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The μ-topological Hausdorff dimension

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Abstract: In 2015, R. Balkaa, Z. Buczolich and M. Elekes introduced the topological Hausdorff dimension which is a combination of the definitions of the topological dimension and the Hausdorff dimension. In our manuscript, we propose to generalize the topological Hausdorff dimension by combining the definitions of the topological dimension and the μ-Hausdorff dimension and we call it the μ-topological Hausdorff dimension. We will present upper and lower bounds for the μ-topological Hausdorff dimension of the Sierpiński carpet valid in a general framework. As an application, we give a large class of measures μ, where the μ-topological Hausdorff dimension of the Sierpiński carpet coincides with the lower and upper bounds.

Key words: Hausdorff dimension, Topological Hausdorff dimension.

AMS *Subject Class.* (2010): 28A78, 28A80.

1. Introduction

Different notions of dimensions have been introduced since the appearance of the Hausdorff dimension by F. Hausdorff in 1918 (see e.g. [10], [16] and [15]), such as the topological dimension, (see e.g. [4] and [7]). In 1975 when Mandelbrot coined the word *fractal* (see [13]). He did so to denote an object whose Hausdorff dimension was strictly greater than its topological dimension, but he abandoned this definition later, (see e.g. [6], [13] and [14]). In the Euclidean space $\mathbb{R}^n$, there has been no generally accepted definition of a fractal, even though fractal sets have been widely used as models for many physical phenomena (see e.g. [9],[11] and [12]). The idea behind these models is that of self-similarity (see e.g. [5] and [12]). Then Billingsley defined the Hausdorff measure in a probability space (see e.g. [2] and [3]). In [1], R. Balka, Z. Buczolich and M. Elekes introduced a new dimension concept for metric spaces, called the topological Hausdorff dimension. It was defined by a very natural combination of the definitions of the topological dimension and the Hausdorff dimension. The value of the topological Hausdorff dimension was

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always between the topological dimension and the Hausdorff dimension. In particular, this dimension was a non-trivial lower estimate for the Hausdorff dimension.

P. Billingsley introduced a dimension defined by a measure μ, (see e.g. [2] and [3]) called the μ-Hausdorff dimension, which was a generalization of the Hausdorff dimension. In the same vein, we propose to generalize the topological Hausdorff dimension by combining the definitions of the topological dimension and the μ-Hausdorff dimension, so-called the μ-topological Hausdorff dimension.

The paper is organized as follows. In Section 2, we recall the topological Hausdorff dimension. Afterwards, we state the basic properties of this dimension and we cite some examples. In Section 3, we introduce the μ-topological Hausdorff dimension. Finally in Section 4, we give an estimation of the μ-topological Hausdorff dimension of the Sierpiński carpet and then we provide a class of measures μ for which we compute the later dimension.

2. Topological Hausdorff dimension

Let (X, d) be a metric space. We denote by $\overline{A}$ the closure of subset A and ∂A its boundary in the metric space X. If $B \subseteq A$ then $\partial_A B$ designates the boundary of subset B in the metric space A with an induced topology.

Let $\mathcal{B}(x, r) = \big\{y \in X : d(x, y) < r\big\}$ be the open ball of radius r centered at point x.

For a bounded subset U of X we denote the diameter by:

$$|U| = \sup\big\{d(x, y) :\ x, y \in U\big\}.$$

For two metric spaces (X, d) and (Y, d'), a function $f : X \to Y$ is Lipschitz if there exists constant $\lambda \in \mathbb{R}_+$ such that $d'\big(f(x), f(y)\big) \leqslant \lambda\ d(x, y)$, for all $x, y \in X$.

We begin by recalling the definition of the Hausdorff dimension.

Definition 2.1. Let A be a subset of a separable metric space X, and α be a positive number. For any $\varepsilon > 0$, we define:

$$\mathcal{H}^\alpha_\varepsilon(A) = \inf\left\{\sum_j |U_j|^\alpha :\ A \subset \bigcup_j U_j,\ |U_j| < \varepsilon\right\}.$$

We also define:

$$\mathcal{H}^\alpha(A) = \lim_{\varepsilon\to 0^+} \mathcal{H}^\alpha_\varepsilon(A).$$

Then the Hausdorff dimension of A is given as follows:

$$\dim_H(A) = \inf\{\alpha > 0 : \mathcal{H}^\alpha(A) < +\infty\}.$$

The topological Hausdorff dimension of a non-empty separable metric space X, introduced in [1], is defined as:

$$\dim_{tH}(X) = \inf \big\{d : X \text{ has basis } \mathcal{U} \text{ such that } \dim_H(\partial U) \leqslant d-1 \\ \text{for every } U \in \mathcal{U}\big\},$$

with convention $\dim_H(\emptyset) = -1$.

In [1], the authors proved that $\dim_t(X) \leqslant \dim_{tH}(X) \leqslant \dim_H(X)$ where $\dim_t$ denotes the topological dimension of a non-empty separable metric space defined by:

$$\dim_t(X) = \inf \big\{d : X \text{ has basis } \mathcal{U} \text{ such that } \dim_t(\partial U) \leqslant d-1 \\ \text{for every } U \in \mathcal{U}\big\}$$

where convention $\dim_t(\emptyset) = -1$, (see [4] or [7]).

Moreover, the authors gave an alternative recursive definition of the topological dimension as follows:

$$\dim_t(X) = \min \big\{d : \text{ there is } A \subseteq X \text{ such that } \dim_t(A) \leqslant d-1 \\ \text{and } \dim_t(X \backslash A) \leqslant 0\big\}.$$

Notice that Balka et al. defined the topological Hausdorff dimension of a subset A of X by considering A as a metric space and equipping it with a dimension induced from that of X.

Next, we recall some properties of the topological Hausdorff dimension of X given in [1].

PROPOSITION 2.2. *Let X be a separable metric space.*

(i) *If $A \subseteq B \subseteq X$, then $\dim_{tH}(A) \leqslant \dim_{tH}(B)$.*

(ii) *If $X = \bigcup_{n\in\mathbb{N}} X_n$, where X_n ($n \in \mathbb{N}$) are closed subsets of X, then:*

$$\dim_{tH}(X) = \sup_{n\in\mathbb{N}} \dim_{tH}(X_n).$$

(iii) *Let Y be a separable metric space and $f : X \to Y$ be a Lipschitz homeomorphism, so:*

$$\dim_{tH}(Y) \leqslant \dim_{tH}(X).$$

Particularly, if f is bi-Lipschitz, then $\dim_{tH}(Y) = \dim_{tH}(X)$.

EXAMPLE 2.3. (a) Let $X = \mathbb{R}$, so we have:

$$\dim_t(\mathbb{R}) = \dim_{tH}(\mathbb{R}) = \dim_H(\mathbb{R}) = 1.$$

In addition, we have:

$$\dim_{tH}(\mathbb{Q}) = \dim_{tH}(\mathbb{R}\backslash\mathbb{Q}) = 0.$$

It is noticeable that:

$$\dim_{tH}(\mathbb{R}) \neq \sup\left(\dim_{tH}(\mathbb{Q}), \dim_{tH}(\mathbb{R}\backslash\mathbb{Q})\right).$$

Indeed, $\mathbb{Q}$ is not a closed set of $\mathbb{R}$.

(b) Let $X = \mathbb{R}^2$.
Let D be the von Koch snowflake curve. Then:

$$\dim_t(D) = \dim_{tH}(D) = 1 < \dim_H(D) = \frac{\ln 4}{\ln 3}.$$

Let S be the Sierpiński triangle. Thereby:

$$\dim_t(S) = \dim_{tH}(S) = 1 < \dim_H(S) = \frac{\ln 3}{\ln 2}.$$

Let T be the Sierpiński carpet. Thus:

$$\dim_t(T) = 1 < \dim_{tH}(T) = \frac{\ln 6}{\ln 3} < \dim_H(T) = \frac{\ln 8}{\ln 3}.$$

Remark. The topological Hausdorff dimension is not a topological notion. Indeed, the following property was established in [1]:

$$\dim_{tH}(X \times [0,1]) = 1 + \dim_H(X). \tag{2.1}$$

Based on property (2.1), we can build two homeomorphic spaces such that their topological Hausdorff dimensions are different as follows:

Consider $X, Y \subseteq [0,1]$ two Cantor sets such that:

$$\dim_H(X) \neq \dim_H(Y).$$

Since these two sets are totally discontinuous then X and Y are homeomorphic to the middle-thirds Cantor set (see [8]). Therefore, there exists homeomorphism $\varphi : X \to Y$, so:

$$\begin{aligned} X \times [0,1] &\longrightarrow Y \times [0,1] \\ (x,t) &\longmapsto (\varphi(x), t) \end{aligned}$$

is a homeomorphism. Thus, using property (2.1), we obtain:

$$\dim_{tH}(X \times [0,1]) \neq \dim_{tH}(Y \times [0,1]).$$

3. μ-Topological Hausdorff dimension

In the following, we propose to generalize the topological Hausdorff dimension. We begin by recalling the dimension defined by Billingsley in [2] and [3]. Let X be a metric space, $\mathcal{F}$ a countable set of subsets of X, and μ a non-negative function defined on $\mathcal{F}$ and satisfying the following property:

$$\begin{gathered} \text{For each } x \in X \text{ and } \varepsilon > 0, \text{ there is } U \in \mathcal{F} \\ \text{such that } x \in U \text{ and } \mu(U) < \varepsilon. \end{gathered} \tag{3.1}$$

Let A be a non-empty subset of X and α be a positive number. For any $\varepsilon > 0$, we define:

$$\mathcal{H}^{\alpha}_{\mu,\varepsilon}(A) = \inf \left\{ \sum_j \mu(U_j)^{\alpha} : \ A \subset \bigcup_j U_j, \ U_j \in \mathcal{F}, \ \mu(U_j) < \varepsilon \right\} \tag{3.2}$$

with convention $0^{\alpha} = 0$.

As ε decreases, the class of permissible covers of A in (3.2) is reduced. Then the infimum $\mathcal{H}^{\alpha}_{\mu,\varepsilon}(A)$ increases, and so approaches a limit as $\varepsilon \to 0$. We write the following:

$$\mathcal{H}^{\alpha}_{\mu}(A) = \lim_{\varepsilon \to 0} \mathcal{H}^{\alpha}_{\mu,\varepsilon}(A).$$

Therefore, the μ-Hausdorff dimension of a non-empty subset A of X relative to $\mathcal{F}$, as defined by Billingsley, is given by:

$$\dim_{\mu}(A) = \inf \left\{ \alpha > 0 \ : \ \mathcal{H}^{\alpha}_{\mu}(A) < +\infty \right\}.$$

It is noted that the μ-Hausdorff dimension and the Hausdorff dimension have the same properties.

Similar to the definition of the topological Hausdorff dimension, we introduce the μ-topological Hausdorff dimension of X relative to $\mathcal{F}$.

DEFINITION 3.1. Let X be a metric space, $\mathcal{F}$ a countable set of subsets of X, and μ a non-negative function defined on $\mathcal{F}$ and satisfying (3.1). Then the μ-topological Hausdorff dimension of X relative to $\mathcal{F}$ is given by:

$$\dim_{t\mu}(X) = \inf \big\{\alpha : X \text{ has basis } \mathcal{U} \text{ such that, for every } U \in \mathcal{U}, \\ \dim_\mu(\partial U) \leqslant \alpha - 1\big\}$$

with convention $\dim_\mu(\emptyset) = -1$.

Notice that the μ-topological Hausdorff dimension of a subset A of X is defined by considering A as a metric space and equipping it with a dimension induced from that of X. Moreover, the μ-topological Hausdorff dimension is monotonous in the sense of inclusion.

Remark. We find the topological Hausdorff dimension in the following particular case: Let $\mathbb{Q}_+\backslash\{0\}$ be the set of all positive rational numbers. If X is a separable metric space, $\mathcal{F} = \{\mathcal{B}(x_n, r) : n \in \mathbb{N}, r \in \mathbb{Q}_+\backslash\{0\}\}$ where $\{x_n : n \in \mathbb{N}\}$ is dense in X, and μ is the function such that $\mu\big(\mathcal{B}(x_n, r)\big) = 2r$. Then:

$$\dim_{t\mu}(X) = \dim_{tH}(X). \tag{3.3}$$

Indeed, let $\varepsilon > 0$ and $x \in X$. We choose $r \in \mathbb{Q}_+\backslash\{0\}$ where $r < \frac{\varepsilon}{2}$. Then there exists $n \in \mathbb{N}$ such that $d(x_n, x) < r$. Hence, $x \in \mathcal{B}(x_n, r)$ and $\mu\big(\mathcal{B}(x_n, r)\big) = 2r < \varepsilon$. As a consequence, μ satisfies (3.1).

Now, to find (3.3), it must be showed that for all subset A of X:

$$\dim_\mu(A) = \dim_H(A).$$

Firstly, it is clear to see that:

$$\dim_\mu(A) \geqslant \dim_H(A).$$

For the second inequality, let $\alpha > 0$ and $\varepsilon > 0$. Consider $\{U_j\}_j$, a ε-cover of A. For all $j \in \mathbb{N}$ we pick $\lambda_j \in \mathbb{Q}_+\backslash\{0\}$ such that:

$$|U_j|^\alpha < \lambda_j^\alpha < |U_j|^\alpha + \frac{\varepsilon}{2^j}.$$

Let $y_j \in U_j$, then there exists $n_j \in \mathbb{N}$ such that $d(y_j, x_{n_j}) < \lambda_j$. Whence:

$$U_j \subset \mathcal{B}(x_{n_j}, 2\lambda_j).$$

Since $A \subset \bigcup\limits_j U_j$, $A \subset \bigcup\limits_j \mathcal{B}(x_{n_j}, 2\lambda_j)$. Thus:

$$\begin{aligned}\mathcal{H}^\alpha_{\mu,4(\varepsilon^\alpha+\varepsilon)^{\frac{1}{\alpha}}}(A) &\leqslant \sum_j \mu\left(\mathcal{B}(x_{n_j}, 2\lambda_j)\right)^\alpha \\ &= 4^\alpha \sum_j \lambda_j^\alpha \\ &< 4^\alpha \sum_j |U_j|^\alpha + 2 \cdot 4^\alpha \varepsilon.\end{aligned}$$

As a result, $\mathcal{H}^\alpha_{\mu,4(\varepsilon^\alpha+\varepsilon)^{\frac{1}{\alpha}}}(A) \leqslant 4^\alpha \mathcal{H}^\alpha_\varepsilon(A) + 2 \cdot 4^\alpha \varepsilon$. When ε approaches to zero, we obtain $\mathcal{H}^\alpha_\mu(A) \leqslant 4^\alpha \mathcal{H}^\alpha(A)$. Finally, $\dim_\mu(A) \leqslant \dim_H(A)$.

4. Calculating μ-topological Hausdorff dimension of Sierpiński carpet

In this section, we give an estimation of the μ-topological Hausdorff dimension of the Sierpiński carpet T. Let $X = \mathbb{R}^2$, $\mathcal{F} = \bigcup_{n \geqslant 1} \mathcal{F}_n$ where $\mathcal{F}_n$ is the triadic squares set of the n-th generation, and μ is a non-negative function defined on $\mathcal{F}$ and satisfying (3.1). Let us recall that a triadic square of the n-th generation is defined by:

$$C = I \times J \subseteq \mathbb{R}^2,$$

where I and J are two triadic intervals of the n-th generation.

4.1. Lower bound of μ-topological Hausdorff dimension of Sierpiński carpet T. Now we establish a lower estimation of the μ-topological Hausdorff dimension of the Sierpiński carpet T. For μ, we associate functions W_1 and W_2, defined on the set of triadic intervals I in $\mathbb{R}$ by:

If I is a triadic interval of the n-th generation, contained in $[0, 1[$, then:

$$W_1(I) = \inf_J \mu(I \times J) \qquad \text{and} \qquad W_2(I) = \inf_J \mu(J \times I) \tag{4.1}$$

where the lower bound is taken on all the triadic intervals J of the n-th generation contained in $[0,1[$. Else:

$$W_1(I) = W_2(I) = 0.$$

Seeing that μ satisfies (3.1), then the W_1(respectively W_2)-Hausdorff dimension is well-defined. Indeed, it must be proved that W_1 and W_2 satisfy (3.1). It is clear that W_1 satisfy (3.1) when $x \notin [0,1[$. Let $x \in [0,1[$ and $\varepsilon > 0$. As μ satisfies (3.1), then for all $y \in [0,1[$ there exists $C \in \mathcal{F}$ such that $(x,y) \in C = I \times J$ and $\mu(C) < \varepsilon$. Therefore, $x \in I$ and $W_1(I) < \varepsilon$. Consequently, W_1 satisfies (3.1), and similarly we prove that W_2 satisfies (3.1).

Thereby, we have the following result.

Theorem 4.1. *We have*

$$\dim_{t\mu}(T) \geqslant 1 + \sup\big(\dim_{W_1}(K), \dim_{W_2}(K)\big)$$

where K is the middle-thirds Cantor set.

Proof. We will establish that $\dim_{t\mu}(T) \geqslant 1 + \dim_{W_1}(K)$. The other inequality $\dim_{t\mu}(T) \geqslant 1 + \dim_{W_2}(K)$ can be proved in a similar way. For this purpose, we need the following intermediate result.

Lemma 4.2. *Let $s < \dim_{W_1}(K)$. Then there exists $x_s \in K$ satisfying*

$$\dim_{W_1}\big(]x_s - r, x_s + r[\cap K\big) > s \qquad \textit{for each } r > 0. \tag{4.2}$$

Proof of Lemma. Assume, on the contrary, that for all $x \in K$, there exists $r_x > 0$ such that $\dim_{W_1}(]x - r_x, x + r_x[\cap K) \leqslant s$. It is clear to see that $K \subset \bigcup_{x \in K}]x - r_x, x + r_x[$. As K is compact and according to the compactness of subsets, we have $K \subset \bigcup_{i=1}^{p}]x_i - r_{x_i}, x_i + r_{x_i}[$. Then the middle-thirds Cantor set can be written as:

$$K = \bigcup_{i=1}^{p} (]x_i - r_{x_i}, x_i + r_{x_i}[\cap K)\,.$$

Hence:

$$\dim_{W_1}(K) = \sup_{1 \leqslant i \leqslant p} \dim_{W_1}(]x_i - r_{x_i}, x_i + r_{x_i}[\cap K) \leqslant s.$$

This contradicts the fact that $\dim_{W_1}(K) > s$. ∎

Now, we return to the proof of Theorem 4.1. For a fixed $s < \dim_{W_1}(K)$, from Lemma 4.2, there exists $x_s \in K$ such that for all $r > 0$ we have $\dim_{W_1}\big(]x_s - r, x_s + r[\cap K\big) > s$.

Let $\mathcal{U}$ be an open basis of T. Since $K \times [0,1] \subset T$ and as we remark that:

$$\dim_{t\mu}(X) = 1 + \inf_{\mathcal{U}} \sup_{U\in\mathcal{U}} \dim_\mu(\partial U) \tag{4.3}$$

where the lower bound is taken on all basis $\mathcal{U}$ of X, then it sufficient to demonstrate that there exists $U \in \mathcal{U}$ such that $\dim_\mu(\partial_T U \cap K \times [0,1]) \geqslant s$, where ∂_T is the boundary in the Sierpiński carpet T.

1. First case: $x_s \in K\backslash\{0\}$.

   The point $(x_s, 1) \in \left]0, \frac{7}{6}\right[\times \left]0, \frac{7}{6}\right[\cap T = \bigcup_i U_i = \mathcal{U}$ with $U_i \in \mathcal{U}$. Consider i such that U_i contains point $(x_s, 1)$. Note U instead of U_i.

   We put $y_s = \inf\{y \leqslant 1 : (x_s, y) \in U\}$, then $(x_s, y_s) \in \partial_T U \cap K \times [0,1]$.

   On the other hand, there exists $r_s > 0$ such that:

   $$]x_s - r_s, x_s + r_s[\times]1 - r_s, 1 + r_s[\cap T \subset U \quad \text{and} \quad x_s - r_s > 0.$$

   Therefore:

   $$]x_s - r_s, x_s + r_s[\cap K \subset P\left(\partial_T U \cap K \times [0,1]\right),$$

   with $P : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$.

   Fix $\varepsilon > 0$ and let $\bigcup_i C_i$ be a covering of $\partial_T U \cap K \times [0,1]$ by triadic squares, where $C_i = I_i \times J_i$ while satisfying $\mu(C_i) < \varepsilon$. It follows that $\bigcup_i I_i$ is a covering of $]x_s - r_s, x_s + r_s[\cap K$ satisfying $W_1(I_i) < \varepsilon$. Thus:

   $$\sum_i \mu\left(C_i\right)^s \geqslant \sum_i W_1(I_i)^s \geqslant \mathcal{H}^s_{W_1,\varepsilon}\left(]x_s - r_s, x_s + r_s[\cap K\right).$$

   As a consequence:

   $$\mathcal{H}^s_{\mu,\varepsilon}\left(\partial_T U \cap K \times [0,1]\right) \geqslant \mathcal{H}^s_{W_1,\varepsilon}\left(]x_s - r_s, x_s + r_s[\cap K\right).$$

   Accordingly, when ε approaches to zero we obtain:

   $$\mathcal{H}^s_{\mu}\left(\partial_T U \cap K \times [0,1]\right) \geqslant \mathcal{H}^s_{W_1}\left(]x_s - r_s, x_s + r_s[\cap K\right).$$

   Based on $\dim_{W_1}\left(]x_s - r_s, x_s + r_s[\cap K\right) > s$, we have:

   $$\dim_\mu\left(\partial_T U \cap K \times [0,1]\right) \geqslant s.$$

2. Second case: $x_s = 0$.

The point $(0,0) \in \left]-\frac{1}{6},1\right[\times \left]-\frac{1}{6},1\right[\cap T = \bigcup_i U_i = \mathcal{U}$ with $U_i \in \mathcal{U}$. Consider i such that U_i contains point $(0,0)$. Note U instead of U_i. Next, we put $y_s = \sup\{y \geqslant 0 : (0,y) \in U\}$, then $(0,y_s) \in \partial_T U \cap K \times [0,1]$. Moreover, there exists $r_s > 0$ such that:

$$]-r_s, r_s[\times]y_s - r_s, y_s + r_s[\cap T \subset U.$$

Hence,

$$]-r_s, r_s[\cap K \subset P\left(\partial_T U \cap K \times [0,1]\right)$$

with $P : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x$. Indeed, if $t = 0 \in]-r_s, r_s[\cap K$, then according to the above reasoning, we have $(0,0) \in U$, and there exists $y_0 \in [0,1]$ such that:

$$(0, y_0) \in \partial_T U \cap K \times [0,1]$$

where $y_0 = \sup\{y \geqslant 0 : (0,y) \in U\}$. Therefore:

$$0 = P(0, y_0) \in P\left(\partial_T U \cap K \times [0,1]\right).$$

Furthermore, if $t \in]-r_s, r_s[\cap K$ and $t \neq 0$, i.e. $t \in]0, r_s[\cap K$, then according to the above reasoning, we have $(t,0) \in U$, and there exists $y_1 \in [0,1]$ such that $(t, y_1) \in \partial_T U \cap K \times [0,1]$, where

$$y_1 = \sup\{y \geqslant 0 : (t,y) \in U\},$$

so $t \in P\left(\partial_T U \cap K \times [0,1]\right)$.

Let $\varepsilon > 0$ and let $\bigcup_i C_i$ be a covering of $\partial_T U \cap K \times [0,1]$ by triadic squares where $C_i = I_i \times J_i$, while satisfying $\mu(C_i) < \varepsilon$. It follows that $\bigcup_i I_i$ is a covering of $]-r_s, r_s[\cap K$ satisfying $W_1(I_i) < \varepsilon$. Thus:

$$\sum_i \mu(C_i)^s \geqslant \sum_i W_1(I_i)^s \geqslant \mathcal{H}^s_{W_1,\varepsilon}\left(]-r_s, r_s[\cap K\right).$$

Hence:

$$\mathcal{H}^s_{\mu,\varepsilon}\left(\partial_T U \cap K \times [0,1]\right) \geqslant \mathcal{H}^s_{W_1,\varepsilon}\left(]-r_s, r_s[\cap K\right).$$

Then when ε approaches to zero, and since $\dim_{W_1}\left(]-r_s, r_s[\cap K\right) > s$, we have:

$$\dim_\mu\left(\partial_T U \cap K \times [0,1]\right) \geqslant s.$$

∎

4.2. Upper bound of μ-topological Hausdorff dimension of Sierpiński carpet T. Now we establish an upper bound of the μ-topological Hausdorff dimension of T. In the following, we assume that μ satisfies this condition:

$$\begin{gathered}\text{For each } (x,y)\in\mathbb{R}^2,\ \varepsilon>0, \text{ and } n\in\mathbb{N}\backslash\{0\}, \text{ there exists}\\ p>n \text{ and } C\in\mathcal{F}_p \text{ verifiying } (x,y)\in C \text{ with } \mu(C)<\varepsilon.\end{gathered}\tag{4.4}$$

It is easy to see that μ satisfies (3.1). We can observe that a finite, non-atomic and Borelian measure μ, defined on $\mathbb{R}^2$, satisfies (4.4).

In what follows, we will essentially consider triadic squares contained in $[0,1[^2$. These squares are usually coded as follows:

Let $\mathcal{A}^n$ be the n-fold Cartesian product of $\mathcal{A}=\{0,1,2\}$ and $\mathcal{A}^*=\bigcup\limits_{n\geqslant 1}\mathcal{A}^n$.

To concatenate two words a and b in $\mathcal{A}^*$, we put b at the end of a. The resulting word is denoted by ab.

For an element a in $\mathcal{A}^*$ we denote by $|a|$ the length of a where $|a|=n$, such that $a\in\mathcal{A}^n$.

Let $i=i_1i_2\dots i_n\in\mathcal{A}^n$, so we associate a triadic interval as follows:

$$I_i=\left[\sum_{k=1}^{n}\frac{i_k}{3^k},\sum_{k=1}^{n}\frac{i_k}{3^k}+\frac{1}{3^n}\right[\subset[0,1[.$$

For all $i,j\in\mathcal{A}^n$, we consider a triadic square of $\mathcal{F}_n$ defined by:

$$C_{i,j}=I_i\times I_j\subset[0,1[^2.$$

For all $i,j\in\mathcal{A}^*$, such that $|i|=|j|$, we consider the following functions $\nu^1_{i,j}$ and $\nu^2_{i,j}$ associated to μ and defined on the set of triadic intervals I in $\mathbb{R}$ as follows: If $I\not\subset[0,1[$, then $\nu^1_{i,j}(I)=\nu^2_{i,j}(I)=0$, and for all $I=I_k\subset[0,1[$, where $k\in\mathcal{A}^n$:

$$\begin{aligned}\nu^1_{i,j}(I_k)&=\mu(C_{i\mathbb{1},jk})\\ \nu^2_{i,j}(I_k)&=\mu(C_{ik,j\mathbb{1}})\end{aligned}\tag{4.5}$$

with $\mathbb{1}=11\dots1\in\mathcal{A}^n$.

We observe that $\nu^1_{i,j}$ and $\nu^2_{i,j}$ satisfy (3.1). Indeed, let $i,j\in\mathcal{A}^*$, which are written as $i=i_1\dots i_{n_0}$ and $j=j_1\dots j_{n_0}$, so case $x\notin[0,1[$ is trivial. We fix $x\in[0,1[$, and we consider the triadic development of x given by:

$$x=\sum_{i=1}^{+\infty}\frac{x_i}{3^i}\qquad\text{where } x\in I_{x_1\dots x_n}\text{ for all } n\in\mathbb{N}^*.$$

Put:

$$x' = \sum_{k=1}^{n_0} \frac{i_k}{3^k} + \frac{1}{2 \cdot 3^{n_0}} \in [0,1[\quad \text{and} \quad y' = \sum_{k=1}^{n_0} \frac{j_k}{3^k} + \sum_{k=n_0+1}^{\infty} \frac{x_{k-n_0}}{3^k} \in [0,1[.$$

Let $\varepsilon > 0$, seeing that μ satisfies (4.4), then there exists a positive integer $p > n_0$ such that $(x', y') \in C_{i\mathbb{1},jx_1\ldots x_{p-n_0}}$ and $\mu(C_{i\mathbb{1},jx_1\ldots x_{p-n_0}}) < \varepsilon$. Thus, $\nu^1_{i,j}(I_{x_1\ldots x_{p-n_0}}) = \mu(C_{i\mathbb{1},jx_1\ldots x_{p-n_0}}) < \varepsilon$. Therefore, $\nu^1_{i,j}$ satisfies (3.1) since $x \in I_{x_1\ldots x_{p-n_0}}$. In the same way, we prove that $\nu^2_{i,j}$ satisfies (3.1).

Consequently, the $\nu^1_{i,j}$ (respectively $\nu^2_{i,j}$)-Hausdorff dimension is well-defined. Hence, we have the following result.

Theorem 4.3. *Given a function μ satisfying* (4.4) *and vanishing on the triadic squares that are not contained in* $[0,1[^2$, *we have:*

$$\dim_{t\mu}(T) \leqslant 1 + \liminf_{n\to+\infty} \sup_{\substack{|i|=|j|=n \\ l=1,2}} \dim_{\nu^l_{i,j}}(K)$$

where K is the middle-thirds Cantor set.

Proof. For $n \in \mathbb{N}\backslash\{0\}$ and $u, v \in \mathbb{Z}$, let (z^u_n, z^v_n) be the center of a triadic square $\left[\frac{u}{3^n}, \frac{u+1}{3^n}\right[\times \left[\frac{v}{3^n}, \frac{v+1}{3^n}\right[$ from $\mathcal{F}_n$.

Denote by $\mathcal{H}_n$ the set of intervals which are written as $\left]z^u_n, z^{u+2}_n\right[$ where $u \in \mathbb{Z}$. Put $\mathcal{U}_n = \left\{I \times J : I, J \in \mathcal{H}_n\right\}$, clearly $\mathcal{U} = \bigcup_{n\geqslant 1} \mathcal{U}_n$ is a countable open basis of $\mathbb{R}^2$.

To establish Theorem 4.3, considering the fact $\partial_T(U \cap T) \subset \partial U \cap T$ for all $U \in \mathbb{R}^2$, where ∂_T is the boundary in the Sierpiński carpet T and taking into account (4.3), it suffices to show that for all $U \in \mathcal{U}_p$ we have:

$$\dim_\mu(T \cap \partial U) \leqslant \inf_{n\geqslant p} \sup_{\substack{|i|=|j|=n \\ l=1,2}} \dim_{\nu^l_{i,j}}(K). \tag{4.6}$$

Let, for $U \in \mathcal{U}_p$, ∂U be the union of four cloisters. We choose $\mathcal{C}$ as one of these cloisters. We first treat the case where $\mathcal{C}$ is a vertical cloister.

Let $n \geqslant p$ and $C_{i,j}$ be a triadic square of the n-th generation such that $C_{i,j} \cap T \cap \mathcal{C} \neq \emptyset$. We also choose $\{I_k\}_k$, a ε-cover of $K \cap [0,1[$ by triadic intervals, i.e. $K \cap [0,1[\subset \bigcup_k I_k$ with $\nu^1_{i,j}(I_k) < \varepsilon$ for each k. It follows that (see Figure 1) $\quad C_{i,j} \cap T \cap \mathcal{C} \subset \bigcup_k C_{i\mathbb{1},jk}$, where $\mathbb{1} = 11\ldots1$ and $|\mathbb{1}| = |k|$.

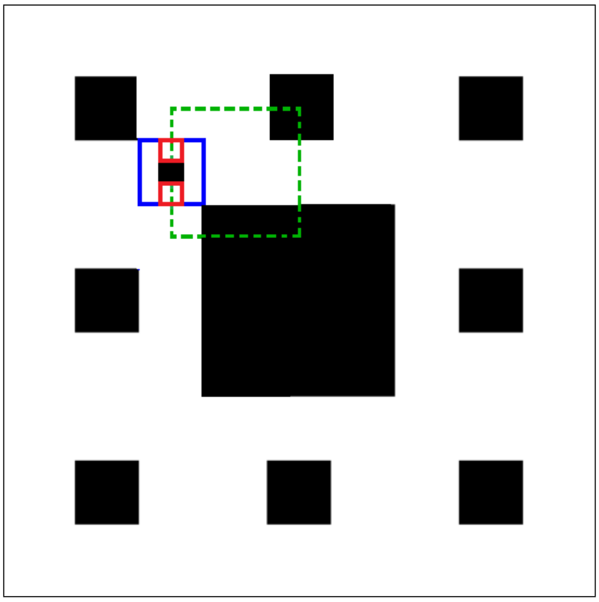

Figure 1: Sierpiński Carpet

Square $U \in \mathcal{U}_2$.
Triadic square $C_{i,j}$ of third generation such that $C_{i,j} \cap \partial U \cap T \neq \emptyset$.
Triadic squares $C_{i1,j0}$ and $C_{i1,j2}$.

Moreover, $\mu\left(C_{i\mathbb{1},jk}\right) = \nu_{i,j}^1(I_k) < \varepsilon$ for all k with $|k| = |\mathbb{1}|$. For $\alpha > 0$, we have:

$$\mathcal{H}^\alpha_{\mu,\varepsilon}(C_{i,j} \cap T \cap \mathcal{C}) \leqslant \sum_k \left(\mu(C_{i\mathbb{1},jk})\right)^\alpha = \sum_k \left(\nu_{i,j}^1(I_k)\right)^\alpha$$

Therefore:

$$\mathcal{H}^\alpha_{\mu,\varepsilon}(C_{i,j} \cap T \cap \mathcal{C}) \leqslant \mathcal{H}^\alpha_{\nu_{i,j}^1,\varepsilon}(K \cap [0,1[),$$

When ε goes to zero, we obtain:

$$\mathcal{H}^\alpha_{\mu}(C_{i,j} \cap T \cap \mathcal{C}) \leqslant \mathcal{H}^\alpha_{\nu_{i,j}^1}(K \cap [0,1[).$$

Subsequently,

$$\dim_\mu(C_{i,j} \cap T \cap \mathcal{C}) \leqslant \dim_{\nu_{i,j}^1}(K \cap [0,1[) \leqslant \dim_{\nu_{i,j}^1}(K).$$

Clearly, taking account of convention $\dim_\mu(\emptyset) = -1$, the previous inequality is still valid if $C_{i,j} \cap T \cap \mathcal{C} = \emptyset$.

On the other hand:

$$T \cap \mathcal{C} \cap [0,1[^2 \subset \bigcup_{|i|=|j|=n} C_{i,j}$$

Thus:

$$T \cap \mathcal{C} \cap [0,1[^2 = \bigcup_{|i|=|j|=n} C_{i,j} \cap T \cap \mathcal{C}$$

Then we obtain:

$$\dim_\mu(T \cap \mathcal{C} \cap [0,1[^2) = \sup_{|i|=|j|=n} \dim_\mu(C_{i,j} \cap T \cap \mathcal{C}) \leqslant \sup_{|i|=|j|=n} \dim_{\nu^1_{i,j}}(K).$$

Since μ vanishes on the triadic squares that are not contained in $[0,1[^2$, then:

$$\dim_\mu(T \cap \mathcal{C}) = \dim_\mu(T \cap \mathcal{C} \cap [0,1[^2).$$

Thus:

$$\dim_\mu(T \cap \mathcal{C}) \leqslant \sup_{|i|=|j|=n} \dim_{\nu^1_{i,j}}(K).$$

As a consequence:

$$\dim_\mu(T \cap \mathcal{C}) \leqslant \inf_{n \geqslant p} \sup_{|i|=|j|=n} \dim_{\nu^1_{i,j}}(K).$$

If $\mathcal{C}$ is a horizontal cloister, we analogously obtain:

$$\dim_\mu(T \cap \mathcal{C}) \leqslant \inf_{n \geqslant p} \sup_{|i|=|j|=n} \dim_{\nu^2_{i,j}}(K).$$

Finally, we have:

$$\dim_\mu(T \cap \partial U) \leqslant \inf_{n \geqslant p} \sup_{\substack{|i|=|j|=n \\ l=1,2}} \dim_{\nu^l_{i,j}}(K).$$

∎

4.3. Equality case. Let us recall that we have proved in Theorem 4.1 and Theorem 4.3 the following inequalities:

$$\begin{aligned} 1 + \sup\left(\dim_{W_1}(K), \dim_{W_2}(K)\right) &\leqslant \dim_{t\mu}(T) \\ &\leqslant 1 + \liminf_{n \to +\infty} \sup_{\substack{|i|=|j|=n \\ l=1,2}} \dim_{\nu^l_{i,j}}(K). \end{aligned} \tag{4.7}$$

In this section, we give an example of measure μ, where the equality holds between the upper and lower bounds of the μ-topological Hausdorff dimension of T. For this purpose, let $\left(p_{i,j}\right)_{i,j\in\mathcal{A}}$ be a square matrix of order 3 such that for each $i,j\in\mathcal{A}$, $p_{i,j}>0$ and $\sum\limits_{0\leqslant i,j\leqslant 2} p_{i,j}=1$.

We consider the Bernoulli measure supported on $[0,1[^2$ and defined by:

$$\mu(C_{i,j})=\prod_{k=1}^{n} p_{i_k,j_k}$$

where $i=i_1i_2\dots i_n$ and $j=j_1j_2\dots j_n$. Choose δ and β as two positive real numbers such that:

$$p_{1,0}^{\delta}+p_{1,2}^{\delta}=1 \qquad \text{and} \qquad p_{0,1}^{\beta}+p_{2,1}^{\beta}=1.$$

Theorem 4.4. (Equality case) *If matrix $(p_{i,j})$ satisfies*

$$\begin{aligned} p_{0,1}&\leqslant \min(p_{0,0},p_{0,2}), \qquad & p_{2,1}&\leqslant \min(p_{2,0},p_{2,2}),\\ p_{1,0}&\leqslant \min(p_{0,0},p_{2,0}), \qquad & p_{1,2}&\leqslant \min(p_{0,2},p_{2,2}), \end{aligned} \tag{4.8}$$

then

$$\dim_{t\mu}(T)=1+\sup(\beta,\delta).$$

Remark. A class of matrices satisfying (4.8) is:

$$A=\begin{pmatrix} \frac{1-4a}{5} & a & \frac{1-4a}{5}\\ a & \frac{1-4a}{5} & a\\ \frac{1-4a}{5} & a & \frac{1-4a}{5} \end{pmatrix},$$

where $0<a\leqslant \frac{1}{9}$. Note that this class of matrices contains Lebesgue measure (case when $a=\frac{1}{9}$.)

Proof. The proof of Theorem 4.4 is split into three steps:

Step 1. We begin by proving that for all $i,j\in\mathcal{A}^*$ such that $|i|=|j|$, we have:

$$\dim_{\nu^1_{i,j}}(K)=\delta \qquad \text{and} \qquad \dim_{\nu^2_{i,j}}(K)=\beta.$$

Let $\tilde{K}$ be the middle-thirds Cantor set deprived of extremities of triadic intervals. Therefore, we obtain:

$$\dim_{\nu^1_{i,j}}(K)=\dim_{\nu^1_{i,j}}(\tilde{K}).$$

Let $i = i_1 i_2 \dots i_q$, $j = j_1 j_2 \dots j_q \in \mathcal{A}^*$. It is observed that if $I_{k_1 k_2 \dots k_n}$ is a triadic interval crossing $\tilde{K}$, then for all $l \in \{1, 2, \dots n\}$, we have $k_l \in \{0, 2\}$ and

$$\left(\nu^1_{i,j}(I_{k_1 k_2 \dots k_n})\right)^\delta = \xi^\delta \; p^\delta_{1,k_1} \; p^\delta_{1,k_2} \; \dots p^\delta_{1,k_n},$$

where $\xi = \prod\limits_{k=1}^{q} p_{i_k, j_k} > 0$. Thus, function $(\nu^1_{i,j})^\delta$ behaves as a measure. Then for all disjoint covering $\{I_s\}_s$ of $\tilde{K}$ by triadic intervals we have:

$$\sum_s \left(\nu^1_{i,j}(I_s)\right)^\delta = \xi^\delta$$

Whence, $\dim_{\nu^1_{i,j}}(K) = \delta$. Similarly, we prove that $\dim_{\nu^2_{i,j}}(K) = \beta$.

Step 2. Now we verify that $\dim_{W_1}(K) = \alpha$ and $\dim_{W_2}(K) = \rho$.

Put for all $k \in \{0, 2\}$:

$$l_k = \min_{0 \leqslant j \leqslant 2} p_{k,j} \qquad \text{and} \qquad t_k = \min_{0 \leqslant j \leqslant 2} p_{j,k}.$$

Choose α and ρ as two positive real numbers such that:

$$l_0^\alpha + l_2^\alpha = 1 \qquad \text{and} \qquad t_0^\rho + t_2^\rho = 1.$$

By step 1, we have:

$$\dim_{W_1}(K) = \dim_{W_1}(\tilde{K}).$$

It is remarkable that if $I_{i_1 i_2 \dots i_n}$ is a triadic interval that crosses $\tilde{K}$, then for all $k \in \{1, 2, \dots n\}$, we have $i_k \in \{0, 2\}$. Therefore:

$$(W_1(I_{i_1 i_2 \dots i_n}))^\alpha = l^\alpha_{i_1} \, l^\alpha_{i_2} \, \dots \, l^\alpha_{i_n}.$$

Hence, W_1^α behaves as a measure. Consequently, if $\{I_s\}_s$ is a covering of $\tilde{K}$ by disjoint triadic intervals, we have:

$$\sum_s (W_1(I_s))^\alpha = 1.$$

It results that $\mathcal{H}^\alpha_{W_1}(\tilde{K}) = 1$. Thus, $\dim_{W_1}(\tilde{K}) = \alpha$, and then:

$$\dim_{W_1}(K) = \alpha.$$

Analogously, we prove that $\dim_{W_2}(K) = \rho$.

Step 3. Finally, from conditions (4.8), we obtain:

$$\begin{aligned} l_0 &= \min_{0\leqslant j\leqslant 2} p_{0,j} = p_{0,1} \\ l_2 &= \min_{0\leqslant j\leqslant 2} p_{2,j} = p_{2,1} \end{aligned} \qquad \text{and} \qquad \begin{aligned} t_0 &= \min_{0\leqslant j\leqslant 2} p_{j,0} = p_{1,0} \\ t_2 &= \min_{0\leqslant j\leqslant 2} p_{j,2} = p_{1,2} \end{aligned}$$

By hypothesis, we have:

$$\begin{aligned} l_0^\alpha + l_2^\alpha &= 1 \\ p_{0,1}^\beta + p_{2,1}^\beta &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} t_0^\rho + t_2^\rho &= 1 \\ p_{1,0}^\delta + p_{1,2}^\delta &= 1 \end{aligned}$$

Then:

$$\alpha = \beta \qquad \text{and} \qquad \rho = \delta.$$

Thus, based on (4.7), we obtain:

$$1 + \sup(\alpha, \rho) \leqslant \dim_{t\mu}(T) \leqslant 1 + \sup(\delta, \beta).$$

Hence, the result yields. ∎

Corollary 4.5.

$$\dim_{tH}(T) = \frac{\ln 6}{\ln 3}.$$

Proof. Matrix $(p_{i,j})_{i,j\in\mathcal{A}}$, where $p_{i,j} = \frac{1}{9}$ for all $i, j \in \mathcal{A}$ satisfies (4.8). Then by Theorem 4.4, and seeing that $\alpha = \rho = \frac{\ln 2}{2\ln 3}$, we have:

$$\dim_{t\mu}(T) = 1 + \frac{\ln 2}{2\ln 3}.$$

Let us recall that triadic squares allow the calculation of the Hausdorff dimension (see [16]).

It is noticeable that if we choose $\mu(C) = \frac{1}{2}|C|^2$, where C is a triadic square in $[0,1[^2$, we have for all $A \subset [0,1[^2$:

$$\dim_H(A) = 2\dim_\mu(A).$$

As a result:

$$\dim_{tH}(T) = 2\dim_{t\mu}(T) - 1.$$

This achieves the proof of this corollary. ∎

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