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Fractional Ostrowski type inequalities for functions whose derivatives are s-preinvex

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Abstract: In this paper, we establish a new integral identity, and then we derive some new fractional Ostrowski type inequalities for functions whose derivatives are s-preinvex.

Key words: integral inequality, -preinvex functions, Hölder inequality, power mean inequality.

AMS *Subject Class.* (2010): 26D15, 26D20, 26A51.

1. Introduction

In 1938, A.M. Ostrowski proved an interesting integral inequality, estimating the absolute value of the derivative of a differentiable function by its integral mean as follows

Theorem 1. ([21]) *Let $I \subseteq \mathbb{R}$ be an interval. Let $f : I \to \mathbb{R}$, be a differentiable mapping in the interior $I^\circ$of I, and $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq M\,(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right]. \tag{1.1}$$

The above inequality has attracted many researchers, various generalizations, refinements, extensions and variants have appeared in the literature see for instance [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19, 20, 24, 25], and references therein.

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Set [25], gave the following fractional Ostrwski inequality for differentiable s-convex functions

$$\left| \left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{b-a} \left(\left(I_{x^-}^{\alpha} f \right)(a) + \left(I_{x^+}^{\alpha} f \right)(b) \right) \right|$$
$$\leq \frac{M}{b-a} \left(1 + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)} \right) \left(\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+s+1} \right)$$

and

$$\left| \left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{b-a} \left(\left(I_{x^-}^{\alpha} f \right)(a) + \left(I_{x^+}^{\alpha} f \right)(b) \right) \right|$$
$$\leq \frac{M}{(1+p\alpha)^{\frac{1}{p}}} \left(\frac{2}{s+1} \right)^{\frac{1}{q}} \left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a} \right).$$

Kirmaci et al. [4], presented some results connected with inequality (1.1)

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2} \right) \right| \leq \frac{b-a}{8} \left(\left| f'(a) \right| + \left| f'(b) \right| \right).$$

Zhu et al. [27], established the following fractional midpoint inequality

$$\left| \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[\left(J_{a^+}^{\alpha} f \right)(b) + \left(J_{b^-}^{\alpha} f \right)(a) \right] - f\left(\frac{a+b}{2} \right) \right|$$
$$\leq \frac{b-a}{4(1+\alpha)} \left(\left| f'(a) \right| + \left| f'(b) \right| \right) \left(\alpha + 3 - \frac{1}{2^{\alpha-1}} \right).$$

Motivated by the above results, in this paper, we establish a new integral identity, and then we derive some new fractional Ostrowski type inequalities for functions whose derivatives are s-preinvex.

2. Preliminaries

In this sections we recall some definitions and lemmas

Definition 1. ([23]) A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2. ([1]) A nonnegative function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 3. ([18]) A set $K \subseteq \mathbb{R}^n$ is said to be an invex with respect to the bifunction $\eta : K \times K \to \mathbb{R}^n$, if for all $x, y \in K$, we have

$$x + t\eta\left(y, x\right) \in K.$$

In what follows we assume that $K \subseteq \mathbb{R}$ be an invex set with respect to the bifunction $\eta : K \times K \to \mathbb{R}$.

Definition 4. ([26]) A function $f : K \to \mathbb{R}$ is said to be preinvex with respect to η, if

$$f\left(x + t\eta\left(y, x\right)\right) \leq \left(1 - t\right) f\left(x\right) + tf(y)$$

holds for all $x, y \in K$ and all $t \in [0, 1]$.

Definition 5. ([5]) A nonnegative function $f : K \subset [0, \infty) \to \mathbb{R}$ is said to be s-preinvex in the second sense with respect to η for some fixed $s \in (0, 1]$, if

$$f\left(x + t\eta\left(y, x\right)\right) \; \leq (1 - t)^s f(x) + t^s f(y)$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Definition 6. ([2, 3, 17]) Let $f \in L_1[a, b]$. The Riemann-Liouville fractional integrals $I^{\alpha}_{a+} f$ and $I^{\alpha}_{b-} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma\left(\alpha\right)} \int_a^x \left(x - t\right)^{\alpha - 1} f(t) dt\,, \qquad x > a\,,$$

$$I^{\alpha}_{b-} f(x) = \frac{1}{\Gamma\left(\alpha\right)} \int_x^b \left(t - x\right)^{\alpha - 1} f(t) dt\,, \qquad b > x\,,$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha - 1} dt$, is the Gamma function and $I^0_{a+} f(x) = I^0_{b-} f(x) = f(x)$.

We also recall that the beta function for any complex numbers and non-positive integers ρ, τ such that $\mathrm{Re}(\rho) > 0$ and $\mathrm{Re}(\tau) > 0$ is defined by

$$B\left(\rho, \tau\right) = \int_0^1 \theta^{\rho - 1} \left(1 - \theta\right)^{\tau - 1} d\theta = \frac{\Gamma\left(\rho\right) \Gamma\left(\tau\right)}{\Gamma\left(\rho + \tau\right)}.$$

The incomplete beta function is given by

$$B_t\left(\rho, \tau\right) = \int_0^t \theta^{\rho - 1} \left(1 - \theta\right)^{\tau - 1} d\theta\,, \qquad 0 < t < 1\,.$$

Lemma 1. ([22]) *For any $0 \leq a < b$ in $\mathbb{R}$ and fixed $p \geq 1$, we have*

$$(b - a)^p \leq b^p - a^p.$$

3. Main results

In what follows, in order to simplify and lighten the writing, we note in all the proofs $\eta(b,a)$ by c.

Lemma 2. *Let $f : [a, a+\eta(b,a)] \to \mathbb{R}$ be a differentiable mapping on $(a, a+\eta(b,a))$ with $\eta(b,a) > 0$, and assume that $f' \in L\big([a, a+\eta(b,a)]\big)$. Then the following equality holds*

$$\begin{aligned} f(x) - \tfrac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)} &\left(\left(I^{\alpha}_{a^+} f\right)(a+\eta(b,a)) + \left(I^{\alpha}_{(a+\eta(b,a))^-} f\right)(a) \right) \\ = \tfrac{\eta(b,a)}{2} &\left(\int_0^1 k f'(a+t\eta(b,a))\,dt \right. \\ &\left. + \int_0^1 \left(t^{\alpha} - (1-t)^{\alpha}\right) f'(a+t\eta(b,a))\,dt \right), \end{aligned} \tag{3.1}$$

where

$$k = \begin{cases} 1 & \text{if } 0 \le t < \frac{x-a}{\eta(b,a)}\,, \\ -1 & \text{if } \frac{x-a}{\eta(b,a)} \le t < 1\,. \end{cases} \tag{3.2}$$

Proof. Let $c = \eta(b,a)$. And let

$$\begin{aligned} I &= \int_0^1 k f'(a+tc)dt + \int_0^1 (t^{\alpha} - (1-t)^{\alpha}) f'(a+tc)dt \\ &= I_1 + I_2\,, \end{aligned} \tag{3.3}$$

where

$$I_1 = \int_0^1 k f'(a+tc)dt\,, \tag{3.4}$$

$$I_2 = \int_0^1 (t^{\alpha} - (1-t)^{\alpha}) f'(a+tc)dt\,, \tag{3.5}$$

and k is defined by (3.2).

Clearly,

$$\begin{aligned} I_1 &= \int_0^1 k f'(a+tc)dt = \int_0^{\frac{x-a}{c}} f'(a+tc)dt - \int_{\frac{x-a}{c}}^1 f'(a+tc)dt \\ &= \tfrac{2}{c} f(x) - \tfrac{1}{c}[f(a) + f(a+c)]. \end{aligned} \tag{3.6}$$

Now, by integration by parts, I_2 gives

$$\begin{aligned}
I_2 &= \tfrac{1}{c} f(a+c) + \tfrac{1}{c} f(a) \\
&\quad - \tfrac{\alpha}{c}\left(\int_0^1 t^{\alpha-1} f(a+tc)\,dt + \int_0^1 (1-t)^{\alpha-1} f(a+tc)\,dt\right) \\
&= \tfrac{1}{c} f(a+c) + \tfrac{1}{c} f(a) \qquad (3.7)\\
&\quad - \tfrac{\alpha}{c^{\alpha+1}}\left(\int_a^{a+c} (u-a)^{\alpha-1} f(u)\,du + \int_a^{a+c} (c+a-u)^{\alpha-1} f(u)\,du\right) \\
&= \tfrac{1}{c} f(a+c) + \tfrac{1}{c} f(a) - \tfrac{\Gamma(\alpha+1)}{c^{\alpha+1}}\left(\left(I_{a^+}^{\alpha} f\right)(a+c) + \left(I_{(a+c)^-}^{\alpha} f\right)(a)\right).
\end{aligned}$$

Substituting (3.6) and (3.7) in (3.3). Multiplying the resulting equality by $\frac{c}{2}$, and then replacing c by $\eta(a,b)$, we get the desired result. ∎

THEOREM 2. *Let $f : [a, a+\eta(b,a)] \to \mathbb{R}$ be a positive function on $[a,b]$ with $\eta(b,a) > 0$ and $f \in L[a,b]$. If $|f'|$ is s-preinvex function with $s \in (0,1]$, then the following fractional inequality holds*

$$\begin{aligned}
&\left| f(x) - \tfrac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left(\left(I_{a^+}^{\alpha} f\right)(a+\eta(b,a)) + \left(I_{(a+\eta(b,a))^-}^{\alpha} f\right)(a)\right)\right| \\
&\qquad \le \tfrac{\eta(b,a)}{2}\left(|f'(a)| + |f'(b)|\right) \qquad (3.8)\\
&\qquad\quad \times \left(\tfrac{1}{s+1} + \tfrac{1-\left(\frac{1}{2}\right)^{\alpha+s}}{\alpha+s+1} + B_{\frac{1}{2}}(s+1,\alpha+1) - B_{\frac{1}{2}}(\alpha+1,s+1)\right).
\end{aligned}$$

where $B_{\frac{1}{2}}(\cdot,\cdot)$ is the incomplete beta function.

Proof. Let $c = \eta(a,b)$. From Lemma 2, and properties of modulus, we have

$$\begin{aligned}
&\left| f(x) - \tfrac{\Gamma(\alpha+1)}{2c^{\alpha}}\left(\left(I_{a^+}^{\alpha} f\right)(a+c) + \left(I_{(a+c)^-}^{\alpha} f\right)(a)\right)\right| \\
&\le \tfrac{c}{2}\Bigg[\int_0^{\frac{x-a}{c}} |f'(a+tc)|\,dt + \int_{\frac{x-a}{c}}^1 |f'(a+tc)|\,dt \\
&\quad + \int_0^{\frac{1}{2}} \left((1-t)^{\alpha} - t^{\alpha}\right)|f'(a+tc)|\,dt + \int_{\frac{1}{2}}^1 \left(t^{\alpha} - (1-t)^{\alpha}\right)|f'(a+tc)|\,dt\Bigg].
\end{aligned}$$

Using the s-preinvexity of $|f'|$, the above inequality gives

$$\begin{aligned}
&\left| f(x)-\tfrac{\Gamma(\alpha+1)}{2c^{\alpha}}\left(\left(I_{a^+}^{\alpha}f\right)(a+c)+\left(I_{(a+c)^-}^{\alpha}f\right)(a)\right)\right| \\
&\leq \tfrac{c}{2}\left(\int_0^{\frac{x-a}{c}}\big((1-t)^s|f'(a)|+t^s|f'(b)|\big)dt\right. \\
&\quad+\int_{\frac{x-a}{c}}^1\big((1-t)^s|f'(a)|+t^s|f'(b)|\big)dt \\
&\quad+\int_0^{\frac{1}{2}}\big((1-t)^{\alpha}-t^{\alpha}\big)\big((1-t)^s|f'(a)|+t^s|f'(b)|\big)dt \\
&\quad\left.+\int_{\frac{1}{2}}^1\big((1-t)^{\alpha}-t^{\alpha}\big)\big((1-t)^s|f'(a)|+t^s|f'(b)|\big)dt\right) \\
&=\tfrac{c}{2}\left[\left(\tfrac{\left(1-\left(1-\frac{x-a}{c}\right)^{s+1}\right)\ |f'(a)|+\left(\frac{x-a}{c}\right)^{s+1}|f'(b)|}{s+1}\right)\right. \\
&\quad+\left(\tfrac{\left(1-\frac{x-a}{c}\right)^{s+1}\ |f'(a)|+\left(1-\left(\frac{x-a}{c}\right)^{s+1}\right)|f'(b)|}{s+1}\right) \\
&\quad+\left|f'(a)\right|\left(\int_0^{\frac{1}{2}}((1-t)^{\alpha+s}-t^{\alpha}(1-t)^s)dt\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^1\left(t^{\alpha}(1-t)^s-(1-t)^{\alpha+s}\right)dt\right) \\
&\quad\left.+\left|f'\left(b\right)\right|\left(\int_0^{\frac{1}{2}}(t^s\left(1-t\right)^{\alpha}-t^{\alpha+s})dt+\int_{\frac{1}{2}}^1(t^{\alpha+s}-t^s(1-t)^{\alpha})dt\right)\right] \\
&=\tfrac{c}{2}\left(\left|f'(a)\right|+\left|f'\left(b\right)\right|\right) \\
&\quad\times\left(\tfrac{1}{s+1}+\int_0^{\frac{1}{2}}((1-t)^{\alpha+s}-t^{\alpha}(1-t)^s)\ dt+\int_0^{\frac{1}{2}}(t^s\left(1-t\right)^{\alpha}-t^{\alpha+s})dt\right) \\
&=\tfrac{c}{2}\left(\left|f'(a)\right|+\left|f'\left(b\right)\right|\right) \\
&\quad\times\left(\tfrac{1}{s+1}+\tfrac{1-\left(\frac{1}{2}\right)^{\alpha+s}}{\alpha+s+1}+B_{\frac{1}{2}}\left(s+1,\alpha+1\right)-B_{\frac{1}{2}}\left(\alpha+1,s+1\right)\right).
\end{aligned}$$

Replacing c by $\eta\left(b,a\right)$ in the above inequality, we get the desired result. The proof is completed. ∎

Corollary 1. *In Theorem 2 if we choose* $x = \frac{2a+\eta(b,a)}{2}$, *we obtain the following fractional midpoint inequality*

$$\left| f\left(\tfrac{2a+\eta(b,a)}{2}\right) - \tfrac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)} \left(\left(I_{a^+}^{\alpha} f\right)(a+\eta(b,a)) + \left(I_{(a+\eta(b,a))^-}^{\alpha} f\right)(a) \right) \right|$$
$$\le \tfrac{\eta(b,a)}{2}\left(\left|f'(a)\right| + \left|f'(b)\right|\right)$$
$$\times \left(\tfrac{1}{s+1} + \tfrac{1-\left(\frac{1}{2}\right)^{\alpha+s}}{\alpha+s+1} + B_{\frac{1}{2}}(s+1,\alpha+1) - B_{\frac{1}{2}}(\alpha+1,s+1) \right).$$

Moreover if we take $\eta(b,a) = b-a$, *we obtain*

$$\left| f\left(\tfrac{a+b}{2}\right) - \tfrac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left(\left(I_{a^+}^{\alpha} f\right)(b) + \left(I_{b^-}^{\alpha} f\right)(a) \right) \right|$$
$$\le \tfrac{b-a}{2}\left(\left|f'(a)\right| + \left|f'(b)\right|\right)$$
$$\times \left(\tfrac{1}{s+1} + \tfrac{1-\left(\frac{1}{2}\right)^{\alpha+s}}{\alpha+s+1} + B_{\frac{1}{2}}(s+1,\alpha+1) - B_{\frac{1}{2}}(\alpha+1,s+1) \right).$$

Corollary 2. *In Theorem 2 if we put* $s = 1$, *we obtain*

$$\left| f(x) - \tfrac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)} \left(\left(I_{a^+}^{\alpha} f\right)(a+\eta(b,a)) + \left(I_{(a+\eta(b,a))^-}^{\alpha} f\right)(a) \right) \right|$$
$$\le \tfrac{\eta(b,a)}{4(\alpha+1)}\left(\left|f'(a)\right| + \left|f'(b)\right|\right)\left(\alpha + 3 - \left(\tfrac{1}{2}\right)^{\alpha-1}\right).$$

Moreover if we choose $x = \frac{2a+\eta(b,a)}{2}$, *we obtain the following fractional midpoint inequality*

$$\left| f\left(\tfrac{2a+\eta(b,a)}{2}\right) - \tfrac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)} \left(\left(I_{a^+}^{\alpha} f\right)(a+\eta(b,a)) + \left(I_{(a+\eta(b,a))^-}^{\alpha} f\right)(a) \right) \right|$$
$$\le \tfrac{\eta(b,a)}{4(\alpha+1)}\left(\left|f'(a)\right| + \left|f'(b)\right|\right)\tfrac{1}{2(\alpha+1)}\left(\alpha + 3 - \left(\tfrac{1}{2}\right)^{\alpha-1}\right).$$

Remark 1. Theorem 2 will be reduced to Theorem 2.3 from [27], if we choose $\eta(b,a) = b-a$, $x = \frac{a+b}{2}$ and $s = 1$.

Theorem 3. *Let* $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ *be a positive function on* $[a,b]$ *with* $a < b$ *and* $f \in L[a,b]$. *If* $|f'|^q$ *is s-preinvex function, where* $s \in (0,1]$ *and*

$q \geq 1$, then the following fractional inequality holds

$$\left|f(x)-\tfrac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left(\left(I_{a^+}^{\alpha}f\right)(a+\eta(b,a))+\left(I_{(a+\eta(b,a))^-}^{\alpha}f\right)(a)\right)\right|$$

$$\leq \tfrac{\eta(b,a)}{2}\left[\left(\tfrac{\left(1-\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1}\right)|f'(a)|^q+\left(\frac{x-a}{\eta(b,a)}\right)^{s+1}|f'(b)|^q d}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+\left(\tfrac{\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1}|f'(a)|^q+\left(1-\left(\frac{x-a}{\eta(b,a)}\right)^{s+1}\right)|f'(b)|^q d}{s+1}\right)^{\frac{1}{q}} \tag{3.9}$$

$$+\left(\tfrac{2^{\alpha}-1}{(\alpha+1)2^{\alpha}}\right)^{1-\frac{1}{q}}\left(\left(\mu_{\alpha,s}^1\left|f'(a)\right|^q+\mu_{\alpha,s}^2\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\mu_{\alpha,s}^2\left|f'(a)\right|^q+\mu_{\alpha,s}^1\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right)\right],$$

where

$$\mu_{\alpha,s}^1=\tfrac{1-\left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1}-B_{\frac{1}{2}}(\alpha+1,s+1) \tag{3.10}$$

and

$$\mu_{\alpha,s}^2=B_{\frac{1}{2}}(s+1,\alpha+1)-\tfrac{\left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1}. \tag{3.11}$$

Proof. Let $c=\eta(b,a)$. From Lemma 2, properties of modulus, and power mean inequality, we have

$$\left|f(x)-\tfrac{\Gamma(\alpha+1)}{2c^{\alpha}}\left(\left(I_{a^+}^{\alpha}f\right)(a+c)+\left(I_{(a+c)^-}^{\alpha}f\right)(a)\right)\right|$$

$$\leq \tfrac{c}{2}\left[\left(\int_0^{\frac{x-a}{c}}\left|f'(a+tc)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{x-a}{c}}^1\left|f'(a+tc)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha}-t^{\alpha}\right)dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha}-t^{\alpha}\right)\left|f'(a+tc)\right|^q dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{1}{2}}^1\left(t^{\alpha}-(1-t)^{\alpha}\right)dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left(t^{\alpha}-(1-t)^{\alpha}\right)\left|f'(a+tc)\right|^q dt\right)^{\frac{1}{q}}\right].$$

Since $|f'|^q$ is s-preinvex function, we deduce

$$\left| f(x) - \frac{\Gamma(\alpha+1)}{2c^\alpha}\left(\left(I^\alpha_{a^+}f\right)(a+c) + \left(I^\alpha_{(a+c)^-}f\right)(a)\right)\right|$$

$$\le \frac{c}{2}\left[\left(\int_0^{\frac{x-a}{c}}(1-t)^s\left|f'(a)\right|^q + t^s\left|f'(b)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{\frac{x-a}{c}}^1(1-t)^s\left|f'(a)\right|^q + t^s\left|f'(b)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\frac{1-\left(\frac{1}{2}\right)^\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left\{\left(\int_0^{\frac{1}{2}}\left((1-t)^\alpha - t^\alpha\right)\left((1-t)^s\left|f'(a)\right|^q + t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_0^{\frac{1}{2}}\left((1-t)^\alpha - t^\alpha\right)\left(t^s\left|f'(a)\right|^q + (1-t)^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right\}\right]$$

$$= \frac{c}{2}\left[\left(\frac{\left(1-\left(1-\frac{x-a}{c}\right)^{s+1}\right)|f'(a)|^q + \left(\frac{x-a}{c}\right)^{s+1}|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{\left(1-\frac{x-a}{c}\right)^{s+1}|f'(a)|^q + \left(1-\left(\frac{x-a}{c}\right)^{s+1}\right)|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1-\left(\frac{1}{2}\right)^\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\left\{\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1} - B_{\frac{1}{2}}(\alpha+1, s+1)\right)\left|f'(a)\right|^q\right.\right.$$

$$\left.+\left(B_{\frac{1}{2}}(s+1,\alpha+1) - \frac{\left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1}\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}$$

$$+\left\{\left(B_{\frac{1}{2}}(s+1,\alpha+1) - \frac{\left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1}\right)\left|f'(a)\right|^q\right.$$

$$\left.\left.\left.+\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1} - B_{\frac{1}{2}}(\alpha+1, s+1)\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right)\right]$$

$$= \frac{c}{2}\left[\left(\frac{\left(1-\left(1-\frac{x-a}{c}\right)^{s+1}\right)|f'(a)|^q+\left(\frac{x-a}{c}\right)^{s+1}|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{-\left(1-\frac{x-a}{c}\right)^{s+1}|f'(a)|^q+\left(1-\left(\frac{x-a}{c}\right)^{s+1}\right)|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}$$

$$+\left(\frac{2^\alpha-1}{(\alpha+1)2^\alpha}\right)^{1-\frac{1}{q}}\left\{\left(\mu^1_{\alpha,s}\left|f'(a)\right|^q+\mu^2_{\alpha,s}\left|f'\left(b\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\mu^2_{\alpha,s}\left|f'(a)\right|^q+\mu^1_{\alpha,s}\left|f'\left(b\right)\right|^q\right)^{\frac{1}{q}}\right\}\right],$$

where $\mu^1_{\alpha,s}$ and $\mu^2_{\alpha,s}$ are defined as in (3.10) and (3.11) respectively. By replacing c by $\eta\left(b,a\right)$ in the above inequality, we get the desired result. The prove is completed. ∎

Corollary 3. *In Theorem 3 if we choose* $x=\frac{2a+\eta(b,a)}{2}$*, we obtain the following fractional midpoint inequality*

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\left(\left(I^\alpha_{a^+}f\right)\left(a+\eta(b,a)\right)+\left(I^\alpha_{(a+\eta(b,a))^-}f\right)(a)\right)\right|$$

$$\leq\frac{\eta(b,a)}{2}\left[\left(\frac{\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f'(a)|^q+\left(\frac{1}{2}\right)^{s+1}|f'(b)|^q d}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{\left(\left(\frac{1}{2}\right)\right)^{s+1}|f'(a)|^q+\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f'(b)|^q d}{s+1}\right)^{\frac{1}{q}}$$

$$+\left(\frac{2^\alpha-1}{(\alpha+1)2^\alpha}\right)^{1-\frac{1}{q}}\left\{\left(\mu^1_{\alpha,s}\left|f'(a)\right|^q+\mu^2_{\alpha,s}\left|f'\left(b\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\mu^2_{\alpha,s}\left|f'(a)\right|^q+\mu^1_{\alpha,s}\left|f'\left(b\right)\right|^q\right)^{\frac{1}{q}}\right\}\right].$$

Moreover if we take $\eta\left(b,a\right)=b-a$*, we obtain*

$$\begin{aligned}&\left|f\left(\tfrac{a+b}{2}\right)-\tfrac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\left(I_{a^+}^{\alpha}f\right)(b)+\left(I_{b^-}^{\alpha}f\right)(a)\right)\right|\\
&\quad\leq\tfrac{b-a}{2}\left[\left(\tfrac{\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f'(a)|^{q}+\left(\frac{1}{2}\right)^{s+1}|f'(b)|^{q}d}{s+1}\right)^{\frac{1}{q}}\right.\\
&\qquad+\left(\tfrac{\left(\left(\frac{1}{2}\right)\right)^{s+1}|f'(a)|^{q}+\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f'(b)|^{q}d}{s+1}\right)^{\frac{1}{q}}\\
&\qquad+\left(\tfrac{2^{\alpha}-1}{(\alpha+1)2^{\alpha}}\right)^{1-\frac{1}{q}}\left\{\left(\mu_{\alpha,s}^{1}\left|f'(a)\right|^{q}+\mu_{\alpha,s}^{2}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\qquad\qquad\left.\left.+\left(\mu_{\alpha,s}^{2}\left|f'(a)\right|^{q}+\mu_{\alpha,s}^{1}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right\}\right].\end{aligned}$$

COROLLARY 4. *In Theorem 3 if we put $s=1$, we obtain*

$$\begin{aligned}&\left|f(x)-\tfrac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left(\left(I_{a^+}^{\alpha}f\right)(a+\eta(b,a))+\left(I_{(a+\eta(b,a))^-}^{\alpha}f\right)(a)\right)\right|\\
&\quad\leq\tfrac{\eta(b,a)}{2}\left[\left(\tfrac{\left(1-\left(1-\frac{x-a}{\eta(b,a)}\right)^{2}\right)|f'(a)|^{q}+\left(\frac{x-a}{\eta(b,a)}\right)^{2}|f'(b)|^{q}d}{2}\right)^{\frac{1}{q}}\right.\\
&\qquad+\left(\tfrac{\left(1-\frac{x-a}{\eta(b,a)}\right)^{2}|f'(a)|^{q}+\left(1-\left(\frac{x-a}{\eta(b,a)}\right)^{2}\right)|f'(b)|^{q}d}{2}\right)^{\frac{1}{q}}\\
&\qquad+\left(\tfrac{2^{\alpha}-1}{(\alpha+1)2^{\alpha}}\right)^{1-\frac{1}{q}}\left\{\left(\mu_{\alpha,s}^{1}\left|f'(a)\right|^{q}+\mu_{\alpha,s}^{2}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\qquad\qquad\left.\left.+\left(\mu_{\alpha,s}^{2}\left|f'(a)\right|^{q}+\mu_{\alpha,s}^{1}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right\}\right],\end{aligned}$$

where

$$\mu_{\alpha,1}^{1}=\tfrac{1}{\alpha+2}-\tfrac{1}{\alpha+1}\left(\tfrac{1}{2}\right)^{\alpha+1} \tag{3.12}$$

and

$$\mu_{\alpha,1}^{2}=\tfrac{1}{(\alpha+1)(\alpha+2)}-\tfrac{1}{\alpha+1}\left(\tfrac{1}{2}\right)^{\alpha+1}. \tag{3.13}$$

Moreover if we choose $x = \frac{2a+\eta(b,a)}{2}$, *we obtain the following fractional midpoint inequality*

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left(\left(I^{\alpha}_{a^{+}}f\right)(a+\eta(b,a)) + \left(I^{\alpha}_{(a+\eta(b,a))^{-}}f\right)(a)\right)\right|$$

$$\leq \frac{\eta(b,a)}{2}\left[\left(\frac{3|f'(a)|^{q}+|f'(b)|^{q}d}{8}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^{q}+3|f'(b)|^{q}d}{8}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{2^{\alpha}-1}{(\alpha+1)2^{\alpha}}\right)^{1-\frac{1}{q}}\left\{\left(\mu^{1}_{\alpha,1}\left|f'(a)\right|^{q} + \mu^{2}_{\alpha,1}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\mu^{2}_{\alpha,1}\left|f'(a)\right|^{q} + \mu^{1}_{\alpha,1}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right\}\right].$$

Additionally if we take $\eta(b,a) = b - a$, *we obtain*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\left(I^{\alpha}_{a^{+}}f\right)(b) + \left(I^{\alpha}_{b^{-}}f\right)(a)\right)\right|$$

$$\leq \frac{b-a}{2}\left[\left(\frac{3|f'(a)|^{q}+|f'(b)|^{q}d}{8}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^{q}+3|f'(b)|^{q}d}{8}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{2^{\alpha}-1}{(\alpha+1)2^{\alpha}}\right)^{1-\frac{1}{q}}\left\{\left(\mu^{1}_{\alpha,1}\left|f'(a)\right|^{q} + \mu^{2}_{\alpha,1}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\mu^{2}_{\alpha,1}\left|f'(a)\right|^{q} + \mu^{1}_{\alpha,1}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right\}\right],$$

where $\mu^{1}_{\alpha,1}$ *and* $\mu^{2}_{\alpha,1}$ *are defined as in* (3.12) *and* (3.13) *respectively.*

THEOREM 4. *Let* $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ *be a positive function on* $[a,b]$ *with* $a < b$ *and* $f \in L[a,b]$. *If* $|f'|^{q}$ *is s-preinvex function, where* $s \in (0,1]$, *and* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following fractional inequality holds*

$$\left| f(x) - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left(\left(I^{\alpha}_{a^{+}}f\right)(a+\eta(b,a)) + \left(I^{\alpha}_{(a+\eta(b,a))^{-}}f\right)(a)\right)\right|$$

$$\leq \frac{\eta(b,a)}{2(s+1)^{\frac{1}{q}}}\left[\left(\left(1-\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1}\right)\left|f'(a)\right|^{q} + \left(\frac{x-a}{\eta(b,a)}\right)^{s+1}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$+\left(\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1}\left|f'(a)\right|^{q} + \left(1-\left(\frac{x-a}{\eta(b,a)}\right)^{s+1}\right)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha p}}{\alpha p+1}\right)^{\frac{1}{p}}\times\left(\left(\frac{\left(2^{s+1}-1\right)|f'(a)|^q+|f'(b)|^q}{2^{s+1}}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+\left(2^{s+1}-1\right)|f'(b)|^q}{2^{s+1}}\right)^{\frac{1}{q}}\right)\Bigg].$$

Proof. Let $c=\eta(b,a)$. From Lemma 2, properties of modulus, Hölder's inequality, and Lemma 1, we have

$$\begin{aligned}
&\left|f(x)-\frac{\Gamma(\alpha+1)}{2c^\alpha}\left(\left(I^\alpha_{a^+}f\right)(a+c)+\left(I^\alpha_{(a+c)^-}f\right)(a)\right)\right|\\
&\leq\frac{c}{2}\left[\left(\int_0^{\frac{x-a}{c}}\left|f'(a+tc)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{x-a}{c}}^1\left|f'(a+tc)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_0^{\frac{1}{2}}\left((1-t)^\alpha-t^\alpha\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(a+tc)\right|^q dt\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{1}{2}}^1\left(t^\alpha-(1-t)^\alpha\right)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|f'(a+tc)|^q dt\right)^{\frac{1}{q}}\right]\\
&\leq\frac{c}{2}\left[\left(\int_0^{\frac{x-a}{c}}\left|f'(a+tc)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{x-a}{c}}^1|f'(a+tc)|^q dt\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha p}-t^{\alpha p}\right)dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f'(a+tc)|^q dt\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{1}{2}}^1\left(t^{\alpha p}-(1-t)^{\alpha p}\right)dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|f'(a+tc)|^q dt\right)^{\frac{1}{q}}\right]\\
&=\frac{c}{2}\left[\left(\int_0^{\frac{x-a}{c}}|f'(a+tc)|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{x-a}{c}}^1|f'(a+tc)|^q dt\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha p}}{\alpha p+1}\right)^{\frac{1}{p}}\\
&\qquad\left.\times\left(\left(\int_0^{\frac{1}{2}}|f'(a+tc)|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1|f'(a+tc)|^q dt\right)^{\frac{1}{q}}\right)\right].
\end{aligned}$$

Since $|f'|^q$ is s-preinvex function, we have

$$\begin{aligned}
&\left| f(x) - \tfrac{\Gamma(\alpha+1)}{2c^{\alpha}} \left(\left(I_{a^+}^{\alpha} f \right)(a+c) + \left(I_{(a+ct)^-}^{\alpha} f \right)(a) \right) \right| \\
&\quad \le \tfrac{c}{2} \left[\left(\int_0^{\frac{x-a}{c}} ((1-t)^s \left| f'(a) \right|^q + t^s |f'(b)|^q) dt \right)^{\frac{1}{q}} \right. \\
&\qquad + \left(\int_{\frac{x-a}{c}}^1 ((1-t)^s |f'(a)|^q + t^s |f'(b)|^q) dt \right)^{\frac{1}{q}} \\
&\qquad + \left(\tfrac{1-\left(\frac{1}{2}\right)^{\alpha p}}{\alpha p+1} \right)^{\frac{1}{p}} \left\{ \left(\int_0^{\frac{1}{2}} ((1-t)^s \left| f'(a) \right|^q + t^s \left| f'(b) \right|^q) dt \right)^{\frac{1}{q}} \right. \\
&\qquad\qquad \left. \left. + \left(\int_{\frac{1}{2}}^1 ((1-t)^s \left| f'(a) \right|^q + t^s \left| f'(b) \right|^q) dt \right)^{\frac{1}{q}} \right\} \right] \\
&\quad = \frac{c}{2(s+1)^{\frac{1}{q}}} \left[\left(\left(1 - \left(1 - \tfrac{x-a}{c} \right)^{s+1} \right) \left| f'(a) \right|^q + \left(\tfrac{x-a}{c} \right)^{s+1} \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right. \\
&\qquad + \left(\left(1 - \tfrac{x-a}{c} \right)^{s+1} \left| f'(a) \right|^q + \left(1 - \left(\tfrac{x-a}{c} \right)^{s+1} \right) \left| f'(b) \right|^q \right)^{\frac{1}{q}} \\
&\qquad + \left(\tfrac{1-\left(\frac{1}{2}\right)^{\alpha p}}{\alpha p+1} \right)^{\frac{1}{p}} \\
&\qquad\quad \left. \times \left(\left(\tfrac{\left(2^{s+1}-1\right)|f'(a)|^q + |f'(b)|^q}{2^{s+1}} \right)^{\frac{1}{q}} + \left(\tfrac{|f'(a)|^q + \left(2^{s+1}-1\right)|f'(b)|^q}{2^{s+1}} \right)^{\frac{1}{q}} \right) \right].
\end{aligned}$$

Replacing c by $\eta(b,a)$ in the above inequality, we get the desired result. ∎

Corollary 5. *In Theorem 4 if we choose* $x = \frac{2a+\eta(b,a)}{2}$, *we obtain the following fractional midpoint inequality*

$$\left| f\left(\tfrac{2a+\eta(b,a)}{2} \right) - \tfrac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)} \left(\left(I_{a^+}^{\alpha} f \right)(a+\eta(b,a)) + \left(I_{(a+\eta(b,a))^-}^{\alpha} f \right)(a) \right) \right|$$

$$\leq \frac{\eta(b,a)}{2(s+1)^{\frac{1}{q}}}\left(1+\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha p}}{\alpha p+1}\right)^{\frac{1}{p}}\right)$$
$$\times\left(\left(\frac{\left(2^{s+1}-1\right)|f'(a)|^{q}+|f'(b)|^{q}}{2^{s+1}}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+\left(2^{s+1}-1\right)|f'(b)|^{q}}{2^{s+1}}\right)^{\frac{1}{q}}\right).$$

Moreover if we take $\eta(b,a)=b-a$, we obtain

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\left(I_{a^{+}}^{\alpha}f\right)(b)+\left(I_{b^{-}}^{\alpha}f\right)(a)\right)\right|$$
$$\leq \frac{b-a}{2(s+1)^{\frac{1}{q}}}\left(1+\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha p}}{\alpha p+1}\right)^{\frac{1}{p}}\right)$$
$$\times\left(\left(\frac{\left(2^{s+1}-1\right)|f'(a)1^{q}+|f'(b)|^{q}}{2^{s+1}}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+\left(2^{s+1}-1\right)|f'(b)|^{q}}{2^{s+1}}\right)^{\frac{1}{q}}\right).$$

COROLLARY 6. *In Theorem 4 if we put $s=1$, we obtain*

$$\left|f(x)-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left(\left(I_{a^{+}}^{\alpha}f\right)(a+\eta(b,a))+\left(I_{(a+\eta(b,a))^{-}}^{\alpha}f\right)(a)\right)\right|$$
$$\leq \frac{\eta(b,a)}{2^{1+\frac{1}{q}}}\left[\left(\left(1-\left(1-\frac{x-a}{\eta(b,a)}\right)^{2}\right)\left|f'(a)\right|^{q}+\left(\frac{x-a}{\eta(b,a)}\right)^{2}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$+\left(\left(1-\frac{x-a}{\eta(b,a)}\right)^{2}\left|f'(a)\right|^{q}+\left(1-\left(\frac{x-a}{\eta(b,a)}\right)^{2}\right)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha p}}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\frac{3|f'(a)|^{q}+|f'(b)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+3|f'(b)|^{q}}{4}\right)^{\frac{1}{q}}\right)\right].$$

Moreover if we choose $x=\frac{2a+\eta(b,a)}{2}$, we obtain the following fractional midpoint inequality

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left(\left(I_{a^{+}}^{\alpha}f\right)(a+\eta(b,a))+\left(I_{(a+\eta(b,a))^{-}}^{\alpha}f\right)(a)\right)\right|$$
$$\leq \frac{\eta(b,a)}{2^{1+\frac{1}{q}}}\left(1+\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha p}}{\alpha p+1}\right)^{\frac{1}{p}}\right)\left(\left(\frac{3|f'(a)|^{q}+|f'(b)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+3|f'(b)|^{q}}{4}\right)^{\frac{1}{q}}\right).$$

Additionally if we take $\eta(b,a) = b-a$, we obtain

$$\left| f\left(\tfrac{a+b}{2}\right) - \tfrac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\left(I_{a^+}^{\alpha} f\right)(b) + \left(I_{b^-}^{\alpha} f\right)(a)\right)\right|$$

$$\leq \tfrac{b-a}{2^{1+\frac{1}{q}}}\left(1+\left(\tfrac{1-\left(\frac{1}{2}\right)^{\alpha p}}{\alpha p+1}\right)^{\frac{1}{p}}\right)\left(\left(\tfrac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\tfrac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

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