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Additivity of elementary maps on gamma rings

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Abstract: Let $\mathfrak{M}$ and $\mathfrak{M}'$ be Gamma rings, respectively. We study the additivity of surjective elementary maps of $\mathfrak{M} \times \mathfrak{M}'$. We prove that if $\mathfrak{M}$ contains a non-trivial γ-idempotent satisfying some conditions, then they are additive.

Key words: Elementary maps, Gamma rings, additivity.

AMS *Subject Class.* (2010): 16Y99.

1. Gamma rings and elementary maps

Let $\mathfrak{M}$ and Γ be two abelian groups. We call $\mathfrak{M}$ a Γ-*ring* if the following conditions are satisfied:

(i) $x\alpha y \in \mathfrak{M}$,

(ii) $(x+y)\alpha z = x\alpha z + y\alpha z,\ x\alpha(y+z) = x\alpha y + x\alpha z$,

(iii) $x(\alpha+\beta)y = x\alpha y + x\beta y$,

(iv) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

for all $x, y, z \in \mathfrak{M}$ and $\alpha, \beta \in \Gamma$.

N. Nobusawa introduced the notion of a Γ-ring, more general than a ring in his paper entitled "On a generalization of the ring theory". For those readers who are not familiar with this language of Γ-rings we recommend "On a generalization of the ring theory" and "On the Γ-rings of Nobusawa" [2] and [1] respectively. Our purpose in this paper is the study of the additivity of a specific application on Γ-rings, for this we will address some preliminary definitions.

A nonzero element $1 \in \mathfrak{M}$ is called a multiplicative γ-identity of $\mathfrak{M}$ or γ-*unity element* (for some $\gamma \in \Gamma$) if $1\gamma x = x\gamma 1 = x$ for all $x \in \mathfrak{M}$. A nonzero element $e_1 \in \mathfrak{M}$ is called a γ_1-*idempotent* (for some $\gamma_1 \in \Gamma$) if $e_1\gamma_1 e_1 = e_1$ and

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a *nontrivial* γ_1*-idempotent* if it is a γ_1-idempotent different from multiplicative γ_1-identity element of $\mathfrak{M}$.

Let Γ, Γ', $\mathfrak{M}$ and $\mathfrak{M}'$ be additive groups such that $\mathfrak{M}$ is a Γ-ring and $\mathfrak{M}'$ is a Γ'-ring. Let $M : \mathfrak{M} \to \mathfrak{M}'$ and $M^* : \mathfrak{M}' \to \mathfrak{M}$ be two maps and $\phi : \Gamma \to \Gamma'$, $\phi^* : \Gamma' \to \Gamma$ two bijective maps. We call the ordered pair (M, M^*) an *elementary map* of $\mathfrak{M} \times \mathfrak{M}'$ if

$$M(a\alpha M^*(x)\beta b) = M(a)\phi(\alpha)x\phi(\beta)M(b),$$
$$M^*(x\mu M(a)\nu y) = M^*(x)\phi^*(\mu)a\phi^*(\nu)M^*(y)$$

for all $\alpha, \beta \in \Gamma$, $a, b \in \mathfrak{M}$, $\mu, \nu \in \Gamma'$ and $x, y \in \mathfrak{M}'$.

We say that the elementary map (M, M^*) of $\mathfrak{M} \times \mathfrak{M}'$ is *additive* (resp., *injective, surjective, bijective*) if both maps M and M^* are additive (resp., injective, surjective, bijective).

Let $\mathfrak{M}$ and Γ be two abelian groups such that $\mathfrak{M}$ is a Γ-ring and $e_1 \in \mathfrak{M}$ a nontrivial γ_1-idempotent. Let us consider $e_2\colon \Gamma \times \mathfrak{M} \to \mathfrak{M}$, $e_2'\colon \mathfrak{M} \times \Gamma \to \mathfrak{M}$ two $\mathfrak{M}$-additive maps such that $e_2(\gamma_1, a) = a - e_1\gamma_1 a$, $e_2'(a, \gamma_1) = a - a\gamma_1 e_1$. Let us denote $e_2\alpha a = e_2(\alpha, a)$, $a\alpha e_2 = e_2'(a, \alpha)$, $1_1\alpha a = e_1\alpha a + e_2\alpha a$, $a\alpha 1_1 = a\alpha e_1 + a\alpha e_2$ and suppose $(a\alpha e_2)\beta b = a\alpha(e_2\beta b)$ for all $\alpha, \beta \in \Gamma$ and $a, b \in \mathfrak{M}$. Then $1_1\gamma_1 a = a\gamma_1 1_1 = a$ and $(a\alpha 1_1)\beta b = a\alpha(1_1\beta b)$, for all $\alpha, \beta \in \Gamma$ and $a, b \in \mathfrak{M}$, and $\mathfrak{M}$ has a Peirce decomposition $\mathfrak{M} = \mathfrak{M}_{11} \oplus \mathfrak{M}_{12} \oplus \mathfrak{M}_{21} \oplus \mathfrak{M}_{22}$, where $\mathfrak{M}_{ij} = e_i\gamma_1\mathfrak{M}\gamma_1 e_j$ $(i, j = 1, 2)$, satisfying the multiplicative relations:

(i) $\mathfrak{M}_{ij}\Gamma\mathfrak{M}_{jl} \subseteq \mathfrak{M}_{il}(i, j, l = 1, 2)$;

(ii) $\mathfrak{M}_{ij}\gamma_1\mathfrak{M}_{kl} = 0$ if $j \neq k$ $(i, j, k, l = 1, 2)$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are subsets of a Γ-ring $\mathfrak{M}$ and $\Theta \subseteq \Gamma$, we denote $\mathfrak{A}\Theta\mathfrak{B}$ the subset of $\mathfrak{M}$ consisting of all finite sums of the form $\sum_i a_i\gamma_i b_i$ where $a_i \in \mathfrak{A}$, $\gamma_i \in \Theta$ and $b_i \in \mathfrak{B}$. A *right ideal* (resp., *left ideal*) of a Γ-ring $\mathfrak{M}$ is an additive subgroup $\mathfrak{I}$ of $\mathfrak{M}$ such that $\mathfrak{I}\Gamma\mathfrak{M} \subseteq \mathfrak{I}$ (resp., $\mathfrak{M}\Gamma\mathfrak{I} \subseteq \mathfrak{I}$). If $\mathfrak{I}$ is both a right and a left ideal of $\mathfrak{M}$, then we say that $\mathfrak{I}$ is an *ideal* or *two-side ideal* of $\mathfrak{M}$.

An ideal $\mathfrak{P}$ of a Γ-ring $\mathfrak{M}$ is called *prime* if for any ideals $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{M}$, $\mathfrak{A}\Gamma\mathfrak{B} \subseteq \mathfrak{P}$ implies that $\mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{B} \subseteq \mathfrak{P}$. A Γ-ring $\mathfrak{M}$ is said to be *prime* if the zero ideal is prime.

THEOREM 1.1. ([9, THEOREM 4]) *If $\mathfrak{M}$ is a Γ-ring, the following conditions are equivalent:*

(i) *$\mathfrak{M}$ is a prime Γ-ring;*

(ii) *if $a, b \in \mathfrak{M}$ and $a\Gamma\mathfrak{M}\Gamma b = 0$, then $a = 0$ or $b = 0$.*

The first result about the additivity of maps on rings was given by Martindale III in an excellent paper [10]. He established a condition on a ring $\mathfrak{M}$ such that every multiplicative bijective map on $\mathfrak{M}$ is additive. Li and Lu [8] also considered this question in the context of prime associative rings containing a nontrivial idempotent. They proved the following theorem.

THEOREM 1.2. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two associative rings. Suppose that $\mathfrak{M}$ is a 2-torsion free ring containing a family $\{e_\alpha : \alpha \in \Lambda\}$ of idempotents which satisfies:*

(i) *If $x \in \mathfrak{M}$ is such that $x\mathfrak{M} = 0$, then $x = 0$;*

(ii) *If $x \in \mathfrak{M}$ is such that $e_\alpha \mathfrak{M} x = 0$ for all $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{M}x = 0$ implies $x = 0$);*

(iii) *For each $\alpha \in \Lambda$ and $x \in \mathfrak{M}$, if $e_\alpha x e_\alpha \mathfrak{M}(1 - e_\alpha) = 0$ then $e_\alpha x e_\alpha = 0$.*

Then every surjective elementary map (M, M^*) of $\mathfrak{M} \times \mathfrak{M}'$ is additive.

During the last decade, many mathematicians devoted to study the additivity of maps on associative rings. However, is very difficult to say anything when these applications are defined on arbitrary rings which are not necessarily associative. For the reader interested in applications defined in non-associative rings we recommend some papers [3, 4, 5, 6, 7]. Thus this motivated us in the present paper takes up the special case of an Γ-ring. We investigate the problem of when a elementary map must be an additive map on the class of Γ-rings.

2. The main result

We will prove that every surjective elementary map (M, M^*) of $\mathfrak{M} \times \mathfrak{M}'$ is additive for this we will assume that $\mathfrak{M}$ contains a family $\{e_\alpha : \alpha \in \Lambda\}$ of γ_α-idempotents satisfying some conditions. Our main result reads as follows.

THEOREM 2.1. *Let Γ, Γ', $\mathfrak{M}$ and $\mathfrak{M}'$ be additive groups such that $\mathfrak{M}$ is a Γ-ring and $\mathfrak{M}'$ is a Γ'-ring. Suppose that $\mathfrak{M}$ contains a family $\{e_\alpha : \alpha \in \Lambda\}$ of γ_α-idempotents which satisfies:*

(i) *If $x \in \mathfrak{M}$ is such that $x\Gamma\mathfrak{M} = 0$, then $x = 0$;*

(ii) *If $x \in \mathfrak{M}$ is such that $e_\alpha \gamma_\alpha \mathfrak{M}\Gamma x = 0$ for all $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{M}\Gamma x = 0$ implies $x = 0$);*

(iii) *For each* $\alpha \in \Lambda$ *and* $x \in \mathfrak{M}$, *if* $(e_\alpha\gamma_\alpha x\gamma_\alpha e_\alpha)\Gamma\mathfrak{M}\Gamma(1_\alpha - e_\alpha) = 0$ *then* $e_\alpha\gamma_\alpha x\gamma_\alpha e_\alpha = 0$.

Then every surjective elementary map (M, M^*) *of* $\mathfrak{M} \times \mathfrak{M}'$ *is additive.*

The following lemmas has the same hypotheses of Theorem 2.1 and we need these lemmas for the proof of this theorem. Thus, let us consider $e_1 \in \{e_\alpha : \alpha \in \Lambda\}$ a nontrivial γ_1-idempotent of $\mathfrak{M}$ and $1_1 = e_1 + e_2$. We begin with the following trivial lemma

Lemma 2.1. $M(0) = 0$ *and* $M^*(0) = 0$.

Proof. $M(0) = M(0\alpha M^*(0)\beta 0) = M(0)\phi(\alpha)0\phi(\beta)M(0) = 0$. Similarly, we have $M^*(0) = 0$. ∎

Lemma 2.2. M *and* M^* *are bijective.*

Proof. It suffices to prove that M and M^* are injective. First show that M is injective. Let x_1 and x_2 be in $\mathfrak{M}$ and suppose that $M(x_1) = M(x_2)$. Since $M^*(u\mu M(x_i)\nu v) = M^*(u)\phi^*(\mu)x_i\phi^*(\nu)M^*(v)$ $(i = 1, 2)$ for all $\mu, \nu \in \Gamma'$ and $u, v \in \mathfrak{M}'$, it follows that $M^*(u)\phi^*(\mu)x_1\phi^*(\nu)M^*(v) = M^*(u)\phi^*(\mu)x_2\phi^*(\nu)M^*(v)$. Hence from the surjectivity of ϕ^* and M^* and conditions (i) and (ii) we conclude that $x_1 = x_2$. Now we turn to proving the injectivity of M^*. Let u_1 and u_2 be in $\mathfrak{M}'$ and suppose $M^*(u_1) = M^*(u_2)$. Since

$$\begin{aligned} M^*M(x\alpha M^*(u_i)\beta y) &= M^*\big(M(x)\phi(\alpha)u_i\phi(\beta)M(y)\big) \\ &= M^*\big(M(x)\phi(\alpha)MM^{-1}(u_i)\phi(\beta)M(y)\big) \\ &= M^*M(x)\phi^*\phi(\alpha)M^{-1}(u_i)\phi^*\phi(\beta)M^*M(y) \end{aligned}$$

for all $\alpha, \beta \in \Gamma$ and $x, y \in \mathfrak{M}$, it follows that

$$\begin{aligned} &M^*M(x)\phi^*\phi(\alpha)M^{-1}(u_1)\phi^*\phi(\beta)M^*M(y) \\ &\qquad = M^*M(x)\phi^*\phi(\alpha)M^{-1}(u_2)\phi^*\phi(\beta)M^*M(y). \end{aligned}$$

Noting that $\phi^*\phi$ and M^*M are also surjective, we see that $M^{-1}(u_1) = M^{-1}(u_2)$, by conditions (i) and (ii). Consequently $u_1 = u_2$. ∎

LEMMA 2.3. *The pair* (M^{*-1}, M^{-1}) *is an elementary map of* $\mathfrak{M} \times \mathfrak{M}'$, *that is, the maps* $M^{*-1} : \mathfrak{M} \to \mathfrak{M}'$ *and* $M^{-1} : \mathfrak{M}' \to \mathfrak{M}$ *satisfy*

$$M^{*-1}\big(a\alpha M^{-1}(x)\beta b\big) = M^{*-1}(a)\phi^{*-1}(\alpha)x\phi^{*-1}(\beta)M^{*-1}(b),$$
$$M^{-1}\big(x\mu M^{*-1}(a)\nu y\big) = M^{-1}(x)\phi^{-1}(\mu)a\phi^{-1}(\nu)M^{-1}(y)$$

for all $\alpha, \beta \in \Gamma$, $\mu, \nu \in \Gamma'$, $a, b \in \mathfrak{M}$ *and* $x, y \in \mathfrak{M}'$.

Proof. The first equality can follow from

$$\begin{aligned} &M^*\big(M^{*-1}(a)\phi^{*-1}(\alpha)x\phi^{*-1}(\beta)M^{*-1}(b)\big) \\ &\qquad = M^*\big(M^{*-1}(a)\phi^{*-1}(\alpha)MM^{-1}(x)\phi^{*-1}(\beta)M^{*-1}(b)\big) \\ &\qquad = a\phi^*\big(\phi^{*-1}(\alpha)\big)M^{-1}(x)\phi^*\big(\phi^{*-1}(\beta)\big)b \\ &\qquad = a\alpha M^{-1}(x)\beta b \end{aligned}$$

and the second equality follows in a similar way. ∎

LEMMA 2.4. *Let* $s, a, b \in \mathfrak{M}$ *such that* $M(s) = M(a) + M(b)$. *Then*

(i) $M(s\alpha x\beta y) = M(a\alpha x\beta y) + M(b\alpha x\beta y)$ *for* $\alpha, \beta \in \Gamma$ *and* $x, y \in \mathfrak{M}$;

(ii) $M(x\alpha y\beta s) = M(x\alpha y\beta a) + M(x\alpha y\beta b)$ *for* $\alpha, \beta \in \Gamma$ *and* $x, y \in \mathfrak{M}$;

(iii) $M^{*-1}(x\alpha s\beta y) = M^{*-1}(x\alpha a\beta y) + M^{*-1}(x\alpha b\beta y)$ *for* $\alpha, \beta \in \Gamma$ *and* $x, y \in \mathfrak{M}$ *for* $x, y \in \mathfrak{M}$.

Proof. (i) Let $\alpha, \beta \in \Gamma$ and $x, y \in \mathfrak{M}$. Then

$$\begin{aligned} M(s\alpha x\beta y) &= M\big(s\alpha M^*M^{*-1}(x)\beta y\big) \\ &= M(s)\phi(\alpha)M^{*-1}(x)\phi(\beta)M(y) \\ &= \big(M(a) + M(b)\big)\phi(\alpha)M^{*-1}(x)\phi(\beta)M(y) \\ &= M(a)\phi(\alpha)M^{*-1}(x)\phi(\beta)M(y) \\ &\quad + M(b)\phi(\alpha)M^{*-1}(x)\phi(\beta)M(y) \\ &= M(a\alpha x\beta y) + M(b\alpha x\beta y). \end{aligned}$$

(ii) The proof is similar to (i).

(iii) Let $x, y \in \mathfrak{M}$. By Lemma 2.3

$$\begin{aligned}
M^{*-1}(x\alpha s\beta y) &= M^{*-1}(x\alpha M^{-1}M(s)\beta y) \\
&= M^{*-1}(x)\phi^{*-1}(\alpha)M(s)\phi^{*-1}(\beta)M^{*-1}(y) \\
&= M^{*-1}(x)\phi^{*-1}(\alpha)\big(M(a) + M(b)\big)\phi^{*-1}(\beta)M^{*-1}(y) \\
&= M^{*-1}(x)\phi^{*-1}(\alpha)M(a)\phi^{*-1}(\beta)M^{*-1}(y) \\
&\quad + M^{*-1}(x)\phi^{*-1}(\alpha)M(b)\phi^{*-1}(\beta)M^{*-1}(y) \\
&= M^{*-1}(x\alpha a\beta y) + M^{*-1}(x\alpha b\beta y).
\end{aligned}$$

The proof is complete. ∎

LEMMA 2.5. *The following are true:*

(i) $M(a_{11} + a_{12} + a_{21} + a_{22}) = M(a_{11}) + M(a_{12}) + M(a_{21}) + M(a_{22})$;

(ii) $M^{*-1}(a_{11} + a_{12} + a_{21} + a_{22}) = M^{*-1}(a_{11}) + M^{*-1}(a_{12}) + M^{*-1}(a_{21}) + M^{*-1}(a_{22})$.

Proof. By the surjectivity of M, there exists $s \in \mathfrak{M}$ such that $M(s) = M(a_{11}) + M(a_{12}) + M(a_{21}) + M(a_{22})$. Now, for arbitrary $\alpha, \beta \in \Gamma$, $x_{i1} \in \mathfrak{M}_{i1}$ and $y_{1j} \in \mathfrak{M}_{1j}$, we have

$$\begin{aligned}
&M^{*-1}(x_{i1}\alpha e_1\gamma_1 s\gamma_1 e_1\beta y_{1j}) \\
&\qquad = M^{*-1}(x_{i1}\alpha e_1\gamma_1 a_{11}\gamma_1 e_1\beta y_{1j}) + M^{*-1}(x_{i1}\alpha e_1\gamma_1 a_{12}\gamma_1 e_1\beta y_{1j}) \\
&\qquad\quad + M^{*-1}(x_{i1}\alpha e_1\gamma_1 a_{21}\gamma_1 e_1\beta y_{1j}) + M^{*-1}(x_{i1}\alpha e_1\gamma_1 a_{22}\gamma_1 e_1\beta y_{1j}) \\
&\qquad = M^{*-1}(x_{i1}\alpha e_1\gamma_1 a_{11}\gamma_1 e_1\beta y_{1j}),
\end{aligned}$$

which implies

$$x_{i1}\alpha e_1\gamma_1\big(s - (a_{11} + a_{12} + a_{21} + a_{22})\big)\gamma_1 e_1\beta y_{1j} = 0. \tag{2.1}$$

In a similar way, for $y_{2j} \in \mathfrak{M}_{2j}$ we get that

$$x_{i1}\alpha e_1\gamma_1\big(s - (a_{11} + a_{12} + a_{21} + a_{22})\big)\gamma_1 e_1\beta y_{2j} = 0. \tag{2.2}$$

From (2.1) and (2.2) we conclude

$$x_{i1}\alpha e_1\gamma_1\big(s - (a_{11} + a_{12} + a_{21} + a_{22})\big)\gamma_1 e_1\beta y = 0.$$

In a similar way, for $y_{1j} \in \mathfrak{M}_{1j}$ and $y_{2j} \in \mathfrak{M}_{2j}$ we get that

$$x_{i1}\alpha e_1\gamma_1\big(s-(a_{11}+a_{12}+a_{21}+a_{22})\big)\gamma_1 e_2\beta y_{1j}=0,$$
$$x_{i1}\alpha e_1\gamma_1\big(s-(a_{11}+a_{12}+a_{21}+a_{22})\big)\gamma_1 e_2\beta y_{2j}=0,$$

respectively, which implies

$$x_{i1}\alpha e_1\gamma_1\big(s-(a_{11}+a_{12}+a_{21}+a_{22})\big)\gamma_1 e_2\beta y=0.$$

Thus,

$$x_{i1}\alpha e_1\gamma_1\big(s-(a_{11}+a_{12}+a_{21}+a_{22})\big)\gamma_1 1_1\beta y=0,$$

for all $\beta \in \Gamma$, $y \in \mathfrak{M}$, that is,

$$x_{i1}\alpha e_1\gamma_1\big(s-(a_{11}+a_{12}+a_{21}+a_{22})\big)\Gamma\mathfrak{M}=0.$$

By condition (i) of the Theorem we have

$$x_{i1}\alpha e_1\gamma_1\big(s-(a_{11}+a_{12}+a_{21}+a_{22})\big)=0.$$

Repeating the above arguments, for $x_{i2} \in \mathfrak{M}_{i2}$ we get that

$$x_{i2}\alpha e_1\gamma_1\big(s-(a_{11}+a_{12}+a_{21}+a_{22})\big)=0$$

which implies

$$x\alpha e_1\gamma_1\big(s-(a_{11}+a_{12}+a_{21}+a_{22})\big)=0.$$

Similarly, we obtain

$$x\alpha e_2\gamma_1\big(s-(a_{11}+a_{12}+a_{21}+a_{22})\big)=0,$$

which yields

$$x\alpha\big(s-(a_{11}+a_{12}+a_{21}+a_{22})\big)=0.$$

Thus, $\mathfrak{M}\Gamma\big(s-(a_{11}+a_{12}+a_{21}+a_{22})\big)=0$ which results in

$$s=a_{11}+a_{12}+a_{21}+a_{22}.$$

The proof of (ii) is similar, since the pair $(M^{*-1}; M^{-1})$ is also an elementary map of $\mathfrak{M} \times \mathfrak{M}'$. ∎

Lemma 2.6. *The following are true:*

(i) $M(a_{12}+b_{12}\alpha a_{22})=M(a_{12})+M(b_{12}\alpha a_{22})$;

(ii) $M(a_{11}+a_{12}\alpha a_{21})=M(a_{11})+M(a_{12}\alpha a_{21})$;

(iii) $M(a_{21}+a_{22}\alpha b_{21})=M(a_{21})+M(a_{22}\alpha b_{21})$.

Proof. (i) From Lemma 2.5-(i) and (ii) we have

$$\begin{aligned}
&M(a_{12}+b_{12}\alpha a_{22})\\
&\qquad=M\big(e_1\gamma_1(e_1+b_{12})\gamma_1(a_{12}+e_2\alpha a_{22})\big)\\
&\qquad=M(e_1)\phi(\gamma_1)M^{*-1}(e_1+b_{12})\phi(\gamma_1)M(a_{12}+e_2\alpha a_{22})\\
&\qquad=M(e_1)\phi(\gamma_1)\big(M^{*-1}(e_1)+M^{*-1}(b_{12})\big)\phi(\gamma_1)\big(M(a_{12})+M(e_2\alpha a_{22})\big)\\
&\qquad=M(e_1)\phi(\gamma_1)M^{*-1}(e_1)\phi(\gamma_1)M(a_{12})\\
&\qquad\quad+M(e_1)\phi(\gamma_1)M^{*-1}(e_1)\phi(\gamma_1)M(e_2\alpha a_{22})\\
&\qquad\quad+M(e_1)\phi(\gamma_1)M^{*-1}(b_{12})\phi(\gamma_1)M(a_{12})\\
&\qquad\quad+M(e_1)\phi(\gamma_1)M^{*-1}(b_{12})\phi(\gamma_1)M(e_2\alpha a_{22})\\
&\qquad=M(e_1\gamma_1e_1\gamma_1a_{12})+M(e_1\gamma_1e_1\gamma_1a_{22})+M(e_1\gamma_1b_{12}\gamma_1a_{12})\\
&\qquad\quad+M(e_1\gamma_1b_{12}\gamma_1e_2\alpha a_{22})\\
&\qquad=M(a_{12})+M(b_{12}\alpha a_{22}).
\end{aligned}$$

Note that we use properties (i) and (ii) in

$$M(a_{12}+e_2\alpha a_{22})=M(a_{12})+M(e_2\alpha a_{22})$$

and

$$M^{*-1}(e_1+b_{12})=M^{*-1}(e_1)+M^{*-1}(b_{12}),$$

respectively. So (i) follows. Observing that

$$\begin{aligned}
a_{11}+a_{12}\alpha a_{21}&=(a_{11}+a_{12}\alpha e_2)\gamma_1(e_1+a_{21})\gamma_1e_1,\\
a_{21}+a_{22}\alpha a_{21}&=(a_{21}+a_{22}\alpha e_2)\gamma_1(e_1+b_{21})\gamma_1e_1,
\end{aligned}$$

then (ii) and (iii) can be proved similarly. ∎

Lemma 2.7. $M(a_{21}\gamma_1a_{12}+a_{22}\gamma_1b_{22})=M(a_{21}\gamma_1a_{12})+M(a_{22}\gamma_1b_{22})$.

Proof. We first claim that

$$\begin{aligned}
&M(a_{21}\gamma_1a_{12}\gamma_1c_{22}+a_{22}\gamma_1b_{22}\gamma_1c_{22})\\
&\qquad=M(a_{21}\gamma_1a_{12}\gamma_1c_{22})+M(a_{22}\gamma_1b_{22}\gamma_1c_{22})
\end{aligned}\tag{2.3}$$

holds for all $c_{22} \in \mathfrak{M}_{22}$. Indeed, from Lemma 2.5-(i) and (ii), we see that

$$\begin{aligned}
&M(a_{21}\gamma_1 a_{12}\alpha c_{22} + a_{22}\gamma_1 b_{22}\alpha c_{22})\\
&\qquad = M\big((a_{21} + a_{22})\gamma_1(a_{12} + b_{22})\alpha c_{22}\big)\\
&\qquad = M(a_{21} + a_{22})\phi(\gamma_1)M^{*-1}(a_{12} + b_{22})\phi(\alpha)M(c_{22})\\
&\qquad = \big(M(a_{21}) + M(a_{22})\big)\phi(\gamma_1)\big(M^{*-1}(a_{12}) + M^{*-1}(b_{22})\big)\phi(\alpha)M(c_{22})\\
&\qquad = M(a_{21})\phi(\gamma_1)M^{*-1}(a_{12})\phi(\alpha)M(c_{22})\\
&\qquad\quad + M(a_{21})\phi(\gamma_1)M^{*-1}(b_{22})\phi(\alpha)M(c_{22})\\
&\qquad\quad + M(a_{22})\phi(\gamma_1)M^{*-1}(a_{12})\phi(\alpha)M(c_{22})\\
&\qquad\quad + M(a_{22})\phi(\gamma_1)M^{*-1}(b_{22})\phi(\alpha)M(c_{22})\\
&\qquad = M(a_{21}\gamma_1 a_{12}\alpha c_{22}) + M(a_{21}\gamma_1 b_{22}\alpha c_{22}) + M(a_{22}\gamma_1 a_{12}\alpha c_{22})\\
&\qquad\quad + M(a_{22}\gamma_1 b_{22}\alpha c_{22})\\
&\qquad = M(a_{21}\gamma_1 a_{12}\alpha c_{22}) + M(a_{22}\gamma_1 b_{22}\alpha c_{22}),
\end{aligned}$$

as desired. Now we find $s \in \mathfrak{M}$ such that $M(s) = M(a_{21}\gamma_1 a_{12}) + M(a_{22}\gamma_1 b_{22})$.

For arbitrary element $x_{21} \in \mathfrak{M}_{21}$, by Lemma 2.4-(i) and Lemma 2.6-(iii),

$$\begin{aligned}
M(s\alpha x_{21}) &= M(s\alpha x_{21}\gamma_1 e_1)\\
&= M\big((a_{21}\gamma_1 a_{12})\alpha x_{21}\gamma_1 e_1\big) + M\big((a_{22}\gamma_1 b_{22})\alpha x_{21}\gamma_1 e_1\big)\\
&= M(a_{21}\gamma_1 a_{12}\alpha x_{21} + a_{22}\gamma_1 b_{22}\alpha x_{21}).
\end{aligned}$$

It follows that

$$\big(s - (a_{21}\gamma_1 a_{12} + a_{22}\gamma_1 b_{22})\big)\alpha x_{21} = 0, \tag{2.4}$$

for all $\alpha \in \Gamma$. Our next step will be to prove that

$$\big(s - (a_{21}\gamma_1 a_{12} + a_{22}\gamma_1 b_{22})\big)\alpha x_{22} = 0 \tag{2.5}$$

holds for every $x_{22} \in \mathfrak{M}_{22}$. First, for y_{21}, by Lemmas 2.4-(i) and Lemma 2.6-(iii)

$$\begin{aligned}
M(s\alpha x_{22}\beta y_{21}) &= M\big((a_{21}\gamma_1 a_{12})\alpha x_{22}\beta y_{21}\big) + M\big((a_{22}\gamma_1 b_{22})\alpha x_{22}\beta y_{21}\big)\\
&= M\big((a_{21}\gamma_1 a_{12})\alpha x_{22}\beta y_{21} + (a_{22}\gamma_1 b_{22})\alpha x_{22}\beta y_{21}\big),
\end{aligned}$$

which implies that $s\alpha x_{22}\beta y_{21} = (a_{21}\gamma_1 a_{12})\alpha x_{22}\beta y_{21} + (a_{22}\gamma_1 b_{22})\alpha x_{22}\beta y_{21}$. It follows that $\big(s - (a_{21}\gamma_1 a_{12} + a_{22}\gamma_1 b_{22})\big)\alpha x_{22} \cdot \beta y_{21} = 0$.

For $y_{22} \in \mathfrak{M}_{22}$, by Lemma 2.4-(i) and (2.3) we have

$$\begin{aligned} M(s\alpha x_{22}\beta y_{22}) &= M\big((a_{21}\gamma_1 a_{12})\alpha x_{22}\beta y_{22}\big) + M\big((a_{22}\gamma_1 b_{22})\alpha x_{22}\beta y_{22}\big) \\ &= M\big((a_{21}\gamma_1 a_{12})\alpha x_{22}\beta y_{22}\big) + M\big((a_{22}\gamma_1 b_{22})\alpha x_{22}\beta y_{22}\big) \\ &= M\big(a_{21}\gamma_1 a_{12}\alpha x_{22}\beta y_{22}\big) + M\big((a_{22}\gamma_1 b_{22})\alpha x_{22}\beta y_{22}\big) \\ &= M\big(a_{21}\gamma_1 a_{12}\alpha x_{22}\beta y_{22} + (a_{22}\gamma_1 b_{22})\alpha x_{22}\beta y_{22}\big) \\ &= M\big((a_{21}\gamma_1 a_{12})\alpha x_{22}\beta y_{22} + (a_{22}\gamma_1 b_{22})\alpha x_{22}\beta y_{22}\big) \end{aligned}$$

yielding that $s\alpha x_{22}\beta y_{22} = (a_{21}\gamma_1 a_{12} + a_{22}\gamma_1 b_{22})\alpha x_{22}\beta y_{22}$. It follows that $\big(s - (a_{21}\gamma_1 a_{12} + a_{22}\gamma_1 b_{22})\big)\alpha x_{22}\beta y_{22} = 0$. Hence, we obtain that

$$\big(s - (a_{21}\gamma_1 a_{12} + a_{22}\gamma_1 b_{22})\big)\alpha x_{22} \cdot \Gamma\mathfrak{M} = 0.$$

So Eq. (2.5) follows by Theorem 2.1 condition (i).

From Eqs. (2.4) and (2.5), we can get that $\big(s-(a_{21}\gamma_1 a_{12}+a_{22}\gamma_1 b_{22})\big)\Gamma\mathfrak{M} = 0$. Therefore, $s = a_{21}\gamma_1 a_{12} + a_{22}\gamma_1 b_{22}$ by Theorem 2.1 condition (i) again. ∎

Taking Lemma 2.3 into account, still hold true when M is replaced by M^{*-1}, that is

Lemma 2.8. *The following are true:*

(i) $M^{*-1}(a_{12} + b_{12}\alpha a_{22}) = M^{*-1}(a_{12}) + M^{*-1}(b_{12}\alpha a_{22})$.
(ii) $M^{*-1}(a_{11} + a_{12}\alpha a_{21}) = M^{*-1}(a_{11}) + M^{*-1}(a_{12}\alpha a_{21})$.
(iii) $M^{*-1}(a_{21} + a_{22}\alpha b_{21}) = M^{*-1}(a_{21}) + M^{*-1}(a_{22}\alpha b_{21})$.
(iv) $M^{*-1}(a_{21}\gamma_1 a_{12} + a_{22}\gamma_1 b_{22}) = M^{*-1}(a_{21}\gamma_1 a_{12}) + M^{*-1}(a_{22}\gamma_1 b_{22})$.

Lemma 2.9. $M(a_{12} + b_{12}) = M(a_{12}) + M(b_{12})$.

Proof. Let $s \in \mathfrak{M}$ such that $M(s) = M(a_{12}) + M(b_{12})$.

For $x_{1j} \in \mathfrak{M}_{1j}$, applying Lemma 2.4-(i),

$$M(s\gamma_1 e_1\alpha x_{1j}) = M(a_{12}\gamma_1 e_1\alpha x_{1j}) + M(b_{12}\gamma_1 e_1\alpha x_{1j}) \;=\; 0.$$

These equations show that $s\gamma_1 e_1\alpha x_{1j} = 0 = (a_{12} + b_{12})\gamma_1 e_1\alpha x_{1j}$. Hence,

$$\big(s - (a_{12} + b_{12})\big)\gamma_1 e_1\alpha x_{1j} = 0.$$

For all $x_{2j} \in \mathfrak{M}_{2j}$

$$M^{*-1}(s\gamma_1 e_1\alpha x_{2j}) = M^{*-1}(a_{12}\gamma_1 e_1\alpha x_{2j}) + M^{*-1}(b_{12}\gamma_1 e_1\alpha x_{2j}) \;=\; 0,$$

which implies that

$$\big(s-(a_{12}+b_{12})\big)\gamma_1 e_1\alpha x_{2j}=0.$$

Thus

$$\big(s-(a_{12}+b_{12})\big)\gamma_1 e_1\alpha x=0,$$

for all $\alpha\in\Gamma$ and $x\in\mathfrak{M}$, which implies

$$\big(s-(a_{12}+b_{12})\big)\gamma_1 e_1\Gamma\mathfrak{M}=0.$$

For $y_{11}\in\mathfrak{M}_{11}$, applying Lemma 2.4-(i),(ii),

$$\begin{aligned}M(y_{11}\beta e_1\gamma_1 s\gamma_1 e_2\alpha x_{22}) &= M(y_{11}\beta e_1\gamma_1 a_{12}\gamma_1 e_2\alpha x_{22})\\ &\quad + M(y_{11}\beta e_1\gamma_1 b_{12}\gamma_1 e_2\alpha x_{22})\\ &= M(y_{11}\beta e_1\gamma_1 a_{12}\gamma_1 e_2\alpha x_{22})\\ &\quad + M(y_{11}\beta e_1\gamma_1 b_{12}\gamma_1 e_2\alpha x_{22})\\ &= M(y_{11}\beta e_1\gamma_1 (a_{12}+b_{12})\gamma_1 e_2\alpha x_{22})\end{aligned}$$

These equations show that

$$y_{11}\beta e_1\gamma_1\big(s-(a_{12}+b_{12})\big)\gamma_1 e_2\alpha x_{22}=0.$$

For all $y_{21}\in\mathfrak{M}_{21}$

$$\begin{aligned}M^{*-1}(y_{21}\beta e_1\gamma_1 s\gamma_1 e_2\alpha y_{21}) &= M^{*-1}(y_{21}\beta e_1\gamma_1 a_{12}\gamma_1 e_2\alpha x_{22})\\ &\quad + M^{*-1}(y_{21}\beta e_1\gamma_1 b_{12}\gamma_1 e_2\alpha x_{22})\\ &= M^{*-1}(y_{21}\beta e_1\gamma_1 a_{12}\gamma_1 e_2\alpha x_{22})\\ &\quad + M^{*-1}(y_{21}\beta e_1\gamma_1 y_{21}\beta e_1\gamma_1 b_{12}\gamma_1 e_2\alpha x_{22})\\ &= M^{*-1}(y_{21}\beta e_1\gamma_1 (a_{12}+b_{12})\gamma_1 e_2\alpha x_{22})\\ &= M^{*-1}\big(y_{21}\beta e_1\gamma_1 (a_{12}+b_{12})\gamma_1 e_2\alpha x_{22}\big)\end{aligned}$$

which implies that

$$y_{21}\beta e_1\gamma_1\big(s-(a_{12}+b_{12})\big)\gamma_1 e_2\alpha x_{22}=0.$$

For $y_{i2}\in\mathfrak{M}_{i2}$, applying Lemma 2.4-(i),(ii),

$$\begin{aligned}M(y_{i2}\beta e_1\gamma_1 s\gamma_1 e_2\alpha x_{22}) &= M(y_{i2}\beta e_1\gamma_1 a_{12}\gamma_1 e_2\alpha x_{22})\\ &\quad + M(y_{i2}\beta e_1\gamma_1 b_{12}\gamma_1 e_2\alpha x_{22}) \;=\; 0,\end{aligned}$$

which implies

$$y_{i2}\beta e_1\gamma_1 s\gamma_1 e_2\alpha x_{22} \;=\; 0 \;=\; y_{i2}\beta e_1\gamma_1(a_{12}+b_{12})\gamma_1 e_2\alpha x_{22},$$
$$y\beta e_1\gamma_1\big(s-(a_{12}+b_{12})\big)\gamma_1 e_2\alpha x_{22} \;=\; 0.$$

For $y_{ij}\in\mathfrak{M}_{ij}$, applying Lemma 2.4-(i),(ii),

$$\begin{aligned} M(y_{ij}\beta e_2\gamma_1 s\gamma_1 e_2\alpha x_{22}) &= M(y_{ij}\beta e_2\gamma_1 a_{12}\gamma_1 e_2\alpha x_{22}) \\ &\quad + M(y_{ij}\beta e_2\gamma_1 b_{12}\gamma_1 e_2\alpha x_{22}) \;=\; 0, \end{aligned}$$
$$y_{ij}\beta e_2\gamma_1 s\gamma_1 e_2\alpha x_{22} \;=\; 0 \;=\; y_{ij}\beta e_2\gamma_1(a_{12}+b_{12})\gamma_1 e_2\alpha x_{22}.$$

These equations show that

$$\begin{aligned} y_{ij}\beta e_2\gamma_1\big(s-(a_{12}+b_{12})\big)\gamma_1 e_2\alpha x_{22} &= 0,\\ y\beta e_2\gamma_1\big(s-(a_{12}+b_{12})\big)\gamma_1 e_2\alpha x_{22} &= 0,\\ \mathfrak{M}\Gamma\big(s-(a_{12}+b_{12})\big)\gamma_1 e_2\alpha x_{22} &= 0,\\ \big(s-(a_{12}+b_{12})\big)\gamma_1 e_2\alpha x_{22} &= 0,\\ \big(s-(a_{12}+b_{12})\big)\gamma_1 e_2\alpha x &= 0,\\ \big(s-(a_{12}+b_{12})\big)\Gamma\mathfrak{M} &= 0,\\ s = a_{12}+b_{12}. \end{aligned}$$

∎

LEMMA 2.10. $M(a_{11}+b_{11}) = M(a_{11}) + M(b_{11})$.

Proof. Choose $s = s_{11}+s_{12}+s_{21}+s_{22}\in\mathfrak{M}$ such that $M(s) = M(a_{11}) + M(b_{11})$.

$$\begin{aligned} M(s) &= M(e_1\gamma_1 a_{11}\gamma_1 e_1) + M(e_1\gamma_1 b_{11}\gamma_1 e_1)\\ &= M(e_1)\phi(\gamma_1)M^{*-1}(a_{11})\phi(\gamma_1)M(e_1)\\ &\quad + M(e_1)\phi(\gamma_1)M^{*-1}(b_{11})\phi(\gamma_1)M(e_1)\\ &= M(e_1)\phi(\gamma_1)\big(M^{*-1}(a_{11}) + M^{*-1}(b_{11})\big)\phi(\gamma_1)M(e_1)\\ &= M(e_1)\phi(\gamma_1)\big(M^{*-1}(e_1\gamma_1 a_{11}\gamma_1 e_1) + M^{*-1}(e_1\gamma_1 b_{11}\gamma_1 e_1)\big)\phi(\gamma_1)M(e_1)\\ &= M(e_1)\phi(\gamma_1)M^{*-1}(e_1\gamma_1 s\gamma_1 e_1)\phi(\gamma_1)M(e_1)\\ &= M(e_1\gamma_1 e_1\gamma_1 s\gamma_1 e_1\gamma_1 e_1)\\ &= M(e_1\gamma_1 s\gamma_1 e_1) \;=\; M(s_{11}). \end{aligned}$$

It follows that $s = s_{11}$. Hence $s - (a_{11} + b_{11}) \in \mathfrak{M}_{11}$.

First we let $x_{11} \in \mathfrak{M}_{12}$ be arbitrary. Applying Lemma 2.4-(i) we get that

$$\begin{aligned}
&M(s\alpha e_1\gamma_1 x_{11}\gamma_1 e_1\beta e_2)\\
&\qquad = M(a_{11}\alpha e_1\gamma_1 x_{11}\gamma_1 e_1\beta e_2) + M(b_{11}\alpha e_1\gamma_1 x_{11}\gamma_1 e_1\beta e_2)\\
&\qquad = M(a_{11}\alpha e_1\gamma_1 x_{11}\gamma_1 e_1\beta e_2 + b_{11}\alpha e_1\gamma_1 x_{11}\gamma_1 e_1\beta e_2)\\
&\qquad = M\big((a_{11} + b_{11})\alpha e_1\gamma_1 x_{11}\gamma_1 e_1\beta e_2\big)
\end{aligned}$$

yielding that $s\alpha e_1\gamma_1 x_{11}\gamma_1 e_1\beta e_2 = (a_{11} + b_{11})\alpha e_1\gamma_1 x_{11}\gamma_1 e_1\beta e_2$. Therefore

$$\big(s - (a_{11} + b_{11})\big)\alpha e_1\gamma_1 x_{11}\gamma_1 e_1\beta e_2 = 0.$$

This implies

$$\big(s - (a_{11} + b_{11})\big)\alpha e_1\gamma_1 x\gamma_1 e_1\beta e_2 = 0. \tag{2.6}$$

Second we let $x_{12} \in \mathfrak{M}_{12}$ be arbitrary. Applying Lemma 2.4-(i) we get that

$$\begin{aligned}
&M(s\alpha e_1\gamma_1 x_{12}\gamma_1 e_2\beta e_2)\\
&\qquad = M(a_{11}\alpha e_1\gamma_1 x_{12}\gamma_1 e_2\beta e_2) + M(b_{11}\alpha e_1\gamma_1 x_{12}\gamma_1 e_2\beta e_2)\\
&\qquad = M(a_{11}\alpha e_1\gamma_1 x_{12}\gamma_1 e_2\beta e_2 + b_{11}\alpha e_1\gamma_1 x_{12}\gamma_1 e_2\beta e_2)\\
&\qquad = M\big((a_{11} + b_{11})\alpha e_1\gamma_1 x_{12}\gamma_1 e_2\beta e_2\big)
\end{aligned}$$

yielding that $s\alpha e_1\gamma_1 x_{12}\gamma_1 e_2\beta e_2 = (a_{11} + b_{11})\alpha e_1\gamma_1 x_{12}\gamma_1 e_2\beta e_2$. Therefore

$$\big(s - (a_{11} + b_{11})\big)\alpha e_1\gamma_1 x_{12}\gamma_1 e_2\beta e_2 = 0.$$

This implies

$$\big(s - (a_{11} + b_{11})\big)\alpha e_1\gamma_1 x\gamma_1 e_2\beta e_2 = 0. \tag{2.7}$$

Third we let $x_{21} \in \mathfrak{M}_{21}$ be arbitrary. Applying Lemma 2.4-(i) we get that

$$\begin{aligned}
&M(s\alpha e_2\gamma_1 x_{21}\gamma_1 e_1\beta e_2)\\
&\qquad = M(a_{11}\alpha e_2\gamma_1 x_{21}\gamma_1 e_1\beta e_2) + M(b_{11}\alpha e_2\gamma_1 x_{21}\gamma_1 e_1\beta e_2)\\
&\qquad = M(a_{11}\alpha e_2\gamma_1 x_{21}\gamma_1 e_1\beta e_2 + b_{11}\alpha e_2\gamma_1 x_{21}\gamma_1 e_1\beta e_2)\\
&\qquad = M\big((a_{11} + b_{11})\alpha e_2\gamma_1 x_{21}\gamma_1 e_1\beta e_2\big)
\end{aligned}$$

yielding that $s\alpha e_2\gamma_1 x_{21}\gamma_1 e_1\beta e_2 = (a_{11} + b_{11})\alpha e_2\gamma_1 x_{21}\gamma_1 e_1\beta e_2$. Therefore

$$\big(s - (a_{11} + b_{11})\big)\alpha e_2\gamma_1 x_{21}\gamma_1 e_1\beta e_2 = 0.$$

This implies
$$\big(s-(a_{11}+b_{11})\big)\alpha e_2\gamma_1 x\gamma_1 e_1\beta e_2 = 0. \tag{2.8}$$

Lastly we let $x_{22}\in\mathfrak{M}_{22}$ be arbitrary. Applying Lemma 2.4-(i) we get that
$$\begin{aligned}
&M(s\alpha e_2\gamma_1 x_{22}\gamma_1 e_2\beta e_2)\\
&\qquad= M(a_{11}\alpha e_2\gamma_1 x_{22}\gamma_1 e_2\beta e_2) + M(b_{11}\alpha e_2\gamma_1 x_{22}\gamma_1 e_2\beta e_2)\\
&\qquad= M(a_{11}\alpha e_2\gamma_1 x_{22}\gamma_1 e_2\beta e_2 + b_{11}\alpha e_2\gamma_1 x_{22}\gamma_1 e_2\beta e_2)\\
&\qquad= M\big((a_{11}+b_{11})\alpha e_2\gamma_1 x_{22}\gamma_1 e_2\beta e_2\big)
\end{aligned}$$
yielding that $s\alpha e_2\gamma_1 x_{22}\gamma_1 e_2\beta e_2 = (a_{11}+b_{11})\alpha e_2\gamma_1 x_{22}\gamma_1 e_2\beta e_2$. Therefore
$$\big(s-(a_{11}+b_{11})\big)\alpha e_2\gamma_1 x_{22}\gamma_1 e_2\beta e_2 = 0.$$
This implies
$$\big(s-(a_{11}+b_{11})\big)\alpha e_2\gamma_1 x\gamma_1 e_2\beta e_2 = 0. \tag{2.9}$$

From (2.6)-(2.9) we have
$$\big(s-(a_{11}+b_{11})\big)\alpha 1_1\gamma_1 x\gamma_1 1_1\beta e_2 = 0,$$
which implies
$$\big(s-(a_{11}+b_{11})\big)\alpha x\beta e_2 = 0,$$
for all $\alpha,\beta\in\Gamma$ and $x\in\mathfrak{M}$, which yields
$$\big(s-(a_{11}+b_{11})\big)\Gamma\mathfrak{M}\Gamma e_2 = 0.$$
So $\big(s-(a_{11}+b_{11})\big)\alpha\mathfrak{M}\beta(1_1-e_1)=0$ which implies
$$\big(e_1\gamma_1(s-(a_{11}+b_{11})\gamma_1 e_1\big)\Gamma\mathfrak{M}\Gamma(1_1-e_1) = 0.$$
It follows, from Theorem 2.1 condition (iii), that $s=a_{11}+b_{11}$. ∎

LEMMA 2.11. *M is additive on $e_1\gamma_1\mathfrak{M}=\mathfrak{M}_{11}+\mathfrak{M}_{12}$.*

Proof. The proof is the same as that of Martindale III (1969, Lemma 5) and is included for the sake of completeness. In fact, let $a_{11},b_{11}\in\mathfrak{M}_{11}$ and $a_{12},b_{12}\in\mathfrak{M}_{12}$. Making use of Lemmas 2.5, 2.9 and 2.10 we can see that
$$\begin{aligned}
M\big((a_{11}+a_{12})+(b_{11}+b_{12})\big) &= M\big((a_{11}+b_{11})+(a_{12}+b_{12})\big)\\
&= M(a_{11}+b_{11})+M(a_{12}+b_{12})\\
&= M(a_{11})+M(b_{11})+M(a_{12})+M(b_{12})\\
&= M(a_{11}+a_{12})+M(b_{11}+b_{12}).
\end{aligned}$$
holds true, as desired. ∎

Proof of Theorem 2.1. Suppose that $a, b \in \mathfrak{M}$ and choose $s \in \mathfrak{M}$ such that $M(s) = M(a) + M(b)$. For all $\alpha \in \Lambda$, M is additive on $e_\alpha \gamma_\alpha \mathfrak{M}$ because of Lemma 2.11. Thus, for every $r \in \mathfrak{M}$, we have

$$\begin{aligned} M(e_\alpha \gamma_\alpha r \mu s) &= M(e_\alpha)\phi(\gamma_\alpha) M^{*-1}(r)\phi(\mu) M(s) \\ &= M(e_\alpha)\phi(\gamma_\alpha) M^{*-1}(r)\phi(\mu)\big(M(a) + M(b)\big) \\ &= M(e_\alpha)\phi(\gamma_\alpha) M^{*-1}(r)\phi(\mu) M(a) \\ &\quad + M(e_\alpha)\phi(\gamma_\alpha) M^{*-1}(r)\phi(\mu) M(b) \\ &= M(e_\alpha \gamma_\alpha r \mu a) + M(e_\alpha \gamma_\alpha r \mu b) \\ &= M(e_\alpha \gamma_\alpha r \mu a + e_\alpha \gamma_\alpha r \mu b) \\ &= M\big(e_\alpha \gamma_\alpha r \mu (a + b)\big). \end{aligned}$$

So $e_\alpha \gamma_\alpha r \mu s = e_\alpha \gamma_\alpha r \mu (a + b)$. Therefore $e_\alpha \gamma_\alpha \mathfrak{M}\Gamma\big(s - (a + b)\big) = 0$ holds for every $\alpha \in \Lambda$. We then conclude that $s = a + b$ from Theorem 2.1 condition (ii). This shows that M is additive on $\mathfrak{M}$.

To prove the additivity of M^*, let $x, y \in \mathfrak{M}'$. For $a, b \in \mathfrak{M}$, by using the additivity of M, we have

$$\begin{aligned} M\big(a\lambda\big(M^*(x) + M^*(y)\big)\mu b\big) &= M\big(a\lambda M^*(x)\mu b\big) + M\big(a\lambda M^*(y)\mu b\big) \\ &= M(a)\phi(\lambda) x \phi(\mu) M(b) \\ &\quad + M(a)\phi(\lambda) y \phi(\mu) M(b) \\ &= M(a)\phi(\lambda)(x + y)\phi(\mu) M(b) \\ &= M\big(a\lambda M^*(x + y)\mu b\big). \end{aligned}$$

It follows that $a\lambda\big(M^*(x) + M^*(y) - M^*(x + y)\big)\mu b = 0$ holds for all $a, b \in \mathfrak{M}$, that is,

$$a\lambda\big(M^*(x) + M^*(y) - M^*(x + y)\big)\Gamma\mathfrak{M} = 0$$

holds for all $a \in \mathfrak{M}$, which implies

$$a\lambda\big(M^*(x) + M^*(y) - M^*(x + y)\big) = 0$$

holds for all $a \in \mathfrak{M}$, which implies

$$\mathfrak{M}\Gamma\big(M^*(x) + M^*(y) - M^*(x + y)\big) = 0$$

which forces $M^*(x + y) = M^*(x) + M^*(y)$ because of Theorem 2.1 conditions (i) and (ii). This completes the proof. ∎

Corollary 2.1. *Let Γ, Γ', $\mathfrak{M}$ and $\mathfrak{M}'$ be additive groups such that $\mathfrak{M}$ is a Γ-ring and $\mathfrak{M}'$ is a Γ'-ring such that $\mathfrak{M}$ is a prime Γ-ring containing a non-trivial γ-idempotent ($\mathfrak{M}$ need not have an γ-identity element), where $\gamma \in \Gamma$. Suppose $e_2 \colon \Gamma \times \mathfrak{M} \to \mathfrak{M}$, $e_2' \colon \mathfrak{M} \times \Gamma \to \mathfrak{M}$ two $\mathfrak{M}$-additive maps such that $e_2(\gamma_1, a) = a - e_1\gamma_1 a$, $e_2'(a, \gamma_1) = a - a\gamma_1 e_1$. Denote $e_2\alpha a = e_2(\alpha, a)$, $a\alpha e_2 = e_2'(a, \alpha)$, $1_1\alpha a = e_1\alpha a + e_2\alpha a$, $a\alpha 1_1 = a\alpha e_1 + a\alpha e_2$ and suppose $(a\alpha e_2)\beta b = a\alpha(e_2\beta b)$ for all $\alpha, \beta \in \Gamma$ and $a, b \in \mathfrak{M}$. Then every surjective elementary map (M, M^*) of $\mathfrak{M} \times \mathfrak{M}'$ is additive.*

Proof. The result follows directly from Theorem 2.1 and Theorem 1.1. ∎

Corollary 2.2. *Let Γ, Γ', $\mathfrak{M}$ and $\mathfrak{M}'$ be additive groups such that $\mathfrak{M}$ is a Γ-ring and $\mathfrak{M}'$ is a Γ'-ring such that $\mathfrak{M}$ is a prime Γ-ring containing a non-trivial γ-idempotent and a γ-unity element, where $\gamma \in \Gamma$. Then every surjective elementary map (M, M^*) of $\mathfrak{M} \times \mathfrak{M}'$ is additive.*

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