further details): If convex sets K_1 and K_2 in $\mathbb{R}^n$ are separated by a hyperplane $H \subset \mathbb{R}^n$, then we say that

6. H *properly* separates K_1 *from* K_2 provided $K_1 \not\subset H$.
7. H *strictly* separates K_1 *from* K_2 provided K_1 is disjoint from H.

The relation between symmetric and asymmetric types of separation is described in the following theorem.

THEOREM 5.1. ([76, Theorem 10.6]) *If K_1 and K_2 are convex sets and H_1 and H_2 are hyperplanes such that H_i properly* (*strictly*) *separates K_i from K_{3-i}, $i = 1, 2$, then there is a hyperplane containing $H_1 \cap H_2$ and properly* (*strictly*) *separating K_1 and K_2.*

The results below describe the hyperplanes which separate a pair of convex sets and have a given direction of normals or contain a given point. An initial step towards description of separating hyperplanes with a given direction of normals consists in reduction to the case of a single convex set.

THEOREM 5.2. ([76, Theorem 10.7]) *Let K_1 and K_2 be convex sets in $\mathbb{R}^n$, and H be a hyperplane of the form* (2.1). *Then the assertions below hold.*

(a) *A translate of H separates* (*properly separates*) *K_1 and K_2 if and only if the subspace* (2.2) *bounds* (*properly bounds*) *$K_1 - K_2$.*
(b) *A translate of H strictly separates at least one of the sets K_1 and K_2 from the other if and only if the subspace* (2.2) *strictly bounds $K_1 - K_2$.*
(c) *A translate of H strongly separates K_1 and K_2 if and only if the subspace* (2.2) *strongly bounds $K_1 - K_2$.*

A combination of Theorems 4.8 – 4.10 and Theorem 5.2 implies the following assertions.

THEOREM 5.3. ([78, 82]) *Given convex sets K_1 and K_2 in $\mathbb{R}^n$, the following assertions hold.*

(a) *There is a hyperplane separating K_1 and K_2 if and only if any of the following equivalent conditions holds:*

$$\operatorname{int} K_1 \cap \operatorname{int} K_2 = \varnothing, \quad o \notin \operatorname{int}(K_1 - K_2), \quad (K_1 - K_2)^\circ \neq \{o\}.$$

Furthermore, a translate of a hyperplane $H \subset \mathbb{R}^n$ separates K_1 and K_2 if and only if H can be expressed in the form (2.1), *where* $e \in (K_1 - K_2)^\circ \setminus \{o\}$.

(b) *There is a hyperplane properly separating K_1 and K_2 if and only if any of the following equivalent conditions holds:*

$$\operatorname{rint} K_1 \cap \operatorname{rint} K_2 = \varnothing, \qquad o \notin \operatorname{rint}(K_1 - K_2),$$
$$(K_1 - K_2)^\circ \setminus \operatorname{lin}(K_1 - K_2)^\circ \neq \varnothing.$$

Furthermore, a translate of a hyperplane $H \subset \mathbb{R}^n$ properly separates K_1 and K_2 if and only if H can be expressed in the form (2.1), *where* $e \in (K_1 - K_2)^\circ \setminus \operatorname{lin}(K_1 - K_2)^\circ$.

(c) *There is a hyperplane definitely separating K_1 and K_2 if and only if either $o \notin \operatorname{cl}(K_1 - K_2)$ or*

$$o \in \operatorname{cl}(K_1 - K_2) \quad \text{and} \quad (K_1 - K_2)^\circ \setminus (\operatorname{ort} K_1 \cup \operatorname{ort} K_2) \neq \varnothing.$$

Furthermore, a translate of a hyperplane $H \subset \mathbb{R}^n$ definitely separates K_1 and K_2 if and only if H can be expressed in the form (2.1) *such that one of the following conditions is satisfied:*

(i) $o \notin \operatorname{cl}(K_1 - K_2)$ *and*

$$e \in \operatorname{rint}(K_1 - K_2)^\circ \cup (\operatorname{rbd}(K_1 - K_2)^\circ \setminus (\operatorname{ort} K_1 \cup \operatorname{ort} K_2)),$$

(ii) $o \in \operatorname{cl}(K_1 - K_2)$ *and* $e \in (K_1 - K_2)^\circ \setminus (\operatorname{ort} K_1 \cup \operatorname{ort} K_2)$.

(d) *There exists some hyperplane strongly separating K_1 and K_2 if and only if $o \notin \operatorname{cl}(K_1 - K_2)$. Furthermore, if $o \notin \operatorname{cl}(K_1 - K_2)$ and $e \in \operatorname{rint}(K_1 - K_2)^\circ$, then a suitable translate of a hyperplane of the form* (2.1) *strongly separates K_1 and K_2.*

A variation of Theorem 5.3 allows us to describe all hyperplanes which separate a pair of convex sets and contain a given point of $\mathbb{R}^n$. This description uses the following auxiliary lemma.

LEMMA 5.4. ([82]) *Convex sets K_1 and K_2 in $\mathbb{R}^n$ are separated (properly separated) by a hyperplane $H \subset \mathbb{R}^n$ through a given point $z \in \mathbb{R}^n$ if and only if the generated cones $C_z(K_1)$ and $C_z(K_2)$ are separated (properly separated) by H.*

THEOREM 5.5. ([82]) *Given convex sets K_1 and K_2 in $\mathbb{R}^n$ and a point $z \in \mathbb{R}^n$, let $D_1 = C_z(K_1) - z$, $D_2 = C_z(K_2) - z$. The assertions below hold.*

(a) *There is a hyperplane through z separating K_1 and K_2 if and only if the cones D_1 and D_2 satisfy any of the following equivalent conditions:*

$$o \notin \operatorname{int}(D_1 - D_2), \qquad (D_1 - D_2)^\circ \neq \{o\}, \qquad D_1^\circ \cap (-D_2^\circ) \neq \{o\}.$$

(b) *There is a hyperplane through z properly separating K_1 and K_2 if and only if the cones D_1 and D_2 satisfy any of the following equivalent conditions:*

$$o \notin \operatorname{rint}(D_1 - D_2), \quad (D_1 - D_2)^\circ \ \text{is not a subspace},$$
$$D_1^\circ \cap (-D_2^\circ) \ \text{is not a subspace}.$$

5.2. GEOMETRIC CONDITIONS ON HYPERPLANE SEPARATION Theorem 5.3 provides a unified description of all hyperplanes separating convex sets K_1 and K_2 in $\mathbb{R}^n$, which is formulated in terms of the polar cone $(K_1 - K_2)^\circ$. Some other types of geometric conditions that guaranty the existence of a desired type of separation are given below.

Proper separation. The condition $\operatorname{rint} K_1 \cap \operatorname{rint} K_2 = \varnothing$ was already mentioned in Theorem 5.3. It was obtained in various forms of generality by Fenchel [30, p. 48], Klee [46], and Rockafellar [72, Theorem 11.3].

A related result on asymmetric type of proper separation is due to Rockafellar [72, Theorem 20.2]: Given convex sets K_1 and K_2 in $\mathbb{R}^n$ such that K_2 is polyhedral, there is a hyperplane properly separating K_1 from K_2 if and only if $\operatorname{rint} K_1 \cap K_2 = \varnothing$.

We observe that the latter assertion does not hold if the set K_2 is not polyhedral. For instance, if K_1 and K_2 are planar circular disks in $\mathbb{R}^3$ given by

$$K_1 = \{(x, y, 0) : x^2 + (y-1)^2 \le 1\},$$
$$K_2 = \{(0, y, z) : y^2 + (z-1)^2 \le 1\},$$

then $\operatorname{rint} K_1 \cap K_2 = \varnothing$, while K_1 is not properly separated from K_2.

Strict separation. The only known geometric result on strict separation of convex sets is attributed to Klee [47]: If none of the disjoint closed convex sets K_1 and K_2 in $\mathbb{R}^n$ is a plane, and none of the sets $\operatorname{rbd} K_1$ and $\operatorname{rbd} K_2$ contains a halfline, then K_1 and K_2 are strictly separated by a hyperplane.

Strong separation. An obvious continuity argument shows that if convex sets K_1 and K_2 are strongly separated by a hyperplane H, then there is a slab of positive width which separates K_1 and K_2 and whose boundary hyperplanes are parallel to H. A natural question here is to determine the maximum possible width of such a slab. The answer to this question was given by Dax [20] (see also [76, Theorem 10.20]).

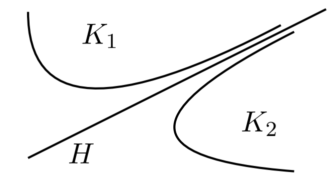


Figure 13: Strict separation of K_1 and K_2.

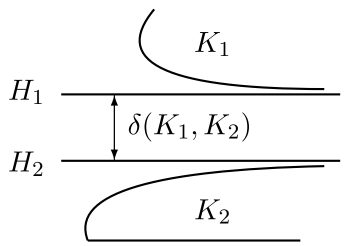


Figure 14: Separation of K_1 and K_2 by a slab of maximum width.

THEOREM 5.6. ([20]) *If convex sets K_1 and K_2 in $\mathbb{R}^n$ are strongly disjoint (that is, $\delta(K_1, K_2) > 0$), then there is a unique pair of parallel hyperplanes H_1 and H_2 in $\mathbb{R}^n$, both separating K_1 and K_2 and satisfying the condition $\delta(H_1, H_2) = \delta(K_1, K_2)$.*

The above equality $\delta(H_1, H_2) = \delta(K_1, K_2)$, without specifying the uniqueness of the pair $\{H_1, H_2\}$, was obtained later by Gabidullina [32] for the case when at least one of the sets K_1 and K_2 is compact.

A similar question on strong separation of convex sets K_1 and K_2 concerns the existence of a pair of nearest points $z_1 \in \operatorname{cl} K_1$ and $z_2 \in \operatorname{cl} K_2$. A simple geometric argument shows that in this case the hyperplanes through z_1 and z_2 orthogonal to $[z_1, z_2]$ form a slab of maximum width separating K_1 and K_2. A sufficient condition for the existence of such a pair $\{z_1, z_2\}$ can be found in [76].

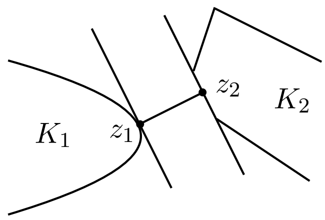


Figure 15: A nearest pair of points in K_1 and K_2.

THEOREM 5.7. ([76, Theorem 10.24]) *If convex sets K_1 and K_2 in $\mathbb{R}^n$ satisfy the condition* $\mathrm{rec}\,(\mathrm{cl}\,K_1) \cap \mathrm{rec}\,(\mathrm{cl}\,K_2) = \{o\}$, *then* $\delta(K_1, K_2) = \|z_1 - z_2\|$ *for suitable points* $z_1 \in \mathrm{cl}\,K_1$ *and* $z_2 \in \mathrm{cl}\,K_2$. *In particular, a nearest pair* $\{z_1, z_2\}$ *exists provided at least one of the sets K_1 and K_2 is bounded.*

A useful result on strong separation of convex sets was obtained by De Wilde [21].

THEOREM 5.8. ([21]) *If K_1 and K_2 are disjoint closed convex sets in $\mathbb{R}^n$, then the following conditions are equivalent:*

(a) *Both K_1 and K_2 are line-free and* $\mathrm{rec}\,K_1 \cap \mathrm{rec}\,K_2 = \{o\}$.

(b) *There are parallel disjoint hyperplanes H_1 and H_2 both separating K_1 and K_2 such that $H_i \cap K_i$ is an exposed point of K_i, $i = 1, 2$.*

Analysis of the proof of Theorem 5.8 shows that the exposed points $H_1 \cap K_1$ and $H_2 \cap K_2$ are not necessarily the nearest. Nevertheless, H_1 and H_2 may be chosen to satisfy the condition $\delta(H_1, H_2) > \delta(K_1, K_2) - \varepsilon$ for any given scalar $\varepsilon > 0$.

Maximal separation. Klee [50] obtained various results regarding strict and strong separation of convex sets by hyperplanes. Given a pair $\{\mathcal{F}, \mathcal{G}\}$ of nonempty families of closed convex sets in $\mathbb{R}^n$, we say that $\mathcal{F}$ is maximal with respect to a certain type S of separation provided it satisfies the following conditions:

1. The sets F and G are S-separated whenever $F \in \mathcal{F}$ and $G \in \mathcal{G}$, with $F \cap G = \varnothing$.

2. For every $F \in \mathcal{F}$, there is $G \in \mathcal{G}$ such that $F \cap G = \varnothing$.
3. For every closed convex set $F \notin \mathcal{F}$, there is $G \in \mathcal{G}$ such that $F \cap G = \varnothing$ but F and G are not S-separated.

Following Gale and Klee [33], we say that a closed convex set $K \subset \mathbb{R}^n$ is *continuous* provided K admits no boundary halfline and no line asymptote. Given disjoint closed convex sets F and G in $\mathbb{R}^n$, the assertions below hold (see [50]).

4. Each of the following conditions implies that F and G are strictly separated and represents a maximal theorem for strict separation:
   (a) F is continuous,
   (b) neither F nor G admits a line asymptote,
   (c) neither F nor G has a boundary halfline.
5. Each of the following conditions implies that F and G are strongly separated and represents a maximal theorem for strong separation:
   (d) F is continuous,
   (e) neither F nor G admits a line asymptote.

An extensive development and generalization of Klee's results on maximal separation is given in the book of Fajardo, Goberna, Rodríguez, and Vicente-Pérez [27]. Theorem 1.2 and Theorem 1.3 from this book give a comprehensive list of various maximal separation assertions for the case of evenly convex sets. (According to Fenchel [29], a convex set in $\mathbb{R}^n$ is called *evenly convex* if it is the intersection of a family of open halfspaces. It is easy to see that every proper closed convex sets is evenly convex.)

5.3. Sharp Separation of Convex Cones If convex cones C_1 and C_2 with a common apex in $\mathbb{R}^n$ are separated by a hyperplane $H \subset \mathbb{R}^n$, then H supports both cones $\operatorname{cl} C_1$ and $\operatorname{cl} C_2$. Consequently, $\operatorname{ap}(\operatorname{cl} C_1) \cup \operatorname{ap}(\operatorname{cl} C_2) \subset H$. In this regard, we will say that H *sharply separates* C_1 *from* C_2 provided $H \cap \operatorname{cl} C_1 = \operatorname{ap}(\operatorname{cl} C_1)$. Similarly, H *sharply separates* C_1 *and* C_2 if

$$H \cap (\operatorname{cl} C_1) = \operatorname{ap}(\operatorname{cl} C_1) \quad \text{and} \quad H \cap \operatorname{cl} C_2 = \operatorname{ap}(\operatorname{cl} C_2).$$

The next two theorems give criteria for sharp separation of cones in terms of their polar cones.

THEOREM 5.9. ([78]) *If C_1 and C_2 are convex cones in $\mathbb{R}^n$ with a common apex $a \in \mathbb{R}^n$, then the following conditions are equivalent.*

(a) *C_1 is sharply separated from C_2.*

(b) *The set $E = \operatorname{rint}(C_1 - a)^\circ \cap (a - C_2)^\circ$ has positive dimension.*

THEOREM 5.10. ([74, 78]) *If C_1 and C_2 are convex cones in $\mathbb{R}^n$ with a common apex $a \in \mathbb{R}^n$, then the following conditions are equivalent.*

(a) *C_1 and C_2 are sharply separated.*

(b) *Each of the cones C_1 and C_2 is sharply separated from the other.*

(c) *The set $D = \operatorname{rint}(C_1 - a)^\circ \cap \operatorname{rint}(a - C_2)^\circ$ has positive dimension.*

Analysis of the proof of Theorem 5.10 reveals a simple corollary: If C_1 is not a plane and is sharply separated from C_2, then C_1 is properly separated from C_2. The converse assertion is not true. For instance, in $\mathbb{R}^2$, the cone $C_1 = \{(x, 0) : x \in \mathbb{R}\}$ is separated sharply but not properly from the cone $C_2 = \{(x, y) : 0 \le x, 0 \le y \le x\}$, while C_2 is separated properly but not sharply from C_1 (see Figure 16).

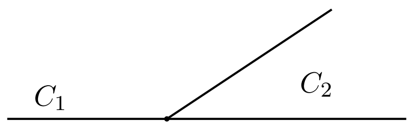


Figure 16: Proper but not sharp separation of cones C_1 and C_2.

A geometric criterion for sharp separation of convex cones is given in the following theorem.

THEOREM 5.11. ([74]) *Let C_1 and C_2 be convex cones in $\mathbb{R}^n$ with a common apex $a \in \mathbb{R}^n$. The conditions below are equivalent.*

(a) *C_1 and C_2 are sharply separated by a hyperplane.*

(b) $\operatorname{cl} C_1 \cap \operatorname{cl} C_2 = \operatorname{ap}(\operatorname{cl} C_1) \cap \operatorname{ap}(\operatorname{cl} C_2)$ *and at least one of the cases below holds:*

  (i) $\dim(C_1 \cup C_2) \leqslant n - 1$,

  (ii) *at least one of the cones C_1 and C_2 is not a plane.*

In terms of continuous linear functionals on a linear topological space, Theorem 5.11, formulated for the case of closed convex cones with a common apex o, was proved earlier by Klee [46] under the assumption $\operatorname{ap} C_1 \cap \operatorname{ap} C_2 = \{o\}$, and by Bair and Gwinner [2] under the condition that $\operatorname{ap} C_1 \cap \operatorname{ap} C_2$ is a subspace.

5.4. Penumbras and Separation Following Rockafellar [72, p. 22], we recall that the *penumbra* of a convex set K_1 with respect to another convex set K_2, denoted below $P(K_1, K_2)$, is defined by

$$\begin{aligned} P(K_1, K_2) &= \cup(\mu K_1 + (1-\mu)K_2 \,:\, \mu \geq 1) \\ &= \{\mu x_1 + (1-\mu)x_2 \,:\, \mu \geq 1, x_1 \in K_1, x_2 \in K_2\}. \end{aligned}$$

Geometrically, $P(K_1, K_2)$ is the union of all closed halflines initiated at the points of K_1 in the directions of vectors from $K_1 - K_2$ (see Fig. 17). It is possible to show (see [80]) that both sets $P(K_1, K_2)$ and $P(K_2, K_1)$ are convex and contain K_1 and K_2, respectively. The following theorem illustrates the role of penumbras in separation of convex sets.

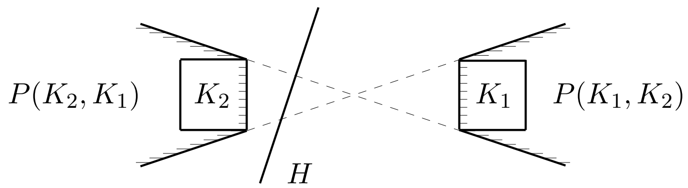


Figure 17: Illustration to Theorem 5.12.

Theorem 5.12. ([80]) *Let K_1 and K_2 be convex sets in $\mathbb{R}^n$. A hyperplane $H \subset \mathbb{R}^n$ separates (respectively, properly, strictly, or strongly) K_1 and K_2 if and only if it separates (respectively, nontrivially, strictly, or strongly) the sets $P(K_1, K_2)$ and $P(K_2, K_1)$.*

Given convex sets K_1 and K_2 in $\mathbb{R}^n$, denote by $\mathcal{H}_1(K_1, K_2)$ (respectively, by $\mathcal{H}_2(K_1, K_2)$ and $\mathcal{H}_3(K_1, K_2)$) the family of all hyperplanes properly (respectively, strictly and strongly) separating K_1 and K_2. Also, let

$$E_i(K_1, K_2) = \cup\,(H : H \in \mathcal{H}_i), \quad i = 1, 2, 3.$$

Theorem 5.13. ([80]) *If convex sets K_1 and K_2 in $\mathbb{R}^n$ satisfy the condition* $\operatorname{rint} K_1 \cap \operatorname{rint} K_2 = \varnothing$, *then*

$$E_1(K_1, K_2) = \mathbb{R}^n \setminus (\operatorname{rint} P(K_1, K_2) \cup \operatorname{rint} P(K_2, K_1)).$$

Furthermore, a hyperplane $H \subset \mathbb{R}^n$ properly separates K_1 and K_2 if and only if $H \subset E_1(K_1, K_2)$ and $H \cap \operatorname{aff}(K_1 \cup K_2) \neq \varnothing$.

Corollary 5.14. ([80]) *If convex sets K_1 and K_2 in $\mathbb{R}^n$ satisfy the condition* $\operatorname{cl} K_1 \cap \operatorname{cl} K_2 = \varnothing$, *then $E_2(K_1, K_2) \subset F_2(K_1, K_2)$, where*

$$F_2(K_1, K_2) = \mathbb{R}^n \setminus (P(\operatorname{cl} K_1, \operatorname{cl} K_2) \cup P(\operatorname{cl} K_2, \operatorname{cl} K_1))).$$

Furthermore, a hyperplane $H \subset \mathbb{R}^n$ strictly separates $\operatorname{cl} K_1$ *and* $\operatorname{cl} K_2$ *if and only if $H \subset F_2(K_1, K_2)$ and $H \cap \operatorname{aff}(K_1 \cup K_2) \neq \varnothing$.*

The inclusion $E_2(K_1, K_2) \subset F_2(K_1, K_2)$ in Corollary 5.14 may be proper. Indeed, consider the closed convex sets

$$K_1 = \{(x, 1) : 0 \le x \le 1\} \qquad \text{and} \qquad K_2 = \{(x, 0) : x \in \mathbb{R}\}.$$

Then $E_2(K_1, K_2) = \{(x, y) : 0 < y < 1\}$, while

$$F_2(K_1, K_2) = E_2(K_1, K_2) \cup \{(x, 1) : x < 0\} \cup \{(x, 1) : x > 1\}.$$

Theorem 5.15. ([80]) *If convex sets K_1 and K_2 in $\mathbb{R}^n$ are strongly disjoint, then*

$$E_3(K_1, K_2) = \mathbb{R}^n \setminus (\operatorname{cl} P(K_1, K_2) \cup \operatorname{cl} P(K_2, K_1)).$$

The following assertions from [80] relate various properties of penumbras to some known classes of convex sets in $\mathbb{R}^n$.

1. If K_1 is compact, then $P(K_1, K_2)$ is an M-predecomposable set.
2. If both K_1 and K_2 are compact and $K_1 \cap K_2 = \varnothing$, then $P(K_1, K_2)$ is an M-decomposable set.
3. If both K_1 and K_2 are polyhedra, then $\operatorname{cl} P(K_1, K_2)$ is a polyhedron.
4. If both K_1 and K_2 are polytopes, then $P(K_1, K_2)$ is a polyhedron.

5.5. Hemispaces The following concept was introduced by Motzkin [66, Lecture III] in three dimensions, and, independently, by Hammer [39] in vector spaces of any dimension: Given a point $v \in \mathbb{R}^n$, any maximal (under inclusion) convex subset of $\mathbb{R}^n \setminus \{v\}$, denoted S_v, is called a *semispace* of $\mathbb{R}^n$ at v (in [43] and [53] these sets are called *hypercones*). The next properties of semispaces can be easily obtained (see [39, 43, 66]).

1. For a semispace $S_v \subset \mathbb{R}^n$, both sets S_v and $\mathbb{R}^n \setminus S_v$ are convex cones with apex v.
2. For a convex set $K \subset \mathbb{R}^n \setminus \{v\}$, there is a semispace $S_v \subset \mathbb{R}^n$ containing K.
3. If $C \subset \mathbb{R}^n$ is a convex cone with improper apex $v \in \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ is a convex set missing v and disjoint from C, then there is a semispace $S_v \subset \mathbb{R}^n$ containing C and disjoint from B (Jamison [43] for the case $v = o$).

Additional properties of semispaces in vector spaces of any dimension can be found in the papers [22, 43, 48, 56, 65].

Theorem 5.16. ([39]) *The family of all semispaces of $\mathbb{R}^n$ is the smallest among all families $\mathcal{F}$ of convex sets in $\mathbb{R}^n$ satisfying the following condition: every proper convex set $K \subset \mathbb{R}^n$ is the intersection of some elements from $\mathcal{F}$.*

The structure of semispaces can be described in different ways. The first one, briefly mentioned by Hammer [40] (see the books [58, Satz 1.10], and [76, Theorem 10.32] for complete proofs), uses a nested family of planes

$$\{v\} = L_0 \subset L_1 \subset \cdots \subset L_{n-1} \subset L_n = \mathbb{R}^n, \qquad \dim L_i = i, \ 0 \le i \le n, \quad (5.6)$$

and their halfplanes $E_1, \ldots, E_n$, where E_i an open halfplane of L_i determined by L_{i-1}, $1 \le i \le n$.

Theorem 5.17. ([40]) *If $S_v \subset \mathbb{R}^n$ is a semispace at $v \in \mathbb{R}^n$, then there is a nested sequence of planes of the form* (5.6) *and a respective sequence of open halfplanes $E_1, \ldots, E_n$ such that $S_v = E_1 \cup \cdots \cup E_n$. Conversely, any set of the form $E_1 \cup \cdots \cup E_n$ is a semispace at v.*

Another way (given by Hammer [39, 40] without proof) is based on the choice of a suitable basis for $\mathbb{R}^n$.

Theorem 5.18. ([39, 40]) *If $S_v \subset \mathbb{R}^n$ is a semispace at $v \in \mathbb{R}^n$, then there is a basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$ such that S_v consists of all vectors of the form*

$v + \alpha_1 e_1 + \cdots + \alpha_n e_n$, where $\alpha_1^2 + \cdots + \alpha_n^2 > 0$ and the first nonzero scalar in the sequence $\alpha_1, \ldots, \alpha_n$ is positive. Conversely, given any basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$, the set of described above vectors is a semispace at v.

The equivalence of description of semispaces in Theorem 5.17 and Theorem 5.18 follows from the simple geometric arguments:

4. If $e_1, \ldots, e_n$ is a basis for $\mathbb{R}^n$, then the open halfplanes E_i from Theorem 5.17 can be chosen as
$$E_i = v + \{\alpha_{n-i+1} e_{n-i+1} + \cdots + \alpha_n e_n \, : \, \alpha_{n-i+1} > 0\}, \;\; 1 \le i \le n. \quad (5.7)$$
5. For any choice of planes (5.6) and of respective halfplanes $E_1, \ldots, E_n$, nonzero vectors
$$e_i \in (E_{n-i+1} - v) \setminus (L_{n-i} - v), \quad 1 \le i \le n,$$
form a basis for $\mathbb{R}^n$ such that the equalities (5.7) hold.

Independently, Martínez-Legaz [59] described a similar separation result, based on lexicographic ordering $\preceq$ of $\mathbb{R}^n$. We recall that for distinct vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, one can write $x \prec y$ if $x_i < y_i$, with i being the first index in $\{1, \ldots, n\}$ for which $x_i \ne y_i$; also, $x \preceq y$ if $x \prec y$ or $x = y$. In a standard way, a invertible $n \times n$ matrix A is orthogonal if $A^{-1} = A^T$.

Theorem 5.19. ([59]) *For a proper convex set $K \subset \mathbb{R}^n$ and a point $x_0 \notin K$, there is an invertible (even orthogonal) $n \times n$ matrix A and a vector $v \in \mathbb{R}^n$ such that $Ax \prec v \preceq Ax_0$ whenever $x \in K$.*

Although Martínez-Legaz [59] made an observation that the sets from Theorem 5.19 are similar in their properties to semispaces, it was Singer [73] who proved the following assertion:

A set $M \subset \mathbb{R}^n$ is a semispace at v if and only if there is an invertible matrix $n \times n$ matrix A such that $M = \{x \in \mathbb{R}^n \, : \, Ax \prec v\}$.

The result below is proved by Tukey [91] (the condition that the vector space E should be normed is superfluous) and, independently, by Stone [85] (see Theorem 7 from Chapter 3).

Theorem 5.20. ([85, 91]) *Any pair of disjoint convex sets K_1 and K_2 in a vector space E can be separated by complementary convex sets Q_1 and Q_2:*
$$K_1 \subset Q_1, \quad K_2 \subset Q_2, \quad Q_1 \cup Q_2 = E, \quad Q_1 \cap Q_2 = \varnothing.$$

We observe that Theorem 5.20 cannot be extended to the case of more than two convex sets. For instance, the convex cones C_1, C_2, and C_3 in the plane, depicted in Figure 18, cannot be enlarged into pairwise disjoint (even pairwise non-overlapping) convex sets whose union is the entire plane.

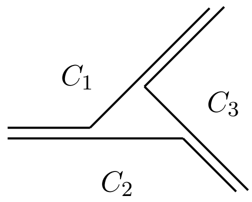


Figure 18: No convex extensions of cones C_1, C_2 and C_3 cover the whole plane.

The following results are similar to those from Theorem 5.20.

6. If $C \subset \mathbb{R}^n$ is a convex cone with apex o such that $C \cap (-C) = \{o\}$, then there is a convex cone C' with apex o satisfying the conditions $C \subset C'$, $C' \cap (-C') = \{o\}$, and $C' \cup (-C') = \mathbb{R}^n$ (Ellis [26]).
7. Let $\mathcal{F}$ ba a commuting family of affine transformations in $\mathbb{R}^n$, and let K_1 and K_2 be disjoint convex sets both invariant with respect to transformations from $\mathcal{F}$. Then there are complementary $\mathcal{F}$-invariant convex sets Q_1 and Q_2 such that $K_1 \subset Q_1$ and $K_2 \subset Q_2$ (Páles [68]).

Following Jamison [42], we say that a proper convex subset Q of $\mathbb{R}^n$ is a *hemispace* provided its complement $\mathbb{R}^n \setminus Q$ is a convex set. In [54] and [76], hemispaces are also called *convex halfspaces*. Various properties of hemispaces in vectors spaces of any dimension can be found in the papers [23, 42, 55] (see [36] for related material). A description of hemispaces in $\mathbb{R}^n$, similar to that of Theorem 5.18, was obtained by Lassak [54].

Theorem 5.21. ([54]) *If Q and Q' are complementary hemispaces in $\mathbb{R}^n$, then there is a point $v \in \mathbb{R}^n$, a (orthogonal) basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$, and an integer $r \geq 1$ such that one of the sets Q and Q' consists of all vectors of the form $v + \alpha_r e_r + \cdots + \alpha_n e_n$, where $\alpha_r^2 + \cdots + \alpha_n^2 > 0$ and the first nonzero scalar in the sequence $\alpha_r, \ldots, \alpha_n$ is positive. Conversely, for any choice of a point $v \in \mathbb{R}^n$, a basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$, and an integer $r \geq 1$, the sets of vectors described above is a hemispace in $\mathbb{R}^n$.*

The next theorem shows that the above description of complementary hemispaces can be reformulated in terms of nested sequences of planes (5.6) and of their halfplanes $E_1, \dots, E_n$.

Theorem 5.22. ([76, Theorem 10.28]) *If Q and Q' are complementary hemispaces in $\mathbb{R}^n$, then there is a sequence of planes of the form* (5.6) *and an integer $1 \leqslant r \leqslant n$ such that either $Q = F_r$ and $Q' = F'_r$, or $Q = F'_r$ and $Q' = F_r$, with*

$$F_r = E_r \cup \cdots \cup E_n, \quad F'_r = L_{r-1} \cup E'_r \cup \cdots \cup E'_n, \quad 1 \leqslant r \leqslant n, \tag{5.8}$$

where E_i, E'_i are complementary open halfplanes of L_i determined by L_{i-1}.

Independently, Martínez-Legaz [59] defined a hemispace in $\mathbb{R}^n$ as the set of vectors $x \in \mathbb{R}^n$ satisfying the condition $Ax \prec v$, where A is an arbitrary (not necessarily invertible) $n \times n$ matrix and $v \in \mathbb{R}^n$. Later, Martínez-Legaz and Singer [60] proved that for any pair of disjoint convex sets K_1 and K_2 in $\mathbb{R}^n$, there exists an orthogonal $n \times n$ matrix A such that $Ax_1 \prec Ax_2$ for all $x_1 \in K_1$ and $x_2 \in K_2$.

The following description of hemispaces in terms of lexicographical order on $\bar{\mathbb{R}}^n$ is due to Martínez-Legaz and Singer[61] (here $\bar{\mathbb{R}}^n$ stands for the Cartesian product of n extended lines $\bar{\mathbb{R}} = [-\infty, \infty]$).

Theorem 5.23. ([61]) *A set $Q \subset \mathbb{R}^n$ is a hemispace if and only if there is an $n \times n$ orthogonal matrix A and a point $v \in \bar{\mathbb{R}}^n$ such that either $Q = \{x \in \mathbb{R}^n : Ax \prec v\}$ or $Q = \{x \in \mathbb{R}^n : Ax \preceq v\}$.*

Acknowledgements

The author is thankful to the referees for the careful reading and considered suggestions leading to a better presented paper.

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