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# Content Semimodules

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*Abstract*: The purpose of this paper is to study content semimodules. We obtain some results on content semimodules similar to the corresponding ones on content modules. We study normally flat content semimodules and multiplication content semimodules. Moreover, we characterize content semimodules over discrete valuation semirings and Boolean algebras.

*Key words*: Semiring, Content semimodule, Multiplication semimodule, Normally flat semimodule.

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## Introduction

Semirings and semimodules have many applications in different branches of mathematics (see [7], [8] and [9]). Semiring is a generalization of ring and bounded distributive lattice. We recall here some definitions:

A semiring is a nonempty set $S$ with two binary operations addition $(+)$ and multiplication $(\cdot)$ such that the following conditions hold:

1) $(S,+)$ is a commutative monoid with identity element 0;
2) $(S,.)$ is a monoid with identity element $1 \neq 0$;
3) $0a = 0 = a0$ for all $a \in S$;
4) $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for every $a, b, c \in S$.

The semiring $S$ is commutative if the monoid $(S,.)$ is commutative. All semirings in this paper are commutative. An ideal $I$ of a semiring $S$ is a nonempty subset of $S$ such that $a + b \in I$ and $sa \in I$ for all $a, b \in I$ and $s \in S$. An ideal $I$ is subtractive if $a + b \in I$ and $b \in I$ imply that $a \in I$ for all $a, b \in S$. A semiring is entire if $ab = 0$ implies that $a = 0$ or $b = 0$. Further, an element $a$ of a semiring $S$ is multiplicatively cancellable (abbreviated as $MC$) if $ab = ac$ implies that $b = c$. If every nonzero element of $S$ is multiplicatively cancellable we say that the semiring $S$ is a semidomain. An element $a$ of a semiring $S$ is multiplicatively idempotent if $a^2 = a$. Let $I^{\times}(S)$ denote the set of all multiplicatively idempotent elements of $S$. We say that $S$ is multiplicatively idempotent, if $I^{\times}(S) = S$.

Let $S$ be a semiring. An $S$-semimodule is an additive abelian monoid $(M, +)$ with additive identity $0_M$ and a function $S \times M \to M$ $((s, m) \mapsto sm)$, called scalar multiplication, such that the following conditions hold for all $s, s' \in S$ and all $m, m' \in M$:

1) $(ss')m = s(s'm)$;
2) $s(m + m') = sm + sm'$;
3) $(s + s')m = sm + s'm$;
4) $1m = m$;
5) $s0_M = 0_M = 0m$;

A subset $N$ of an $S$-semimodule $M$ is a subsemimodule of $M$ if $N$ is closed under addition and scalar multiplication.

We say that a subsemimodule $N$ of an $S$-semimodule $M$ is subtractive if $m+m' \in N$ and $m \in N$ imply that $m' \in N$ for all $m, m' \in M$. Let $M$ and $M'$ be $S$-semimodules. Then a function $\alpha$ from $M$ to $M'$ is an $S$-homomorphism if $\alpha(m + m') = \alpha(m) + \alpha(m')$ for all $m, m' \in M$ and $\alpha(sm) = s(\alpha(m))$ for all $m \in M$ and $s \in S$. The kernel of $\alpha$ is $\ker(\alpha) = \alpha^{-1}\{0\}$. Then $\ker(\alpha)$ is a subtractive $S$-semimodule of $M$. The set $\alpha(M) = \{\alpha(m) \mid m \in M\}$ is a subsemimodule of $M'$. An $S$-homomorphism $\alpha : M \to M'$ is an $S$-monomorphism if $\alpha\beta = \alpha\beta'$ implies $\beta = \beta'$ for all $S$-semimodule $K$ and all $S$-homomorphisms $\beta, \beta' : K \to M$. If $\alpha$ is an $S$-monomorphism, then $\ker(\alpha) = 0$. But the converse need not be true. For example, let $S$ be an entire semiring and $b \in S$ such that it is not multiplicatively cancellative. Thus there exists $a \neq a' \in S$ such that $ab = a'b$. Define a map $\phi : S \to Sb$ by $s \mapsto sb$. Then $\phi$ is an $S$-homomorphism with $\ker(\phi) = 0$. But $\phi$ is not injective, since $\phi(a) = \phi(a')$. An $S$-homomorphism $\alpha : M \to M'$ is surjective if $\alpha(M) = M'$.

Let $S$ be a semiring and $M$ an $S$-semimodule. For any $x \in M$, we define $c(x) = \cap\{I \mid I \text{ is an ideal of } S \text{ and } x \in IM\}$. Then $c$ is a function from $M$ to the set of ideals of $S$ and it is called the content function. An $S$-semimodule $M$ is called a content semimodule if for every $x \in M$, $x \in c(x)M$. In this paper, we study content semimodules and extend some results of [14] to semimodules over semirings. In Section 1, we recall some properties of content semimodules from [12] and we show that projective semimodules are content semimodules. We study normally flat content semimodules in Section 2. In Section 3, we characterize content $S$-semimodules over discrete valuation semirings. In Section 4, we investigate some properties of faithful multiplication content semimodules, as a generalization of faithful multiplication modules. In the last section, we prove that if every subsemimodule of a content $S$-semimodule is a content $S$-semimodule with restricted content function, then $S$ is a multiplicatively regular semiring. We also characterize content semimodules over Boolean algebras.

## 1. Content semimodules

The concepts of content modules and content algebras were introduced in [14]. The concept of content semimodules is studied in [12].

Let $S$ be a semiring and $M$ an $S$-semimodule. For any $x \in M$, we define the content of $x$ by,

$$c_{S,M}(x) = \cap\big\{I \;\mid\; I \text{ is an ideal of } S \text{ and } x \in IM\big\}.$$

Therefore $c_{S,M}$ is a function from $M$ to the set of ideals of $S$ which is called the content function. If $N$ is any non-empty subset of $M$, we define $c_{S,M}(N)$ to be the ideal $\sum_{x\in N} c_{S,M}(x)$. Whenever there is no fear of ambiguity, either or both of the subscripts $S$ and $M$ will be omitted.

Definition 1. Let $S$ be a semiring. An $S$-semimodule $M$ will be called a *content $S$-semimodule* if for every $x \in M$, $x \in c(x)M$.

Example 2. Let $S$ be a multiplicatively idempotent semiring, $J$ an ideal of $S$ and $x \in J$. It is clear that $(x) \subseteq c(x) = \cap\{I \mid I \subseteq S,\ x \in IJ\}$. Thus $(x)J \subseteq c(x)J$. But $x = x^2 \in (x)J$ and hence $x \in c(x)J$. Therefore $J$ is a content $S$-semimodule.

Now, we recall next results from [12], which will be used repeatedly.

Theorem 3. *Let $M$ be an $S$-semimodule. Then the following statements are equivalent:*

1) *$M$ is a content $S$-semimodule.*
2) *For every set of ideals $\{I_i\}$ of $S$, $(\cap I_i)M = \cap(I_iM)$.*
3) *For every set of finitely generated ideals $\{I_i\}$ of $S$, $(\cap I_i)M = \cap(I_iM)$.*

4) *There exists a function $f$ from $M$ to the set of ideals of $S$ such that for every $x \in M$ and every ideal $I$ of $S$, $x \in IM$ if and only if $f(x) \subseteq I$.*

*Moreover, if $M$ is a content $S$-semimodule and $x \in M$, then $c(x)$ is a finitely generated ideal.*

THEOREM 4. *Let $M$ be a content $S$-semimodule, and $N$ a subsemimodule of $M$. Then the following statements are equivalent:*

1) *$IM \cap N = IN$ for every ideal $I$ of $S$.*
2) *For every $x \in N$, $x \in c_M(x)N$.*
3) *$N$ is a content $S$-semimodule and $c_M$ restricted to $N$ is $c_N$.*

We know that every free module and every projective module as a direct summand of a free module, are content modules by [14, Corollary 1.4]. Moreover every free semimodule is a content semimodule by [12, Corollary 26]. But not all projective semimodules are direct summands of free semimodules (cf. [3, Example 2.3]). We can prove that every projective semimodule is a content semimodule as follows:

THEOREM 5. *Any projective semimodule is a content semimodule.*

*Proof.* Let $S$ be a semiring and $M$ a projective $S$-semimodule. Then by [18, Theorem 3.4.12], there exist $\{m_i\}_{i \in I} \subseteq M$ and $\{f_i\}_{i \in I} \subseteq \mathrm{Hom}_S(M, S)$ such that for any $x \in M$, $f_i(x) = 0$ for almost all $i \in I$, and $x = \sum_i f_i(x) m_i$. Suppose that $x \in M$. Then $x = \sum_{i=1}^n f_i(x) m_i$ and hence $x \in (f_1(x), \ldots, f_n(x))M$. Thus $c(x) \subseteq (f_1(x), \ldots, f_n(x))$. Now assume that $x \in IM$ for some ideal $I$ of $S$. Then there exist $m \in \mathbb{N}$, $r_1, \ldots, r_m \in I$ and $x_1, \ldots, x_m \in M$ such that $x = \sum_{i=1}^m r_i x_i$. For each $1 \leq j \leq n$, $f_j(x) = \sum_{i=1}^m r_i f_j(x_i)$, and hence $f_j(x) \in (r_1, \ldots, r_m)$. Therefore $(f_1(x), \ldots, f_n(x)) \subseteq (r_1, \ldots, r_m) \subseteq I$. This implies $(f_1(x), \ldots, f_n(x)) \subseteq c(x)$, by definition of content function. Therefore $(f_1(x), \ldots, f_n(x)) = c(x)$, and $x \in (f_1(x), \ldots, f_n(x))M = c(x)M$. ∎

Let $M$ be an $S$-semimodule, $N$ a subsemimodule of $M$ and $I$ an ideal of $S$. Put
$$(N :_M I) = \{x \mid x \in M \text{ and } Ix \subseteq N\}.$$
Then $(N :_M I)$ is a subsemimodule of $M$.

THEOREM 6. *Let $M$ be a content $S$-semimodule, and let $s \in S$. Then the following statements are equivalent:*

1) *$s(c(x)) = c(sx)$ for all $x \in M$.*

2) *$(I :_S s)M = (IM :_M s)$ for every ideal $I$ of $S$.*

*Proof.* (1) $\Rightarrow$ (2): The proof is similar to [14, Theorem 1.5].
(2) $\Rightarrow$ (1): Let $x \in M$. Then $x \in c(x)M$ since $M$ is a content $S$-semimodule. Thus, $sx \in sc(x)M$. This implies that $c(sx) \subseteq sc(x)$. Now by (2), $(c(sx) :_S s)M = (c(sx)M :_M s)$. On the other hand, since $M$ is a content semimodule, $sx \in c(sx)M$. This implies $x \in (c(sx)M :_M s) = (c(sx) :_S s)M$. So $c(x) \subseteq (c(sx) : s)$ and hence $sc(x) \subseteq c(sx)$. ∎

Definition 7. Let $S$ be a semidomain. An $S$-semimodule $M$ is said to be torsionfree if for any $0 \neq a \in S$, multiplication by $a$ on $M$ is injective, i.e., if $ax = ay$ for some $x, y \in M$, then $x = y$.

Now we give the following theorem for content torsionfree semimodules over a semidomain.

Theorem 8. *Let $S$ be a semidomain and $M$ a content torsionfree $S$-semimodule. Then for every $s \in S$ and $x \in M$, $s(c(x)) = c(sx)$.*

*Proof.* Since $M$ is a content $S$-semimodule, $x \in c(x)M$. Therefore $sx \in sc(x)M$, which implies $c(sx) \subseteq sc(x)$. Now $sx \in sM$, implies that $c(sx) \subseteq (s)$. Therefore $c(sx) = (s)J$, where $J = (c(sx) : s)$. Since $M$ is a content semimodule, $sx \in c(sx)M = sJM$. Therefore $sx = sz$, for some element $z \in JM$. Then $x = z$, since $M$ is torsionfree. Thus $x \in JM$ and hence $c(x) \subseteq J$. Therefore $sc(x) \subseteq sJ = c(sx)$. ∎

Theorem 9. *Let $S$ be a semiring such that any $0 \neq s \in S$ is in at most finitely many ideals, and let $M$ be an $S$-semimodule such that for all ideals $I, J$ of $S$, $(I \cap J)M = IM \cap JM$. Then $M$ is a content semimodule if and only if $\cap(I_iM) = 0$, whenever $\{I_i\}$ is an infinite set of ideals of $S$.*

*Proof.* Let $M$ be a content $S$-semimodule and let $\{I_i\}$ be an infinite set of ideals of $S$. Then by Theorem 3, $\cap(I_iM) = (\cap I_i)M = 0$. Conversely, let $\{I_i\}_{i \in I}$ be a set of ideals of $S$. If $I$ is finite, then $\cap(I_iM) = (\cap I_i)M$ by assumption. Now if $I$ is infinite, then $\cap(I_iM) = (\cap I_i)M = 0$. Therefore $M$ is a content $S$-semimodule. ∎

## 2. Normally flat semimodules and content semimodules

In this section, we investigate normally flat content semimodules. The concept of normally flat semimodules was introduced in [2]. Let us recall some definitions.

Let $M$ and $N$ be two $S$-semimodules. An $S$-balanced map $g : M \times N \to G$, where $G$ is an Abelian monoid, is a bilinear map such that $g(ms, n) = g(m, sn)$ for all $m \in M$, $s \in S$ and $n \in N$.

A commutative monoid $M \otimes_S N$ together with an $S$-balanced map $\tau : M \times N \to M \otimes_S N$ is called a tensor product of $M$ and $N$ over $S$ if for every Abelian monoid $G$ with an $S$-balanced map $\beta : M \times N \to G$, there exists a unique morphism of monoids $\gamma : M \otimes_S N \to G$ that $\gamma \circ \tau = \beta$. For more details on tensor product of semimodules see [10], [1] and [15].

Definition 10. Assume that $M$ is an $S$-semimodule. We say that a subsemimodule $N \leq_S M$ is a normal subsemimodule, and write $N \leq_S^n M$, if the embedding $N \hookrightarrow M$ is a normal monomorphism, that is, $N = \ker(f)$ for some $S$-homomorphism $f : M \to L$ and some $S$-semimodule $L$. Note that $N \leq_S^n M$ if and only if $N = \overline{N}$, the normal closure of $N$, defined by $\overline{N} := \{m \in M \mid m + n_1 = n_2 \text{ for some } n_1, n_2 \in N\}$. Therefore $N \leq_S^n M$ if and only if $N$ is a subtractive subsemimodule of $M$.

Definition 11. Let $F$ and $M$ be $S$-semimodules. We say that $F$ is normally flat with respect to $M$ (or normally $M$-flat) if $N \otimes_S F \leq_{\mathbb{N}}^n M \otimes_S F$ for every $N \leq_S^n M$. We say that $F$ is normally flat, if $F$ is normally $M$-flat for every $S$-semimodule $M$.

Assume that $R$ is a domain. It is well known that if $M$ is a flat $R$-module, then $M$ is torsionfree [4, Chapter I, §2.5, Proposition 3]. We have the following result for normally flat semimodules.

Theorem 12. *Let $S$ be a semidomain such that every principal ideal of $S$ is subtractive and let $M$ be a normally flat $S$-semimodule. Then $M$ is a torsionfree $S$-semimodule.*

*Proof.* Let $0 \neq a \in S$. We should show that for all $x, y \in M$, $ax = ay$ implies $x = y$. Define a map $f : S \to S$ by $f : s \mapsto as$. If $as = as'$ for some $s, s' \in S$, then $s = s'$ since $a$ is an $MC$ element. Therefore $f$ is an injective $S$-homomorphism. Moreover, $f(S) = Sa$ is a subtractive subsemimodule of $S$. Since $M$ is normally flat, $\bar{f} : S \otimes_S M \to S \otimes_S M$ where $\bar{f} : s \otimes m \mapsto as \otimes m$, is injective. But $S \otimes_S M \overset{\theta}{\cong} M$ by [15, Theorem 7.6]. Thus $\theta \circ \bar{f} \circ \theta^{-1} : m \mapsto am$ is injective. ∎

In [14, Corollary 1.6], it is proved that a content module is a flat module if and only if for every $s \in S$ and $x \in M$, $s(c(x)) = c(sx)$. Now from Theorem 12, we have the following result.

Corollary 13. *Let $S$ be a semidomain such that every principal ideal of $S$ is subtractive and let $M$ be a content normally flat $S$-semimodule. Then for every $s \in S$ and $x \in M$, $s(c(x)) = c(sx)$.*

*Proof.* By Theorem 12, $M$ is a torsionfree $S$-semimodule. Thus by Theorem 8, for every $s \in S$ and $x \in M$, $s(c(x)) = c(sx)$. ∎

Theorem 14. *Let $S$ be a semiring and $M$ a content $S$-semimodule such that for every $s \in S$ and every ideal $I$ of $S$, $(I :_S s)M = (IM :_M s)$. Then $(I :_S J)M = (IM :_M J)$ for every pair of ideals $I$, $J$ of $S$.*

*Proof.* Since $M$ is a content $S$-semimodule, by Theorem 3, $(I : J)M = (\cap_{s \in J}(I : s))M = \cap_{s \in J}(I : s)M$. But $\cap_{s \in J}(I : s)M = \cap_{s \in J}(IM : s) = (IM : J)$. ∎

Corollary 15. *Assume that $S$ is a semidomain such that every principal ideal of $S$ is subtractive and let $M$ be a content normally flat $S$-semimodule. Then $(I :_S J)M = (IM :_M J)$ for every pair of ideals $I$, $J$ of $S$.*

*Proof.* By Theorem 13, for every $s \in S$ and $x \in M$, $s(c(x)) = c(sx)$ and by Theorem 6, $(I :_S s)M = (IM :_M s)$ for every ideal $I$ of $S$ and $s \in S$. Thus by Theorem 14, $(I :_S J)M = (IM :_M J)$. ∎

## 3. Content semimodules over discrete valuation semirings

Discrete valuation semiring was introduced and studied in [13]. Similar to [14, Proposition 2.1], we will obtain a characterization of content $S$-semimodules over a discrete valuation semiring. First, we recall some definitions and results from [13].

Let $(M, +, 0, <)$ be a totally ordered commutative monoid (abbreviated as tomonoid) with no greatest element and let $+\infty$ be an element such that $+\infty \notin M$. Put $M_\infty = M \cup \{+\infty\}$. Now set $m < +\infty$ for all $m \in M$ and

$m + (+\infty) = (+\infty) + m = +\infty$ for all $m \in M_\infty$. Then $M_\infty$ is a tomonoid with the greatest element $+\infty$.

Definition 16. A map $v : S \to M_\infty$ is an $M$-valuation on $S$ if the following properties hold:

1) $S$ is a semiring and $M_\infty$ is a tomonoid with the greatest element $+\infty$, which has been obtained from the tomonoid $M$ with no greatest element,
2) $v(xy) = v(x) + v(y)$ for all $x, y \in S$,
3) $v(x + y) \geq \min\{v(x), v(y)\}$, whenever $x, y \in S$,
4) $v(1) = 0$ and $v(0) = +\infty$.

If in the above $M = \mathbb{Z}$, we will say that $v$ is a discrete valuation on $S$.

Definition 17. Let $S$ be a semiring. If there exists an $M$-valuation $v$ on $S$, then it is obvious that $S_v = \{s \in S \mid v(s) \geq 0\}$ is a subsemiring of $S$. In this case we say that "$S_v$ is a $V$-semiring with respect to the triple $(S, v, M)$".

An element $s$ of a semiring $S$ is a unit if there exists an element $s'$ of $S$ such that $ss' = 1$. We say that $S$ is a semifield if every nonzero element of $S$ is a unit.

Definition 18. A semiring $S$ is called discrete valuation semiring, if $S = K_v$ is a $V$-semiring with respect to the triple $(K, v, \mathbb{Z})$, where $K$ is a semifield and $v$ is surjective.

A semiring $S$ is called a local semiring if it has a unique maximal ideal. Note that by [13, Theorem 3.6] every discrete valuation semiring is a local semiring.

Theorem 19. *Let $(S, m)$ be a discrete valuation semiring and let $M$ be an $S$-semimodule. Then $M$ is a content $S$-semimodule if and only if $\cap\{m^iM \mid i = 1, 2, \cdots\} = 0$.*

*Proof.* Let $M$ be a content $S$-semimodule. By [13, Theorem 3.6], $\cap_{i=1}^{\infty} m^i = 0$. Thus by Theorem 3, $\cap_{i=1}^{\infty}(m^iM) = (\cap_{i=1}^{\infty} m^i)M = 0$.

Now let $0 \neq x \in M$. Since every ideal of $S$ is of the form $m^i (i \in \mathbb{N})$ [13, Lemma 3.3], $c(x) = \cap\{m^i \mid x \in m^iM\}$. But $\cap\{m^iM \mid i = 1, 2, \cdots\} = 0$. So there exists a positive integer $n$ such that $x \in m^nM$ and $x \notin m^iM$ for all $i > n$. Therefore $c(x) = m^n$. Hence $x \in c(x)M$. ∎

We know that free semimodules, and more generally, projective semimodules are examples of normally flat semimodules (see [1]), and in Section 1 we proved that these semimodules are content semimodules. Now, we give an example of a content semimodule which is not normally flat. First, we recall the following definition:

Let $M$ be an $S$-semimodule and $N$ a subsemimodule of $M$. Then we can define a congruence relation on $M$ as follows: $m \equiv_N n$ iff $m + a = n + b$ for some $a, b \in N$. The set of equivalence classes is an $S$-semimodule and denoted by $M/N$. The equivalence class of $m \in M$ is denoted by $m/N$.

EXAMPLE 20. Let $S = (\mathbb{N} \cup \{+\infty\}, min, +, +\infty, 0)$. Then $S$ is a semidomain.

Let $J = S \backslash \{1_S\} = \{-\infty\} \cup \{1, 2, \cdots\}$. We show that $J$ is a principal ideal of $S$. If $x, y \in J$ and $s \in S$, then $x \oplus y = \min\{x, y\} \in J$ and $0 \neq s + x = s \odot x \in J$. Now let $0_S \neq a \in J$. Then $a = 1 + \overset{a}{\cdots} + 1 = 1 \odot \overset{a}{\cdots} \odot 1 = 1^a \in (1)$. Therefore $J = (1)$ and $J$ is the unique maximal ideal of $S$.

If $I$ is an ideal of $S$, then $I$ is a power of $J$. Let $I$ be an ideal of $S$, $0_S \neq n \in I$ the smallest element in $I$ and $0_S \neq x \neq n \in I$. Then $x \geq n$ and hence $x - n \in S$. Thus $x = x - n + n = (x - n) \odot n \in (n)$. Therefore $I = (n)$. Moreover, $n = 1 + \cdots + 1 = 1 \odot \cdots \odot 1 = 1^n$ and hence $I = (n) = (1^n) = (1)^n = J^n$. Thus $S$ is a discrete valuation semiring by [13, Theorem 3.6].

Now let $I = (n)$ be an ideal of $S$ and $x, y \in S$ such that $x + y$, $y \in I$. If $y \geq x$, then $x + y = \min\{y, x\} = x \in I$. If $x \geq y$, then $x - y \in S$ and so $x = x - y + y = x - y \odot y \in I$. Therefore $I$ is subtractive. This implies $S/I \neq 0$. Now consider the $S$-semimodule $M = S/J^2$. Since $J^2 M = 0$, $\cap\{J^i M \mid i = 1, 2, \cdots\} = 0$. Thus by Theorem 19, $M$ is a content $S$-semimodule. Note that $S$ is a semidomain such that every ideal of $S$ is subtractive and $M$ is not torsionfree. Thus from Theorem 12, $M$ is not a normally flat $S$-semimodule.

## 4. MULTIPLICATION SEMIMODULES AND CONTENT SEMIMODULES

In this section we study the relation between multiplication semimodules and content semimodules and give some results about multiplication semimodules. It is known that every faithful multiplication module is a content module. Here we investigate faithful multiplication content semimodules and extend some results of [6] to semimodules.

If $N$ and $L$ are subsemimodules of an $S$-semimodule $M$, we set $(N : L) = \{s \in S \mid sL \subseteq N\}$. Then $(N : L)$ is an ideal of $S$.

Definition 21. Let $S$ be a semiring and $M$ an $S$-semimodule. Then $M$ is called a multiplication semimodule if for each subsemimodule $N$ of $M$ there exists an ideal $I$ of $S$ such that $N = IM$.

In this situation we can prove that $N = (N : M)M$. Cyclic semimodules are examples of multiplication semimodules [19, Example 2].

Theorem 22. *Suppose that $M$ is a content $S$-semimodule and for any subsemimodule $N$ of $M$ and ideal $I$ of $S$ such that $N \subset IM$ there exists an ideal $J$ of $S$ such that $J \subset I$ and $N \subseteq JM$. Then $M$ is a multiplication $S$-semimodule.*

*Proof.* The proof is similar to [6, Theorem 1.6]. ∎

We recall the following results from [16].

Let $M$ be an $S$-semimodule and $P$ a maximal ideal of $S$. We say that $M$ is $P$-cyclic if there exist $m \in M$, $t \in S$ and $q \in P$ such that $t + q = 1$ and $tM \subseteq Sm$.

Theorem 23. *Let $M$ be an $S$-semimodule. If $M$ is a multiplication semimodule, then for every maximal ideal $P$ of $S$ either $M = \{m \in M \mid m = qm \text{ for some } q \in P\}$ or $M$ is $P$-cyclic* [16, Theorem 1.6].

Definition 24. An element $m$ of an $S$-semimodule $M$ is cancellable if $m+m' = m+m''$ implies that $m' = m''$. The $S$-semimodule $M$ is cancellative if every element of $M$ is cancellable.

A semiring $S$ is yoked if for all $a, b \in S$, there exists an element $t$ of $S$ such that $a + t = b$ or $b + t = a$. Now, we give the following theorem for yoked semirings.

Theorem 25. *Let $S$ be a yoked semiring such that every maximal ideal of $S$ is subtractive and let $M$ be a cancellative faithful multiplication $S$-semimodule. Then $M$ is a content $S$-semimodule.*

*Proof.* Let $\{I_\lambda\}(\lambda \in \Lambda)$ be any non-empty collection of ideals of $S$. Put $I = \cap_{\lambda\in\Lambda} I_\lambda$. Clearly $IM \subseteq \cap_{\lambda\in\Lambda}(I_\lambda M)$. Let $x \in \cap_{\lambda\in\Lambda}(I_\lambda M)$ and let $K = \{r \in S \mid rx \in IM\}$. If $K \neq S$, then there exists a maximal ideal $Q$ of $S$ such that $K \subseteq Q$. Suppose that $M = \{m \in M \mid m = pm \text{ for some } p \in Q\}$. Then $x = px$ for some $p \in Q$. Since $S$ is a yoked semiring, there exists $t \in S$ such that $t + p = 1$ or $1 + t = p$. Suppose that $t + p = 1$. Then $px + tx = x$.

Since $M$ is a cancellative $S$-semimodule, $tx = 0$ and hence $t \in K \subseteq Q$ which is a contradiction. Now suppose that $1 + t = p$. Then $x + tx = px$. Since $M$ is a cancellative semimodule, $tx = 0$ and hence $t \in K \subseteq Q$. On the other hand, since $Q$ is a subtractive ideal, $1 \in Q$ which is a contradiction. Therefore by Theorem 23, $M$ is $Q$-cyclic. Hence there exist $m \in M$, $t \in S$ and $q \in Q$ such that $t + q = 1$ and $tM \subseteq Sm$. Then $tx \in \cap_{\lambda\in\Lambda}(I_\lambda m)$. Thus for each $\lambda \in \Lambda$, there exists $a_\lambda \in I_\lambda$ such that $tx = a_\lambda m$. Choose $\alpha \in \Lambda$. Then $a_\alpha m = a_\lambda m$ for each $\lambda \in \Lambda$. Since $S$ is a yoked semiring, there exists $r_\lambda \in S$ such that $a_\alpha + r_\lambda = a_\lambda$ or $a_\lambda + r_\lambda = a_\alpha$. Suppose that $a_\alpha + r_\lambda = a_\lambda$. Then $a_\alpha m + r_\lambda m = a_\lambda m$ and hence $r_\lambda m = 0$. Thus $tr_\lambda M \subseteq r_\lambda(Sm) = 0$. Since $M$ is a faithful semimodule, $tr_\lambda = 0$. But $ta_\alpha + tr_\lambda = ta_\lambda$ and hence $ta_\alpha = ta_\lambda$. Now suppose that $a_\lambda + r_\lambda = a_\alpha$. A similar argument shows that $ta_\alpha = ta_\lambda$. Thus in any case $ta_\alpha \in I_\lambda$ for each $\lambda \in \Lambda$ and hence $ta_\alpha \in I$. Therefore $t^2x = ta_\alpha m \in IM$. This implies that $t^2 \in K \subseteq Q$ which is a contradiction. Therefore $K = S$ and hence $x \in IM$. ∎

We call an $S$-semimodule $M$ multiplicatively cancellative (abbreviated as $MC$) if for any $s, s' \in S$ and $0 \neq m \in M$, $sm = s'm$ implies $s = s'$ [5].

Theorem 26. *Let $M$ be an $MC$ multiplication $S$-semimodule. Then $M$ is a content $S$-semimodule.*

*Proof.* By [16, Theorem 2.9], $M$ is a projective $S$-semimodule and by Corollary 5, $M$ is a content $S$-semimodule. ∎

Now, by using [16, Corollary 2.10], we get the following result.

Corollary 27. *Let $S$ be a yoked entire semiring and $M$ a cancellative faithful multiplication $S$-semimodule. Then $M$ is a content $S$-semimodule.*

We say that a subsemimodule $E$ of an $S$-semimodule $M$ is an essential subsemimodule, if for any nonzero subsemimodule $N \subseteq M$, $E \cap N \neq 0$ [11]. Let $S$ be a semiring and $M$ a faithful multiplication content $S$-semimodule. Then similar to [6, Theorem 2.13] we can prove that a subsemimodule $N$ of $M$ is essential if and only if there exists an essential ideal $E$ of $S$ such that $N = EM$.

Assume that $M$ is an $S$-semimodule. We define the socle of $M$, denoted by $Soc(M)$, to be $Soc(M) = \bigcap\{N \mid N \subseteq_e M\}$ (see also [11]). Now if $M$ is a faithful multiplication content $S$-semimodule, then similar to [6, Corollary 2.14], we conclude that $Soc(M) = Soc(S)M$.

An $S$-semimodule $M$ is called finitely cogenerated if for every set $A$ of subsemimodules of $M$, $\bigcap A = 0$ if and only if $\bigcap F = 0$ for some finite set $F \subseteq A$ [11]. The semiring $S$ is called finitely cogenerated if it is finitely cogenerated as an $S$-semimodule. Let $S$ be a semiring and $M$ a faithful multiplication content $S$-semimodule. Then with a similar proof for [6, Corollary 1.8], we can show that, $M$ is finitely cogenerated if and only if $S$ is finitely cogenerated.

Assume that $M$ is an $S$-semimodule. Now we give some properties of the ideal $c(M)$.

Theorem 28. (see [14, Corollary 1.6]) *Let $M$ be a content $S$-semimodule. Then $c(M) = S$ iff $\mathfrak{m}M \neq M$ for every maximal ideal $\mathfrak{m}$ of $S$.*

*Proof.* ($\Rightarrow$) Let $c(M) = S$ and $\mathfrak{m}$ a maximal ideal of $S$ such that $\mathfrak{m}M = M$. If $x \in M = \mathfrak{m}M$, then $c(x) \subseteq \mathfrak{m}$. Hence $c(M) \subseteq \mathfrak{m}$ which is a contradiction.

($\Leftarrow$) Let $\mathfrak{m}$ be a maximal ideal of $S$ and $\mathfrak{m}M \neq M$. Then there exists $x \in M \backslash \mathfrak{m}M$. Thus $c(x) \nsubseteq \mathfrak{m}$ since $x \in c(x)M$. Therefore $c(M) \nsubseteq \mathfrak{m}$. Since for all maximal ideal $\mathfrak{m}$ of $S$, $c(M) \nsubseteq \mathfrak{m}$, we have $c(M) = S$. ∎

Let $S$ be a semiring and $M$ an $S$-semimodule. Put $A = \{I \subseteq S | M = IM\}$ and $\tau(M) = \cap_{I \in A} I$. Then $\tau(M)$ is an ideal of $S$.

Theorem 29. *Let $M$ be a content $S$-semimodule. Then $c(M) = \tau(M)$.*

*Proof.* If $x \in M$, then $x \in c(x)M \subseteq c(M)M$. Therefore $c(M)M = M$ and hence $\tau(M) \subseteq c(M)$. Now let $I$ be an ideal of $S$ such that $M = IM$. Then for each $x \in M = IM$, $c(x) \subseteq I$ and hence $c(M) \subseteq I$. Therefore $c(M) \subseteq \tau(M)$. ∎

Theorem 30. *Let $M$ be a faithful multiplication content $S$-semimodule and $I = \tau(M)$. Then:*

1) *$m \in Im$ for each $m \in M$;*
2) *$I^2 = I$;*
3) *$ann(I) = 0$.*

*Proof.* The proof is similar to [6, Lemma 3.2]. ∎

## 5. Regular semirings and content semimodules

An element $a$ of a semiring $S$ is multiplicatively regular if there exists an element $x$ of $S$ such that $axa = a$. A semiring $S$ is multiplicatively regular if every element of $S$ is multiplicatively regular. Bounded distributive lattices, and in particular, Boolean algebras are multiplicatively regular semirings.

Example 31. Let $S$ be a semifield and $A$ a nonempty set. Suppose that $f \in S^A$. Define a map $g : A \to S$ by $g(a) = f(a)^{-1}$ if $f(a) \neq 0$, and $g(a) = 0$ if $f(a) = 0$. Then $f = fgf$. Therefore $S^A$ is a multiplicatively regular semiring.

Theorem 32. *Let $S$ be a multiplicatively regular semiring. Then every ideal of $S$ is generated by idempotents.*

*Proof.* Let $I$ be an ideal of $S$ and $x \in I$. Then $x = x^2 s$ for some $s \in S$. Thus $xs \in I^{\times}(S)$ and $(x) = (xs)$. Therefore $I = \sum_{x \in I} Sx$ is generated by idempotents. ∎

In [14], it is shown that a ring $R$ is regular if and only if every submodule of a content $R$-module is a content module with restricted content function. We can extend this result to multiplicatively regular semirings as follows:

Theorem 33. *Assume that $S$ is a semiring. If every subsemimodule of a content $S$-semimodule is a content $S$-semimodule with restricted content function, then $S$ is a multiplicatively regular semiring.*

*Proof.* By [17, Proposition 1], it is sufficient to show that every principal ideal of $S$ is generated by an idempotent. Suppose that $a \in S$. Then by Theorem 4, $(a)S \cap (a) = (a^2)$. Thus there exists $r \in S$ such that $a = ra^2$. Hence $ar \in I^{\times}(S)$ and $(a) = (ar)$. ∎

Let $S$ be a semiring. An element $a$ of $S$ is complemented if there exists an element $c$ of $S$ such that $ac = 0$ and $a + c = 1$. Let $comp(S)$ denote the set of all complemented elements of $S$. Note that $comp(S) \subseteq I^{\times}(S)$. Since if $a \in comp(S)$, then $a = a1 = a(a + c) = a^2 + ac = a^2$.

Theorem 34. *Let $S$ be a semiring such that $comp(S) = I^{\times}(S)$. Let $I = (e, f)$ be an ideal of $S$ such that $e, f \in I^{\times}(S)$. Then $I = (g)$ for some $g \in I^{\times}(S)$.*

*Proof.* Since $e, f \in comp(S)$, there exist elements $x, y \in I^{\times}(S)$ such that $x + e = 1$, $y + f = 1$, $xe = 0$ and $yf = 0$. Then $1 = xy + ye + xf + fe$. Put $g = ye+xf+fe \in I$. Then $1 = xy+g$ and $g^2 = g$. Moreover $e = exy+eg = eg$ and $f = fxy + fg = fg$. Therefore $I = (e, f) \subseteq (g) \subseteq I$. Hence $I = (g)$. ∎

EXAMPLE 35. Let $S$ be a semiring such that $I^{\times}(S) = \{0, 1\}$. Let $A$ be a nonempty set and $f \in I^{\times}(S^A)$. Then for each $a \in A$, $f(a) \in I^{\times}(S) = \{0, 1\}$. Define a map $g : A \to S$ by $g(a) = 1$ if $f(a) = 0$, and $g(a) = 0$ if $f(a) = 1$. Then $f + g = 1_{S^A}$, and $fg = 0$. Thus $f \in comp(S^A)$. Therefore $S^A$ is a semiring such that $comp(S^A) = I^{\times}(S^A)$.

THEOREM 36. *Let $S$ be a semiring. If every finitely generated ideal in $S$ is generated by an idempotent, then every subsemimodule of a content $S$-semimodule is a content $S$-semimodule with restricted content function.*

*Proof.* Let $M$ be a content $S$-semimodule and $N \subseteq M$. By Theorem 4, we should show that for every $x \in N$, $x \in c_M(x)N$. Let $x \in N$. Then $x \in c_M(x)M$. Since $M$ is a content $S$-semimodule, $c_M(x)$ is a finitely generated ideal. Hence there exists an element $e \in I^{\times}(S)$ such that $c_M(x) = (e)$. Thus $x \in (e)M$ and hence there exists $m \in M$ such that $x = em = e^2m = ex$. Therefore $x \in c_M(x)N$. ∎

Here we study content semimodules over Boolean algebras (see [14, section 4]). Note that, by Theorem 34, every finitely generated ideal of a Boolean algebra is generated by an idempotent.

LEMMA 37. *Let $S$ be a semiring and $M$ an $S$-semimodule. If every finitely generated ideal of $S$ is generated by an idempotent then for all ideals $I, J \subseteq S$, $(I \cap J)M = IM \cap JM$.*

*Proof.* It is clear that $(I \cap J)M \subseteq IM \cap JM$. Suppose that $x \in IM \cap JM$. Then $x = \sum_{i=1}^{m} r_i m_i = \sum_{j=1}^{n} s_j m'_j$ , where $m_i, m'_j \in M$, $r_i \in I$ and $s_j \in J$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Put $I' = (r_1, \ldots, r_m)$, $J' = (s_1, \ldots, s_n)$. Then there exist $e, u \in I^{\times}(S)$ such that $I' = (e) \subseteq I$ and $J' = (u) \subseteq J$. Then for each $i$, $1 \leq i \leq m$, there exists $u_i \in S$ such that $r_i = eu_i$. Moreover, for each $j$, $1 \leq j \leq n$, there exists $t_j \in S$ such that $s_j = ut_j$. Hence $x = \sum_{i=1}^{m} r_i m_i = \sum_{i=1}^{m} eu_i m_i = e \sum_{i=1}^{m} eu_i m_i = ex$. Similarly $x = \sum_{j=1}^{n} s_j m'_j = \sum_{j=1}^{m} ut_j m'_j = u \sum_{j=1}^{m} ut_j m'_j = ux$. Thus $x = ex = eux$. Therefore $x \in (I \cap J)M$. ∎

Lemma 38. *Let $S$ be a Boolean algebra and $M$ an $S$-semimodule. Then for all $s \in S$, $(0 :_S s)M = (0 :_M s)$.*

*Proof.* Clearly, $(0 :_S s)M \subseteq (0 :_M s)$. Let $x \in M$ such that $sx = 0$. Since $S$ is a Boolean algebra, there exists an element $t \in S$ such that $t + s = 1$ and $ts = 0$. Thus $x = tx \in (0 :_S s)M$. Therefore $(0 :_S s)M = (0 :_M s)$. ∎

Lemma 39. *Let $S$ be a Boolean algebra, $M$ an $S$-semimodule and $x \in M$. Then $c(x) = ann(ann(x))$.*

*Proof.* Let $I$ be a finitely generated ideal of $S$ such that $x \in IM$. Then $annI \subseteq ann(x)$ and hence $ann(ann(x)) \subseteq ann(ann(I))$. But $ann(ann(I)) = I$ and so $ann(ann(x)) \subseteq I$. By Theorem 3, $ann(ann(x)) \subseteq c(x)$.

Conversely, let $s \in S$ such that $sx = 0$. Then $x \in (0 :_M s) = (0 :_S s)M$ by Theorem 38. Thus $c(x) \subseteq (0 :_S s)$ and hence $c(x) \subseteq \bigcap_{s \in ann(x)}(0 :_S s) = ann(ann(x))$. ∎

In the following theorem we characterize content $S$-semimodules over Boolean algebras.

Theorem 40. *Let $S$ be a Boolean algebra and $M$ an $S$-semimodule. Then $M$ is a content $S$-semimodule if and only if for all $x \in M$, $ann(x)$ is a finitely generated ideal.*

*Proof.* Suppose that $M$ is a content $S$-semimodule and $x \in M$. Then $c(x) = ann(ann(x))$ is a finitely generated ideal. Thus there exists $e \in S$ such that $ann(ann(x)) = (e)$. Moreover there exists $u \in S$ such that $ue = 0$ and $u + e = 1$. We show that $ann(x) = (u)$. Since $M$ is a content $S$-semimodule, $x \in c(x)M = (e)M$. Thus there exists $z \in M$ such that $x = ez$. Hence $ux = uez = 0$. Therefore $u \in ann(x)$ and hence $(u) \subseteq ann(x)$.

For the reverse inclusion, let $r \in ann(x)$. Then $r = ur + er = ur$. Thus $r \in (u)$ and hence $ann(x) \subseteq (u)$.

Now suppose that $x \in M$ and $ann(x)$ is a finitely generated ideal. Note that by Lemma 37, $(I \cap J)M = IM \cap JM$ for all ideals $I, J \subseteq S$. Let $ann(x) = (s_1, \ldots, s_n)$. Then $x \in \bigcap_{i=1}^n (0 :_M s_i) = \bigcap_{i=1}^n (0 :_S s_i)M = (\bigcap_{i=1}^n (0 :_S s_i))M = (ann(ann(x)))M$. Thus by Theorem 39, $x \in c(x)M$ and hence $M$ is a content $S$-semimodule. ∎

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