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Geometry of Foliated Manifolds

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Abstract: In this paper some results of the authors on geometry of foliated manifolds are stated and results on geometry of Riemannian (metric) foliations are discussed.

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1. Introduction

A manifold, with some fixed foliation on it, is called a foliated manifold. Theory of foliated manifolds is one of new fields of mathematics. It appeared in the intersection of Differential Equations, Differential Geometry and Differential Topology in the second part of 20^{th} century.

In formation and development of the theory of foliations the big contribution was made by famous mathematicians such as C. Ehresmann [1], G. Reeb [15], A. Haefliger [3], R. Langevin [8], C. Lamoureux [7].

Further development of the geometrical theory of foliations is connected with known works of R. Hermann [4, 5], P. Molino [9], B.L. Reinhart [16], Ph. Tondeur [17].

At present, the theory of foliations (the theory of foliated manifolds) is intensively developing and has wide applications in many fields of science and technique. In the theory of foliations, it is possible to get acquainted with the latest scientific works in work Ph. Tondeur [18], where the bibliography consisting of more than 2500 works on the theory of foliations is provided. In work [10] applications of the theory of foliations in the qualitative theory of optimal control are discussed.

In this paper some results of the authors on geometry of foliated manifolds are stated and results on geometry of Riemannian (metric) foliations are discussed.

At first we will give some necessary definitions and examples.

Let (M, g) be a smooth Riemannian manifold of dimension n, where g is a Riemannian metric and $0 < k < n$.

Definition 1. A family $F = \{L_\alpha \subset M : \alpha \in B\}$ of pathwise connected subsets M is called a k-dimensional smooth foliation if it satisfies the following three conditions:

(F_1): $\bigcup\limits_{\alpha \in B} L_\alpha = M$;

(F_2): for all $\alpha, \beta \in B$ if $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \emptyset$;

(F_3): for any point $p \in M$ there is a neighborhood U_p and a coordinate chart $\left(x^1, x^2, \ldots, x^n\right)$ such that if $U_p \bigcap L_\alpha \neq \emptyset$ for some $\alpha \in B$, then pathwise connected components of the set $U_p \cap L_\alpha$ are given by the equations: $x^{k+1} = c^{k+1}$, $x^{k+2} = c^{k+2}$, $\ldots$, $x^n = c^n$, where numbers c^{k+1}, c^{k+2}, $\ldots$, c^n are constant on components of pathwise connectedness.

The set L_α is called a leaf of a foliation F. In the described situation a k-dimensional C^r-foliation is also called C^r-foliation of codimension $q = n - k$.

Existence of a foliation F in a manifold M is expressed by a symbol (M, F). Conditions (F_1), (F_2) mean that M consists of mutually disjoint leaves. The condition (F_3) means that locally leaves are arranged as the parallel planes. The neighborhood U in the definition is called a foliated neighborhood.

The simplest foliations from the point of view of geometry are the foliations generated by submersions, in particular the family of level surfaces of differentiable functions.

Definition 2. A differentiable mapping $f : M \to B$ of maximal rank, where M, B are smooth manifolds of dimension n, m respectively, and $n > m$, is called a submersion.

For submersions the following theorem holds.

Theorem 1. *Let $f : M \to B$ be a submersion, where M is a smooth manifold of dimension m, $n > m$. Then for each point $q \in B$ the set $L_q = \{p \in M : f(p) = q\}$ is a manifold of dimension $(n - m)$ and partition of M into connected components of the fibers is $k = (n-m)$-dimensional foliations.*

Thus, the submersion $f : M \to B$ generates a foliation F of dimension on $k = (n - m)$ on the manifold M, leaves of which are connected components of fibers $L_q = f^{-1}(q)$, $q \in B$.

Numerous researches [10-14], [17] are devoted to studying of geometry and topology of foliations generated by submersions.

Let F be a smooth foliation of dimension k on M. By $L(p)$ denote the leaf of the foliation F passing through a point p, T_pL is a tangent space of the leaf $L(p)$ at the point p, $H(p)$ is an orthogonal complement of T_pL in T_pM, $p \in M$.

We get two sub-bundles $TF = \{T_pL : p \in M\}$, $H = \{H\,(p) : p \in M\}$ of a tangent bundle TM such that $TM = TF \oplus H$, where H is an orthogonal complement of TF. In this case each vector field $X \in V\,(M)$ can be represented in the form $X = X_v + X_h$, where X_v, X_h are orthogonal projections of X on TF, H respectively.

If $X \in V\,(F)$ (i.e., $X_h = 0$), then X is called a vertical field. If $X \in V(H)$ $(X_\nu = 0)$, then X is a horizontal field.

Definition 3. A submersion $f : M \to B$ is Riemannian, if the differential of a mapping df preserves the length of horizontal vectors.

Definition 4. A foliation on a Riemannian manifold is called Riemannian if every geodesic, orthogonal to a leaf of the foliation F remains orthogonal to all leaves in all its points.

For the first time a Riemannian foliation was entered in work [16] and was shown that Riemannian submersions generate Riemannian foliations.

This class of foliations plays very important role in the theory of foliations and is substantial from the point of view of geometry. There is a large number of works devoted to geometry of Riemannian foliations.

A Riemannian foliations with singularity were introduced by P. Molino [9], and studied in A. Narmanov's works [10], [14] and other authors.

2. Previous results

An important class of foliations of codimension one are the foliations generated by level surfaces of differentiable functions without critical points.

Function $f : M^n \to R^1$ on a riemannian manifold M^n, whose length of a gradient vector is constant on each level surface (i.e., for each vertical vector field X it holds $X\left(|\mathrm{grad} f|^2\right) = 0$), is called metric. For the first time the geometry of foliations generated by surfaces of metric functions is studied in work [17].

The following theorem shows that metric functions are included into a class of Riemannian submersions.

Theorem 2. *Let $f : M \to R^1$ be a metric function. Then on R^1 there is a Riemannian metric such that $f : M \to R^1$ is a Riemannian submersion.*

Therefore, level surfaces of metric function generate a Riemannian foliation.

Riemannian foliations generated by metric functions are studied in works of A.Ya. Narmanov, A.M. Bayturayev [11], A.Ya. Narmanov, G.Kh. Kaipnazarova [12], Ph. Tondeur [17].

We remind that by definition the gradient vector $X = \text{grad} f$ of the function f given on Riemannian manifold depends not only on the function f, but also on a Riemannian metric. The integral curve of the gradient vector field is called the gradient line of function f.

By A.Ya. Narmanov and G.Kh. Kaipnazarova in work [12] it is shown that if for each vertical vector field the equality $X\left(|\text{grad} f|^2\right) = 0$ holds, then each gradient line is the geodesic line of Riemannian manifold.

In work [12] geometry of foliations is studied generated by level surfaces of metric functions and the whole classification appears in the next form:

Theorem 3. *Let f metric function is defined in R^n. Then the level surface of function makes F surface that has one of these types of n:*

1) *foliations F consists of parallel hyperplanes;*
2) *foliations F consists of concentric hyperspheres and a point (that is the center of spheres);*
3) *foliations F consists of concentric cylinders in the form $S^{n-k-1} \times R^k$ and singular foliation R^k (that occurs when sphere S^{n-k-1} shrinks and becomes a point), where k is minimal dimension of critical level surfaces and $1 \leq k \leq n-2$.*

In work [11] the following theorem is proved.

Theorem 4. *Let M be a smooth complete and connected Riemannian manifold of constant non-negative section curvature, $f : M \to R^1$ metric function without critical points. Then, level surfaces of function f generate completely geodesic foliation F on M, whose leaves are mutually isometric.*

3. Main part

Before formulating the following theorem about curvature of leaves, we will recall the Gaussian curvature of a submanifold.

The Riemannian metric on the manifold M induces a Riemannian metric $\tilde{g}$ on a leaf L_p. The canonical injection $i : L_p \to M$ is an isometric immersion with respect to this metric. Connection ∇ induces a connection $\tilde{\nabla}$ on L_p which coincides with the connection determined by the Riemannian metric $\tilde{\nabla}$ [6].

Let Z be a horizontal vector field. For each vertical vector field we will define a vector field

$$S(X, Z) = (\nabla_X Z)^v,$$

where ∇ is the Levi-Civita connection defined by the Riemannian metric g. At the fixed horizontal field we obtain a tensor field of type (1,1)

$$S_Z X = S(X, Z).$$

With the help of this tensor field the bilinear form

$$l_Z(X, Y) = g(S_Z X, Y)$$

is defined, where $g(X, Y)$ is the scalar product defined by the Riemannian metric g.

The defined tensor field S_Z is called the second main tensor, and a form $l_Z(X, Y)$ is called the second main form with respect to a horizontal field Z.

The mapping $S_Z : T_qF \to T_qF$ determined by the formula $X_q \to S(X, Z)_q$ is a self-conjugate endomorphism with respect to a scalar product, determined by a Riemannian metric $\tilde{g}$.

If the vector field Z is a field of unit vectors, then eigenvalues of this endomorphism are called the main curvatures of the manifold L_p at a point q, and the corresponding eigenvectors are called the main directions. By the main curvatures the Gaussian curvature $K_Z = \det S_Z$ is defined.

We will prove that level surfaces of Riemannian submersions are surfaces of constant Gaussian curvature.

Theorem 5. *Let M be a Riemannian manifold of constant non-negative curvature, $f : M \to R^1$ a Riemannian submersion. Then each leaf of a foliation F generated by Riemannian submersion (connected components of the level surfaces of the function f) is a manifold of constant Gaussian curvature.*

Proof. As is known the Hessian is given by

$$h_f\left(X,Y\right)=l_Z\left(X,Y\right)=\left\langle \nabla_X Z, Y\right\rangle$$

where $Z = \mathrm{grad} f$, ∇- the Levi-Civita connection defined by Riemannian metric g.

The map $X \to h_f\left(X\right) = \nabla_X Z$ (Hesse tensor) is a linear operator and is given by a symmetric matrix A:

$$h_f\left(X\right)=\nabla_X Z = AX.$$

We denote by $\chi(\lambda)$ the characteristic polynomial of the matrix A with a free term $(-1)^n \det A$ and define a new polynomial $\rho(\lambda)$ by the equation

$$\lambda\rho(\lambda)=\det A-(-1)^n\chi(\lambda)\,.$$

Since $\chi\left(A\right)=0$ we have that $A\rho\left(A\right)=\det A\cdot E$, where E is the identity matrix. The elements of the matrix $\rho\left(A\right)$ are cofactors of the matrix A. This matrix is denoted by H_f^c.

It is well known that the Gaussian curvature of the surface is calculated by the formula [2, p. 110]

$$K=\det S=\frac{1}{|\mathrm{grad} f|^{n+1}}\left\langle H_f^c\left(\mathrm{grad} f\right),\mathrm{grad} f\right\rangle.$$

To prove the theorem it suffices to show that $X\left(K\right)=0$ for each vertical vector field X at any point q of a leaf L_p.

By hypothesis of the theorem differential df preserves the length of $|\mathrm{grad} f|$. Therefore, we have

$$X\left(|\mathrm{grad} f|^2\right)=0$$

and so

$$X\left(\frac{1}{|\mathrm{grad} f|^{n+1}}\right)=0\,.$$

Therefore we need to show that

$$\left\langle \nabla_X H_f^c Z, Z\right\rangle+\left\langle H_f^c Z,\nabla_X Z\right\rangle=0\,.$$

We know that if $X(|\mathrm{grad} f|^2)=0$ for each vertical vector field X, each gradient line of f is a geodesic line of Riemannian manifold [12]. By definition, the gradient line is a geodesic if and only if $\nabla_N N=0$, where $N=\frac{Z}{|Z|}$.

We calculate the covariant differential

$$\nabla_N N = \frac{1}{|Z|}\nabla_Z N = \frac{1}{|Z|}\left(\frac{1}{|Z|}\nabla_Z Z + Z\left(\frac{1}{|Z|}\right)Z\right) = 0$$

and get $\nabla_Z Z = \lambda Z$, where $\lambda = -|Z|Z\left(\frac{1}{|Z|}\right)$. This means that the gradient vector Z is the eigenvector of matrix A.

Let $X_1^0, X_2^0, \ldots, X_{n-1}^0, Z^0$-be mutually orthogonal eigenvectors of A at the point $q \in L_p$ such that $X_1^0, X_2^0, \ldots, X_{n-1}^0$ the unit vectors, Z^0 - the value of the gradient field at a point q. Locally, they can be extended to the vector fields $X_1, X_2, \ldots, X_{n-1}, Z$ to a neighborhood of (say U) point q so that they formed at each point of an orthogonal basis consisting of eigenvectors. We construct the Riemannian normal system of coordinates $(x_1, x_2, \ldots, x_n)$ in a neighborhood U via vectors $X_1^0, X_2^0, \ldots, X_{n-1}^0, Z^0$ [2, p. 112].

The components g_{ij} of the metric g and the connection components Γ_{ij}^k in the normal coordinate system satisfies the conditions of [2, p. 132]:

$$g_{ij}(q) = \delta_{ij}\,, \quad \Gamma_{ij}^k(q) = 0\,.$$

We show that $X(\lambda) = 0$ for each vertical field X. From the equality

$$X(\lambda) = -X(|Z|)\,Z\left(\frac{1}{|Z|}\right) - |Z|\,X\left(Z\left(\frac{1}{|Z|}\right)\right)$$

and from the condition $X(|Z|) = 0$ follows equality

$$X\left(Z\left(\frac{1}{|Z|}\right)\right) = X\left(Z\left(\phi\right)\right) = [X, Z]\left(\phi\right) - Z\left(X\left(\phi\right)\right),$$

where $\phi = \frac{1}{|Z|}$, $[X, Z]$-Lie bracket of vector fields X, Z.

From the condition of the theorem follows $X(\phi) = 0$. In [17] it is shown that $X\left(|\mathrm{grad} f|^2\right) = 0$ for each of the vertical vector field$X$ if and only if $[X, Z]$ a vertical field. Therefore $[X, Z]\left(\phi\right) = 0$. Thus, λ is a constant function on the leaf L.

Now we denote by $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ the eigenvalues of the matrix A corresponding to the eigenvectors $X_1, X_2, \ldots, X_{n-1}$. Then in the basis $X_1, X_2, \ldots,$ X_{n-1}, Z matrix A has the form:

$$A = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

By hypothesis of the theorem, the vector field $\nabla_X Z$ is vertical field. It follows Codazzi equations have the form [6, p. 29]

$$(\nabla_X A)Y = (\nabla_Y A)X.$$

From this equation we get

$$\nabla_{X_i} AX_j = \nabla_{X_j} AX_i\,, \quad \nabla_{X_i} AZ = \nabla_Z AX_i \tag{1}$$

at any point of U for each vector field X_i. From first equation of (1) we take following equality

$$X_i(\lambda_j)X_j + \lambda_j \nabla_{X_i} X_j = X_j(\lambda_i)X_i + \lambda_i \nabla_{X_j} X_i\,. \tag{2}$$

Since $\nabla_{X_i} X_j = \Gamma^k_{ij} X_k = 0$ at the point q by properties of normal coordinate system, from (2) follows equality

$$X_i(\lambda_j)X_j = X_j(\lambda_i)X_i\,. \tag{3}$$

By the linear independence $X_1, X_2, \cdots, X_{n-1}$,we have that

$$X_i(\lambda_j) = 0 \qquad \text{for } i \neq j\,.$$

From second equation of (1) we take following

$$X_i(\lambda)Z + \lambda \nabla_{X_i} Z = Z(\lambda_i)X_i + \lambda_i \nabla_Z X_i\,. \tag{4}$$

Since

$$\nabla_{X_i} Z = \nabla_Z X_i = 0$$

at the point q from the linear independence of vectors X_i, Z we have that

$$X_i(\lambda) = 0\,, \quad Z(\lambda_i) = 0 \qquad \text{for all } i\,.$$

On the other hand

$$\begin{aligned} \nabla_Z AX_i &= Z(\lambda_i)X_i + \lambda_i \nabla_Z X_i\,, \\ \nabla_{X_i} AZ &= \nabla_Z AX_i\,, \\ \nabla_{X_i} Z &= \lambda_i X_i\,. \end{aligned} \tag{5}$$

From (5) we get that

$$\lambda_i^2 X_i + Z(\lambda_i)X_i = X_i(\lambda)Z + \lambda\lambda_i X_i\,. \tag{6}$$

Since $Z(\lambda_i) = 0$, $X_i(\lambda) = 0$ from the (6) we get

$$\lambda_i^2 X_i = \lambda\lambda_i X_i\,. \tag{7}$$

Since X_i is nonzero vector, from the (7) follows that $\lambda_i^2 = \lambda\lambda_i$. From this equality follows if $\lambda = 0$, then $\lambda_i = 0$. If $\lambda_i \neq 0$ then $\lambda_i = \lambda$ and $X(\lambda_i) = X(\lambda) = 0$, $Z(\lambda) = Z(\lambda_i) = 0$ for all i. Thus, in the neighborhood U of the point q non-zero eigenvalues of the matrix A are constant and equal λ.

Given this fact we compute $X(K)$. We denote by m the number of zero eigenvalues of A. If $m = 0$, all the eigenvalues are equal to the number λ. In this case, by the definition of the matrix H_f^c we get that $H_f^c Z = \lambda^{n-1} Z$ and

$$\nabla_X H_f^c Z = X\left(\lambda^{n-1}\right) Z + \lambda^{n-1}\nabla_X Z\,.$$

As mentioned above field $\nabla_X \mathrm{grad} f$ is a vertical vector field for each vertical vector field X (the field AX is vertical). From this equalities follows $\left\langle \nabla_X H_f^c(\mathrm{grad} f), \mathrm{grad} f\right\rangle = 0$ at the point q.

Consider the case when $m > 0$. If $m > 1$, then $H_f^c = 0$. If $m = 1$ than $\lambda_i = 0$ for some i and $AX_i = \nabla_{X_i} Z = 0$. This means that the vector field Z is parallel along the integral curve of a vector field X_i (along i-coordinate line). If $i = n$ we have $\lambda = \lambda_i = 0$ for all i and $H_f^c = 0$.

Without loss of generality we assume that $i < n$. In this case vector $H_f^c Z$ have only one nonzero component b_i and $H_f^c Z = b_i \frac{\partial}{\partial x_i}$. In this case we get

$$\nabla_X H_f^c Z = X(b_i)\frac{\partial}{\partial x_i} + b_i \nabla_X \frac{\partial}{\partial x_i}\,.$$

As we know that $X_i = \frac{\partial}{\partial x_i}$ vertical and $\nabla_X \frac{\partial}{\partial x_i} = 0$. Thus in the case $m = 1$ we have $\left\langle \nabla_X H_f^c(\mathrm{grad} f), \mathrm{grad} f\right\rangle = 0$. The Theorem 5 is proved. ∎

Example 1. Let $M = R^3 \setminus \{(x,y,z) : x = 0,\ y = 0\}$, $f(x,y,z) = x^2 + y^2$. Level surfaces of this submersion are manifolds of zero Gaussian curvature.

Example 2. Let $M = R^3 \setminus \{(0,0,0)\}$, $f(x,y,z) = x^2 + y^2 + z^2$. Level surfaces of this submersion are concentric spheres, Gaussian curvature of which is positive.

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