

## The Differences Between Birkhoff and Isosceles Orthogonalities in Radon Planes

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*Abstract:* The notion of orthogonality for vectors in inner product spaces is simple, interesting and fruitful. When moving to normed spaces, we have many possibilities to extend this notion. We consider Birkhoff orthogonality and isosceles orthogonality. Recently the constants which measure the difference between these orthogonalities have been investigated. The usual orthogonality in inner product spaces and isosceles orthogonality in normed spaces are symmetric. However, Birkhoff orthogonality in normed spaces is not symmetric in general. A two-dimensional normed space in which Birkhoff orthogonality is symmetric is called a Radon plane. In this paper, we consider the difference between Birkhoff and isosceles orthogonalities in Radon planes.

*Key words:* Birkhoff orthogonality, Isosceles orthogonality, Minkowski plane, Minkowski geometry, Radon plane.

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### 1. INTRODUCTION

We denote by  $X$  a real normed space with the norm  $\|\cdot\|$ , the unit ball  $B_X$  and the unit sphere  $S_X$ . Throughout this paper, we assume that the dimension of  $X$  is at least two. In case of that  $X$  is an inner product space, an element  $x \in X$  is said to be orthogonal to  $y \in X$  (denoted by  $x \perp y$ ) if the inner product  $\langle x, y \rangle$  is zero. In the general setting of normed spaces, many notions of orthogonality have been introduced by means of equivalent propositions to the usual orthogonality in inner product spaces. For example, Roberts [20] introduced Roberts orthogonality: for any  $x, y \in X$ ,  $x$  is said to be Roberts orthogonal to  $y$  (denoted by  $x \perp_R y$ ) if

$$\|x + ty\| = \|x - ty\| \quad \text{for all } t \in \mathbb{R}.$$

Birkhoff [4] introduced Birkhoff orthogonality:  $x$  is said to be Birkhoff orthogonal to  $y$  (denoted by  $x \perp_B y$ ) if

$$\|x + ty\| \geq \|x\| \quad \text{for all } t \in \mathbb{R}.$$

James [4] introduced isosceles orthogonality:  $x$  is said to be isosceles orthogonal to  $y$  (denoted by  $x \perp_I y$ ) if

$$\|x + y\| = \|x - y\|.$$

These generalized orthogonality types have been studied in a lot of papers ([1], [2], [8] and so on).

Recently, quantitative studies of the difference between two orthogonality types have been performed:

$$\begin{aligned} D(X) &= \inf \left\{ \inf_{\lambda \in \mathbb{R}} \|x + \lambda y\| : x, y \in S_X, x \perp_I y \right\}, \\ D'(X) &= \sup \{ \|x + y\| - \|x - y\| : x, y \in S_X, x \perp_B y \}, \\ BR(X) &= \sup_{\alpha > 0} \left\{ \frac{\|x + \alpha y\| - \|x - \alpha y\|}{\alpha} : x, y \in S_X, x \perp_B y \right\} \\ &= \sup \left\{ \frac{\|x + y\| - \|x - y\|}{\|y\|} : x, y \in X, x, y \neq 0, x \perp_B y \right\}, \\ BI(X) &= \sup \left\{ \frac{\|x + y\| - \|x - y\|}{\|x\|} : x, y \in X, x, y \neq 0, x \perp_B y \right\}, \\ IB(X) &= \inf \left\{ \frac{\inf_{\lambda \in \mathbb{R}} \|x + \lambda y\|}{\|x\|} : x, y \in X, x, y \neq 0, x \perp_I y \right\}. \end{aligned}$$

(see [10], [14], [19]).

An orthogonality notion " $\perp$ " is called *symmetric* if  $x \perp y$  implies  $y \perp x$ . The usual orthogonality in inner product spaces is, of course symmetric. By the definition, isosceles orthogonality in normed spaces is symmetric, too. However Birkhoff orthogonality is not symmetric in general. Birkhoff [4] proved that if Birkhoff orthogonality is symmetric in a strictly convex normed space whose dimension is at least three, then the space is an inner product space. Day [6] and James [9] showed that the assumption of strict convexity in Birkhoff's result can be released.

**THEOREM 1.1.** ([2], [6], [9]) *A normed space  $X$  whose dimension is at least three is an inner product space if and only if Birkhoff orthogonality is symmetric in  $X$ .*

The assumption of the dimension of the space in the above theorem cannot be omitted. A two-dimensional normed space in which Birkhoff orthogonality is symmetric is called a Radon plane.

In this paper, we consider the constant  $IB(X)$  in Radon planes. The inequality  $1/2 \leq IB(X) \leq 1$  holds for any normed space  $X$ . Under the assumption that the space  $X$  is a Radon plane, an inequality  $8/9 \leq IB(X) \leq 1$  is proved, and the Radon plane in which  $IB(X) = 8/9$  is characterized. On the other hand, a Radon plane is made by connecting the unit sphere of a two-dimensional normed space and its dual ([6], [12], [13]). A collection of normed spaces in which  $IB(X) < 8/9$  holds and that constant of the induced Radon plane is equal to  $8/9$  is obtained.

## 2. THE DIFFERENCE BETWEEN TWO ORTHOGONALITY TYPES IN RADON PLANES

To consider the difference between Birkhoff and isosceles orthogonalities, the results obtained by James in [7] are important.

PROPOSITION 2.1. ([7])

- (i) *If  $x (\neq 0)$  and  $y$  are isosceles orthogonal elements in a normed space, then  $\|x + ky\| > \frac{1}{2}\|x\|$  for all  $k$ .*
- (ii) *If  $x (\neq 0)$  and  $y$  are isosceles orthogonal elements in a normed space, and  $\|y\| \leq \|x\|$ , then  $\|x + ky\| \geq 2(\sqrt{2} - 1)\|x\|$  for all  $k$ .*

From this, one can have  $1/2 \leq IB(X) \leq 1$  and  $2(\sqrt{2} - 1) \leq D(X) \leq 1$  for any normed space.

For two elements  $x, y$  in the unit sphere in a normed space  $X$ , the sine function  $s(x, y)$  is defined by

$$s(x, y) = \inf_{t \in \mathbb{R}} \|x + ty\|$$

([22]). V. Balestro, H. Martini, and R. Teixeira [3] showed the following

PROPOSITION 2.2. ([3]) *A two dimensional normed space  $X$  is a Radon plane if and only if its associated sine function is symmetric.*

Thus for elements  $x, y$  in the unit sphere in a Radon plane  $X$  with  $x \perp_I y$  we have  $\inf_{\lambda \in \mathbb{R}} \|x + \lambda y\| = \inf_{\mu \in \mathbb{R}} \|y + \mu x\|$ . Hence the inequality  $2(\sqrt{2} - 1) \leq IB(X) \leq 1$  holds for a Radon plane  $X$ .

Using Proposition 2.2 again, we start to consider the lower bound of  $IB(X)$  in a Radon plane.

PROPOSITION 2.3. *Let  $X$  be a Radon plane, an element  $x \in S_X$  be isosceles orthogonality to  $\alpha y$  for another element  $y \in S_X$  and a real number  $\alpha \in \mathbb{R}$ . Take numbers  $k, l \in \mathbb{R}$  such that  $\|x + ky\| = \min_{\lambda \in \mathbb{R}} \|x + \lambda y\| = \min_{\mu \in \mathbb{R}} \|y + \mu x\| = \|y + lx\|$ . Then, in the estimation of the constant  $IB(X)$ , we may only consider the situation  $0 \leq \alpha \leq 1$ ,  $0 \leq k$  and  $0 \leq l$ . In this case,  $k \leq \min\{1/2, \alpha\}$  and  $l \leq 1/2$  hold.*

*Proof.* Since  $x \perp_I \alpha y$  implies  $x \perp_I -\alpha y$  and  $y \perp_I x/\alpha$ , we can suppose  $0 \leq \alpha \leq 1$ . From the assumption  $\|x + ky\| = \min_{\lambda \in \mathbb{R}} \|x + \lambda y\| = \min_{\mu \in \mathbb{R}} \|y + \mu x\| = \|y + lx\|$ , we can also suppose  $0 \leq k$  and  $0 \leq l$ . Then it follows from  $x \perp_I \alpha y$  and  $\|x + ky\| = \min_{\lambda \in \mathbb{R}} \|x + \lambda y\|$  that  $k \leq \alpha$ .

The assumption  $\|x + ky\| = \min_{\lambda \in \mathbb{R}} \|x + \lambda y\|$  implies that  $x + ky$  is Birkhoff orthogonal to  $y$ . From the symmetry of Birkhoff orthogonality in a Radon plane,  $y$  is Birkhoff orthogonal to  $x + ky$ . Using this fact, one has

$$\begin{aligned} \alpha + k &\leq \|x + ky - (\alpha + k)y\| \\ &= \|x - \alpha y\| = \|x + \alpha y\| \\ &= \|x + ky + (\alpha - k)y\| \\ &\leq \|x + ky\| + \alpha - k \end{aligned}$$

and hence  $2k \leq \|x + ky\| = \min_{\lambda \in \mathbb{R}} \|x + \lambda y\| \leq 1$ .

In a similar way, from the fact that  $x$  is Birkhoff orthogonal to  $y + lx$ , we have  $2l \leq \|y + lx\| \leq 1$ . ■

PROPOSITION 2.4. *Let  $X$  be a Radon plane, an element  $x \in S_X$  be isosceles orthogonality to  $\alpha y$  for another element  $y \in S_X$  and a number  $\alpha \in [0, 1]$ . Take numbers  $k \in [0, \min\{1/2, \alpha\}]$  and  $l \in [0, 1/2]$  such that  $\|x + ky\| = \min_{\lambda \in \mathbb{R}} \|x + \lambda y\| = \min_{\mu \in \mathbb{R}} \|y + \mu x\| = \|y + lx\|$ . Then*

$$\|x + ky\| \geq \max \left\{ \frac{(\alpha + k)(1 - kl)}{(\alpha + k)(1 - kl) + k(1 - l)(\alpha - k)}, \frac{(1 + \alpha l)(1 - kl)}{(1 + \alpha l)(1 - kl) + l(1 - k)(1 - \alpha l)} \right\}.$$

*Proof.* It follows from

$$x = \frac{\alpha(x + ky) + k(x - \alpha y)}{\alpha + k}$$

and  $x \perp_I \alpha y$  that

$$\alpha + k \leq \alpha \|x + ky\| + k \|x - \alpha y\| = \alpha \|x + ky\| + k \|x + \alpha y\|.$$

For

$$c = \frac{\alpha - k}{1 + \alpha - k - \alpha l} \quad \text{and} \quad d = \frac{1 - kl}{1 + \alpha - k - \alpha l},$$

the equality  $d(x + \alpha y) = (1 - c)(x + ky) + c(y + lx)$  holds, and hence one has

$$\|x + \alpha y\| \leq \frac{\|x + ky\|}{d} = \frac{1 + \alpha - k - \alpha l}{1 - kl} \|x + ky\|.$$

Thus, we obtain

$$\begin{aligned} \alpha + k &\leq \left( \alpha + k \cdot \frac{1 + \alpha - k - \alpha l}{1 - kl} \right) \|x + ky\| \\ &= \frac{(\alpha + k)(1 - kl) + k(\alpha - k - \alpha l + kl)}{1 - kl} \|x + ky\| \\ &= \frac{(\alpha + k)(1 - kl) + k(1 - l)(\alpha - k)}{1 - kl} \|x + ky\|. \end{aligned}$$

Meanwhile, from the equality

$$y = \frac{l(-x + \alpha y) + y + lx}{1 + \alpha l},$$

we obtain

$$\begin{aligned} 1 + \alpha l &\leq \|y + lx\| + l \| -x + \alpha y \| \\ &= \|x + ky\| + l \|x + \alpha y\| \\ &\leq \left( 1 + l \cdot \frac{1 + \alpha - k - \alpha l}{1 - kl} \right) \|x + ky\| \\ &= \frac{(1 + \alpha l)(1 - kl) + l(1 - k)(1 - \alpha l)}{1 - kl} \|x + ky\|. \end{aligned}$$

■

Let

$$F(\alpha, k, l) = \frac{k(1 - l)(\alpha - k)}{(\alpha + k)(1 - kl)} \quad \text{and} \quad G(\alpha, k, l) = \frac{l(1 - k)(1 - \alpha l)}{(1 + \alpha l)(1 - kl)}.$$

From the above proposition, the inequality

$$\|x + ky\|^{-1} \leq 1 + \min \{F(\alpha, k, l), G(\alpha, k, l)\} \quad (2.1)$$

holds.

It follows from

$$\frac{1-l}{1-kl} = \frac{1}{k} + \frac{1-k}{k(kl-1)}$$

that the function  $F(\alpha, k, l)$  is decreasing on  $l$  in the interval  $[0, 1]$ . In a similar way,  $G(\alpha, k, l)$  is decreasing on  $k$  in the interval  $[0, 1]$ .

Let us consider the upper bound of  $\min\{F(\alpha, k, l), G(\alpha, k, l)\}$ .

LEMMA 2.5. *Let  $0 \leq \alpha \leq 1$ ,  $0 \leq k \leq \min\{\alpha, 1/2\}$  and  $k \leq l \leq 1/2$ . Then*

$$\min \{F(\alpha, k, l), G(\alpha, k, l)\} = F(\alpha, k, l) \leq \frac{k(1-k)}{(1+k)^2}.$$

*Proof.* Let  $0 \leq \alpha \leq 1$ ,  $0 \leq k \leq \min\{\alpha, 1/2\}$  and  $k \leq l \leq 1/2$ . For the function

$$H(\alpha, k, l) := (G(\alpha, k, l) - F(\alpha, k, l))(1 - kl),$$

we have

$$H(\alpha, k, l) = l(1-k) \frac{1-\alpha l}{1+\alpha l} - k(1-l) \frac{\alpha-k}{\alpha+k}$$

and hence

$$\begin{aligned} \frac{\partial H}{\partial \alpha} &= l(1-k) \frac{\partial}{\partial \alpha} \left( \frac{1-\alpha l}{1+\alpha l} \right) - k(1-l) \frac{\partial}{\partial \alpha} \left( \frac{\alpha-k}{\alpha+k} \right) \\ &= -\frac{2l^2(1-k)}{(1+\alpha l)^2} - \frac{2k^2(1-l)}{(\alpha+k)^2} \leq 0. \end{aligned}$$

This implies that  $H$  is decreasing on  $\alpha$ . Thus we obtain the inequality

$$\begin{aligned} H(\alpha, k, l) &\geq H(1, k, l) = l(1-k) \frac{1-l}{1+l} - k(1-l) \frac{1-k}{1+k} \\ &= \frac{(1-k)(1-l)(l-k)}{(1+k)(1+l)} \geq 0, \end{aligned}$$

and so  $F(\alpha, k, l) \leq G(\alpha, k, l)$  holds.

Using the fact that  $F(\alpha, k, l)$  is a decreasing function on  $l$ ,

$$\min \{F(\alpha, k, l), G(\alpha, k, l)\} = F(\alpha, k, l) \leq F(\alpha, k, k) = \frac{k(\alpha - k)}{(1+k)(\alpha + k)}.$$

From the fact that the function  $(\alpha - k)/(\alpha + k)$  is increasing on  $\alpha$ , it follows that

$$\frac{k(\alpha - k)}{(1+k)(\alpha + k)} \leq \frac{k(1 - k)}{(1+k)^2},$$

which completes the proof. ■

LEMMA 2.6. *Let  $0 \leq \alpha \leq 1$ ,  $0 \leq k \leq \min\{\alpha, 1/3\}$  and  $0 \leq l < k$ . Then*

$$\min \{F(\alpha, k, l), G(\alpha, k, l)\} \leq \frac{k(1 - k)}{(1+k)^2}.$$

*Proof.* Let  $0 \leq \alpha \leq 1$ ,  $0 \leq k \leq \min\{\alpha, 1/3\}$  and  $0 \leq l < k$ . Then

$$\begin{aligned} \min \{F(\alpha, k, l), G(\alpha, k, l)\} &\leq \frac{(1 - k)l F(\alpha, k, l) + (1 - l)k G(\alpha, k, l)}{(1 - k)l + (1 - l)k} \\ &= \frac{2\alpha(1 - k)k(1 - l)l}{(\alpha + k)(1 + \alpha l)((1 - k)l + (1 - l)k)} = (*). \end{aligned}$$

We have that  $(*) \leq \frac{k(1-k)}{(1+k)^2}$  if and only if the function

$$f(\alpha, k, l) := 2\alpha(1 + k)^2(1 - l)l - (\alpha + k)(1 + \alpha l)((1 - k)l + (1 - l)k)$$

is negative. One can has

$$\begin{aligned} f(\alpha, k, l) &= ((2 + \alpha)l - 1)k^2 \\ &\quad + (\alpha(1 - l)(4l - 1 - \alpha l) - l(1 + \alpha l)(1 - \alpha))k \\ &\quad + \alpha l(1 - (2 + \alpha)l) \end{aligned}$$

and hence  $\frac{\partial f}{\partial k} = 2Ak + B$ , where  $A = (2 + \alpha)l - 1$  and

$$B = \alpha(1 - l)(4l - 1 - \alpha l) - l(1 + \alpha l)(1 - \alpha).$$

From the fact  $l < k \leq 1/3$ , we obtain  $A \leq (3l - 1) \leq 0$  and

$$\begin{aligned} B &\leq \alpha(1 - l)(4l - 1 - \alpha l) - l(1 - \alpha) \\ &\leq \alpha(1 - l)(l - \alpha l) - l(1 - \alpha) \\ &= l(1 - \alpha)(\alpha(1 - l) - 1) \leq 0. \end{aligned}$$

Thus the function  $f$  is decreasing with respect to  $k$  and hence

$$\begin{aligned} f(\alpha, k, l) &\leq f(\alpha, l, l) \\ &= 2\alpha(1+l)^2(1-l)l - 2(\alpha+l)(1+\alpha l)(1-l)l \\ &= 2(1-l)l(\alpha(1+l)^2 - (\alpha+l)(1+\alpha l)) \\ &= -2(1-l)l^2(1-\alpha)^2 \leq 0. \end{aligned}$$

This completes the proof. ■

Under the assumption  $1/3 < k$  and  $l < k$ , we consider the upper bound of (\*), too.

LEMMA 2.7. *Let  $0 \leq \alpha \leq 1$ ,  $1/3 < k \leq \min\{\alpha, 1/2\}$  and  $0 \leq l < k$ . Then*

$$\min\{F(\alpha, k, l), G(\alpha, k, l)\} \leq \frac{2k(1-k)(\sqrt{2(1-k)} - \sqrt{k})^2}{(1+k)(\sqrt{2(1-2k)} + \sqrt{k(1-k)})^2}.$$

*Proof.* As in the above lemma,  $\min\{F(\alpha, k, l), G(\alpha, k, l)\}$  is less than

$$\frac{2\alpha(1-k)k(1-l)l}{(\alpha+k)(1+\alpha l)((1-k)l + (1-l)k)} = (*).$$

The inequality

$$(*) \leq \frac{2k(1-k)(\sqrt{2(1-k)} - \sqrt{k})^2}{(1+k)(\sqrt{2(1-2k)} + \sqrt{k(1-k)})^2}$$

is equivalent to

$$\begin{aligned} g(\alpha, k, l) &:= \frac{\alpha(1-l)l}{(\alpha+k)(1+\alpha l)((1-k)l + (1-l)k)} \\ &\leq \frac{(\sqrt{2(1-k)} - \sqrt{k})^2}{(1+k)(\sqrt{2(1-2k)} + \sqrt{k(1-k)})^2}. \end{aligned}$$

On this function  $g$ , one can see

$$\begin{aligned} \frac{\partial g}{\partial \alpha} &= \frac{(1-l)l}{l(1-k) + k(1-l)} \times \frac{\partial}{\partial \alpha} \left( \frac{\alpha}{(\alpha+k)(1+\alpha l)} \right) \\ &= \frac{(1-l)l}{l(1-k) + k(1-l)} \times \frac{k - \alpha^2 l}{(\alpha+k)^2(1+\alpha l)^2}. \end{aligned}$$

From the assumption  $k > l$ , the function  $g$  is increasing on  $\alpha$  and so

$$g(\alpha, k, l) \leq g(1, k, l) = \frac{(1-l)l}{(1+k)(1+l)(l(1-k) + k(1-l))}.$$

We have that

$$g(1, k, l) \leq \frac{(\sqrt{2(1-k)} - \sqrt{k})^2}{(1+k)(\sqrt{2(1-2k)} + \sqrt{k(1-k)})^2}$$

if and only if

$$\begin{aligned} P_k(l) := & (\sqrt{2(1-2k)} + \sqrt{k(1-k)})^2(1-l)l \\ & - (\sqrt{2(1-k)} - \sqrt{k})^2(1+l)(l(1-k) + k(1-l)) \leq 0. \end{aligned}$$

Letting

$$l_k = \frac{k}{k + \sqrt{2k(1-k)}},$$

we have

$$l_k(1-k) + k(1-l_k) = k + (1-2k)l_k = \frac{1-k + \sqrt{2k(1-k)}}{k + \sqrt{2k(1-k)}}k,$$

and hence

$$\begin{aligned} \frac{(1+l_k)(k + (1-2k)l_k)}{(1-l_k)l_k} &= \frac{(2k + \sqrt{2k(1-k)})\{1-k + \sqrt{2k(1-k)}\}}{\sqrt{2k(1-k)}} \\ &= 2\sqrt{2k(1-k)} + 1 + k \\ &= (\sqrt{1-k} + \sqrt{2k})^2. \end{aligned}$$

Meanwhile one can easily check

$$(\sqrt{1-k} + \sqrt{2k})(\sqrt{2(1-k)} - \sqrt{k}) = \sqrt{2(1-2k)} + \sqrt{k(1-k)}.$$

Thus we obtain

$$\begin{aligned} \frac{(1+l_k)(l_k(1-k) + k(1-l_k))}{(1-l_k)l_k} &= (\sqrt{1-k} + \sqrt{2k})^2 \\ &= \frac{(\sqrt{2(1-2k)} + \sqrt{k(1-k)})^2}{(\sqrt{2(1-k)} - \sqrt{k})^2}, \end{aligned}$$

which implies  $P_k(l_k) = 0$ .

We consider the derivation

$$P'_k(l) = (\sqrt{2}(1-2k) + \sqrt{k(1-k)})^2(1-2l) \\ - (\sqrt{2(1-k)} - \sqrt{k})^2((1-k) + 2(1-2k)l),$$

too. For  $l_k$ , we have

$$1 - k + 2(1 - 2k)l_k = 1 - k + \frac{2(1 - 2k)k}{k + \sqrt{2k(1 - k)}} \\ = \frac{3k - 5k^2 + (1 - k)\sqrt{2k(1 - k)}}{k + \sqrt{2k(1 - k)}},$$

and hence

$$\frac{1 - k + 2(1 - 2k)l_k}{1 - 2l_k} = \frac{3k - 5k^2 + (1 - k)\sqrt{2k(1 - k)}}{-k + \sqrt{2k(1 - k)}}.$$

On the other hand, a equality

$$(\sqrt{1-k} + \sqrt{2k})^2(-k + \sqrt{2k(1-k)}) \\ = (1 + k + 2\sqrt{2k(1-k)})(-k + \sqrt{2k(1-k)}) \\ = 3k - 5k^2 + (1 - k)\sqrt{2k(1 - k)}$$

holds. Thus we have

$$\frac{1 - k + 2(1 - 2k)l_k}{1 - 2l_k} = (\sqrt{1-k} + \sqrt{2k})^2 \\ = \frac{(\sqrt{2}(1-2k) + \sqrt{k(1-k)})^2}{(\sqrt{2(1-k)} - \sqrt{k})^2}.$$

This implies  $P'_k(l_k) = 0$ .

Combining the fact  $P_k(0) = -k(\sqrt{2(1-k)} - \sqrt{k})^2 \leq 0$  with  $P_k(l_k) = 0$  and  $P'_k(l_k) = 0$ , one can see that  $P_k(l) \leq 0$  for any real number  $l$ . Therefore the inequality

$$\min \{F(\alpha, k, l), G(\alpha, k, l)\} \leq \frac{2k(1-k)(\sqrt{2(1-k)} - \sqrt{k})^2}{(1+k)(\sqrt{2(1-2k)} + \sqrt{k(1-k)})^2}$$

holds. ■

A fundamental derivation implies that the function  $k(1-k)/(1+k)^2$  takes maximum  $1/8$  at  $k = 1/3$ . Now we let

$$\begin{aligned} h(k) &= \frac{k(1-k)(\sqrt{2(1-k)} - \sqrt{k})^2}{(1+k)(\sqrt{2(1-2k)} + \sqrt{k(1-k)})^2} \\ &= \frac{k(\sqrt{2(1-k)} - \sqrt{k(1-k)})^2}{(1+k)(\sqrt{2(1-2k)} + \sqrt{k(1-k)})^2} \end{aligned}$$

and consider the maximum of  $h(k)$ .

LEMMA 2.8. *The function  $h(k)$  in the interval  $[0, 1/2]$  takes maximum  $1/16$  at  $k = 1/3$ .*

*Proof.* We can consider the derivation  $h'(k)$  as follows:

$$\begin{aligned} &(1+k)^2(\sqrt{2(1-2k)} + \sqrt{k(1-k)})^4 h'(k) \\ &= \left[ (\sqrt{2(1-k)} - \sqrt{k(1-k)})^2 \right. \\ &\quad \left. + 2k(\sqrt{2(1-k)} - \sqrt{k(1-k)}) \left( -\sqrt{2} - \frac{1-2k}{2\sqrt{k(1-k)}} \right) \right] \\ &\quad \times (1+k)(\sqrt{2(1-2k)} + \sqrt{k(1-k)})^2 - k(\sqrt{2(1-k)} - \sqrt{k(1-k)})^2 \\ &\quad \times \left[ (\sqrt{2(1-2k)} + \sqrt{k(1-k)})^2 \right. \\ &\quad \left. + 2(1+k)(\sqrt{2(1-2k)} + \sqrt{k(1-k)}) \left( -2\sqrt{2} + \frac{1-2k}{2\sqrt{k(1-k)}} \right) \right]. \end{aligned}$$

Thus we obtain

$$\begin{aligned} &\sqrt{k(1-k)}(1+k)^2(\sqrt{2(1-k)} + \sqrt{k(1-k)})^{-1} \\ &\quad \times (\sqrt{2(1-2k)} + \sqrt{k(1-k)})^3 h'(k) \\ &= ((1-k)(2-5k) + \sqrt{2k}\sqrt{k(1-k)})\sqrt{k(1-k)} \\ &\quad + k(1+k)(4\sqrt{k(1-k)} - \sqrt{2(2-k)}) \\ &= (9k^2 - 3k + 2)\sqrt{k(1-k)} - 2\sqrt{2}k \end{aligned}$$

and hence

$$\begin{aligned} & \sqrt{1-k}(1+k)^2(\sqrt{2}(1-k) + \sqrt{k(1-k)})^{-1}(\sqrt{2}(1-2k) + \sqrt{k(1-k)})^3 h'(k) \\ &= (9k^2 - 3k + 2)\sqrt{1-k} - 2\sqrt{2k}. \end{aligned}$$

We note that  $(9k^2 - 3k + 2)\sqrt{1-k} - 2\sqrt{2k}$  is positive if and only if  $(9k^2 - 3k + 2)^2(1-k) - 8k$  is so. Meanwhile, one has

$$\begin{aligned} & (9k^2 - 3k + 2)^2(1-k) - 8k \\ &= (3k(3k-1) + 2)^2(1-k) - 8k \\ &= 9k^2(3k-1)^2(1-k) + 12k(3k-1)(1-k) + 4(1-k) - 8k \\ &= (3k-1)(9k^2(3k-1)(1-k) + 12k(3k-1)(1-k) - 4) \\ &= -(3k-1)(1+3k^2)(2-3k)^2. \end{aligned}$$

Therefore we obtain that the function  $h(k)$  takes maximum at  $k = 1/3$ . One can easily have  $h(1/3) = 1/16$ , which completes the proof. ■

From the inequality (2.1) and the above lemmas we have

**THEOREM 2.9.** *Let  $X$  be a Radon plane. Then  $8/9 \leq IB(X) \leq 1$ .*

In addition, we are able to characterize a Radon plane  $X$  satisfying  $IB(X) = 8/9$ . For simplicity, we use the notation  $\hat{z}$  as  $z/\|z\|$  for any nonzero  $z \in X$ .

**THEOREM 2.10.** *Let  $X$  be a Radon plane. Then  $IB(X) = 8/9$  if and only if its unit sphere is an affine regular hexagon.*

*Proof.* Suppose that  $X$  is a Radon plane and the equality  $IB(X) = 8/9$  holds. Then there exist elements  $x, y \in S_X$  and a real number  $\alpha$  such that  $\|x + \alpha y\| = \|x - \alpha y\|$  and  $\min_{\lambda \in \mathbb{R}} \|x + \lambda y\| = \min_{\mu \in \mathbb{R}} \|y + \mu x\| = 8/9$ . For  $k$  and  $l$  in the above lemmas, all inequalities in the proofs have to turn into equalities and hence  $k = l = 1/3$ . As one of them, the inequality

$$\alpha + k \leq \alpha\|x + ky\| + k\|x - \alpha y\| = \alpha\|x + ky\| + k\|x + \alpha y\|$$

also becomes an equality for  $\alpha = 1$  and  $k = 1/3$ . This implies

$$\frac{4}{3} = \left\|x + \frac{1}{3}y\right\| + \frac{1}{3}\|x - y\| = \frac{8}{9} + \frac{1}{3}\|x - y\|$$

and hence  $\|x + y\| = \|x - y\| = 4/3$ .

Using these facts, one has

$$\begin{aligned}\widehat{x + y} &= \frac{3}{4}(x + y) = \frac{9}{16} \left( \left(x + \frac{1}{3}y\right) + \left(y + \frac{1}{3}x\right) \right) \\ &= \frac{1}{2} \left( \widehat{x + \frac{1}{3}y} + \widehat{y + \frac{1}{3}x} \right).\end{aligned}$$

This implies

$$\left\| \frac{1}{2} \left( \widehat{x + \frac{1}{3}y} + \widehat{y + \frac{1}{3}x} \right) \right\| = \|\widehat{x + y}\| = 1.$$

On the other hand, for

$$x = \frac{3}{4} \left(x + \frac{1}{3}y\right) + \frac{1}{4}(x - y),$$

from

$$\left\| x + \frac{1}{3}y \right\| = \frac{8}{9} \quad \text{and} \quad \|x - y\| = \frac{4}{3}$$

we have

$$x = \frac{2}{3} \left(x + \frac{1}{3}y\right) + \frac{1}{3}(\widehat{x - y})$$

and hence

$$\left\| \frac{2}{3} \left(x + \frac{1}{3}y\right) + \frac{1}{3}(\widehat{x - y}) \right\| = \|x\| = 1.$$

In a similar way, the equality

$$\left\| \frac{2}{3} \left(y + \frac{1}{3}x\right) + \frac{1}{3}(\widehat{-x + y}) \right\| = \|y\| = 1$$

holds. Thus the three segments

$$\left[ \widehat{x - y}, \widehat{x + \frac{1}{3}y} \right], \quad \left[ \widehat{x + \frac{1}{3}y}, \widehat{y + \frac{1}{3}x} \right] \quad \text{and} \quad \left[ \widehat{y + \frac{1}{3}x}, \widehat{-x + y} \right]$$

are contained in the unit sphere  $S_X$ .

Moreover we obtain

$$(\widehat{x - y}) + \left(\widehat{y + \frac{1}{3}x}\right) = \frac{3}{4}(x - y) + \frac{9}{8} \left(y + \frac{1}{3}x\right) = \frac{9}{8} \left(x + \frac{1}{3}y\right) = \widehat{x + \frac{1}{3}y}.$$

Therefore, the unit sphere  $S_X$  is an affine regular hexagon.

Conversely, suppose that  $S_X$  is an affine regular hexagon (and therefore  $X$  is a Radon plane). Then there exist  $u, v \in S_X$  such that  $\pm u, \pm v$  and  $\pm(u+v)$  are the vertices of  $S_X$ . Letting

$$x = u + \frac{1}{3}v \quad \text{and} \quad y = -\frac{1}{3}u - v,$$

we have

$$x + y = \frac{2}{3}(u - v) \quad \text{and} \quad x - y = \frac{4}{3}(u + v).$$

Thus  $\|x + y\| = 4/3 = \|x - y\|$  and hence  $x \perp_I y$ .

Meanwhile, one has

$$x + \frac{1}{3}y = u + \frac{1}{3}v + \frac{1}{3}\left(-\frac{1}{3}u - v\right) = \frac{8}{9}u.$$

Therefore, the inequality

$$IB(X) = \inf \left\{ \frac{\inf_{\lambda \in \mathbb{R}} \|x + \lambda y\|}{\|x\|} : x, y \in X, x, y \neq 0, x \perp_I y \right\} \leq \frac{8}{9}$$

holds. This implies  $IB(X) = 8/9$ . ■

### 3. PRACTICAL RADON PLANES AND A CALCULATION

A Radon plane is made by connecting the unit sphere of a normed plane and its dual ([6]). Hereafter, we make a collection of the space  $X$  in which the unit sphere  $S_X$  is a hexagon, the constant  $IB(X)$  is less than  $8/9$  and that of the induced Radon plane coincides with  $8/9$ .

A norm  $\|\cdot\|$  on  $\mathbb{R}^2$  is said to be absolute if  $\|(a, b)\| = \|(|a|, |b|)\|$  for any  $(a, b) \in \mathbb{R}^2$ , and normalized if  $\|(1, 0)\| = \|(0, 1)\| = 1$ . Let  $AN_2$  denote the family of all absolute normalized norm on  $\mathbb{R}^2$ , and  $\Psi_2$  denote the family of all continuous convex function  $\psi$  on  $[0, 1]$  such that  $\max\{1-t, t\} \leq \psi(t) \leq 1$  for all  $t \in [0, 1]$ . As in [5, 21], it is well known that  $AN_2$  and  $\Psi_2$  are in a one-to-one correspondence under the equation  $\psi(t) = \|(1-t, t)\|$  for  $t \in [0, 1]$  and

$$\|(a, b)\|_\psi = \begin{cases} (|a| + |b|)\psi\left(\frac{|b|}{|a| + |b|}\right) & \text{if } (a, b) \neq (0, 0), \\ 0 & \text{if } (a, b) = (0, 0). \end{cases}$$

Let  $\|\cdot\|_\psi$  denote an absolute normalized norm associated with a convex function  $\psi \in \Psi_2$ .

For  $\psi \in \Psi_2$ , the dual function  $\psi^*$  on  $[0, 1]$  is defined by

$$\psi^*(s) = \sup \left\{ \frac{(1-t)(1-s) + ts}{\psi(t)} : t \in [0, 1] \right\}$$

for  $s \in [0, 1]$ . It is known that  $\psi^* \in \Psi_2$  and that  $\|\cdot\|_{\psi^*} \in AN_2$  is the dual norm of  $\|\cdot\|_{\psi}$ , that is,  $(\mathbb{R}^2, \|\cdot\|_{\psi})^*$  is identified with  $(\mathbb{R}^2, \|\cdot\|_{\psi^*})$  (cf. [16, 17, 18]). Meanwhile, for  $\psi \in \Psi_2$ , the function  $\tilde{\psi} \in \Psi_2$  is defined by  $\tilde{\psi}(t) = \psi(1-t)$  for any  $t \in [0, 1]$ . One can easily check  $(\tilde{\psi}^*) = (\tilde{\psi})^*$ . So we write it  $\tilde{\psi}^*$ . According to [6], [12] and [13], for any  $\psi \in \Psi_2$ , the Day-James space  $\ell_{\psi-\tilde{\psi}^*}$  becomes a Radon plane.

For any  $c \in [0, 1]$ , let

$$\psi_c(t) = \begin{cases} -ct + 1 & \text{if } 0 \leq t \leq (1+c)^{-1}, \\ t & \text{if } (1+c)^{-1} \leq t \leq 1. \end{cases}$$

Then the norm of  $(a, b) \in \mathbb{R}^2$  is computed by

$$\|(a, b)\|_{\psi_c} = \begin{cases} |a| + (1-c)|b| & \text{if } |a| \geq c|b|, \\ |b| & \text{if } |a| \leq c|b|. \end{cases}$$

The dual function is calculated as follows:

PROPOSITION 3.1. *Let  $c \in [0, 1]$ . Then*

$$\psi_c^*(s) = \begin{cases} 1-s & \text{if } 0 \leq s \leq \frac{1-c}{2-c}, \\ (1-c)s + c & \text{if } \frac{1-c}{2-c} \leq s \leq 1. \end{cases}$$

*Proof.* Fix  $s \in [0, 1]$ . We define the function  $f_{c,s}(t)$  from  $[0, 1]$  into  $\mathbb{R}$  by

$$f_{c,s}(t) = \frac{(1-t)(1-s) + ts}{\psi_c(t)}.$$

We note that  $\psi_c^*(s) = \max\{f_{c,s}(t) : 0 \leq t \leq 1\}$  and calculate the maximum of  $f_{c,s}$  on  $[0, 1]$ . By the definition of  $\psi_c$ , we have

$$f_{c,s}(t) = \begin{cases} \frac{1-s+(2s-1)t}{-ct+1} & \text{if } 0 \leq t \leq (1+c)^{-1}, \\ 2s-1+\frac{1-s}{t} & \text{if } (1+c)^{-1} \leq t \leq 1. \end{cases}$$

The function  $2s - 1 + (1 - s)/t$  is clearly decreasing on  $t$ .

If  $0 \leq s \leq (1 - c)/(2 - c)$ , then the function  $f_{c,s}(t)$  is decreasing on  $[0, (1 + c)^{-1}]$ . Hence we have  $\psi_c^*(s) = f_{c,s}(0) = 1 - s$ .

Suppose that  $(1 - c)/(2 - c) \leq s \leq 1$ . Then the function  $f_{c,s}(t)$  is increasing on  $[0, (1 + c)^{-1}]$ . Thus we have

$$\psi_c^*(s) = f_{c,s}\left(\frac{1}{1 + c}\right) = (1 - c)s + c.$$

Therefore we obtain this proposition. ■

From this result, one has

PROPOSITION 3.2. *Let  $c \in [0, 1]$ . Then*

$$\|(a, b)\|_{\psi_c^*} = \begin{cases} |a| & \text{if } |b| \leq (1 - c)|a|, \\ c|a| + |b| & \text{if } (1 - c)|a| \leq |b|. \end{cases}$$

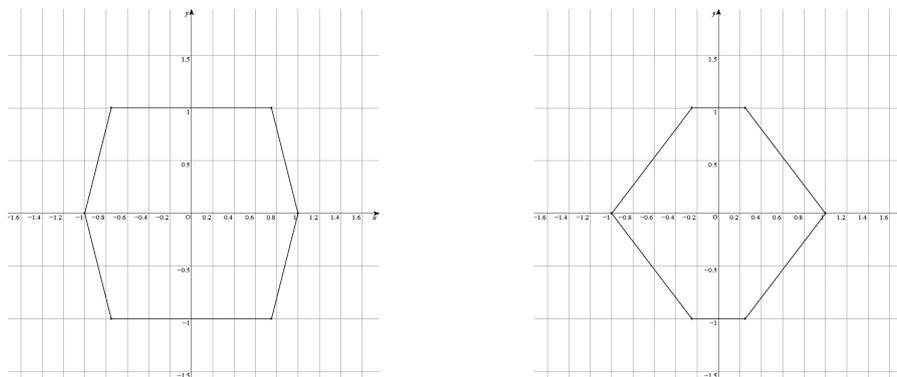
Thus the Radon plane  $\ell_{\psi_c} - \ell_{\tilde{\psi}_c^*}$  induced by  $\psi_c$  is the space  $\mathbb{R}^2$  with the norm

$$\|(a, b)\|_{\psi_c, \tilde{\psi}_c^*} = \begin{cases} |a| + (1 - c)|b| & \text{if } c|b| \leq |a| \text{ and } ab \geq 0, \\ |b| & \text{if } -(1 - c)|b| \leq a \leq c|b| \text{ and } b \geq 0, \\ |b| & \text{if } -(1 - c)|b| \leq -a \leq c|b| \text{ and } b \leq 0, \\ |a| + c|b| & \text{if } (1 - c)|b| \leq |a| \text{ and } ab \leq 0. \end{cases}$$

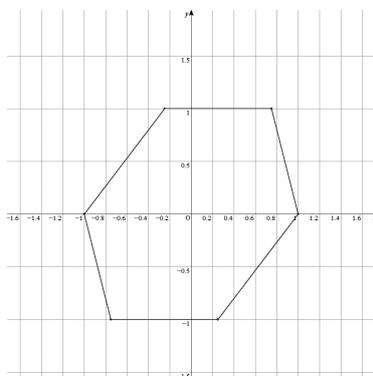
Therefore the unit sphere of this space is an affine regular hexagon with the vertices  $\pm(1, 0)$ ,  $\pm(1 - c, 1)$ ,  $\pm(-c, 1)$  and hence the constant  $IB(\ell_{\psi_c} - \ell_{\tilde{\psi}_c^*})$  coincide with  $8/9$  by the Theorem 2.10.

Next, we calculate the constants  $IB((\mathbb{R}^2, \|\cdot\|_{\psi_c}))$  and  $IB((\mathbb{R}^2, \|\cdot\|_{\tilde{\psi}_c^*}))$ . Then we obtain that the values are smaller than  $IB(\ell_{\psi_c} - \ell_{\tilde{\psi}_c^*}) = 8/9$  and equal to  $8/9$  only when  $c = 1/2$ . We note that  $\tilde{\psi}_c^* = \psi_{1-c}$  and it is enough to calculate  $IB((\mathbb{R}^2, \|\cdot\|_{\psi_c}))$  for  $c \in [0, 1]$ . To do this, we need to recall the Dunkl-Williams constant defined in [11]:

$$\begin{aligned} DW(X) &= \sup \left\{ \frac{\|x\| + \|y\|}{\|x - y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| : x, y \in X, x, y \neq 0, x \neq y \right\} \\ &= \sup \left\{ \frac{\|u + v\|}{\|(1 - t)u + tv\|} : u, v \in S_X, 0 \leq t \leq 1 \right\}. \end{aligned}$$



The unit sphere of  $(\mathbb{R}^2, \|\cdot\|_{\psi_c})$  and  $(\mathbb{R}^2, \|\cdot\|_{\tilde{\psi}_c^*})$ .



The unit sphere of Radon plane  $\ell_{\psi_c} - \ell_{\tilde{\psi}_c^*}$ .

For any normed space, the equality  $2 \leq DW(X) \leq 4$  holds. In [14], it is shown that the equality  $IB(X)DW(X) = 2$  holds for any normed space  $X$ . One can find a formula to calculate this constant in the paper [15]. For each  $x \in S_X$  and for each  $y \in X$  with  $x \perp_B y$ , we put

$$m(x, y) = \sup \left\{ \left\| x + \frac{\lambda + \mu}{2} y \right\| : \lambda \leq 0 \leq \mu, \|x + \lambda y\| = \|x + \mu y\| \right\}.$$

We define the positive number  $M(x)$  by

$$M(x) = \sup \{ m(x, y) : x \perp_B y \}.$$

Using these notions, the Dunkl-Williams constant can be calculated as

$$DW(X) = 2 \sup \{ M(x) : x \in S_X \} = 2 \sup \{ M(x) : x \in \text{fr}(B_X) \},$$

where  $\text{fr}(B_X)$  is the frame of unit ball. An element  $x \in S_X$  is called an extreme point of  $B_X$  if  $y, z \in S_X$  and  $x = (y + z)/2$  implies  $x = y = z$ . The set of all extreme points of  $B_X$  is denoted by  $\text{ext}(B_X)$ . Suppose that the space  $X$  has two-dimension. Then the above calculation method is turned into

$$DW(X) = 2 \sup \{M(x) : x \in \text{ext}(B_X)\}.$$

Here, we reduce the amount of calculation a little more. As in Section 2, we use the notation  $\hat{z}$ .

PROPOSITION 3.3. *Let  $X$  be a two-dimensional normed space. Then*

$$DW(X) = \sup \left\{ \frac{\|u + v\|}{\|(1-t)u + tv\|} : u \in \text{ext}(B_X), v \in S_X, 0 \leq t \leq 1 \right\}.$$

*Proof.* Take arbitrary elements  $u, v \in S_X \setminus \text{ext}(B_X)$ . If the segment  $[u, v]$  belongs to the unit sphere  $S_X$ , then

$$\frac{\|u + v\|}{\|(1-t)u + tv\|} = 2$$

for any  $t \in [0, 1]$ . So we may assume  $[u, v] \not\subset S_X$ . Then we have  $t_0 \in [0, 1]$  such that

$$\min_{0 \leq t \leq 1} \|(1-t)u + tv\| = \|(1-t_0)u + t_0v\|.$$

Letting  $x = (1 - \widehat{t_0})u + t_0v$  and  $y = \widehat{u - v}$ , we have four elements  $u_1, u_2, v_1, v_2 \in S_X$  such that at least two elements among them belong to  $\text{ext}(B_X)$  and satisfying  $u \in [u_1, u_2] \subset S_X, v \in [v_1, v_2] \subset S_X$  and  $\widehat{u_1 - v_1} = y = \widehat{u_2 - v_2}$ . For these elements, from the fact that three vectors  $u - v, u_1 - v_1$  and  $u_2 - v_2$  are parallel each other, we can take numbers  $s_0 \in (0, 1)$  satisfying  $u = (1-s_0)u_1 + s_0u_2, v = (1-s_0)v_1 + s_0v_2$ . Meanwhile there exist  $t_1, t_2 \in (0, 1)$  such that

$$\min_{0 \leq t \leq 1} \|(1-t)u_1 + tv_1\| = \|(1-t_1)u_1 + t_1v_1\|,$$

$$\min_{0 \leq t \leq 1} \|(1-t)u_2 + tv_2\| = \|(1-t_2)u_2 + t_2v_2\|.$$

It follow from  $x \perp_B y$  and  $\widehat{u_1 - v_1} = y = \widehat{u_2 - v_2}$  that  $(1-t_2)\widehat{u_2} + t_2v_2 = x$  and  $(1-t_1)\widehat{u_1} + t_1v_1 = \pm x$ . In case of  $(1-t_1)\widehat{u_1} + t_1v_1 = -x$ , the element  $-u_1$  belongs to the arc between  $v_1$  and  $x$ . Letting  $v_3 = -u_1$ , we

can take element  $u_3$  satisfying  $\widehat{u_3 - v_3} = y$ , again. Hence we may consider  $(1 - t_1)\widehat{u_1} + t_1v_1 = x$ .

Then the equalities

$$(1 - t_0)u + t_0v = (1 - s_0)((1 - t_1)u_1 + t_1v_1) + s_0((1 - t_2)u_2 + t_2v_2)$$

and

$$\|(1 - t_0)u + t_0v\| = (1 - s_0)\|(1 - t_1)u_1 + t_1v_1\| + s_0\|(1 - t_2)u_2 + t_2v_2\|$$

holds. Thus, using triangle inequality and the fact that an inequality

$$\frac{(1 - \alpha)a + \alpha b}{(1 - \alpha)c + \alpha d} \leq \max \left\{ \frac{a}{c}, \frac{b}{d} \right\}$$

holds for  $\alpha \in [0, 1]$  and positive numbers  $a, b, c, d$ , we obtain

$$\begin{aligned} \frac{\|u + v\|}{\|(1 - t_0)u + t_0v\|} &= \frac{\|(1 - s_0)u_1 + s_0u_2 + (1 - s_0)v_1 + s_0v_2\|}{(1 - s_0)\|(1 - t_1)u_1 + t_1v_1\| + s_0\|(1 - t_2)u_2 + t_2v_2\|} \\ &\leq \frac{(1 - s_0)\|u_1 + v_1\| + s_0\|u_2 + v_2\|}{(1 - s_0)\|(1 - t_1)u_1 + t_1v_1\| + s_0\|(1 - t_2)u_2 + t_2v_2\|} \\ &\leq \max \left\{ \frac{\|u_1 + v_1\|}{\|(1 - t_1)u_1 + t_1v_1\|}, \frac{\|u_2 + v_2\|}{\|(1 - t_2)u_2 + t_2v_2\|} \right\} \\ &\leq \sup \left\{ \frac{\|u + v\|}{\|(1 - t)u + tv\|} : u \in \text{ext}(B_X), v \in S_X, 0 \leq t \leq 1 \right\}. \end{aligned}$$

This completes the proof. ■

Thus, to obtain the value of the Dunkl-Williams constant, in the above calculation method, for  $x \in \text{ext}(B_X)$  and  $y \in X$  with  $x \perp_B y$ , the value  $m(x, y)$  can be computed as

$$m(x, y) = \sup \left\{ \left\| x + \frac{\lambda + \mu}{2}y \right\| : \lambda \leq 0 \leq \mu, \widehat{x + \lambda y} \in \text{ext}(B_X) \right\}.$$

4. THE CONSTANT  $IB(X)$  IN HEXAGONAL PLANES

Now, we start to compute  $DW((\mathbb{R}^2, \|\cdot\|_{\psi_c}))$  and  $IB((\mathbb{R}^2, \|\cdot\|_{\psi_c}))$  for  $c \in [0, 1]$ . For simplicity we write  $X_c$  and  $\|\cdot\|$  for  $(\mathbb{R}^2, \|\cdot\|_{\psi_c})$  and  $\|\cdot\|_{\psi_c}$ , respectively. First we suppose  $1/2 \leq c$ . Let  $e_1 = (1, 0)$ ,  $u = (c, 1)$ . Then, by [15, Proposition 2.5],  $DW(X_c) = 2 \max\{M(e_1), M(u)\}$ . Putting  $v_t = (-t, 1)$  and

$$w_t = (1-t)(-e_1) + t(-c, 1) = (-1+t-ct, t)$$

for  $t \in [0, 1]$ , we have  $e_1 \perp_B v_t$  for  $t \in [0, 1-c]$ ,  $u \perp_B v_t$  for  $t \in [1-c, c]$  and  $u \perp_B w_t$  for  $t \in [0, 1]$ . By [15, Theorem 2.9 and Corollary 2.10], one has

$$M(e_1) = \sup \{m(e_1, v_t) : t \in (0, 1-c)\}$$

and

$$M(u) = \max \left\{ \sup \{m(u, v_t) : t \in (1-c, c)\}, \right. \\ \left. \sup \{m(u, w_t) : t \in (0, 1) \setminus \{1/2\}\} \right\}.$$

LEMMA 4.1. *Let  $c \in [1/2, 1]$ . Then, in  $X_c$ ,*

$$M(e_1) = 1 + \frac{1-c}{(1+\sqrt{2c})^2}.$$

*Proof.* Let  $t \in (0, 1-c)$ . Then the norm of  $e_1 + \lambda v_t$  is computed as

$$\|e_1 + \lambda v_t\| = \begin{cases} -\lambda & \text{if } \lambda \leq -(c-t)^{-1}, \\ 1 - (1-c+t)\lambda & \text{if } -(c-t)^{-1} \leq \lambda \leq 0, \\ 1 + (1-c-t)\lambda & \text{if } 0 \leq \lambda \leq (c+t)^{-1}, \\ \lambda & \text{if } (c+t)^{-1} \leq \lambda. \end{cases}$$

From the inequality

$$\left\| e_1 + \frac{1}{c+t} v_t \right\| = 1 + \frac{1-c-t}{c+t} < 1 + \frac{1-c+t}{c-t} = \left\| e_1 - \frac{1}{c-t} v_t \right\|,$$

we can find real numbers  $p_t \in (-(c-t)^{-1}, 0)$  and  $q_t$  more than  $(c+t)^{-1}$  such that

$$\|e_1 + p_t v_t\| = \left\| e_1 + \frac{1}{c+t} v_t \right\| \quad \text{and} \quad \|e_1 + q_t v_t\| = \left\| e_1 - \frac{1}{c-t} v_t \right\|,$$

respectively. To obtain  $m(e_1, v_t)$ , it is enough to consider

$$\left\| e_1 + \frac{1}{2} \left( p_t + \frac{1}{c+t} \right) v_t \right\| \quad \text{and} \quad \left\| e_1 + \frac{1}{2} \left( -\frac{1}{c+t} + q_t \right) v_t \right\|.$$

Since the equality

$$q_t = \|e_1 + q_t v_t\| = \left\| e_1 - \frac{1}{c-t} v_t \right\| = 1 + \frac{1-c+t}{c-t} = \frac{1}{c-t}$$

holds, one has  $(-(c-t)^{-1} + q_t)/2 = 0$ . On the other hand, from the equality

$$1 - (1-c+t)p_t = \|e_1 + p_t v_t\| = \left\| e_1 + \frac{1}{c+t} v_t \right\| = 1 + \frac{1-c-t}{c+t},$$

we have

$$p_t = -\frac{1-c-t}{(1-c+t)(c+t)} \quad \text{and hence} \quad \frac{1}{2} \left( p_t + \frac{1}{c+t} \right) = \frac{t}{(1-c+t)(c+t)}.$$

It follows from

$$0 < \frac{t}{(1-c+t)(c+t)} = \frac{1}{2} \left( p_t + \frac{1}{c+t} \right) < \frac{1}{c+t}$$

that

$$\left\| e_1 + \frac{1}{2} \left( p_t + \frac{1}{c+t} \right) v_t \right\| = 1 + \frac{(1-c-t)t}{(1-c+t)(c+t)}.$$

This implies that

$$m(e_1, v_t) = 1 + \frac{(1-c-t)t}{(1-c+t)(c+t)}.$$

Letting

$$F_c(t) = \frac{(1-c-t)t}{(1-c+t)(c+t)},$$

one can figure out

$$\begin{aligned} & (1-c+t)^2(c+t)^2 F_c'(t) \\ &= (-2t+1-c)(1-c+t)(c+t) - (2t+1)(1-c-t)t \\ &= -(2-c)t^2 - 2c(1-c)t + c(1-c)^2. \end{aligned}$$

Let  $t_0$  be the larger solution of the equation  $-(2-c)t^2 - 2c(1-c)t + c(1-c)^2 = 0$ . Then

$$t_0 = \frac{c(1-c)}{\sqrt{2c+c}} \in (0, 1-c)$$

and  $F_c$  takes maximum at  $t_0$ . This  $t_0$  satisfies the equality

$$(-2t_0 + 1 - c)(1 - c + t_0)(c + t_0) = (1 - c - t_0)t_0(2t_0 + 1),$$

too. Thus we obtain

$$\begin{aligned} M(e_1) &= 1 + \frac{(1-c-t_0)t_0}{(1-c+t_0)(c+t_0)} \\ &= 1 + \frac{-2t_0 + 1 - c}{2t_0 + 1} \\ &= 1 + \frac{-2c(1-c) + (\sqrt{2c+c})(1-c)}{2c(1-c) + \sqrt{2c+c}} \\ &= 1 + \frac{1-c}{(1+\sqrt{2c})^2}. \end{aligned}$$

■

LEMMA 4.2. Let  $c \in [1/2, 1]$ . Then, in  $X_c$ ,

$$\sup \{m(u, v_t) : t \in (1-c, c)\} = 2c.$$

*Proof.* Let  $t \in (1-c, c)$ . Then the norm of  $u + \lambda v_t$  is calculated by

$$\|u + \lambda v_t\| = \begin{cases} -(1+\lambda) & \text{if } \lambda \leq -2c/(c-t), \\ 2c-1 - \{t + (1-c)\}\lambda & \text{if } -2c/(c-t) \leq \lambda \leq -1, \\ 1 - \{t - (1-c)\}\lambda & \text{if } -1 \leq \lambda \leq 0, \\ 1 + \lambda & \text{if } 0 \leq \lambda. \end{cases}$$

There exist two real numbers  $\alpha_t, \beta_t$  satisfying  $0 < \alpha_t < \beta_t$ ,  $\|u + \alpha_t v_t\| = \|u - v_t\|$  and

$$\|u + \beta_t v_t\| = \left\| u - \frac{2c}{c-t} v_t \right\|.$$

It is enough to consider  $\|u + \frac{1}{2}(-1 + \alpha_t)v_t\|$  and

$$\left\| u + \frac{1}{2} \left( -\frac{2c}{c-t} + \beta_t \right) v_t \right\|.$$

From the equality

$$1 + \alpha_t = \|u + \alpha_t v_t\| = \|u - v_t\| = 1 + (t - (1 - c)),$$

we have  $\alpha_t = t - (1 - c)$  and hence  $(-1 + \alpha_t)/2 = -(2 - (t + c))/2$ . Meanwhile, it follows from

$$1 + \beta_t = \|u + \beta_t v_t\| = \left\| u - \frac{2c}{c-t} v_t \right\| = - \left( 1 - \frac{2c}{c-t} \right)$$

that

$$\frac{1}{2} \left( -\frac{2c}{c-t} + \beta_t \right) = -1.$$

By the inequality

$$\frac{1}{2} \left( -\frac{2c}{c-t} + \beta_t \right) = -1 < -(2 - (t + c))/2 = (-1 + \alpha_t)/2 < 0,$$

we obtain  $m(u, v_t) = \|u - v_t\| = t + c$  and hence  $\sup\{m(u, v_t) : t \in (1 - c, c)\} = 2c$ . ■

Next, for  $t \in (0, 1)$ , the norm of  $u + \lambda w_t$  is calculated by

$$\|u + \lambda w_t\| = \begin{cases} 2c - 1 - \lambda & \text{if } \lambda \leq -1/t, \\ 1 - \{1 - 2(1 - c)t\}\lambda & \text{if } -1/t \leq \lambda \leq 0, \\ 1 + t\lambda & \text{if } 0 \leq \lambda \leq 2c/(1 - t), \\ -(2c - 1) + \lambda & \text{if } 2c/(1 - t) \leq \lambda. \end{cases}$$

In particular we have

$$\left\| u + \frac{2c}{1-t} w_t \right\| = 1 + \frac{2c}{1-t} t,$$

$$\left\| u - \frac{1}{t} w_t \right\| = 1 + \frac{1 - 2(1 - c)t}{t},$$

and hence

$$\begin{aligned} \left\| u - \frac{1}{t} w_t \right\| - \left\| u + \frac{2c}{1-t} w_t \right\| &= \frac{(1 - t)(1 - 2(1 - c)t) - 2ct^2}{t(1 - t)} \\ &= \frac{(1 - 2t)(1 + (2c - 1)t)}{t(1 - t)}. \end{aligned}$$

From this equality, we obtain that if  $t \in (0, 1/2)$  then

$$\left\| u + \frac{2c}{1-t} w_t \right\| < \left\| u - \frac{1}{t} w_t \right\|$$

and that if  $t \in (1/2, 1)$  then

$$\left\| u - \frac{1}{t} w_t \right\| < \left\| u + \frac{2c}{1-t} w_t \right\|.$$

LEMMA 4.3. *Let  $c \in [1/2, (1 + \sqrt{5})/4]$ . Then, in  $X_c$ ,*

$$\sup \{m(u, w_t) : t \in (0, 1/2)\} = \max \left\{ \frac{1}{2} + c, 1 + \frac{c}{\{1 + \sqrt{2(1-c)}\}^2} \right\}.$$

*Proof.* Let  $t \in (0, 1/2)$ . Then there exist two numbers  $\gamma_t \in (-1/t, 0)$  and  $\delta_t$  greater than  $2c/(1-t)$  satisfying

$$\|u + \gamma_t w_t\| = \left\| u + \frac{2c}{1-t} w_t \right\| \quad \text{and} \quad \|u + \delta_t w_t\| = \left\| u - \frac{1}{t} w_t \right\|,$$

respectively. To obtain  $m(u, w_t)$  it is enough to consider

$$\left\| u + \frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) w_t \right\| \quad \text{and} \quad \left\| u + \frac{1}{2} \left( -\frac{1}{t} + \delta_t \right) w_t \right\|.$$

From the equality

$$-(2c-1) + \delta_t = \|u + \delta_t w_t\| = \left\| u - \frac{1}{t} w_t \right\| = 2c-1 - \left( -\frac{1}{t} \right),$$

one has

$$\frac{1}{2} \left( -\frac{1}{t} + \delta_t \right) = 2c-1.$$

It is easy to check  $2c-1 < 2c/(1-t)$  and hence we obtain

$$\left\| u + \frac{1}{2} \left( -\frac{1}{t} + \delta_t \right) w_t \right\| = 1 + (2c-1)t.$$

Under the assumption  $t \in (0, 1/2)$ , this function takes the supremum  $1/2 + c$ .

Meanwhile, it follows from

$$1 - (1 - 2(1-c)t)\gamma_t = \|u + \gamma_t w_t\| = \left\| u + \frac{2c}{1-t} w_t \right\| = 1 + \frac{2ct}{1-t}$$

that

$$\gamma_t = -\frac{2ct}{(1-2(1-c)t)(1-t)}.$$

Hence we have

$$\frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) = \frac{c(1-(3-2c)t)}{(1-t)(1-2(1-c)t)}$$

and

$$\left\| u + \frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) w_t \right\| = 1 + \frac{ct(1-(3-2c)t)}{(1-t)(1-2(1-c)t)}.$$

We note that  $1-(3-2c)t > 1-(3-2c)/2 = c-1/2 > 0$ .

Letting

$$G_c(t) = \frac{t(1-(3-2c)t)}{(1-t)(1-2(1-c)t)}$$

in the interval  $[0, 1/2]$ , we have

$$\begin{aligned} & (1-2(1-c)t)^2(1-t)^2 G_c'(t) \\ &= (-2(3-2c)t+1)(1-t)(1-2(1-c)t) \\ &\quad - (4(1-c)t-(3-2c))t(1-(3-2c)t) \\ &= ((3-2c)^2-2(1-c))t^2-2(3-2c)t+1 \end{aligned}$$

We note that  $(3-2c)^2-2(1-c) = 4c^2-10c+7 = 4(c-5/4)^2+3/4 > 0$ . Let  $t_1$  be the smaller solution of equality  $((3-2c)^2-2(1-c))t^2-2(3-2c)t+1 = 0$ , i.e.,

$$t_1 = \frac{1}{3-2c+\sqrt{2(1-c)}}.$$

If  $c < (1+\sqrt{5})/4$ , then this  $t_1$  belongs to the interval  $(0, 1/2)$ . Thus  $G_c(t)$  takes the maximum

$$G_c(t_1) = \frac{1}{(1+\sqrt{2(1-c)})^2}.$$

This implies that

$$\left\| u + \frac{1}{2} \left( \gamma + \frac{2c}{1-t} \right) w_t \right\|$$

takes maximum

$$1 + \frac{c}{(1+\sqrt{2(1-c)})^2}. \quad \blacksquare$$

In case of  $(1 + \sqrt{5})/4 \leq c$ , the solution  $t_1$  is more than  $1/2$  and hence the function  $G_c(t)$  takes the maximum at  $t = 1/2$ . One can follow the above proof except for this part, and obtain the following:

LEMMA 4.4. *Let  $c \in [(1 + \sqrt{5})/4, 1]$ . Then, in  $X_c$ ,*

$$\sup \{m(u, w_t) : t \in (0, 1/2)\} = 1/2 + c.$$

Next we consider  $\sup\{m(u, w_t) : t \in (1/2, 1)\}$ .

LEMMA 4.5. *Let  $c \in [1/2, 1]$ . Then, in  $X_c$ ,*

$$\sup \{m(u, w_t) : t \in (1/2, 1)\} = \begin{cases} \frac{9}{8} & \text{if } \frac{1}{2} \leq c \leq \frac{9}{16}, \\ 2c & \text{if } \frac{9}{16} \leq c \leq 1. \end{cases}$$

*Proof.* Let  $t \in (1/2, 1)$ . Then from

$$\left\| u - \frac{1}{t}w_t \right\| < \left\| u + \frac{2c}{1-t}w_t \right\|$$

one can take  $\gamma_t$  less than  $-1/t$  and  $\delta_t \in (0, 2c/(1-t))$  satisfying

$$\|u + \gamma_t w_t\| = \left\| u + \frac{2c}{1-t}w_t \right\| \quad \text{and} \quad \|u + \delta_t w_t\| = \left\| u - \frac{1}{t}w_t \right\|,$$

respectively. From the equality

$$2c - 1 - \gamma_t = \|u + \gamma_t w_t\| = \left\| u + \frac{2c}{1-t}w_t \right\| = -(2c - 1) + \frac{2c}{1-t},$$

we have

$$\frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) = 2c - 1.$$

The fact

$$0 \leq 2c - 1 = \frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) < \frac{2c}{1-t}$$

implies that

$$\left\| u + \frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) w_t \right\| = \|u + (2c - 1)w_t\| = 1 + (2c - 1)t.$$

It is clear that the function  $1 + (2c - 1)t$  takes the supremum  $2c$ .

On the other hand, it follows from

$$1 + t\delta_t = \|u + \delta_t w_t\| = \left\| u - \frac{1}{t} w_t \right\| = 1 + \frac{1 - 2(1 - c)t}{t}$$

that

$$\delta_t = \frac{1 - 2(1 - c)t}{t^2}$$

and hence

$$\frac{1}{2} \left( -\frac{1}{t} + \delta_t \right) = \frac{1 - (3 - 2c)t}{2t^2}.$$

Under the assumption  $c \in [1/2, 1]$ , one can easily check  $(3 - 2c)^{-1} \in (1/2, 1)$ .

In case of  $t \in (1/2, (3 - 2c)^{-1}]$ , from the inequality

$$0 \leq \frac{1 - (3 - 2c)t}{2t^2} = \frac{1}{2} \left( -\frac{1}{t} + \delta_t \right) < \delta_t < \frac{2c}{1 - t},$$

we have

$$\left\| u + \frac{1}{2} \left( -\frac{1}{t} + \delta_t \right) w_t \right\| = 1 + \frac{1 - (3 - 2c)t}{2t}.$$

It is easy to check that this function takes the supremum  $1/2 + c$ .

Suppose that  $t \in ((3 - 2c)^{-1}, 1)$ . Then, from the inequality

$$-\frac{1}{t} < \frac{1}{2} \left( -\frac{1}{t} + \delta_t \right) = -\frac{(3 - 2c)t - 1}{2t^2} < 0,$$

one has

$$\left\| u + \frac{1}{2} \left( -\frac{1}{t} + \delta_t \right) w_t \right\| = 1 + \frac{(1 - 2(1 - c)t)((3 - 2c)t - 1)}{2t^2}.$$

Considering the function  $H_c(t)$  in the interval  $[1/2, 1]$  defined by

$$H_c(t) = \frac{(1 - 2(1 - c)t)((3 - 2c)t - 1)}{t^2},$$

we figure out

$$\begin{aligned} t^4 H_c'(t) &= (-4(1 - c)(3 - 2c)t + 5 - 4c)t^2 \\ &\quad - 2(1 - 2(1 - c)t)((3 - 2c)t - 1)t \end{aligned}$$

and hence

$$\begin{aligned} t^3 H'_c(t) &= (-4(1-c)(3-2c)t + 5-4c)t \\ &\quad - 2(1-2(1-c)t)((3-2c)t-1) \\ &= -(5-4c)t + 2. \end{aligned}$$

Since the function  $-(5-4c)t + 2$  is decreasing, we have the following: If  $c < 3/4$ , then one has  $2/(5-4c) \in ((3-2c)^{-1}, 1)$  and hence

$$\begin{aligned} \max \{H_c(t) : t \in ((3-2c)^{-1}, 1)\} \\ &= H_c\left(\frac{2}{5-4c}\right) \\ &= \frac{((5-4c) - 4(1-c))(2(3-2c) - (5-4c))}{4} = \frac{1}{4}. \end{aligned}$$

This implies that

$$\left\| u + \frac{1}{2} \left( -\frac{1}{t} + \delta_t \right) w_t \right\|$$

takes the maximum  $9/8$  at  $t = 2/(5-4c)$ . Meanwhile,  $9/8$  is greater than  $2c$  only when  $c < 9/16$ .

In case of  $3/4 \leq c$ , from  $1 \leq 2/(5-4c)$  one has that  $H_c(t)$  is increasing. Hence we have

$$\max \{H_c(t) : t \in ((3-2c)^{-1}, 1)\} = H(1) = 2(1-c)(2c-1)$$

This implies that

$$\left\| u + \frac{1}{2} \left( -\frac{1}{t} + \delta_t \right) w_t \right\|$$

takes the supremum  $1 + (1-c)(2c-1)$ . We note that  $1 + (1-c)(2c-1) < 1 + (2c-1) = 2c$ . ■

Therefore we obtain the following proposition.

PROPOSITION 4.6. *Let  $c \in [1/2, 1]$ . Then  $IB(X_c)^{-1} = DW(X_c)/2$  coincide with*

$$\begin{cases} \max \left\{ 1 + \frac{1-c}{\{1+\sqrt{2c}\}^2}, 1 + \frac{c}{\{1+\sqrt{2(1-c)}\}^2}, \frac{9}{8} \right\} & \text{if } \frac{1}{2} \leq c \leq \frac{9}{16}, \\ \max \left\{ 1 + \frac{1-c}{\{1+\sqrt{2c}\}^2}, 1 + \frac{c}{\{1+\sqrt{2(1-c)}\}^2}, 2c \right\} & \text{if } \frac{9}{16} < c < \frac{1+\sqrt{5}}{4}, \\ \max \left\{ 1 + \frac{1-c}{\{1+\sqrt{2c}\}^2}, 2c \right\} & \text{if } \frac{1+\sqrt{5}}{4} \leq c \leq 1. \end{cases}$$

Hereafter we suppose  $c < 1/2$ . Similarly to the above paragraph,  $DW(X_c) = 2 \max\{M(e_1), M(u)\}$  holds. On the other hand, for  $v_t$  and  $w_t$ , Birkhoff orthogonality relations differ from the above paragraph. We have  $e_1 \perp_B v_t$  for  $t \in [0, c]$ ,  $e_1 \perp_B w_t$  for  $t \in [1/2(1-c), 1]$  and  $u \perp_B w_t$  for  $t \in [0, 1/2(1-c)]$ .

By [15, Theorem 2.9 and Corollary 2.10], one figure out

$$M(e_1) = \max \left\{ \sup\{m(e_1, v_t) : t \in (0, c)\}, \sup\{m(e_1, w_t) : t \in (1/2(1-c), 1)\} \right\}$$

and

$$M(u) = \sup \{m(u, w_t) : t \in (0, 1/2(1-c)) \setminus \{1/2\}\}.$$

LEMMA 4.7. *Let  $c \in [0, (3 - \sqrt{5})/4]$ . Then in  $X_c$ ,*

$$\sup \{m(e_1, v_t) : t \in (0, c)\} = \frac{3}{2} - c.$$

*Proof.* Let  $t \in (0, c)$ . Then in a similar way to the proof of Lemma 4.1, we have

$$m(e_1, v_t) = 1 + \frac{(1-c-t)t}{(1-c+t)(c+t)}.$$

Moreover letting

$$F_c(t) = \frac{(1-c-t)t}{(1-c+t)(c+t)},$$

we also have

$$(1-c+t)^2(c+t)^2 F'_c(t) = -(2-c)t^2 - 2c(1-c)t + c(1-c)^2$$

again. From  $c \in [0, (3 - \sqrt{5})/4]$ , it is more than

$$-(2 - c)c^2 - 2c(1 - c)c + c(1 - c)^2 = c(4c^2 - 6c + 1) \geq 0.$$

From this fact,  $F_c(t)$  increases and hence

$$\sup \{m(e_1, v_t) : t \in (0, c)\} = 1 + F_c(c) = \frac{3}{2} - c. \quad \blacksquare$$

Suppose that  $c \in ((3 - \sqrt{5})/4, 1/2)$ . Then for  $t_0$  defined in the same formula to the proof of Lemma 4.1, we have  $t_0 \in (0, c)$  and  $F'_c(t_0) = 0$ . Hence we obtain

LEMMA 4.8. *Let  $c \in ((3 - \sqrt{5})/4, 1/2)$ . Then in  $X_c$ ,*

$$\sup \{m(e_1, v_t) : t \in (0, c)\} = 1 + \frac{1 - c}{(1 + \sqrt{2c})^2}.$$

LEMMA 4.9. *Let  $c \in [0, 1/2)$ . Then in  $X_c$ ,*

$$\sup \{m(e_1, w_t) : t \in (1/2(1 - c), 1)\} = 2(1 - c).$$

*Proof.* Let  $t \in (1/2(1 - c), 1)$ . Then the norm of  $e_1 + \lambda w_t$  is calculated as

$$\|e_1 + \lambda w_t\| = \begin{cases} 1 - \lambda & \text{if } \lambda \leq 0, \\ 1 + (2(1 - c)t - 1)\lambda & \text{if } 0 \leq \lambda \leq (1 - (1 - 2c)t)^{-1}, \\ t\lambda & \text{if } (1 - (1 - 2c)t)^{-1} \leq \lambda \leq (1 - t)^{-1}, \\ -1 + \lambda & \text{if } (1 - t)^{-1} \leq \lambda. \end{cases}$$

One can take two real numbers  $s_t, r_t$  satisfying  $s_t < r_t < 0$ ,  $\|e_1 + r_t w_t\| = \|e_1 + (1 - (1 - 2c)t)^{-1} w_t\|$  and  $\|e_1 + s_t w_t\| = \|e_1 + (1 - t)^{-1} w_t\|$ . It is enough to consider  $\|e_1 + \frac{1}{2}(r_t + (1 - (1 - 2c)t)^{-1}) w_t\|$  and  $\|e_1 + \frac{1}{2}(s_t + (1 - t)^{-1}) w_t\|$ . From the equality

$$1 - r_t = \|e_1 + r_t w_t\| = \|e_1 + (1 - (1 - 2c)t)^{-1} w_t\| = 1 + \frac{2(1 - c)t - 1}{1 - (1 - 2c)t},$$

one has

$$r_t = -\frac{2(1 - c)t - 1}{1 - (1 - 2c)t}$$

and hence

$$\frac{1}{2} \left( r_t + \frac{1}{1 - (1 - 2c)t} \right) = \frac{1 - (1 - c)t}{1 - (1 - 2c)t}.$$

It follows from

$$1 - s_t = \|e_1 + s_t w_t\| = \|e_1 + (1 - t)^{-1} w_t\| = -1 + \frac{1}{1 - t}$$

that  $\frac{1}{2}(s_t + (1 - t)^{-1}) = 1$ . Since the inequality

$$0 < \frac{1 - (1 - c)t}{1 - (1 - 2c)t} < 1 < \frac{1}{1 - (1 - 2c)t}$$

holds, we obtain

$$m(e_1, w_t) = \left\| e_1 + \left( s_t + \frac{1}{1 - t} \right) w_t \right\| = 1 + (2(1 - c)t - 1) \times 1 = 2(1 - c)t.$$

This implies

$$\sup \{m(e_1, w_t) : t \in (1/2(1 - c), 1)\} = 2(1 - c). \quad \blacksquare$$

For  $t \in (0, 1/2(1 - c))$  the norm of  $u + \lambda w_t$  is calculated in a similar way to the case of  $c \in [1/2, 1]$ . Now we suppose  $c \in [0, 1/2)$  and so  $1/2 \leq 1/2(1 - c) < 1$  holds. Thus we have to consider the following two cases again:

If  $t \in (0, 1/2)$  then

$$\left\| u + \frac{2c}{1 - t} w_t \right\| < \left\| u - \frac{1}{t} w_t \right\|.$$

If  $t \in (1/2, 1)$  then

$$\left\| u - \frac{1}{t} w_t \right\| < \left\| u + \frac{2c}{1 - t} w_t \right\|.$$

LEMMA 4.10. *Let  $c \in [0, 1/2)$ . Then, in  $X_c$*

$$\sup \{m(u, w_t) : t \in (0, 1/2)\} = \max \left\{ 2(1 - c), 1 + \frac{c}{(1 + \sqrt{2(1 - c)})^2} \right\}.$$

*Proof.* Let  $t \in (0, 1/2)$ . In a similar way to Lemma 4.3, one can take  $\delta_t$  and figure out that this constant satisfy  $\|u + \frac{1}{2}(-1/t + \delta_t)w_t\| = 1 + (1 - 2c)(1 - 2(1 - c)t)$  and that this function of  $t$  takes the supremum  $2(1 - c)$ . We also have  $\gamma_t$  and that

$$\frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) = \frac{c(1 - (3 - 2c)t)}{(1-t)(1 - 2(1 - c)t)}.$$

Now we are considering the case of  $c \in [0, 1/2)$  and so  $1/(3 - 2c) \in (0, 1/2)$ . If  $t \in (0, 1/(3 - 2c))$ , then we have

$$0 < \frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) < \frac{2c}{1-t}$$

and hence

$$\left\| u + \frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) w_t \right\| = 1 + \frac{ct(1 - (3 - 2c)t)}{(1-t)(1 - 2(1 - c)t)}.$$

For  $t_1$  defined by same formula to Lemma 4.3, we have  $t_1 \in (0, 1/(3 - 2c))$  and that the function

$$\left\| u + \frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) w_t \right\|$$

takes maximum  $1 + c/(1 + \sqrt{2(1 - c)})^2$  at  $t_1$ .

Assume that  $t \in (1/(3 - 2c), 1/2)$ . Then from the inequality

$$-\frac{1}{t} < \gamma_t < \frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) = -\frac{c((3 - 2c)t - 1)}{(1-t)(1 - 2(1 - c)t)} < 0,$$

we obtain

$$\left\| u + \frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) w_t \right\| = 1 + \frac{c((3 - 2c)t - 1)}{1-t}.$$

This function of  $t$  is increasing and hence less than

$$1 + \frac{c((3 - 2c)/2 - 1)}{1 - 1/2} = (1 - c)(1 + 2c) < 2(1 - c).$$

Thus we obtain

$$\left\| u + \frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) w_t \right\| < 2(1 - c),$$

which completes the proof. ■

LEMMA 4.11. *Let  $c \in [0, 1/2)$ . Then, in  $X_c$ ,*

$$\begin{aligned} &\sup \{m(u, w_t) : t \in (1/2, 1/(2-c))\} \\ &= \begin{cases} (1-c)(1+2c) & \text{if } 0 < c \leq \frac{1}{4}, \\ \max \left\{ (1-c)(1+2c), \frac{9}{8} \right\} & \text{if } \frac{1}{4} < c < \frac{1}{2}. \end{cases} \end{aligned}$$

*Proof.* Let  $t \in (1/2, 1/(2-c))$ . In a similar way to Lemma 4.5, we take  $\gamma_t$  less than  $-1/t$  and  $\delta_t \in (0, 2c/(1-t))$ . Then we have

$$\frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) = -(1-2c)$$

It follows from  $-1/t < -2(1-c) < -(1-2c) < 0$  that

$$\left\| u + \frac{1}{2} \left( \gamma_t + \frac{2c}{1-t} \right) w_t \right\| = 1 + (1-2c)(1-2(1-c)t)$$

In the situation  $t \in (1/2, 1/(2-c))$ , it takes supremum  $1 + c(1-2c) = (1-c)(1+2c)$ . In addition, we have

$$-\frac{1}{t} < \frac{1}{2} \left( -\frac{1}{t} + \delta_t \right) = -\frac{(3-2c)t-1}{2t^2} < 0.$$

Hence the equality

$$\left\| u + \frac{1}{2} \left( -\frac{1}{t} + \delta_t \right) w_t \right\| = 1 + \frac{(1-2(1-c)t)((3-2c)t-1)}{2t^2}$$

holds. As in Lemma 4.5, one can consider the following two cases:

If  $0 < c < 1/4$ , then the above function is decreasing and hence takes the supremum

$$1 + \frac{(1-2(1-c)/2)((3-2c)/2-1)}{2(1/2)^2} = 1 + c(1-2c) = (1-c)(1+2c)$$

When  $1/4 \leq c < 1/2$ , we have that the above function takes maximum  $9/8$  at  $t = 2/(5-4c)$ . ■

Indeed,  $(1-c)(1+2c)$  is less than  $2(1-c)$  for any  $c \in [0, 1/2)$ . Meanwhile, it is easy to see that  $2(1-c) < 9/8$  only if  $c > 7/16$ . Therefore we have

PROPOSITION 4.12. *Let  $c \in [0, 1/2]$ . Then  $IB(X_c)^{-1} = DW(X_c)/2$  coincide with*

$$\left\{ \begin{array}{ll} \max \left\{ 2(1-c), 1 + \frac{c}{(1 + \sqrt{2(1-c)})^2} \right\} & \text{if } 0 \leq c \leq \frac{3 - \sqrt{5}}{4}, \\ \max \left\{ 2(1-c), 1 + \frac{c}{(1 + \sqrt{2(1-c)})^2}, \right. \\ \qquad \qquad \qquad \left. 1 + \frac{1-c}{(1 + \sqrt{2c})^2} \right\} & \text{if } \frac{3 - \sqrt{5}}{4} < c < \frac{7}{16}, \\ \max \left\{ 1 + \frac{c}{(1 + \sqrt{2(1-c)})^2}, 1 + \frac{1-c}{(1 + \sqrt{2c})^2}, \frac{9}{8} \right\} & \text{if } \frac{7}{16} \leq c \leq \frac{1}{2}. \end{array} \right.$$

Considering the symmetry of the functions

$$\frac{c}{(1 + \sqrt{2(1-c)})^2} \quad \text{and} \quad \frac{1-c}{(1 + \sqrt{2c})^2}$$

and that these function takes value  $1/8$  at  $t = 1/2$ , we finally obtain

THEOREM 4.13. *Let  $c \in [0, 1]$  and put  $d = \max\{c, 1 - c\}$ . Then both  $DW(X_c)$  and  $DW(X_c^*)$  coincide with*

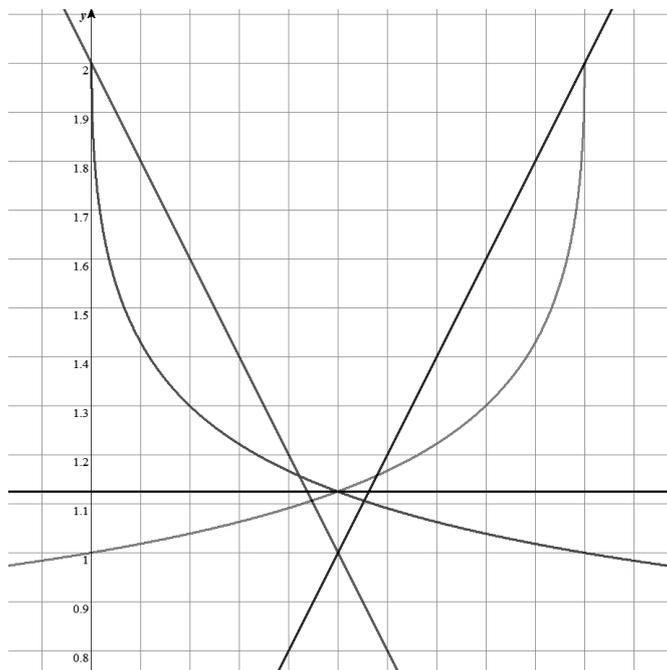
$$2 \max \left\{ 2d, 1 + \frac{d}{(1 + \sqrt{2(1-d)})^2} \right\}.$$

*This is more than  $DW(\ell_{\psi_c} - \ell_{\tilde{\psi}_c^*}) = 9/4$  and the equality holds only when  $c = 1/2$ .*

THEOREM 4.14. *Let  $c \in [0, 1]$  and put  $d = \max\{c, 1 - c\}$ . Then both  $IB(X_c)$  and  $IB(X_c^*)$  coincide with*

$$\min \left\{ \frac{1}{2d}, \frac{(1 + \sqrt{2(1-d)})^2}{d + (1 + \sqrt{2(1-d)})^2} \right\}.$$

*This is less than  $IB(\ell_{\psi_c} - \ell_{\tilde{\psi}_c^*}) = 8/9$  and the equality holds only when  $c = 1/2$ .*



The graphs of the functions  $y = 2x$ ,  $y = 2(1 - x)$ ,  $y = 1 + \frac{x}{(1 + \sqrt{2(1 - x)})^2}$ ,  
 $y = 1 + \frac{1 - x}{(1 + \sqrt{2x})^2}$  and  $y = \frac{9}{8}$  on the interval  $[0, 1]$ .

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