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Rosenthal ℓ_∞-theorem revisited

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Abstract: A remarkable Rosenthal ℓ_∞-theorem is extended to operators $T : \ell_\infty(\Gamma, E) \to F$, where Γ is an infinite set, E a locally bounded (for instance, normed or p-normed) space, and F any topological vector space.

Key words: Vector-valued ℓ_∞-function space, locally bounded space, p-normed space, quasi-normed space, isomorphism, Rosenthal ℓ_∞-theorem.

MSC (2020): 46A16.

1. Introduction and the standard Rosenthal ℓ_∞-theorem

I begin by quoting my antique result from 1975 which I dare to call Standard Rosenthal ℓ_∞-Theorem. It was first placed in [4] and, after a short time, with a much refined proof, inspired very essentially by ideas from a work of J. Kupka [11](1974), in [3] (the order of appearance happened to be just the reverse). It was quite a surprising common extension of the original $\ell_\infty(\Gamma)$ and $c_0(\Gamma)$ results of Haskell P. Rosenthal [18, Proposition 1.2; Theorem 3.4 and Remark 1 after it] (announced in [17]) proved for operators acting from these spaces to any Banach space F, and of similar in spirit results of N.J. Kalton [8, Theorem 3.2;3.3, 4.3; Theorem 2.3] for $\Gamma = \mathbb{N}$ and any TVS F. Consult [4, a comment after (R:**N**) on p. 209] for more precise information. Also see N.J. Kalton [9]. In the ℓ_∞-case Rosenthal imposed a much stronger condition on T than (R-0), namely, that "$T|_{c_0(\Gamma)}$ is an isomorphism", while in the c_0-case his condition was precisely (R-0). I now return to that old work of mine with further extensions. The previous variants of Rosenthal's theorem have found significant applications to Banach spaces [18, 1, 14] and to more general spaces ([6, Theorem 1.2], [7, p. 314 and ff.], [5]), and to vector measures ([12],

*To my wife Krystyna and our daughters Monika and Karolina

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[13], [2, Lemma 4.1.37, Lemma 4.1.39, Lemma 4.1.41], [15, Proposition 3]). It is therefore natural to expect that the new, more general variants will have an even wider range of applicability. A feature that all the Rosenthal-type results encountered in this paper share is that they concern operators $T : X \to F$ which send 'primitive' vectors in X that are away from zero to vectors that are away from zero in F. In an unbelievable way 'primitive' can be removed (though, in general, there is no good functional analytic road in X from "primitive" vectors to those arbitrary ones), leading to the conclusion that the restriction of T to a large subspace of X must be an isomorphism. This means that these results are of a fundamentally basic nature.

THEOREM 1.1. (STANDARD ROSENTHAL ℓ_∞-THEOREM) *Let F be a* TVS *and X be a subspace of $\ell_\infty(\Gamma)$ such that $e_\gamma \in X$ for all $\gamma \in \Gamma$. Also, let $T : X \to F$ be an operator such that*

(R-0) *for some* 0*-neighborhood U in F and all $\gamma \in \Gamma$: $T(e_\gamma) \notin U$.*

Then Γ has a subset A of the same cardinality $\mathfrak{m}$ as Γ for which the restriction $T|_{X(A)}$ is an isomorphism.

1.1. NOTATION, CONVENTIONS, TERMINOLOGY, SOME BASIC FACTS. To ensure proper understanding of was said above, and of the new material, I clarify here a few points. The acronym TVS stands for (nonzero and Hausdorff) *topological vector space* over the field of scalars $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and *operator* always means continuous linear operator.

For a TVS E and an infinite set Γ, $\ell_\infty(\Gamma, E)$ is the space of bounded functions $\boldsymbol{x} : \Gamma \to E$ with the topology of uniform convergence on Γ. Thus for any 0-neighborhood B in E the corresponding 0-neighborhood in $\ell_\infty(\Gamma, E)$ is $B^\bullet := \{\boldsymbol{x} \in \ell_\infty(\Gamma, E) : \boldsymbol{x}(\Gamma) \subset B\}$. Clearly, if $E = (E, \|\cdot\|)$ is an F-normed space, then this topology is defined by the F-norm $\|\cdot\|_\infty$ given by $\|\boldsymbol{x}\|_\infty = \sup_{\gamma\in\Gamma} \|\boldsymbol{x}(\gamma)\|$. The notation B^c and $B^{\bullet c}$ for the complement of B in E, and of $B^\bullet$ in $\ell_\infty(\Gamma, E)$, resp., will also be used in the sequel. $c_0(\Gamma, E)$ is the closed subspace of $\ell_\infty(\Gamma, E)$ consisting of those $\boldsymbol{x}$ that have limit 0 along the filter of cofinite subsets of Γ. If $\boldsymbol{x}, \boldsymbol{y} \in \ell_\infty(\Gamma, E)$ and $\operatorname{supp}(\boldsymbol{x}) \cap \operatorname{supp}(\boldsymbol{y}) = \emptyset$, then $\boldsymbol{x}, \boldsymbol{y}$ are said to be *disjoint*, and this is indicated by writing $\boldsymbol{x} \perp \boldsymbol{y}$. Notice that whenever $\boldsymbol{x}, \boldsymbol{y} \in B^\bullet$ and $\boldsymbol{x} \perp \boldsymbol{y}$, also $\boldsymbol{x} + \boldsymbol{y} \in B^\bullet$. This property will be essential in what follows. If X is a set of functions $\boldsymbol{x} : \Gamma \to E$ and $A \subset \Gamma$, then $X(A) := \{\boldsymbol{x} \in X : \operatorname{supp}(\boldsymbol{x}) \subset A\}$; 1_A stands for the characteristic function of A. If $E = (\mathbb{K}, |\cdot|)$, then the notation for the spaces is simplified to $\ell_\infty(\Gamma)$ and $c_0(\Gamma)$. For $\gamma \in \Gamma$, $e_\gamma = 1_{\{\gamma\}}$ are the *standard unit vectors* in $\ell_\infty(\Gamma)$. In the general case, their role will be taken over by the *primitive vectors* xe_γ ($x \in E$, $\gamma \in \Gamma$). No completeness or convexity assumptions will be imposed; they are simply not needed in this paper. Finishing, we recall that by a well known Aoki-Rolewicz theorem (see [16, Theorem 3.2.1]), the topology of any locally bounded space E may be defined by some p-norm $\|\cdot\|$ $(0 < p \leqslant 1)$. ($\|ax\| = |a|^p\|x\|$ for all $x \in E, a \in \mathbb{K}$). See also [10, Chapter I.2] for a thorough discussion of this and other types of normlike functionals (e.g., quasi-norms) that are often used to define the topology of a locally bounded space, and for the relevant terminology concerning spaces used in such contexts. The author's preference are p-norms. In principle, however, it is a matter of taste and/or convenience (or, sometimes, necessity), whether one works with locally bounded spaces or with, for instance, p-normed spaces. A more general type of vector-valued ℓ_∞-spaces will appear in Theorem 2.4. Additional notation of temporary character will be introduced when needed.

2. New extensions of Rosenthal ℓ_∞-theorem

As the main result (also in view of the versatility of its proof) of the paper I consider the following.

Theorem 2.1. *Let Γ be an infinite set, E a locally bounded* TVS, *F any* TVS, *and X a subspace of $\ell_\infty(\Gamma, E)$ such that for all $(\gamma, x) \in \Gamma \times E$: $xe_\gamma \in X$. Also, let $T : X \to F$ be an operator and assume that*

(R-1) *for some bounded 0-neighborhood B in E there is a 0-neighborhood U in F such that for all $(\gamma, x) \in \Gamma \times B^c : T(xe_\gamma) \notin U$.*

Then there exists a subset A of Γ of cardinality $\mathfrak{m} = |\Gamma|$ such that $T|_{X(A)}$ is an isomorphism.

Proof. The reasoning is an adaptation, with appropriate modifications and a very cautious treatment of the most 'delicate' points, of the proof of Theorem 1.1 given in [3].

Step 1: Denote $Z := X \cap B^\bullet$ and notice the following:

(i) $T(Z)$ is a bounded subset of F.

(ii) $\forall(\gamma, x) \in \Gamma \times B^c : xe_\gamma \in X(\Gamma) \cap B^{\bullet c}$, hence $T(xe_\gamma) \notin U$.

(iii) If $\boldsymbol{x}, \boldsymbol{y} \in Z$ and $\boldsymbol{x} \perp \boldsymbol{y}$, then $\boldsymbol{x} + \boldsymbol{y} \in Z$.

(iv) If $\boldsymbol{x} \in Z, \gamma \in \Gamma$ and $x_\gamma := \boldsymbol{x}(\gamma)$, then $\boldsymbol{x} - x_\gamma e_\gamma \in Z$.

Step 2: Next, fix a base $\mathfrak{U}$ of balanced neighborhoods of zero in F, choose $V \in \mathfrak{U}$ with $V + V \subset U$, then $r \in \mathbb{N}$ such that $T(Z) \subset rV$, and finally $W \in \mathfrak{U}$ for which $W^{(r)} := W + \overset{r)}{\cdots} + W \ \subset rV$. The assertion of the theorem will be reached once we show that:

(A1) There exists a set $A \subset \Gamma$ such that $|A| = |\Gamma| = \mathfrak{m}$ and $T\boldsymbol{x} \notin W$ for all $\boldsymbol{x} \in X(A) \cap B^{\bullet c}$. Beware: The validity of the statement concerning the role of (A1) crucially depends on the local boundedness of $X(A)$ implied by the assumption that E has this property.

Before proceeding, observe that (A1) is equivalent to the following:

(A2) There exists a set $\Delta \subset \Gamma$ with $|\Delta| = \mathfrak{m}$ such that for any $\Delta' \subset \Delta$ with $|\Delta'| = \mathfrak{m}$ one can find $(\alpha, s) \in \Delta' \times B^c$ with the property: $\forall \boldsymbol{x} \in X(\Delta \setminus \Delta') \cap B^{\bullet}$: $T(\boldsymbol{x} + se_\alpha) \notin W$.

(A1) $\Rightarrow$(A2): Simply take $\Delta = A$.

(A2) $\Rightarrow$(A1): Since $\mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$, one can partition Δ into $\mathfrak{m}$ (disjoint) sets C_i ($i \in I$, $|I| = \mathfrak{m}$), each of cardinality $\mathfrak{m}$. According to (A2), for each $i \in I$ one can find $(\gamma_i, s) \in C_i \times B^c$ such that $T(\boldsymbol{x} + se_{\gamma_i}) \notin W$ for all $\boldsymbol{x} \in Z(\Delta \setminus C_i)$. Then $A := \{\gamma_i : i \in I\}$ is as required in (A1).

Now we are ready to show that (A1) or, equivalently, (A2) does indeed hold. Suppose (A2) is false so that:

> non(A2) For every set $\Delta \subset \Gamma$ with $|\Delta| = \mathfrak{m}$ there is $\Delta' \subset \Delta$ with $|\Delta'| = \mathfrak{m}$ and such that $\forall (\alpha, s) \in \Delta' \times B^c$: one can find $\boldsymbol{x} \in Z(\Delta \setminus \Delta')$ with the property: $T(\boldsymbol{x} + se_\alpha) \in W$.

Then, applying non(A2) r times, one arrives at a decreasing sequence of sets $\Gamma = \Delta_0 \supset \Delta_1 \supset \ldots \supset \Delta_r$ such that, for every $k \in [r] = \{1, \ldots, r\}$, $|\Delta_k| = \mathfrak{m}$, and for each $\alpha \in \Delta_k$ and $s \in B^c$ there exists $\boldsymbol{x} \in Z(\Delta_{k-1} \setminus \Delta_k)$ for which $T(\boldsymbol{x} + se_\alpha) = T\boldsymbol{x} + T(se_\alpha) \in W$. Now, fix $\alpha \in \Delta_r$ and $s \in B^c$, and use them in each of the r steps above. This will give for each $k \in [r]$: $\boldsymbol{x}_k \in Z(\Delta_{k-1} \setminus \Delta_k)$ so that $T(\boldsymbol{x}_k + se_\alpha) = T\boldsymbol{x}_k + T(se_\alpha) \in W$. Summation over $k \in [r]$ leads to the following:

$$rT(se_\alpha) + T\biggl(\sum_{k=1}^{r} \boldsymbol{x}_k\biggr) \in W^{(r)} \subset rV.$$

Since the $\boldsymbol{x}_k$'s are pairwise disjoint, $\boldsymbol{x} := \sum_{k=1}^{r} \boldsymbol{x}_k \in Z$, hence $T\boldsymbol{x} \in rV$. It follows that $rT(se_\alpha) \in rV + rV \subset rU$. Hence $T(se_\alpha) \in U$, contradicting condition (R-1). ∎

Remarks 2.2. (i) In condition (R-1) any other bounded neighborhood B' of zero in E can be used as well. In fact, if U is 'good' for B, then since $B \subset kB'$ for some $k \in \mathbb{N}$, $k^{-1}U$ is good for B'.

(ii) The most important examples of subspaces X are $\ell_\infty(\Gamma)$ and $c_0(\Gamma)$ and, of somewhat lesser importance, their further subspaces consisting of elements $\boldsymbol{x}$ with either $|\operatorname{supp}(\boldsymbol{x})| \leqslant \mathfrak{n}$ or $|\operatorname{supp}(\boldsymbol{x})| < \mathfrak{n}$, where $\mathfrak{n}$ is any infinite cardinal number $\leqslant \mathfrak{m}$. Note that for these X, if $A \subset \Gamma$ and $|A| = |\Gamma|$, then $X(A)$ is isomorphic to X.

(iii) The theorem admits a seemingly stronger form, with condition (R-1) requiring that for a bounded 0-neighborhood B in E there is a 0-neighborhood U in E such that the set $\Gamma' := \{\gamma \in \Gamma : \ \forall x \in B^c : T(xe_\gamma) \notin U\}$ is infinite, and the assertion saying that then there exists a subset A of Γ' of the same cardinality as Γ' such that $T|_{X(A)}$ is an isomorphism. In reality, as easily understood, both versions are of the same strength. A similar remark applies to Theorem 2.3 below.

It is obvious that Theorem 2.1 can be equivalently restated for E being a p-normed space $(E, \|\cdot\|)$ $(0 < p \leqslant 1)$, using the ball $B = \{x \in E : \|x\| \leqslant \varepsilon\}$ for some $\varepsilon > 0$ in condition (R-1). We won't do that here. Yet two other variants of Theorem 2.1, formulated in the framework of p-normed spaces are worth explicit stating. In the first (and in its proof), close in spirit to the standard Rosenthal Theorem we will use the following notation. $B := \{x \in E : \|x\| \leqslant 1\}$, $S := \{x \in E : \|x\| = 1\}$, $\|\cdot\|_\infty$ for the associated p-norm in $\ell_\infty(\Gamma, E)$, B_∞ for the closed unit ball, and S_∞ for the unit sphere in $\ell_\infty(\Gamma, E)$, resp. Additionally, we let $\hat{S}_\infty := \{\boldsymbol{x} \in S_\infty : \boldsymbol{x}(\gamma) \in S \text{ for some } \gamma \in \Gamma\}$.

Theorem 2.3. *Let $E = (E, \|\cdot\|)$ be a p-normed space ($0 < p \leqslant 1$), F a* TVS, *and Γ an infinite set. Further, let X be a subspace of the p-normed space $\ell_\infty(\Gamma, E)$ such that $\forall (\gamma, x) \in \Gamma \times E$: $xe_\gamma \in X$. Also, let $T : X \to F$ be an operator such that*

(R-2) *For some 0-neighborhood U in F, $T(se_\gamma) \notin U$ for all $(\gamma, s) \in \Gamma \times S$.*

Then Γ has a subset A of the same cardinality $\mathfrak{m}$ as Γ for which $T|_{X(A)}$ is an isomorphism.

Proof. The result follows from Theorem 2.1. This will be achieved by showing that the present condition (R-2) implies condition (R-1) in that theorem. Fix $\varepsilon > 0$, and take any $x \in E$ with $\lambda = \|x\| > 1 + \varepsilon$. Next, let $y = x/\lambda^{1/p}$. Then for any $\gamma \in \Gamma$: $T(ye_\gamma) \notin U$. Hence $T(xe_\gamma) \notin \lambda^{1/p}U \supset (1+\varepsilon)^{1/p}U =: U'$.

Thus (R-1) in Theorem 2.1 holds for the bounded 0-neighborhood $(1+\varepsilon)B_\infty$ in E and the 0-neighborhood U' in F. To complete the story, let us yet see that also Theorem 2.3 implies Theorem 2.1: Assume that in (R-1) of the latter theorem we want to use $\frac{1}{2}B$ in place of B (cf. Remark 2.2(1)). So take any $\gamma \in \Gamma$ and $x \in E$ with $\lambda = \|x\| > \frac{1}{2}$. Set $y = x/\lambda^{1/p}$. Then $\|y\| = 1$, so that $T(ye_\gamma) \notin U$ (U as in (R-2)). Hence $T(xe_\gamma) \notin \lambda^{1/p}U \supset \frac{1}{2^p}U =: U'$. Thus $T(xe_\gamma) \notin U'$ and (R-2) is satisfied with U' in place of U. ∎

It may be of some interest to have an independent and direct proof of the last result. Here it goes.

Another proof. Proceed precisely as in the proof of Theorem 2.1, replacing $B^\bullet$ and $B^{\bullet c}$ by B_∞ and $\hat{S}_\infty$, keeping Steps 1 and 2 unchanged, and then replacing conditions (A1), (A2) by the following two, resp.

(B1) There exists a set $A \subset \Gamma$ such that $|A| = |\Gamma| = \mathfrak{m}$ and $T\boldsymbol{x} \notin W$ for all $\boldsymbol{x} \in Z(A) \cap \hat{S}_\infty$.

(B2) There exists a set $\Delta \subset \Gamma$ with $|\Delta| = \mathfrak{m}$ such that for any $\Delta' \subset \Delta$ with $\Delta'| = \mathfrak{m}$ one can find $(\alpha, s) \in \Delta' \times S$ with the property: $\forall \boldsymbol{x} \in Z(\Delta \setminus \Delta')$: $T(\boldsymbol{x} + se_\alpha) \notin W$, verifying that they are equivalent, and after that showing that under the assumptions of the theorem (B1) does indeed hold. Having done that select $W' \in \mathcal{U}$ with $W' + W' \subset W$, take the smallest $m \in \mathbb{N}$ with $m \geqslant 1/p$, and next select $0 < \varepsilon < 1$ so that for $\eta := m^p\varepsilon^p$ there holds $T((\eta B_\infty) \cap X) \subset W'$ (by the continuity of T), and then proceed to verify that

($\widehat{\mathrm{B}}1$) For the set A from (B1) one has: $T\boldsymbol{x} \notin W'$ for all $\boldsymbol{x} \in Z(A) \cap S_\infty$.

So, take any $\boldsymbol{x} \in Z(A) \cap S_\infty$, and then choose $\alpha \in A$ so that writing $x_\alpha = \boldsymbol{x}(\alpha)$ and $\lambda = \|x_\alpha\|$, $\lambda > 1 - \varepsilon$. Then let $y_\alpha = x_\alpha/\lambda^{1/p}$, and define $\boldsymbol{y} := \boldsymbol{x} - x_\alpha e_\alpha + y_\alpha e_\alpha$. Clearly, $\boldsymbol{y} \in Z(A) \cap \hat{S}_\infty$ and hence $T\boldsymbol{y} \notin W$. Since $\boldsymbol{z} := \boldsymbol{x} - \boldsymbol{y} = (x_\alpha - y_\alpha)e_\alpha \in X(A)$,

$$\begin{aligned}\|\boldsymbol{z}\|_\infty = \|x_\alpha - y_\alpha\| &= (1-\lambda^{1/p})^p \leqslant (1-\lambda^m)^p \\ &= [(1-\lambda)(1+\lambda+\cdots+\lambda^{m-1})]^p \\ &< m^p\varepsilon^p = \eta.\end{aligned}$$

It follows that $T\boldsymbol{z} \in W'$. Since $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{z}$ and $T\boldsymbol{y} \notin W$ while $T\boldsymbol{z} \in W'$, we conclude that $T\boldsymbol{x} \notin W'$. To finish, take any $\boldsymbol{x} \in X(A)$ with $\lambda := \|\boldsymbol{x}\|_\infty > 1$. Then $\boldsymbol{y} := \lambda^{-1/p}\boldsymbol{x} \in S_\infty$. By ($\widehat{\mathrm{B}}1$), $T\boldsymbol{y} = \lambda^{-1/p}T\boldsymbol{x} \notin W'$. Hence $T\boldsymbol{x} \notin \lambda^{1/p}W'$, and $T\boldsymbol{x} \notin W'$, too. From this the assertion follows. ∎

The second promised variant, suggested by the referee, deals with "multi-space" type ℓ_∞-spaces.

THEOREM 2.4. *Let Γ be an infinite set and, for a fixed $p \in (0,1]$, let $\boldsymbol{E}_\Gamma = (E_\gamma : \gamma \in \Gamma)$ be a family of p-normed spaces, each E_γ with its own p-norm $\|\cdot\|_\gamma$. Consider the vector subspace $\ell_\infty(\boldsymbol{E}_\Gamma)$ of the product $\prod_{\gamma\in\Gamma} E_\gamma$ consisting of elements $\boldsymbol{x} = (x_\gamma : \gamma \in \Gamma)$ (with x_γ written often as $\boldsymbol{x}(\gamma)$) such that $\|\boldsymbol{x}\|_\infty := \sup_\gamma \|x_\gamma\|_\gamma < \infty$, and equip it with the p-norm $\|\cdot\|_\infty$ defined by this equality. Next, let X be a subspace of $\ell_\infty(\boldsymbol{E}_\Gamma)$ such that $xe_\gamma \in X$ for all $\gamma \in \Gamma$ and all $x \in E_\gamma$. Further, let F be any* TVS, *$T : X \to F$ be an operator, and assume that*

(R-3) *for some $\varepsilon > 0$ there is a 0-neighborhood U in F such that $T(xe_\gamma) \notin U$ for all $\gamma \in \Gamma$ and all $x \in E_\gamma$ with $\|x\|_\gamma > \varepsilon$.*

Then there exists a subset A of Γ of cardinality $\mathfrak{m} = |\Gamma|$ such that $T|_{X(A)}$ is an isomorphism.

Proof. Again, we argue as in the proof of Theorem 2.1, with necessary changes. To avoid any ambiguity, all details are given. We first fix the additional notation to be used in the present proof. Thus, we denote $B(E_\gamma) =: \{x \in E_\gamma : \|x\|_\gamma \leqslant \varepsilon\}$, $B^c(E_\gamma) =: \{x \in E_\gamma : \|x\|_\gamma\| > \varepsilon\}$, $B^\bullet =: \{\boldsymbol{x} \in \ell_\infty(\boldsymbol{E}_\Gamma) : \|\boldsymbol{x}\|_\infty \leqslant \varepsilon\}$, $B^{\bullet c} =: \{\boldsymbol{x} \in \ell_\infty(\boldsymbol{E}_\Gamma) : \|\boldsymbol{x}\|_\infty > \varepsilon\}$. Next, keep *Steps 1* and *2* unchanged, and replace conditions (A1), (A2) by the following two, resp.

(C1) There exists a set $A \subset \Gamma$ such that $|A| = |\Gamma| = \mathfrak{m}$ and $T\boldsymbol{x} \notin W$ for all $\boldsymbol{x} \in X(A) \cap B^{\bullet c}$.

(C2) There exists a set $\Delta \subset \Gamma$ with $|\Delta| = \mathfrak{m}$ such that for any $\Delta' \subset \Delta$ with $|\Delta'| = \mathfrak{m}$ one can find $(\alpha, s) \in \Delta' \times B^c(E_\alpha)$ with the property: $\forall \boldsymbol{x} \in X(\Delta \setminus \Delta') \cap B^\bullet$: $T(\boldsymbol{x} + se_\alpha) \notin W$.

Then we show that (C1) $\Longleftrightarrow$ (C2).

(C1) $\Rightarrow$(C2): Simply take $\Delta = A$.

(C2) $\Rightarrow$(C1): Since $\mathfrak{m}\cdot\mathfrak{m} = \mathfrak{m}$, one can partition Δ into $\mathfrak{m}$ (disjoint) sets C_i ($i \in I$, $|I| = \mathfrak{m}$), each of cardinality $\mathfrak{m}$. According to (C2), for each $i \in I$ one can find $(\gamma_i, s) \in C_i \times B^c(E_{\gamma_i})$ such that $T(\boldsymbol{x}+se_{\gamma_i}) \notin W$ for all $\boldsymbol{x} \in Z(\Delta \setminus C_i)$. Then $A := \{\gamma_i : i \in I\}$ is as required in (C1).

Then we go on to show that (C1) or, equivalently, (C2) does indeed hold. Suppose (C2) is false so that:

non(C2) For every set $\Delta \subset \Gamma$ with $|\Delta| = \mathfrak{m}$ there is $\Delta' \subset \Delta$ with $|\Delta'| = \mathfrak{m}$ and such that $\forall(\alpha, s) \in \Delta' \times B^c(E_\alpha)$: one can find $\boldsymbol{x} \in Z(\Delta \setminus \Delta')$ with the property: $T(\boldsymbol{x} + se_\alpha) \in W$.

Then, applying non(C2) r times, one arrives at a decreasing sequence of sets $\Gamma = \Delta_0 \supset \Delta_1 \supset \ldots \supset \Delta_r$ such that, for every $k \in [r] = \{1, \ldots, r\}$, $|\Delta_k| = \mathfrak{m}$, and for each $\alpha \in \Delta_k$ and $s \in B^c(E_\alpha)$ there exists $\boldsymbol{x} \in Z(\Delta_{k-1} \setminus \Delta_k)$ for which $T(\boldsymbol{x} + se_\alpha) = T\boldsymbol{x} + T(se_\alpha) \in W$. Now, fix $\alpha \in \Delta_r$ and $s \in B^c(E_\alpha)$, and use them in each of the r steps above. This will give for each $k \in [r]$: $\boldsymbol{x}_k \in Z(\Delta_{k-1} \setminus \Delta_k)$ so that $T(\boldsymbol{x}_k + se_\alpha) = T\boldsymbol{x}_k + T(se_\alpha) \in W$. Summation over $k \in [r]$ leads to the following:

$$rT(se_\alpha) + T\left(\sum_{k=1}^{r} \boldsymbol{x}_k\right) \in W^{(r)} \subset rV.$$

Since the $\boldsymbol{x}_k$'s are pairwise disjoint, $\boldsymbol{x} := \sum_{k=1}^{r} \boldsymbol{x}_k \in Z$, hence $T\boldsymbol{x} \in rV$. It follows that $rT(se_\alpha) \in rV + rV \subset rU$. Hence $T(se_\alpha) \in U$, contradicting condition (R-3). ∎

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