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Browder essential approximate pseudospectrum and defect pseudospectrum on a Banach space

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Abstract: In this paper, we introduce and study the Browder essential approximate pseudospectrum and the Browder essential defect pseudospectrum of bounded linear operators on a Banach space. Moreover, we characterize these spectra and will give some results concerning the stability of them under suitable perturbations.

Key words: Pseudospectrum, Browder essential approximate pseudospectrum, Browder essential defect pseudospectrum.

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1. Introduction

Let X be an infinite-dimensional Banach space, let $\mathcal{L}(X)$ be the set of all bounded linear operators acting on X, and let $\mathcal{K}(X)$ be its ideal of compact operators on X.

Let $T \in \mathcal{L}(X)$. Then $\mathcal{D}(T)$, $\mathcal{N}(T)$, $\alpha(T)$, $\mathcal{R}(T)$, $\beta(T)$, $T^{'}$ and $\sigma(T)$ are, respectively, used to denote the domain, the kernel, the nullity, the range, the defect, the adjoint and the spectrum of T. If the range $\mathcal{R}(T)$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$) then T is said to be an upper semi-Fredholm operator (resp. a lower semi-Fredholm operator). The set of upper semi-Fredholm operators (resp. lower semi-Fredholm operators) is denoted by $\Phi_+(X)$ (resp. $\Phi_-(X)$). The set of all semi-Fredholm operators is defined by

$$\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X), \quad \text{and}$$

the class $\Phi(X)$ of all Fredholm operators is defined by

$$\Phi(X) := \Phi_+(X) \cap \Phi_-(X).$$

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The index of a semi-Fredholm operator T is defined by

$$i(T) = \alpha(T) - \beta(T).$$

An operator $F \in \mathcal{L}(X)$ is called a Fredholm perturbation if $T + F \in \Phi(X)$ whenever $T \in \Phi(X)$. The set of Fredholm perturbations is denoted by $\mathcal{F}(X)$. An operator $F \in \mathcal{L}(X)$ is called an upper semi-Fredholm perturbation (resp. a lower semi-Fredholm perturbation) if $T + F \in \Phi_+(X)$ (resp. $T + F \in \Phi_-(X)$) whenever $T \in \Phi_+(X)$ (resp. $T \in \Phi_-(X)$). The set of upper semi-Fredholm perturbations (resp. lower semi-Fredholm perturbations) is denoted by $\mathcal{F}^+(X)$ (resp. $\mathcal{F}^-(X)$). Now, we define the minimum modulus

$$m(T) := \inf\big\{\|Tx\| : x \in \mathcal{D}(X),\ \|x\| = 1\big\},$$

and the defect modulus

$$q(T) := \sup\big\{r > 0 : rB_X \subset TB_X\big\},$$

where B_X is the closed unit ball of X. For more information see [16] and [20]. Note that $m(T) > 0$ if and only if T is bounded below, i.e. T is injective and T has closed range and $q(T) > 0$ if and only if T is surjective. Recall also that $m(T^*) = q(T)$ and $q(T^*) = m(T)$.

The ascent (resp. descent) of $T \in \mathcal{L}(X)$ is the smallest nonnegative integer $\mathrm{a} := \mathrm{asc}(T)$ (resp. $\mathrm{d} := \mathrm{desc}(T)$) such that $\mathcal{N}(T^{\mathrm{a}}) = \mathcal{N}(T^{\mathrm{a}+1})$ (resp. $\mathcal{R}(T^{\mathrm{d}}) = \mathcal{R}(T^{\mathrm{d}+1})$). If such an integer does not exist, then $\mathrm{asc}(T) = \infty$ (resp. $\mathrm{desc}(T) = \infty$). We also introduce some special parts of pseudospectrum having valuable spectral properties such as

$$\sigma_{ap}(T) := \big\{\lambda \in \mathbb{C} : m(\lambda - T) = 0\big\},$$
$$\sigma_{\delta}(T) := \big\{\lambda \in \mathbb{C} : q(\lambda - T) = 0\big\}.$$

The spectrum $\sigma_{ap}(T)$ (resp. $\sigma_{\delta}(T)$) is called the approximate spectrum (resp. defect spectrum). The Browder essential spectrum of T is defined as

$$\sigma_{eb}(T) := \bigcap_{\mathcal{K}_T(X)} \sigma(T + K), \tag{1.1}$$

the Browder essential approximate point spectrum of T is defined as

$$\sigma_{eab}(T) := \bigcap_{\mathcal{K}_T(X)} \sigma_{ap}(T + K), \tag{1.2}$$

and the Browder essential defect spectrum of T is defined as

$$\sigma_{e\delta b}(T) := \bigcap_{\mathcal{K}_T(X)} \sigma_\delta(T+K), \tag{1.3}$$

where

$$\mathcal{K}_T(X) := \{K \in \mathcal{K}(X) : TK = KT\}.$$

For more information on the Browder essential approximate spectrum and his essential defect spectrum one may refer to [1, 9, 17, 18]. It is clear that

$$\sigma_{eb}(T) = \sigma_{eab}(T) \cup \sigma_{e\delta b}(T).$$

The pseudospectrum of bounded linear operators T on a Banach space X can be split into subsets in many different ways, depending on the purpose one has in mind. We may refer to [2, 3, 5, 7, 13] as examples.

Definition 1.1. Let $T \in \mathcal{L}(X)$ and $\varepsilon > 0$. We define the following sets:

(i) the pseudospectrum

$$\sigma_\varepsilon(T) = \bigcup_{\mathcal{D}_T(X)} \sigma(T+D),$$

(ii) the approximate pseudospectrum

$$\sigma_{ap,\varepsilon}(T) = \bigcup_{\mathcal{D}_T(X)} \sigma_{ap}(T+D),$$

(iii) the defect pseudospectrum

$$\sigma_{\delta,\varepsilon}(T) = \bigcup_{\mathcal{D}_T(X)} \sigma_\delta(T+D),$$

where

$$\mathcal{D}_T(X) = \{D \in \mathcal{L}(X) : \|D\| < \varepsilon,\ TD = DT\}.$$

In this paper we study some parts of the pseudospectrum of bounded linear operators on a Banach space from the viewpoint of Fredholm theory. In particular, we study the Browder essential approximate pseudospectrum and the Browder essential defect pseudospectrum. We have already mentioned that (1.1), (1.2) and (1.3) inherit ε-versions, which are the Browder essential pseudospectrum $\sigma_{eb,\varepsilon}(\cdot)$, the Browder essential approximate pseudospectrum $\sigma_{eab,\varepsilon}(\cdot)$ and the Browder essential defect pseudospectrum $\sigma_{e\delta b,\varepsilon}(\cdot)$ defined by

$$\sigma_{eb,\varepsilon}(T) = \bigcap_{\mathcal{K}_T(X)} \sigma_\varepsilon(T+K),$$

$$\sigma_{eab,\varepsilon}(T) = \bigcap_{\mathcal{K}_T(X)} \sigma_{ap,\varepsilon}(T+K),$$

$$\sigma_{e\delta b,\varepsilon}(T) = \bigcap_{\mathcal{K}_T(X)} \sigma_{\delta,\varepsilon}(T+K).$$

This paper is divided into three sections. In the second one we recall some facts which are helpful to prove the main results. Throughout the third section we characterize Browder essential approximate pseudospectrum and the Browder essential defect pseudospectrum. Finally, we prove the invariance of the Browder essential approximate pseudospectrum and his essential defect pseudospectrum and establish some results of perturbation on the context of linear operators on a Banach space.

2. Auxiliary results

In order to prove our main results we begin by introducing some well known perturbation results

LEMMA 2.1. ([16, Theorem 9]) *Let $T, K \in \mathcal{L}(X)$. We have*

(i) *If $T \in \Phi_+(X)$ and $K \in \mathcal{K}(X)$ then $T+K \in \Phi_+(X)$ and $i(T+K) = i(T)$.*

(ii) *If $T \in \Phi_-(X)$ and $K \in \mathcal{K}(X)$ then $T+K \in \Phi_-(X)$ and $i(T+K) = i(T)$.*

The following result was proved in [11].

LEMMA 2.2. *Let $T \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that K commutes with T. We have*

(i) *If $T \in \Phi_+(X)$ then $\operatorname{asc}(T) < \infty$ if, and only if, $\operatorname{asc}(T+K) < \infty$.*

(ii) *If $T \in \Phi_-(X)$ then $\operatorname{desc}(T) < \infty$ if, and only if, $\operatorname{desc}(T+K) < \infty$.*

A bounded operator $R \in \mathcal{L}(X)$ on a Banach space X is said to be a Riesz operator if $\lambda - T \in \Phi(X)$ for every $\lambda \in \mathbb{C}\backslash\{0\}$. The class of all Riesz operators is denoted by $\mathcal{R}(X)$.

LEMMA 2.3. ([15, Theorem 3.5]) *Let* $R \in \mathcal{R}(X)$ *which commutes with* T. *We have*

(i) *If* $T \in \Phi_+(X)$ *then* $\mathrm{asc}(T) < \infty$ *if and only if* $\mathrm{asc}(T+R) < \infty$.

(ii) *If* $T \in \Phi_-(X)$ *then* $\mathrm{desc}(T) < \infty$ *if and only if* $\mathrm{desc}(T+R) < \infty$.

LEMMA 2.4. ([14, Theorem 3.9]) *Let* $T \in \Phi_+(X)$. *The following statements are equivalent:*

(i) $i(T) \leq 0$.

(ii) T *can be expressed in the form* $T = S + K$ *where* $K \in \mathcal{K}(X)$ *and* $S \in \mathcal{L}(X)$ *is an operator with closed range and* $\alpha(S) = 0$.

3. Main results

In this section we establish an useful result for the Browder essential approximate pseudospectrum and the Browder essential defect pseudospectrum. We start our characterization with the following theorem:

THEOREM 3.1. *Let* $T \in \mathcal{L}(X)$ *and* $\varepsilon > 0$. *Then*

(i) $\lambda \notin \sigma_{eab,\varepsilon}(T)$ *if and only if, for all* $D \in \mathcal{L}(X)$ *such that* $\|D\| < \varepsilon$, *we have* $\lambda - T - D \in \Phi_+(X)$, $i(\lambda - T - D) \leq 0$ *and* $\mathrm{asc}(\lambda - T - D) < \infty$.

(ii) $\lambda \notin \sigma_{e\delta b,\varepsilon}(T)$ *if and only if, for all* $D \in \mathcal{L}(X)$ *such that* $\|D\| < \varepsilon$, *we have* $\lambda - T - D \in \Phi_-(X)$, $i(\lambda - T - D) \geq 0$ *and* $\mathrm{desc}(\lambda - T - D) < \infty$.

Proof. (i) Let $\lambda \notin \sigma_{eap,\varepsilon}(T)$. Then there exists a compact operator K on X such that $TK = KT$ and $\lambda \notin \sigma_{ap,\varepsilon}(T+K)$. According to the Definition 1.1, we obtain that $\lambda \notin \sigma_{ap}(T + D + K)$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and D commutes with $T + K$. Therefore,

$$\lambda - T - D - K \in \Phi_+(X), \quad i(\lambda - T - D - K) \leq 0 \quad \text{and} \quad \mathrm{asc}(\lambda - T - D - K) = 0$$

for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Since K commutes with $\lambda - T - D - K$, from Lemma 2.2 we obtain $\mathrm{asc}(\lambda - T - D) < \infty$. Using Lemma 2.1, we deduce that

$$\lambda - T - D \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - D) \leq 0.$$

To prove the converse, suppose that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have

$$\lambda - T - D \in \Phi_+(X), \quad i(\lambda - T - D) \leq 0 \quad \text{and} \quad \operatorname{asc}(\lambda - T - D) < \infty.$$

There are two possible cases:

1^{st}*case* : If $\lambda \notin \sigma_{ap,\varepsilon}(T)$ then $\lambda \notin \sigma_{ap,\varepsilon}(T+K)$, so the proof is completed.

2^{nd}*case* : If $\lambda \in \sigma_{ap,\varepsilon}(T)$ then from [16, Theorem 10] we infer that the space X is decomposed into a direct sum of two closed subspaces X_0 and X_1 such that $\dim X_0 < \infty$, $(\lambda - T - D)(X_i) \subseteq X_i$ for $i \in \{1, 2\}$, $(\lambda - T - D)\backslash_{X_0}$ is nilpotent operator and $(\lambda - T - D)\backslash_{X_1}$ is injective operator. Let K be the finite rank operator defined by

$$\begin{cases} K = I & \text{on } X_0, \\ K = 0 & \text{on } X_1. \end{cases}$$

It is clear that K is a compact operator commuting with T and D such that $\lambda - T - D - K$ is an injective operator (i.e. $\alpha(\lambda - T - D - K) = 0$). Then, from Lemma 2.4 there exists a constant $c > 0$ such that

$$\|(\lambda - T - D - K)x\| \geq c\|x\|, \text{ for all } x \in \mathcal{D}(T).$$

This proves that $\inf_{x \in X,\ \|x\|=1} \|(\lambda - T - D - K)x\| \geq c > 0$. Thus $\lambda \notin \sigma_{ap}(T+D+K)$. Moreover, $(T+D)K = K(T+D)$ and by using Definition 1.1 we infer that $\lambda \notin \sigma_{ap,\varepsilon}(T+K)$. Hence $\lambda \notin \sigma_{eab,\varepsilon}(T)$.

(ii) Reasoning in the same way as (i), it suffices to replace $\Phi_+(\cdot)$, $\sigma_{eab,\varepsilon}(\cdot)$, $\sigma_{ap,\varepsilon}(\cdot)$, $i(\cdot) \leq 0$ and $(\lambda - T - D)\backslash_{X_1}$, which is injective, by $\Phi_-(\cdot)$, $\sigma_{e\delta b,\varepsilon}(\cdot)$, $\sigma_{\delta,\varepsilon}(\cdot)$, $i(\cdot) \geq 0$ and $(\lambda - T - D)\backslash_{X_1}$, which is surjective, respectively. ∎

Remark 3.1. It follows immediately from Theorem 3.1 (i) that

$$\sigma_{eab,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{eab}(T+D).$$

Moreover, it follows from Theorem 3.1 (ii) that

$$\sigma_{e\delta b,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{e\delta b}(T+D).$$

Next, the Browder essential approximate pseudospectrum and the Browder essential defect pseudospectrum will be characterized by means of semi-Fredholm perturbation. We set

$$\mathcal{F}_T^+(X) = \{F \in \mathcal{F}^+(X) : TF = FT\},$$

and

$$\mathcal{F}_T^-(X) = \{F \in \mathcal{F}^-(X) : TF = FT\}.$$

THEOREM 3.2. *Let $T \in \mathcal{L}(X)$ and $\varepsilon > 0$. Then*

(i) $\sigma_{eab,\varepsilon}(T) = \bigcap_{F \in \mathcal{F}_T^+(X)} \sigma_{ap,\varepsilon}(T+F)$.

(ii) $\sigma_{e\delta b,\varepsilon}(T) = \bigcap_{F \in \mathcal{F}_T^-(X)} \sigma_{\delta,\varepsilon}(T+F)$.

Proof. (i) For the first inclusion, it is clear that $\mathcal{K}_T(X) \subset \mathcal{F}_T^+(X)$. Then,

$$\bigcap_{F \in \mathcal{F}_T^+(X)} \sigma_{ap,\varepsilon}(T+F) \subset \bigcap_{F \in \mathcal{K}_T(X)} \sigma_{ap,\varepsilon}(T+F) := \sigma_{eab,\varepsilon}(T).$$

For the second inclusion, let $\lambda \notin \bigcap_{F \in \mathcal{F}_T^+(X)} \sigma_{ap,\varepsilon}(T+F)$, then there exists $F \in \mathcal{F}^+(X)$ such that $TF = FT$ and $\lambda \notin \sigma_{ap,\varepsilon}(T+F)$. Using Definition 1.1, we have $\lambda \notin \sigma_{ap}(T+F+D)$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and D commutes with T and F. Hence,

$$\lambda - T - D - F \in \Phi_+(X), \quad i(\lambda - T - D - F) \leq 0 \quad \text{and} \quad \operatorname{asc}(\lambda - T - D - F) = 0.$$

Since F commutes with $\lambda - T - D - F$, from Lemma 2.3, it follows that

$$\operatorname{asc}(\lambda - T - D) < \infty$$

and from Lemma 2.1 we deduce that

$$\lambda - T - D \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - D) \leq 0.$$

Hence $\lambda \notin \sigma_{eab,\varepsilon}(T)$.

(ii) The proof is similar to that of the first part. ∎

If we set

$$\mathcal{R}_T(X) = \{R \in \mathcal{R}(X) : TR = RT\},$$

then Theorem 3.2 remains true if $\mathcal{F}_T^+(X)$ and $\mathcal{F}_T^-(X)$ are replaced by $\mathcal{R}_T(X)$. We then have

$$\sigma_{eab,\varepsilon}(T) = \bigcap_{\mathcal{R}_T(X)} \sigma_{ap,\varepsilon}(T+R) \quad \text{and} \quad \sigma_{e\delta b,\varepsilon}(T) = \bigcap_{\mathcal{R}_T(X)} \sigma_{\delta,\varepsilon}(T+R).$$

Definition 3.1. An operator $T \in \mathcal{L}(X)$ is said to be quasi-compact operator ($T \in \mathcal{QK}(X)$) if there exists a compact operator K and an integer m such that

$$\|T^m - K\| < 1.$$

If $T \in \mathcal{L}(X)$, we define the set

$$\mathcal{QK}_T(X) = \{K \in \mathcal{QK}(X) : TK = KT\}$$

We invite the reader to [6] for more information about the quasi-compactness operators. We have the following inclusions

$$\mathcal{K}_T(X) \subset \mathcal{R}_T(X) \subset \mathcal{QK}_T(X).$$

If $T \in \mathcal{L}(X)$ we define the sets

$$\mathcal{S}_T^\varepsilon(X) = \Big\{K \in \mathcal{L}(X) : K \text{ commutes with } T+D \text{ and } (\lambda - T - D - K)^{-1}K \in \mathcal{QK}_T(X) \text{ for all } D \in \mathcal{L}(X) \text{ such that } \|D\| < \varepsilon \text{ and } \lambda \in \rho(T+D+K)\Big\},$$

and

$$\mathcal{L}_T^\varepsilon(X) = \Big\{K \in \mathcal{L}(X) : K \text{ commutes with } T+D \text{ and } K(\lambda - T - D - K)^{-1} \in \mathcal{QK}_T(X) \text{ for all } D \in \mathcal{L}(X) \text{ such that } \|D\| < \varepsilon \text{ and } \lambda \in \rho(T+D+K)\Big\}.$$

Theorem 3.3. *Let $T \in \mathcal{L}(X)$ with nonempty resolvent set. Then,*

$$\sigma_{eab,\varepsilon}(T) = \bigcap_{K \in \mathcal{S}_T^\varepsilon(X)} \sigma_{ap,\varepsilon}(T+K).$$

Proof. Let $\lambda \notin \bigcap_{K\in\mathcal{S}_T^{\varepsilon}(X)} \sigma_{ap,\varepsilon}(T+K)$, then there exists $K \in \mathcal{S}_T^{\varepsilon}(X)$ such that for every $\|D\| < \varepsilon$ and $\lambda \in \rho(T+D+K)$, we have

$$(\lambda - T - D - K)^{-1}K \in \mathcal{QK}_T(X) \quad \text{and} \quad \lambda \notin \sigma_{ap,\varepsilon}(T+K).$$

Using [6, Theorem 1.6] we obtain that

$$I + (\lambda - T - D - K)^{-1}K \in \Phi(X) \quad \text{and} \quad i\big(I + (\lambda - T - D - K)^{-1}K\big) = 0.$$

Since we can write

$$\lambda - T - D = (\lambda - T - D - K)\big(I + (\lambda - T - D - K)^{-1}K\big).$$

According to Definition 1.1, we have for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$,

$$(T+K)D = D(T+K) \quad \text{and} \quad \lambda \notin \sigma_{ap}(T+D+K).$$

We conclude for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ that $\lambda - T - D \in \Phi_+(X)$. Also, we have

$$i(\lambda - T - D) = i(\lambda - T - D - K) \leq 0.$$

It remains to show that $asc(\lambda - T - D) < 0$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Let K commutes with $T+D$, then K commutes with $\lambda - T - D$ for every $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}(\lambda - T - D)^n &= (\lambda - T - D - K)^n\big(I + (\lambda - T - D - K)^{-1}K\big)^n \\ &= \big(I + (\lambda - T - D - K)^{-1}K\big)^n(\lambda - T - D - K)^n\end{aligned}$$

for every $n \in \mathbb{N}$. Use the fact that $(\lambda - T - D)^n$ is injective $\big($i.e., 0 belongs to $\mathcal{N}((\lambda - T - D)^n)\big)$, This implies that $\lambda - T - D$ is injective $\big(\mathcal{N}(\lambda - T - D) \subset \mathcal{N}((\lambda - T - D)^n)$ for every $n\big)$. Consequently, the ascent of $\lambda - T - D$ is 0. Then $\mathrm{asc}(\lambda - T - D) < \infty$. This prove that $\lambda \notin \sigma_{eab,\varepsilon}(T)$. The opposite inclusion follows from $\mathcal{K}_T(X) \subseteq \mathcal{S}_T^{\varepsilon}(X)$. Then

$$\bigcap_{K\in\ \mathcal{S}_T^{\varepsilon}(X)} \sigma_{ap,\varepsilon}(T+K) \subseteq \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{ap,\varepsilon}(T+K).$$

∎

Corollary 3.1. *Let $T \in \mathcal{L}(X)$ with nonempty resolvent set. Then,*

$$\sigma_{eab,\varepsilon}(T) = \bigcap_{K\in\ \mathcal{L}_T^{\varepsilon}(X)} \sigma_{ap,\varepsilon}(T+K).$$

PROPOSITION 3.1. *Let $T \in \mathcal{L}(X)$ with nonempty resolvent set. Then,*

$$\sigma_{e\delta,\varepsilon}(T) = \bigcap_{K\in \mathcal{S}_T^\varepsilon(X)} \sigma_{\delta,\varepsilon}(T+K) = \bigcap_{K\in \mathcal{L}_T^\varepsilon(X)} \sigma_{\delta,\varepsilon}(T+K).$$

Remark 3.2. Let $T \in \mathcal{L}(X)$ and $\varepsilon > 0$.

(i) Let $\mathcal{U}_T(X)$, (resp. $\mathcal{V}_T(X)$) be a subset of $\mathcal{L}(X)$. If $\mathcal{K}_T(X) \subset \mathcal{U}_T(X) \subset \mathcal{S}_T^\varepsilon(X)$, (resp. $\mathcal{K}_T(X) \subset \mathcal{V}_T(X) \subset \mathcal{L}_T^\varepsilon(X)$) then

$$\sigma_{eab,\varepsilon}(T) = \bigcap_{K\in\mathcal{U}_T(X)} \sigma_{ap,\varepsilon}(T+K) = \bigcap_{K\in\mathcal{V}_T(X)} \sigma_{ap,\varepsilon}(T+K).$$

$$\Big(\text{resp. } \sigma_{e\delta,\varepsilon}(T) = \bigcap_{K\in\mathcal{U}_T(X)} \sigma_{\delta,\varepsilon}(T+K) = \bigcap_{K\in\mathcal{V}_T(X)} \sigma_{\delta,\varepsilon}(T+K)\Big).$$

(ii) If for all $J, J_2 \in \mathcal{U}_T(X)$ (resp. $\mathcal{V}_T(X)$) we have $J \pm J_2 \in \mathcal{U}_T(X)$ (resp. $\mathcal{V}_T(X)$) then for each $J \in \mathcal{U}_T(X)$ (resp. $\mathcal{V}_T(X)$) we have

$$\sigma_{eab,\varepsilon}(T+J) = \sigma_{eab,\varepsilon}(T) \quad \text{and} \quad \sigma_{e\delta,\varepsilon}(T+J) = \sigma_{e\delta,\varepsilon}(T).$$

In the next theorem we will give a fine characterization of $\sigma_{eab,\varepsilon}(\cdot)$ and $\sigma_{e\delta,\varepsilon}(\cdot)$ by means of $T+D$-bounded perturbations.

DEFINITION 3.2. An operator $T \in \mathcal{L}(X)$ is called T-bounded if there exist $c > 0$ such that

$$\|Bx\| \le c(\|x\| + \|Tx\|) \quad \text{for all } \ x \in \mathcal{D}(T) \subset \mathcal{D}(B).$$

We define for all $T \in \mathcal{L}(X)$ the set

$$\mathcal{H}_T^\varepsilon(X) = \big\{K \in \mathcal{S}_T^\varepsilon(X) :\ K \text{ is } (T+D)\text{-bounded}\big\}.$$

THEOREM 3.4. *Let $T \in \mathcal{L}(X)$ and $\varepsilon > 0$. Then,*

$$\sigma_{eab,\varepsilon}(T) = \bigcap_{K\in\mathcal{H}_T^\varepsilon(X)} \sigma_{ap,\varepsilon}(T+K).$$

Proof. Because $\mathcal{K}_T(X) \subseteq \mathcal{H}_T^\varepsilon(X)$, then

$$\bigcap_{K\in\mathcal{H}_T^\varepsilon(X)} \sigma_{ap,\varepsilon}(T+K) \subseteq \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{ap,\varepsilon}(T+K) := \sigma_{eab,\varepsilon}(T).$$

Conversely, let $\lambda \notin \bigcap_{K \in \mathcal{H}_T^\varepsilon(X)} \sigma_{ap,\varepsilon}(T+K)$, then there exists $K \in \mathcal{H}_T^\varepsilon(X)$ such that

$$\lambda \notin \sigma_{ap,\varepsilon}(T+K),$$

which means that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $\lambda - T - D - K$ is injective. Using [6, Theorem 1.6] we obtain that

$$I + (\lambda - T - D - K)^{-1}K \in \Phi(X) \quad \text{and} \quad i\big(I + (\lambda - T - D - K)^{-1}K\big) = 0.$$

We can write

$$\lambda - T - D = (\lambda - T - D - K)\big(I + (\lambda - T - D - K)^{-1}K\big).$$

The proof of our statement is then obtained by using the same argument of the proof of Theorem 3.3. ∎

By using analogous arguments to those of the proof of Theorem 3.4 we obtain:

Theorem 3.5. *Let $T \in \mathcal{L}(X)$ and $\varepsilon > 0$. Then,*

$$\sigma_{e\delta,\varepsilon}(T) = \bigcap_{K \in \mathcal{H}_T^\varepsilon(X)} \sigma_{\delta,\varepsilon}(T+K).$$

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