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Extreme and exposed points of $\mathcal{L}({}^n l_\infty^2)$ and $\mathcal{L}_s({}^n l_\infty^2)$

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Abstract: For every $n \geq 2$ this paper is devoted to the description of the sets of extreme and exposed points of the closed unit balls of $\mathcal{L}({}^n l_\infty^2)$ and $\mathcal{L}_s({}^n l_\infty^2)$, where $\mathcal{L}({}^n l_\infty^2)$ is the space of n-linear forms on $\mathbb{R}^2$ with the supremum norm, and $\mathcal{L}_s({}^n l_\infty^2)$ is the subspace of $\mathcal{L}({}^n l_\infty^2)$ consisting of symmetric n-linear forms. First we classify the extreme points of the closed unit balls of $\mathcal{L}({}^n l_\infty^2)$ and $\mathcal{L}_s({}^n l_\infty^2)$, correspondingly. As corollaries we obtain $|\operatorname{ext} B_{\mathcal{L}({}^n l_\infty^2)}| = 2^{(2^n)}$ and $|\operatorname{ext} B_{\mathcal{L}_s({}^n l_\infty^2)}| = 2^{n+1}$. We also show that $\exp B_{\mathcal{L}({}^n l_\infty^2)} = \operatorname{ext} B_{\mathcal{L}({}^n l_\infty^2)}$ and $\exp B_{\mathcal{L}_s({}^n l_\infty^2)} = \operatorname{ext} B_{\mathcal{L}_s({}^n l_\infty^2)}$.

Key words: n-linear forms, symmetric n-linear forms, extreme points, exposed points.

AMS *Subject Class.* (2010): 46A22.

1. Introduction

Let $n \in \mathbb{N}, n \geq 2$. We write B_E for the unit ball of a real Banach space E and the dual space of E is denoted by E^*. An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. We denote by $\operatorname{ext} B_E$ the set of all the extreme points of B_E. An element $x \in B_E$ is called an *exposed point* of B_E if there is a $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $\exp B_E$ the set of exposed points of B_E. We denote by $\mathcal{L}({}^n E)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \cdots, x_n)|$. $\mathcal{L}_s({}^n E)$ denote the closed subspace of all continuous symmetric n-linear forms on E.

Let us say about the history of the classifications of extreme and exposed points of the unit ball of continuous (symmetric) multilinear forms on a Banach space. Kim [1] initiated and classified $\operatorname{ext} B_{\mathcal{L}_s({}^2 l_\infty^2)}$ and $\exp B_{\mathcal{L}_s({}^2 l_\infty^2)}$, where $l_\infty^n = \mathbb{R}^n$ with the supremum norm. It was shown that $\operatorname{ext} B_{\mathcal{L}_s({}^2 l_\infty^2)} = \exp B_{\mathcal{L}_s({}^2 l_\infty^2)}$. Kim [2, 3, 4, 5] classified $\operatorname{ext} B_{\mathcal{L}_s({}^2 d_*(1,w)^2)}$, $\operatorname{ext} B_{\mathcal{L}({}^2 d_*(1,w)^2)}$, $\exp B_{\mathcal{L}_s({}^2 d_*(1,w)^2)}$, and $\exp B_{\mathcal{L}({}^2 d_*(1,w)^2)}$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm $\|(x,y)\|_w = \max\left\{|x|, |y|, \frac{|x|+|y|}{1+w}\right\}$. Kim [6, 7] classified $\operatorname{ext} B_{\mathcal{L}_s({}^2 \mathbb{R}^2_{h(w)})}$

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and ext $B_{\mathcal{L}(^2\mathbb{R}^2_{h(w)})}$, where where $\mathbb{R}^2_{h(w)} = \mathbb{R}^2$ with the hexagonal norm $\|(x,y)\|_{h(w)} = \max\{|y|, |x| + (1-w)|y|\}$. Kim [8, 9, 10] classified ext $B_{\mathcal{L}_s(^2l^3_\infty)}$, ext $B_{\mathcal{L}_s(^3l^2_\infty)}$ and ext $B_{\mathcal{L}(^3l^2_\infty)}$. It was shown that every extreme point is exposed in each space. Kim [11] characterized ext $B_{\mathcal{L}(^2l^n_\infty)}$ and ext $B_{\mathcal{L}_s(^2l^n_\infty)}$. Recently, Kim [12] classified ext $B_{\mathcal{L}(^2l^3_\infty)}$ and showed exp $B_{\mathcal{L}(^2l^3_\infty)} = \text{ext}\, B_{\mathcal{L}(^2l^3_\infty)}$.

2. The extreme and exposed points of the unit ball of $\mathcal{L}(^n l^2_\infty)$

Let $l^2_\infty = \{(x,y) \in \mathbb{R}^2 : \|(x,y)\|_\infty = \max(|x|,|y|)\}$. For $n \geq 2$, we denote

$$\mathcal{W}_n := \{[(1,w_1),\ldots,(1,w_n)] : w_j = \pm 1 \text{ for } j = 1,\ldots,n\}.$$

Note that $\mathcal{W}_n$ has 2^n elements in $S_{l^2_\infty} \times \cdots \times S_{l^2_\infty}$.

Recall that the Krein-Milman Theorem [13] say that every nonempty compact convex subset of a Housdorff locally convex space is the closed convex hull of its set of extreme points. Hence, the unit ball of l^2_∞ is the closed convex hull of

$$\{(1,1),(-1,1),(1,-1),(-1,-1)\}.$$

Theorem 2.1. *Let $n \geq 2$ and $T \in \mathcal{L}(^n l^2_\infty)$. Then,*

$$\|T\| = \sup_{W \in \mathcal{W}_n} |T(W)|.$$

Proof. It follows that from the Krein-Milman theorem and multilinearity of T. ∎

Let $Z_1,\ldots,Z_{2^n}$ be an ordering of the monomials $x_{l_1}\cdots x_{l_j} y_{k_1}\cdots y_{k_{n-j}}$ with $\{l_1,\cdots,l_j,k_1,\cdots,k_{n-j}\} = \{1,\cdots,n\}$. Note that $\{Z_1,\ldots,Z_{2^n}\}$ is a basis for $\mathcal{L}(^n l^2_\infty)$. Hence, $\dim(\mathcal{L}(^n l^2_\infty)) = 2^n$. If $T \in \mathcal{L}(^n l^2_\infty)$, then,

$$T = \sum_{k=1}^{2^n} a_k Z_k$$

for some $a_1,\ldots,a_{2^n} \in \mathbb{R}$. By simplicity, we denote $T = (a_1,\cdots,a_{2^n})^t$. Let $W_1,\ldots,W_{2^n}$ be an ordering of the elements of $\mathcal{W}_n$. Let

$$M(Z_1,\ldots,Z_{2^n}; W_1,\ldots,W_{2^n}) = [Z_i(W_j)]$$

be the $2^n \times 2^n$ matrix. Note that, for every $T \in \mathcal{L}(^n l^2_\infty)$,

$$M(Z_1,\ldots,Z_{2^n}; W_1,\ldots,W_{2^n})T = (T(W_1),\ldots,T(W_{2^n}))^t.$$

Here, $(\epsilon_1,\ldots,\epsilon_{2^n})^t$ denote the transpose of $(\epsilon_1,\ldots,\epsilon_{2^n})$.

THEOREM 2.2. *Let $n \geq 2$. Then,*

$$\operatorname{ext} B_{\mathcal{L}(^n l_\infty^2)} = \big\{ M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})^{-1}(\epsilon_1, \ldots, \epsilon_{2^n})^t \\ : \epsilon_j = \pm 1,\ j = 1, \ldots, 2^n \big\}.$$

Proof. CLAIM 1: $M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})$ is invertible.

Consider the equation

$$M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})(t_1, \ldots, t_{2^n})^t = (0, \ldots, 0)^t. \qquad (*)$$

Let $a_1, \cdots, a_{2^n}$ be a solution of (*) and let $T = \sum_{k=1}^{2^n} a_k Z_k \in \mathcal{L}(^n l_\infty^2)$. Then,

$$T(W_j) = 0 \qquad j = 1, \ldots, 2^n.$$

By Theorem 2.1, $\|T\| = 0$, hence $T = 0$. Since $Z_1, \ldots, Z_{2^n}$ are linearly independent in $\mathcal{L}(^n l_\infty^2)$, we have $a_j = 0$ for all $j = 1, \ldots, 2^n$. Hence, the equation (*) has only zero solution. Therefore, $M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})$ is invertible.

CLAIM 2: $M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})^{-1}(\epsilon_1, \ldots, \epsilon_{2^n})^t$ is an extreme point for $\epsilon_j = \pm 1, (j = 1, \ldots, 2^n)$.

Let

$$T := M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})^{-1}(\epsilon_1, \ldots, \epsilon_{2^n})^t.$$

Since

$$M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})T = (\epsilon_1, \ldots, \epsilon_{2^n})^t,$$

$T(W_j) = \epsilon_j$ for $j = 1, \ldots, 2^n$. By Theorem 2.1,

$$\|T\| = \sup_{1 \leq j \leq 2^n} |T(W_j)| = \sup_{1 \leq j \leq 2^n} |\epsilon_j| = 1.$$

Suppose that $T = \frac{1}{2}(T_1 + T_2)$ for some $T_k \in B_{\mathcal{L}(^n l_\infty^2)}$ $(k = 1, 2)$. We may write

$$T_1 = M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})^{-1}(\epsilon_1, \ldots, \epsilon_{2^n})^t + (\delta_1, \ldots, \delta_{2^n})^t$$

and

$$T_2 = M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})^{-1}(\epsilon_1, \ldots, \epsilon_{2^n})^t - (\delta_1, \ldots, \delta_{2^n})^t$$

for some $\delta_j \in \mathbb{R}$ $(j = 1, \ldots, 2^n)$. Note that

$$(T_k(W_1), \ldots, T_k(W_{2^n}))^t = M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})T_k \qquad \text{for } k = 1, 2.$$

Therefore,

$$(T_1(W_1), \dots, T_1(W_{2^n}))^t = (\epsilon_1, \dots, \epsilon_{2^n})^t + M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})(\delta_1, \dots, \delta_{2^n})^t$$

and

$$(T_2(W_1), \dots, T_2(W_{2^n}))^t = (\epsilon_1, \dots, \epsilon_{2^n})^t - M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})(\delta_1, \dots, \delta_{2^n})^t.$$

Hence, for $j = 1, \dots, 2^n$,

$$T_1(W_j) = \epsilon_j + (Z_1(W_j), \dots, Z_{2^n}(W_j))(\delta_1, \dots, \delta_{2^n})^t,$$

and

$$T_2(W_j) = \epsilon_j - (Z_1(W_j), \dots, Z_{2^n}(W_j))(\delta_1, \dots, \delta_{2^n})^t.$$

It follows that, for $j = 1, \dots, 2^n$,

$$\begin{aligned} 1 &\geq \max\{|T_1(W_j)|, |T_2(W_j)|\} \\ &= |\epsilon_j| + |(Z_1(W_j), \dots, Z_{2^n}(W_j))(\delta_1, \dots, \delta_{2^n})^t| \\ &= 1 + |(Z_1(W_j), \dots, Z_{2^n}(W_j))(\delta_1, \dots, \delta_{2^n})^t|, \end{aligned}$$

which shows that

$$(Z_1(W_j), \dots, Z_{2^n}(W_j))(\delta_1, \dots, \delta_{2^n})^t = 0 \qquad \text{for } j = 1, \dots, 2^n.$$

Hence,

$$M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})(\delta_1, \dots, \delta_{2^n})^t = 0\,.$$

Therefore,

$$\begin{aligned} (\delta_1, \dots, \delta_{2^n})^t &= M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1}(0, \dots, 0)^t \\ &= (0, \dots, 0)^t. \end{aligned}$$

Hence, $T_k = T$ for $k = 1, 2$. Therefore, T is extreme.

Suppose that $T \in \operatorname{ext} B_{\mathcal{L}(^n l_\infty^2)}$. Note that

$$(T(W_1), \dots, T(W_{2^n}))^t = M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})T.$$

CLAIM 3: $|T(W_j)| = 1$ for all $j = 1, \ldots, 2^n$.

If not. There exists $1 \leq j_0 \leq 2^n$ such that $|T(W_{j_0})| < 1$. Let $\delta_0 > 0$ such that $|T(W_{j_0})| + \delta_0 < 1$. Let

$$\begin{aligned} T_1 = M&(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})^{-1} \\ &\times (T(W_1), \ldots, T(W_{j_0-1}), T(W_{j_0}) + \delta_0, T(W_{j_0+1}), \ldots, T(W_{2^n}))^t \end{aligned}$$

and

$$\begin{aligned} T_2 = M&(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})^{-1} \\ &\times (T(W_1), \ldots, T(W_{j_0-1}), T(W_{j_0}) - \delta_0, T(W_{j_0+1}), \ldots, T(W_{2^n}))^t. \end{aligned}$$

Hence,

$$\begin{aligned} T_1(W_{j_0}) &= T(W_{j_0}) + \delta_0, T_2(W_{j_0}) \\ &= T(W_{j_0}) - \delta_0, T_1(W_j) = T_2(W_j) = T(W_j) \qquad (j \neq j_0). \end{aligned}$$

Obviously, $T \neq T_k$ for $k = 1, 2$. By Theorem 2.1, $\|T_k\| = 1$ for $k = 1, 2$ and $T = \frac{1}{2}(T_1 + T_2)$, which is a contradiction. Therefore,

$$T = M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})^{-1}(T(W_1), \ldots, T(W_{2^n}))^t$$

with $|T(W_j)| = 1$ for all $j = 1, \ldots, 2^n$. ∎

Kim [10] characterized $\operatorname{ext} B_{\mathcal{L}(^3 l_\infty^2)}$. Notice that using Wolfram Mathematica 8 and Theorem 2.2, we can exclusively describe $\operatorname{ext} B_{\mathcal{L}(^n l_\infty^2)}$ for a given $n \geq 2$.

For every $T \in \mathcal{L}(^n l_\infty^2)$, we let

$$\operatorname{Norm}(\mathrm{T}) := \big\{[(1, w_1), \ldots, (1, w_n)] \in \mathcal{W}_n \,:\, |T((1, w_1), \ldots, (1, w_n))| = \|T\|\big\}.$$

We call Norm(T) the set of *the norming points* of T.

COROLLARY 2.3. (a) *Let $n \geq 2$. $\operatorname{ext} B_{\mathcal{L}(^n l_\infty^2)}$ has exactly $2^{(2^n)}$ elements.*

(b) *Let $n \geq 2$ and $T \in \mathcal{L}(^n l_\infty^2)$ with $\|T\| = 1$. Then $T \in \operatorname{ext} B_{\mathcal{L}(^n l_\infty^2)}$ if and only if Norm(T) $= \mathcal{W}_n$.*

THEOREM 2.4. ([4]) *Let E be a real Banach space such that $\operatorname{ext} B_E$ is finite. Suppose that $x \in \operatorname{ext} B_E$ satisfies that there exists an $f \in E^*$ with $f(x) = 1 = \|f\|$ and $|f(y)| < 1$ for every $y \in \operatorname{ext} B_E \backslash \{\pm x\}$. Then $x \in \exp B_E$.*

THEOREM 2.5. *Let $n \geq 2$. Then,* $\exp B_{\mathcal{L}(^n l_\infty^2)} = \operatorname{ext} B_{\mathcal{L}(^n l_\infty^2)}$.

Proof. Let $T \in \operatorname{ext} B_{\mathcal{L}(^n l_\infty^2)}$ and let

$$f := \frac{1}{2^n} \sum_{1 \leq j \leq 2^n} \operatorname{sign}(T(W_j))\delta_{W_j} \in \mathcal{L}(^n l_\infty^2)^*.$$

Note that $1 = \|f\| = f(T)$. Let $S \in \operatorname{ext} B_{\mathcal{L}(^n l_\infty^2)}$ be such that $|f(S)| = 1$. We will show that $S = T$ or $S = -T$. It follows that

$$\begin{aligned} 1 = |f(S)| &= \Big|\frac{1}{2^n} \sum_{1 \leq j \leq 2^n} \operatorname{sign}(T(W_j))S(W_j)\Big| \\ &\leq \frac{1}{2^n} \sum_{1 \leq j \leq 2^n} |S(W_j)| \\ &\leq 1, \end{aligned}$$

which shows that

$$S(W_j) = \operatorname{sign}(T(W_j)) \qquad (1 \leq j \leq 2^n)$$

or

$$S(W_j) = -\operatorname{sign}(T(W_j)) \qquad (1 \leq j \leq 2^n).$$

Suppose that

$$S(W_j) = -\operatorname{sign}(T(W_j)) \qquad (1 \leq j \leq 2^n).$$

It follows that

$$\begin{aligned} S &= M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})^{-1}(S(W_1), \ldots, S(W_{2^n}))^t \\ &= M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})^{-1}(-\operatorname{sign}(T(W_1)), \ldots, -\operatorname{sign}(T(W_{2^n})))^t \\ &= M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})^{-1}(-T(W_1), \ldots, -T(W_{2^n}))^t \\ &= -T. \end{aligned}$$

Note that if $S(W_j) = \operatorname{sign}(T(W_j))$ $(1 \leq j \leq 2^n)$, then $S = T$. By Theorem 2.4, T is exposed. ∎

3. The extreme and exposed points of the unit ball of $\mathcal{L}_s(^n l^2_\infty)$

Let $n \geq 2$ and

$$\begin{aligned}\mathcal{U}_n := \big\{ & [(1,1),(1,1),\ldots,(1,1)], [(1,-1),(1,1),\ldots,(1,1)],\\ & [(1,-1),(1,-1),(1,1),\ldots,(1,1)],\\ & [(1,-1),(1,-1),(1-1),(1,1),\ldots,(1,1)],\\ & \ldots,[(1,-1),(1,-1),\ldots,(1,-1),(1,1)],\\ & [(1,-1),(1,-1),\ldots,(1,-1),(1,-1)]\big\}.\end{aligned}$$

Note that $\mathcal{U}_n$ has $n+1$ elements in $S_{l^2_\infty} \times \cdots \times S_{l^2_\infty}$.

Theorem 3.1. *Let $n \geq 2$ and $T \in \mathcal{L}_s(^n l^2_\infty)$. Then,*

$$\|T\| = \sup_{U \in \mathcal{U}_n} |T(U)|.$$

Proof. It follows that from Theorem 2.1 and symmetry of T. ∎

For $j = 0, \ldots, n$, we let

$$F_j = \sum_{\{l_1,\cdots,l_j,k_1,\cdots,k_{n-j}\}=\{1,\cdots,n\}} x_{l_1}\cdots x_{l_j} y_{k_1}\cdots y_{k_{n-j}}.$$

Then, $\{F_0, \ldots, F_n\}$ is a basis for $\mathcal{L}_s(^n l^2_\infty)$. Hence, $\dim(\mathcal{L}_s(^n l^2_\infty)) = n+1$. If $T \in \mathcal{L}_s(^n l^2_\infty)$, then,

$$T = \sum_{j=0}^{n} b_j F_j$$

for some $b_0, \ldots, b_n \in \mathbb{R}$. By simplicity, we denote $T = (b_0, \cdots, b_n)^t$. For $j = 0, \ldots, n$, we let

$$U_j = [(1,u_1), \ldots, (1,u_n)] \in \mathcal{U}_n,$$

where $u_k = -1$ for $1 \leq k \leq j$ and $u_k = 1$ for $j+1 \leq k \leq n$. Let

$$M(F_0, \ldots, F_n; U_0, \ldots, U_n) = [F_i(U_j)]$$

be the $(n+1) \times (n+1)$ matrix. Note that, for every $T \in \mathcal{L}_s(^n l^2_\infty)$,

$$M(F_0, \ldots, F_n; U_0, \ldots, U_n)T = (T(U_0), \ldots, T(U_n))^t.$$

By analogous arguments in the claim 1 of Theorem 2.2, $M(F_0, \ldots, F_n; U_0, \ldots, U_n)$ is invertible.

THEOREM 3.2. *Let $n \geq 2$. Then,*

$$\operatorname{ext} B_{\mathcal{L}_s(^n l_\infty^2)} = \big\{ M(F_0, \ldots, F_n; U_0, \ldots, U_n)^{-1} (\epsilon_0, \ldots, \epsilon_n)^t \\ : \epsilon_j = \pm 1, j = 0, \ldots, n \big\}.$$

Proof. It follows by Theorem 3.1 and analogous arguments in the claims 2 and 3 of Theorem 2.2. ∎

Notice that using Wolfram Mathematica 8 and Theorem 3.2, we can exclusively describe $\operatorname{ext} B_{\mathcal{L}_s(^n l_\infty^2)}$ for a given $n \geq 2$.

For every $T \in \mathcal{L}_s(^n l_\infty^2)$, we let

$$\text{Norm(T)} := \big\{ [(1, u_1), \ldots, (1, u_n)] \in \mathcal{U}_n \; : \; |T((1, u_1), \ldots, (1, u_n))| = \|T\| \big\}.$$

We call Norm(T) the set of *the norming points* of T.

COROLLARY 3.3. (a) *Let $n \geq 2$. $\operatorname{ext} B_{\mathcal{L}_s(^n l_\infty^2)}$ has exactly 2^{n+1} elements.*

(b) *Let $n \geq 2$ and $T \in \mathcal{L}_s(^n l_\infty^2)$ with $\|T\| = 1$. Then $T \in \operatorname{ext} B_{\mathcal{L}_s(^n l_\infty^2)}$ if and only if Norm(T) $= \mathcal{U}_n$.*

THEOREM 3.4. *Let $n \geq 2$. Then,* $\exp B_{\mathcal{L}_s(^n l_\infty^2)} = \operatorname{ext} B_{\mathcal{L}_s(^n l_\infty^2)}$.

Proof. Let $T \in \operatorname{ext} B_{\mathcal{L}_s(^n l_\infty^2)}$ and let

$$f := \frac{1}{n+1} \sum_{0 \leq j \leq n} \operatorname{sign}(T(U_j)) \delta_{U_j} \in \mathcal{L}_s(^n l_\infty^2)^*.$$

Note that $1 = \|f\| = f(T)$. By analogous arguments in the proof of Theorem 2.5, f exposes T. Therefore, T is exposed. ∎

QUESTIONS. (a) Let $n \geq 2$ and $\epsilon_1, \ldots, \epsilon_{2^n}$ be fixed with $\epsilon_j = \pm 1, (j = 1, \ldots, 2^n)$. Is it true that

$$\operatorname{ext} B_{\mathcal{L}(^n l_\infty^2)} = \big\{ M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})^{-1} (\epsilon_1, \ldots, \epsilon_{2^n})^t \\ : Z_1, \ldots, Z_{2^n}, W_1, \ldots, W_{2^n} \text{ are any ordering} \big\}?$$

(b) By Theorem 2.2, $M(Z_1, \ldots, Z_{2^n}; W_1, \ldots, W_{2^n})^{-1} (\epsilon_1, \ldots, \epsilon_{2^n})^t$ is extreme if $Z_1, \ldots, Z_{2^n}$, $W_1, \ldots, W_{2^n}$ are any ordering. Similarly, we may ask the following: Let $n \geq 2$ and $\delta_0, \ldots, \delta_n$ be fixed with $\delta_k = \pm 1$, $(k = 0, \ldots, n)$. Is it true that

$$\operatorname{ext} B_{\mathcal{L}_s(^n l_\infty^2)} = \big\{ M(F_0, \ldots, F_n; U_0, \ldots, U_n)^{-1} (\delta_0, \ldots, \delta_n)^t \\ : F_0, \ldots, F_n, U_0, \ldots, U_n \text{ are any ordering} \big\}?$$

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