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On $H_3(1)$ Hankel determinant for certain subclass of analytic functions

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Abstract: The objective of this paper is to obtain an upper bound to Hankel determinant of third order for any function f, when it belongs to certain subclass of analytic functions, defined on the open unit disc in the complex plane.

Key words: Analytic function, upper bound, third Hankel determinant, positive real function.

AMS *Subject Class.* (2010): 30C45, 30C50.

1. Introduction

Let A denotes the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture also called as Coefficient conjecture, which states that for a univalent function its n^{th}- Taylor's coefficient is bounded by n (see [4]). The bounds for the coefficients of these functions give information about their geometric properties. For example, the n^{th}-coefficient gives information about the area where as the second coefficient of functions in the family S yields the growth and distortion properties of the function. A typical problem in geometric function theory is to study a functional made up of combinations of the coefficients of the original function. The Hankel determinant of f for

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$q \geq 1$ and $n \geq 1$ was defined by Pommerenke [20], which has been investigated by many authors, as follows.

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

It is worth of citing some of them. Ehrenborg [7] studied the Hankel determinant of exponential polynomials. Noor [18] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with bounded boundary rotation. The Hankel transform of an integer sequence and some of its properties were discussed by Layman (see [13]). It is observed that $H_2(1)$, the Fekete-Szegö functional is the classical problem settled by Fekete-Szegö [8] is to find for each $\lambda \in [0, 1]$, the maximum value of the coefficient functional, defined by $\phi_\lambda(f) := |a_3 - \lambda a_2^2|$ over the class S and was proved by using Loewner method. Ali [1] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ when $f^{-1} \in \widetilde{ST}(\alpha)$, the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$. In recent years, the research on Hankel determinants has focused on the estimation of $|H_2(2)|$, where

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2,$$

known as the second Hankel determinant obtained for $q = 2$ and $n = 2$ in (1.2). Many authors obtained an upper bound to the functional $|a_2 a_4 - a_3^2|$ for various subclasses of univalent and multivalent analytic functions. It is worth citing a few of them. The exact (sharp) estimates of $|H_2(2)|$ for the subclasses of S namely, bounded turning, starlike and convex functions denoted by $\mathcal{R}$, S^* and $\mathcal{K}$ respectively in the open unit disc E, that is, functions satisfying the conditions $\text{Re} f'(z) > 0$, $\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ and $\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$ were proved by Janteng et al. [11, 10] and determined the bounds as 4/9, 1 and 1/8 respectively. For the class $S^*(\psi)$ of Ma-Minda starlike functions, the exact bound of the second Hankel determinant was obtained by Lee et al. [15]. Choosing $q = 2$ and $n = p + 1$ in (1.2), we obtain the second Hankel determinant for the p-valent function (see [24]), as follows.

$$H_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1} a_{p+3} - a_{p+2}^2,$$

The case $q = 3$ appears to be much more difficult than the case $q = 2$. Very few papers have been devoted to the third order Hankel determinant denoted by $H_3(1)$, obtained for $q = 3$ and $n = 1$ in (1.2), also called as Hankel determinant of third kind, namely

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} (a_1 = 1).$$

Expanding the determinant, we have

$$H_3(1) = a_1(a_3a_5 - a_4^2) + a_2(a_3a_4 - a_2a_5) + a_3(a_2a_4 - a_3^2), \tag{1.3}$$

equivalently

$$H_3(1) = H_2(3) + a_2J_2 + a_3H_2(2),$$

where $J_2 = (a_3a_4 - a_2a_5)$ and $H_2(3) = (a_3a_5 - a_4^2)$.

Babalola [2] is the first one, who tried to estimate an upper bound for $|H_3(1)|$ for the classes $\mathcal{R}$, S^* and $\mathcal{K}$. As a result of this paper, Raza and Malik [22] obtained an upper bound to the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli. Sudharsan et al. [23] derived an upper bound to the third kind Hankel determinant for a subclass of analytic functions. Bansal et al. [3] improved the upper bound for $|H_3(1)|$ for some of the classes estimated by Babalola [2] to some extent. Recently, Zaprawa [25] improved all the results obtained by Babalola [2]. Further, Orhan and Zaprawa [19] obtained an upper bound to the third kind Hankel determinant for the classes S^* and $\mathcal{K}$ functions of order alpha. Very recently, Kowalczyk et al. [12] calculated sharp upper bound to $|H_3(1)|$ for the class of convex functions $\mathcal{K}$ and showed as $|H_3(1)| \leq \frac{4}{135}$, which is far better than the bound obtained by Zaprawa [25]. Lecko et al. [14] determined sharp bound to the third order Hankel determinant for starlike functions of order 1/2. Motivated by the results obtained by different authors mentioned above and who are working in this direction (see [5]), in this paper, we are making an attempt to obtain an upper bound to the functional $|H_3(1)|$ for the function f belonging to the class, defined as follows.

Definition 1.1. A function $f(z) \in A$ is said to be in the class $Q(\alpha, \beta, \gamma)$ with α, $\beta > 0$ and $0 \leq \gamma < \alpha + \beta \leq 1$, if it satisfies the condition that

$$\operatorname{Re}\left\{\alpha \frac{f(z)}{z} + \beta f'(z)\right\} \geq \gamma, \qquad z \in E. \tag{1.4}$$

This class was considered and studied by Zhi- Gang Wang et al. [26].

In obtaining our results, we require a few sharp estimates in the form of lemmas valid for functions with positive real part.

Let $\mathscr{P}$ denotes the class of functions consisting of g, such that

$$g(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.5}$$

which are analytic in E and $\mathrm{Re} g(z) > 0$ for $z \in E$. Here g is called the Caratheodòry function [6].

Lemma 1.2. ([9]) *If $g \in \mathscr{P}$, then the sharp estimate $|c_k - \mu c_k c_{n-k}| \leq 2$, holds for $n, k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, with $n > k$ and $\mu \in [0, 1]$.*

Lemma 1.3. ([17]) *If $g \in \mathscr{P}$, then the sharp estimate $|c_k - c_k c_{n-k}| \leq 2$, holds for $n, k \in \mathbb{N}$, with $n > k$.*

Lemma 1.4. ([21]) *If $g \in \mathscr{P}$ then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $g(z) = \frac{1+z}{1-z}$, $z \in E$.*

In order to obtain our result, we refer to the classical method devised by Libera and Zlotkiewicz [16], used by several authors.

2. Main result

Theorem 2.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Q(\alpha, \beta, \gamma)$, ($\alpha, \beta > 0$ and $0 \leq \gamma < \alpha + \beta \leq 1$) then*

$$|H_3(1)| \leq 4t_1^2 \left[\frac{k_1\alpha^6 + k_2\alpha^5 + k_3\alpha^4\beta + k_4\alpha^3\beta^2 + k_5\alpha^2\beta^3 + k_6\alpha\beta^4 + k_7\beta^5}{(\alpha+2\beta)^2(\alpha+3\beta)^3(\alpha+4\beta)^2(\alpha+5\beta)} \right],$$

where $k_1 = 2$, $k_2 = 2(18\beta + 1)$, $k_3 = 2(132\beta + 15)$, $k_4 = 2(511\beta + 87)$, $k_5 = (2179\beta + 490)$, $k_6 = 12(203\beta + 56)$, $k_7 = 12(93\beta + 30)$ and $t_1 = (\alpha + \beta - \gamma)$.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Q(\alpha, \beta, \gamma)$. By virtue of Definition 1.1, there exists an analytic function $g \in \mathscr{P}$ in the open unit disc E with $g(0) = 1$ and $\mathrm{Re}\{g(z)\} > 0$ such that

$$\frac{1}{\alpha + \beta - \gamma} \left\{ \alpha \frac{f(z)}{z} + \beta f'(z) - \gamma \right\} = g(z) \tag{2.1}$$

Using the series representation for f and g in (2.1), upon simplification, we obtain

$$\sum_{n=2}^{\infty}(\alpha+n\beta)a_n z^{n-2}=(\alpha+\beta-\gamma)\sum_{n=1}^{\infty}c_n z^{n-1}. \tag{2.2}$$

The coefficient of z^{t-2}, where t is an integer with $t\geq 2$ in (2.2) is given by

$$a_t=\frac{(\alpha+\beta-\gamma)c_{t-1}}{(\alpha+t\beta)},\ \text{with}\ \ t\geq 2. \tag{2.3}$$

Substituting the values of a_2, a_3, a_4 and a_5 from (2.3) in the functional given in (1.3), it simplifies to

$$\begin{aligned}H_3(1)=(\alpha+\beta-\gamma)^2\Bigg[&\frac{c_2c_4}{(\alpha+3\beta)(\alpha+5\beta)}-\frac{(\alpha+\beta-\gamma)c_2^3}{(\alpha+3\beta)^3}-\frac{c_3^2}{(\alpha+4\beta)^2}\\&-\frac{(\alpha+\beta-\gamma)c_1^2c_4}{(\alpha+2\beta)^2(\alpha+5\beta)}+\frac{2(\alpha+\beta-\gamma)c_1c_2c_3}{(\alpha+2\beta)(\alpha+3\beta)(\alpha+4\beta)}\Bigg].\end{aligned} \tag{2.4}$$

On grouping the terms in the expression (2.4), in order to apply the lemmas, we have

$$\begin{aligned}H_3(1)=t_1^2\Bigg[&\frac{c_4(c_2-t_1c_1^2)}{(\alpha+2\beta)^2(\alpha+5\beta)}-\frac{c_3}{(\alpha+4\beta)^2}\left\{c_3-\frac{t_1(\alpha+4\beta)c_1c_2}{(\alpha+2\beta)(\alpha+3\beta)}\right\}\\&+\frac{c_2(c_4-t_1c_2^2)}{(\alpha+3\beta)^3}-\frac{c_2}{(\alpha+3\beta)(\alpha+4\beta)^2}\left\{c_4-\frac{t_1(\alpha+4\beta)c_1c_3}{(\alpha+2\beta)(\alpha+4\beta)}\right\}\\&+\frac{(d_1\alpha^6+d_2\alpha^5+d_3\alpha^4\beta+d_4\alpha^3\beta^2+d_5\alpha^2\beta^3+d_6\alpha\beta^4+d_7\beta^5)c_2c_4}{(\alpha+2\beta)^2(\alpha+3\beta)^3(\alpha+4\beta)^2(\alpha+5\beta)}\Bigg],\end{aligned} \tag{2.5}$$

with $d_1=1$, $d_2=(18\beta-1)$, $d_3=(133\beta-19)$, $d_4=4(129\beta-35)$, $d_5=2(554\beta-249)$, $d_6=8(156\beta-107)$, $d_7=4(144\beta-143)$ and $t_1=(\alpha+\beta-\gamma)$. On applying the triangle inequality in (2.5), we have

$$\begin{aligned}\Big|H_3(1)\Big|\leq t_1^2\Bigg[&\frac{|c_4||(c_2-t_1c_1^2)|}{(\alpha+2\beta)^2(\alpha+5\beta)}+\frac{|c_3|}{(\alpha+4\beta)^2}\left|c_3-\frac{t_1(\alpha+4\beta)c_1c_2}{(\alpha+2\beta)(\alpha+3\beta)}\right|\\&+\frac{|c_2||(c_4-t_1c_2^2)|}{(\alpha+3\beta)^3}+\frac{|c_2|}{(\alpha+3\beta)(\alpha+4\beta)^2}\left|c_4-\frac{t_1(\alpha+4\beta)c_1c_3}{(\alpha+2\beta)(\alpha+4\beta)}\right|\\&+\frac{|d_1\alpha^6+d_2\alpha^5+d_3\alpha^4\beta+\ d_4\alpha^3\beta^2+d_5\alpha^2\beta^3+d_6\alpha\beta^4+d_7\beta^5||c_2||c_4|}{(\alpha+2\beta)^2(\alpha+3\beta)^3(\alpha+4\beta)^2(\alpha+5\beta)}\Bigg].\end{aligned} \tag{2.6}$$

Upon using the lemmas given in (1.2), (1.3) and (1.4) in the inequality (2.6), it simplifies to

$$|H_3(1)| \leq 4t_1^2\left[\frac{k_1\alpha^6+k_2\alpha^5+k_3\alpha^4\beta+k_4\alpha^3\beta^2+k_5\alpha^2\beta^3+k_6\alpha\beta^4+k_7\beta^5}{(\alpha+2\beta)^2(\alpha+3\beta)^3(\alpha+4\beta)^2(\alpha+5\beta)}\right], \quad (2.7)$$

with $k_1 = 2$, $k_2 = 2(18\beta + 1)$, $k_3 = 2(132\beta + 15)$, $k_4 = 2(511\beta + 87)$, $k_5 = (2179\beta + 490)$, $k_6 = 12(203\beta + 56)$, $k_7 = 12(93\beta + 30)$ and $t_1 = (\alpha + \beta - \gamma)$. This completes the proof of the theorem. ∎

Remark 2.2. For the values $\alpha = 1-\sigma$, $\beta = \sigma$, $\gamma = 0$, so that $(\alpha+\beta-\gamma) = 1$ in (2.7), we obtain

$$|H_3(1)| \leq 4\left[\frac{63\sigma^6+312\sigma^5+411\sigma^4+414\sigma^3+188\sigma^2+44\sigma+4}{(1+\sigma)^2(1+2\sigma)^3(1+3\sigma)^2(1+4\sigma)}\right]. \quad (2.8)$$

Remark 2.3. Choosing $\sigma = 1$ in the expression (2.8), it coincides with the result obtained by Zaprawa [25].

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