EXTRACTA MATHEMATICAE Vol. **31**, Núm. 1, 1 – 10 (2016)

On Small Combination of Slices in Banach Spaces

Sudeshna Basu, T. S. S. R. K. Rao

Department of Mathematics, George Washington University,
Washington DC 20052, USA

sbasu@gwu.edu, sudeshna66@gmail.com

Stat-Math Division, Indian Statistical Institute, R. V. College,
P. O. Bangalore 560059, India

tss@isibang.ac.in, srin@fulbrightmail.org

Presented by Pier L. Papini Received August 13, 2015

Abstract: The notion of Small Combination of Slices (SCS) in the unit ball of a Banach space was first introduced in [4] and subsequently analyzed in detail in [12] and [13]. In this work, we introduce the notion of BSCSP, which can be seen as a generalization of dentability in terms of SCS. We study certain stability results for the w^*-BSCSP leading to a discussion on BSCSP in the context of ideals of Banach spaces. We prove that the w^*-BSCSP can be lifted from a M-ideal to the whole Banach Space. We also prove similar results for strict ideals and U-subspaces of a Banach space. We note that the space $C(K,X)^*$ has w^*-BSCSP when K is dispersed and X^* has the w^*-BSCSP.

Key words: Small combination of slices, M-Ideals, Strict ideals, U-Subspaces.

AMS *Subject Class.* (2010): 46B20, 46B28.

1. Introduction

Let X be a real Banach space and X^* its dual. We will denote by B_X, S_X and $B_X(x,r)$ the closed unit ball, the unit sphere and the closed ball of radius $r > 0$ and center x. We refer to the monograph [2] for notions of convexity theory that we will be using here.

Definition 1. (i) We say $A \subseteq B_{X^*}$ is a norming set for X if $\|x\| = \sup\{|x^*(x)| : x^* \in A\}$, for all $x \in X$. A closed subspace $F \subseteq X^*$ is a norming subspace if B_F is a norming set for X.
(ii) Let $f \in X^*$, $\alpha > 0$ and $C \subseteq X$. Then the set

$$S(C, f, \alpha) = \{x \in C : f(x) > \sup f(C) - \alpha\}$$

is called the open slice determined by f and α. We assume without loss of generality that $\|f\| = 1$. One can analogously define w^* slices in X^*

(iii) A point $x \neq 0$ in a convex set $K \subseteq X$ is called a SCS (small combination of slices) point of K, if for every $\varepsilon > 0$, there exist slices S_i of K, and a convex combination $S = \sum_{i=1}^{n} \lambda_i S_i$ such that $x \in S$ and $\mathrm{diam}(S) < \varepsilon$. One can analogously define w^*-SCS point in X^*.

We introduce the following definition analogous to that of a unit ball being dentable, see [2].

Definition 2. A Banach Space is said to have Ball-Small Combination of Slices Property (BSCSP) if the unit ball has small combination of slices of arbitrarily small diameter. Analogously we can define w^*-BSCSP in a dual space.

Remark 3. (i) It is clear that if B_X has a SCS point, then it has BSCSP.
(ii) Strongly Regular spaces studied in [4] and [13] were referred to as Small Combination of Slices Property (SCSP) in [12].

SCS points were first introduced in [4] as a "slice generalization" of the notion PC (i.e. points for which the identity mapping on the unit ball, from weak topology to norm topology is continuous). It was proved in [4] that X is strongly regular (respectively, X^* is w^*-strongly regular) if and only if every non empty bounded convex set K in X (respectively K in X^*) is contained in the norm closure (respectively, w^*-closure) of $SCS(K)$ (respectively w^*-SCS(K)), i.e. the SCS points (w^*-SCS points) of K. Later, it was proved in [13] that Banach space has Radon Nikodym Property (RNP) if and only if it is strongly regular and has the Krein-Milman Property (KMP). Subsequently, the concept of SCS points was used in [12] to investigate the structure of non dentable closed bounded convex sets in Banach spaces. In this work, we study certain stability results for w^*-BSCSP leading to a discussion on BSCSP in the context of ideals of Banach spaces, see [5] and [12]. We use various techniques from the geometric theory of Banach spaces to achieve this. The spaces that we will be considering have been well studied in the literature. A large class of function spaces like the Bloch spaces, Lorentz and Orlicz spaces, spaces of vector-valued functions and spaces of compact operators are examples of the spaces we will be considering: for details, see [6]. We provide some descriptions of w^*-SCS points in Banach spaces in different contexts. We need the following definition.

DEFINITION 4. Let X be a Banach space.

(i) A linear projection P on X is called an M-*projection* if

$$\|x\| = \max\{\|Px\|, \|x - Px\|\},$$

for all $x \in X$; A linear projection P on X is called an L-*projection* if

$$\|x\| = \|Px\| + \|x - Px\|$$

for all $x \in X$.

(ii) A subspace $M \subseteq X$ is called an M-summand if it is the range of an M-projection. A closed subspace $M \subseteq X$ is called an L-summand if it is the range of an L-projection.

(iii) A subspace $M \subseteq X$ is called an M-ideal if $M^\perp$ is the kernel of an L-projection in X^*

We recall from [6, Chapter I] that when $M \subset X$ is an M-ideal, elements of M^* have unique norm-preserving extension to X^* and one has the identification, $X^* = M^* \oplus_1 M^\perp$. Several examples from among function spaces and spaces of operators that satisfy these geometric properties can be found in the monograph [6], see also [8]. First, we prove that for an L-summand $M \subset X$, if a SCS point of B_X has a non-zero component $m \in M$, then m is a SCS point of B_M. For an M- ideal $M \subset X$, this yields: any w^*-SCS point of B_{X^*}, if its restriction to M, say m^*, has the same norm, then m^* it is a w^*-SCS point of B_{M^*} . We prove a similar result for a U-subspace of a Banach space of X. We prove a converse statement for a strict ideal $Y \subset X$ (see Section 2 for the definition) i.e., we prove that a w^*-SCS point of a strict ideal continues to be so in the bigger space. We also prove corresponding results for the BSCSP.

2. Stability results

We will use the standard notation of $\oplus_1$, $\oplus_\infty$ to denote the ℓ^1 and ℓ^∞-direct sum of two or more Banach spaces.

PROPOSITION 5. *Suppose X, Y, Z are Banach spaces such that $Z = X \oplus_1 Y$; suppose $z_0 = (x_0, y_0) \in B_Z$ is a SCS point of B_Z with both the components non-zero, then x_0 and y_0 are SCS points of B_X and B_Y respectively.*

Proof. Since z_0 is a SCS point of B_Z, we have for any $\varepsilon > 0$, $z_0 = \sum_{i=1}^n \lambda_i z_i$, where $z_i \in S_i$ and for $z_i^* = (x_i^*, y_i^*)$ with $1 = \|z_i^*\| = \max\{\|x_i^*\|, \|y_i^*\|\}$, $S_i = \{z \in B_Z / z_i^*(z) > 1 - \varepsilon_i\}$ and $\operatorname{diam}(\sum_{i=1}^n \lambda_i S_i) < \varepsilon$,

$$S_i = \{z \in B_Z / z_i^*(x,y) > 1 - \varepsilon_i\} = \{z \in B_Z / x_i^*(x) + y_i^*(y) > 1 - \varepsilon_i\}.$$

Since $z_i = (x_i, y_i) \in S_i$, then $x_i^*(x_i) + y_i^*(y_i) > 1 - \varepsilon_i$.

Case 1 : $\|z_i^*\| = \|x_i^*\| = 1$. Then,

$$\begin{aligned} x_i^*(x_i) + y_i^*(y_i) > 1 - \varepsilon_i = \|x_i^*\| - \varepsilon_i, & \\ \implies x_i^*(x_i) > \|x_i^*\| - \varepsilon_i - y_i^*(y_i), & \\ \implies 1 \geq x_i^*(x_i) > \|x_i^*\| - \beta_i, \text{ where } \beta_i = \varepsilon_i + y_i^*(y_i), & \\ \implies \varepsilon_i + y_i^*(y_i) > 0. & \end{aligned}$$

So we have, $x_i \in S_{iX} = \{x \in B_X / x_i^*(x) > 1 - \beta_i\}$. Then $(x_i, y_i) \in S_{iX} \times \{y_i\} \subseteq S_i$.

Case 2: $\|z_i^*\| = \|y_i^*\| = 1$. We may assume that $0 < \|x_i^*\| < 1$, and let $\delta_i = \|y_i^*\| - \|x_i^*\|$. Then,

$$\begin{aligned} x_i^*(x_i) + y_i^*(y_i) > 1 - \varepsilon_i = \|y_i^*\| - \varepsilon_i = \|x_i^*\| + \delta_i - \varepsilon_i & \\ \implies x_i^*(x_i) > \|x_i^*\| + \delta_i - \varepsilon_i - y_i^*(y_i), & \\ \implies \|x_i^*\| \geq x_i^*(x_i) > \|x_i^*\| - r_i, \text{ where } r_i = \delta_i - \varepsilon_i - y_i^*(y_i) > 0, & \\ \implies x_i \in S_{iX} = \{x \in B_X / x_i^*(x) > 1 - r_i\}. & \end{aligned}$$

Then $(x_i, y_i) \in S_{iX} \times \{y_i\} \subseteq S_i$.

Let $x_0 = \sum_{i=1}^n \lambda_i x_i$ and $y_0 = \sum_{i=1}^n \lambda_i y_i$. Now $x_0 \in \sum_{i=1}^n \lambda_i S_{iX}$. Also,

$$\begin{aligned} \sum_{i=1}^n \lambda_i [S_{iX} \times y_i] \subseteq \sum_{i=1}^n \lambda_i S_i, & \\ \implies \sum_{i=1}^n \lambda_i [S_{iX}] \times \{y_0\} \subseteq \sum_{i=1}^n \lambda_i [S_{iX} \times y_i] \subseteq \sum_{i=1}^n \lambda_i S_i, & \\ \implies \operatorname{diam}\left(\sum_{i=1}^n \lambda_i S_{iX} \right) < \varepsilon, & \\ \implies x_0 \text{ is a SCS point of } B_X. & \end{aligned}$$

Similarly it follows that y_0 is a SCS point of B_Y. ∎

Arguments similar to the ones given above in the context of a ℓ^∞-sum yield the following corollary.

Corollary 6. *Suppose X, Y, Z are Banach spaces such that $Z = X \oplus_\infty Y$, suppose $z^* = (x^*, y^*) \in B_{Z^*}$ is a w^*-SCS point of B_{Z^*} with both the components non-zero, then x^* and y^* are w^*-SCS points of B_{X^*} and B_{Y^*} respectively.*

Remark 7. Since in the sequence space ℓ^∞ any weakly open set has norm diameter 2, by taking $X = c_0$ and $Y = \ell^1$, $Z = X \oplus_\infty Y$, any w^*-SCS point of B_{Z^*} has its second component 0. We thank the referee for this observation.

Definition 8. We recall that a closed subspace Y of a Banach space X is called a U-subspace if for $y^* \in Y^*$ there exists a unique norm preserving extension of y^* in X^*. We continue to denote the unique extension also by y^*.

See the discussion on [6, page 44] and the references in that monograph for several examples of U-subspaces from among classical function spaces and spaces of operators.

Before the next result we also need a definition from [5]. See also [11] for more information and several examples from spaces of operators and tensor product spaces.

Definition 9. A closed subspace Y of a Banach Space X is said to be an ideal of X if there is a linear projection $P : X^* \to X^*$ of norm one such that $\ker(P) = Y^\perp$.

For $x^* \in X^*$ since $P(x^*) - x^* = 0$ on Y, as $\|P\| = 1$, we see that $P(x^*)$ is a norm-preserving extension of $x^*|Y$.

Theorem 10. *Suppose Y is an ideal which is also a U-subspace of X. If $y^* \in S_{Y^*}$ is a w^*-SCS point of B_{X^*}, then y^* is a w^*-SCS point of B_{Y^*}.*

Proof. Let $y_0^* \in S_{Y^*}$ be a w^*-SCS point of B_{X^*} , hence for any $\varepsilon > 0$ there exist w^* slices S_i, $0 \le \lambda_i \le 1$, $i = 1, 2, \ldots, n$, $S_i = \{x^* \in B_{X^*}/x^*(x_i) > 1 - \alpha_i\}$ and $\operatorname{diam}(\sum_{i=1}^n \lambda_i S_i) < \varepsilon$ and $y_0^* = \sum \lambda_i x_{0i}^*$. Since $y_0^* \in S_{Y^*}$ and Y is a U-subspace, y_0^* has unique norm preserving extension in X^*. Let $P : X^* \longrightarrow X^*$ be the canonical projection. Then $\|P(y_0^*)\| = \|y_0^*\| = 1$, Also,

$$1 = \|y_0^*\| = \left\| \sum_{i=1}^n \lambda_i x_{0i}^* \right\| \le \sum_{i=1}^n \lambda_i \|P(x_{0i}^*)\| \le 1.$$

This implies $\|P(x_{0i}^*)\| = \|x_{0i}{}^*\| = 1$ for all $i = 1, \dots, n$. Thus by hypothesis, $P(x_{0i}^*)$ and the restriction of x_{0i}^* to Y are denoted by y_{0i}^*. Now $y_{0i}^* \in S_i$, then $y_{0i}^*(x_i) > 1 - \alpha_i$. Also, since Y is an ideal, there exists an operator $T : \text{span}\{x_i\} \longrightarrow Y$ such that $\|T(x_i)\| \leq (1+\varepsilon)\|x_i\| = 1 + \varepsilon$.

Let $y_i = T(x_i)$. Hence,

$$\begin{aligned} y_{0i}^*(x_i) > 1 - \alpha_i &\implies y_{0i}^*(y_i - y_i + x_i) > 1 - \alpha_i, \\ &\implies y_{0i}^*(y_i) + y_{0i}^*(x_i - y_i) > 1 - \alpha_i, \\ &\implies y_{0i}^*(y_i) > 1 - \alpha_i - y_{0i}^*(x_i - y_i). \end{aligned}$$

Case 1: $\|y_i\|$ =1. So we have

$$\begin{aligned} 1 > y_{0i}^*(y_i) > 1 - \alpha_i - & y_{0i}^*(x_i - y_i) = 1 - \beta_i, \\ &\implies y_{0i}^* \in S_{iY} = \{y^* \in B_{Y^*} / y^*(y_i) > 1 - \beta_i\}. \end{aligned}$$

Case 2: $\|y_i\| < 1$. Let $\|y_i\| = 1 - \delta_i$. Then

$$\begin{aligned} \|y_i\| > y_{0i}^*(y_i) > \|y_i\| + \delta_i - \beta_i = \|y_i\| - (\beta_i - \delta_i) = \|y_i\| - \gamma_i, \gamma_i > 0, \\ \implies y_{0i}^* \in S_{iY} = \{y^* \in B_{Y^*} / y^*(y_i) > \|y_i\| - \gamma_i\}. \end{aligned}$$

Case 3: $\|y_i\| = 1 + \delta_i$. Then

$$\begin{aligned} 1 + \delta_i > y_{0i}^*(y_i) > 1 - \beta_i = 1 + \delta_i - (\beta_i + \delta_i), \\ \implies y_{0i}^* \in S_{iY} = \{y^* \in B_{Y^*} / y^*(y_i) > \|y_i\| - (\beta_i + \delta_i\}. \end{aligned}$$

Hence

$$y_0^* = \sum_{i=1}^n \lambda_i y_{0i}^* \in \sum_{i=1}^n \lambda_i S_{iY} \subseteq \sum_{i=1}^n \lambda_i S_i.$$

Hence

$$\text{diam}\left(\sum_{i=1}^n \lambda_i S_{iY}\right) < \text{diam}\left(\sum_{i=1}^n \lambda_i S_i\right) < \varepsilon.$$

Thus y_0^* is w^*-SCS point of B_{Y^*}. ∎

Let $M \subseteq X$ be an M-ideal. It follows from the results in [6, Chapter I] that any $x^* \in X^*$, if $\|m^*\| = \|x^*|_M\| = \|x^*\|$, then x^* is the unique norm preserving extension of m^*. For notational convenience we denote both the functionals by m^*. Clearly any M-ideal is also an ideal. Thus the following corollary answers a natural question in this context for w^*-SCS points of the unit sphere. We omit its easy proof.

COROLLARY 11. *Suppose $M \subseteq X$ is a M-ideal in X. If $m^* \in S_{X^*}$ is w^*-SCS point of B_{X^*}, then $m^* \in S_{M^*}$ is a w^*-SCS point of B_{M^*}.*

Remark 12. The referee has kindly pointed out an independent proof to show that for $Z = X \oplus_1 Y$, Z has the BSCSP if and only if X or Y has the BSCSP.

Arguments similar to the ones given during the proof of Proposition 5 can be used to show that for $Z = X \oplus_\infty Y$, if X^* or Y^* has the w^*-BSCSP then so does Z^*.

In the case of an M-ideal $M \subset X$, for the sake of completeness we give a detailed proof of the following result.

PROPOSITION 13. *Let $M \subseteq X$ be a M-ideal, then if M^* has the w^*-BSCSP then X^* has the w^*-BSCSP.*

Proof. Suppose M^* has the w^*-BSCSP, then for any $\varepsilon > 0$ there exists slices S_{iM} and $0 \leq \lambda_i \leq 1, i = 1, 2, \ldots, n$, $S_{iM} = \{m^* \in B_{M^*}/m^*(m_i) > 1 - \alpha_i\}$ and $\operatorname{diam}(\sum_{i=1}^n \lambda_i S_{iM}) < \varepsilon$. Since M is an M- ideal, for any $x^* \in X^*$ we have the unique decomposition, $x^* = m^* + m^\perp$, where $m^* \in M^*$ and $m^\perp \in M^\perp$. Suppose we have $0 < \mu_i < \alpha_i$. Then

$$\begin{aligned} S_{iX} &= \{x^* \in B_{X^*}/x^*(m_i) > 1 - \mu_i\} \\ &= \{x^* \in B_{X^*}/m^*(m_i) + m^\perp(m_i) > 1 - \mu_i\}, \\ &\subseteq S_{iM} \times \mu_i B_{M^\perp}, \\ &\implies \sum_{i=1}^n \lambda_i S_{iX} \subseteq \sum_{i=1}^n \lambda_i S_{iM} \times \mu_i B_{M^\perp}. \end{aligned}$$

Choose $\beta_i = \min(\mu_i, \varepsilon)$. Then

$$\begin{aligned} S'_{iX} &= \{x^* \in B_{X^*}/x^*(m_i) > 1 - \beta_i\} \subseteq S_{iX} \times \beta_i B_{M^\perp}, \\ &\implies \sum_{i=1}^n \lambda_i S'_{iX} \subseteq \left(\sum_{i=1}^n \lambda_i S_{iM} \times \beta_i B_{M^\perp} \right) \\ &\implies \sum_{i=1}^n \lambda_i S'_{iX} \subseteq \left(\sum_{i=1}^n \lambda_i S_{iM} \times \beta_i B_{M^\perp} \right). \end{aligned}$$

Thus $\operatorname{diam}(\sum_{i=1}^n \lambda_i S'_{iX}) \leq \operatorname{diam}(\sum_{i=1}^n \lambda_i S_{iM}) + 2\varepsilon < \varepsilon + 2\varepsilon = 3\varepsilon$. Also, since $\|m_i\| = 1$, there exists $m_i^* \in B_{M^*}$ such that $m_i^*(m_i) > 1 - \beta_i$. Hence $m_i^* \in S'_{iX}$. Similarly, $\sum_{i=1}^n \lambda_i m_i^* \in \sum_{i=1}^n \lambda_i S'_{iX} \implies \sum_{i=1}^n \lambda_i S'_i \neq \emptyset$. ∎

Since any summand in a ℓ^∞-direct sum is in particular an M-ideal of the sum, the following corollary is easy to prove.

Corollary 14. *Suppose $X = \oplus_{\ell^\infty} X_i$. If X_i^* has the w^*-BSCSP for some i, then X^* has the w^*-BSCSP.*

The above arguments extend easily to vector-valued continuous functions. We recall that for a compact Hausdorff space K, $C(K, X)$ denotes the space of continuous X-valued functions on K, equipped with the supremum norm. We recall from [9] that dispersed compact Hausdorff spaces have isolated points.

Corollary 15. *Suppose K is a compact Hausdorff space with an isolated point. If X^* has the w^*-BSCSP, then $C(K, X)^*$ has the w^*-BSCSP.*

Proof. Suppose X^* has the w^*-BSCSP. For an isolated point $k_0 \in K$, the map $F \to \chi_{k_0} F$ is an M-projection in $C(K, X)$ whose range is isometric to X. Hence we see that $C(K, X)^*$ has the w^*-BSCSP. ∎

We recall that an ideal Y is said to be a strict ideal if for a projection $P : X^* \to X^*$ with $\|P\| = 1$, $\ker(P) = Y^\perp$, one also has $B_{P(X^*)}$ is w^*-dense in B_{X^*} or in other words $B_{P(X^*)}$ is a norming set for X.

In the case of an ideal also one has that Y^* embeds (though there may not be uniqueness of norm-preserving extensions) as $P(X^*)$. Thus we continue to write $X^* = Y^* \oplus Y^\perp$. In what follows we use a result from [11], that identifies strict ideals as those for which $Y \subset X \subset Y^{**}$ under the canonical embedding of Y in Y^{**}.

Proposition 16. *Suppose Y is a strict ideal of X. If $y^* \in B_{Y^*}$ is a w^*-SCS point of B_{Y^*}, then y^* is a w^*-SCS point of B_{X^*}.*

Proof. Since $y^* \in B_{Y^*}$ is a w^*-SCS point of B_{Y^*}, for any $\varepsilon > 0$ there exists w^* slices S_i and $0 \le \lambda_i \le 1$, $i = 1, 2, \ldots, n$, $S_i = \{y^* \in B_{Y^*} / y^*(y_i) > 1 - \alpha_i\}$ and $\operatorname{diam}(\sum_{i=1}^n \lambda_i S_i) < \varepsilon$. Since Y is a strict ideal in X, we have $B_{X^*} = \overline{B_{Y^*}}^{w^*}$, hence we have the following:

$$S_i' = \{x^* \in B_{X^*} / x^*(x_i) > 1 - \alpha_i\} = \{x^* \in \overline{B_{Y^*}}^{w^*} / x^*(x_i) > 1 - \alpha_i\},$$

$$\implies \operatorname{diam}\left(\sum_{i=1}^n \lambda_i S_i'\right) \subseteq \operatorname{diam}\left(\sum_{i=1}^n \lambda_i S_i\right) < \varepsilon,$$

$$\implies \operatorname{diam}\left(\sum_{i=1}^n \lambda_i S_i'\right) < \varepsilon.$$

Hence y^* is a w^*-SCS point of B_{Y^*}. ∎

Arguing similarly it follows that:

PROPOSITION 17. *Suppose Y is a strict ideal of X. If Y^* has w^*-BSCSP then X^* has w^*-BSCSP.*

Remark 18. A prime example of a strict ideal is a Banach space X under its canonical embedding in X^{**}. It is known that any w^*-denting point of $B_{X^{**}}$ is a point of X. Now let $x^{**} \in B_{X^{**}}$ be a w^*-SCS point. The referee has kindly pointed out that since B_X is weak* dense in $B_{X^{**}}$, for any $\epsilon > 0$, there is a convex combination $\sum_{i=1}^{n} \lambda_i x_i$ of vectors in X so that $\|x^{**} - \sum_{i=1}^{n} \lambda_i x_i\| \leq \epsilon$. Hence $x^{**} \in X$.

We conclude the paper with a set of remarks and questions. See also the recent paper [1] for other possible geometric connections. Let us consider the following densities of w^*-SCS points of B_{X^*}.

(i) All points of S_{X^*} are w^*-SCS points of B_{X^*}.

(ii) The w^*-SCS points of B_{X^*} are dense in S_{X^*}.

(iii) B_{X^*} is contained in the closure of w^*-SCS points of B_{X^*}.

(iv) B_{X^*} is the closed convex hull of w^*-SCS points of B_{X^*}.

(v) X^* is the closed linear span of w^*-SCS points of B_{X^*}.

Questions:

(i) How can each of these properties be realized as a ball separation property considered in [3]?

(ii) What stability results will hold for these properties?

ACKNOWLEDGEMENTS

This work was done when the first author was visiting Indian Statistical Institute Bangalore Center. She would like to express her deep gratitude to Professor T. S. S. R. K. Rao and everyone at ISI Bangalore Center, for the warm hospitality provided during her stay. The second author currently is a Fulbright-Nehru Academic and Professional Excellence Fellow, at the Department of Mathematical Sciences, University of Memphis. The authors would also like to thank Professor Pradipta Bandyopadhyay for some discussions they had with him. The authors thank the referee for pointing out inaccuracies in earlier versions and other comments for reorganizing and improving the presentation.

References

[1] M. D. Acosta, A. Kamińska, M. Mastyło, The Daugavet property in rearrangement invariant spaces, *Trans. Amer. Math. Soc.* **367** (6) (2015), 4061 – 4078.

[2] R. D. Bourgin, "Geometric Aspects of Convex Sets with the Radon-Nikodym Property", Lecture Notes in Mathematics 993, Springer-Verlag, Berlin, 1983.

[3] D. Chen, B. L. Lin, Ball topology on Banach spaces, *Houston J. Math.* **22** (2) (1996), 821 – 833.

[4] N. Ghoussoub, G. Godefroy, B. Maurey, W. Schachermayer, "Some Topological and Geometrical Structures in Banach Spaces", *Mem. Amer. Math. Soc.* **70** (378), 1987.

[5] G. Godefroy, N. J. Kalton, P. D. Saphar, Unconditional ideals in Banach spaces, *Studia Math.* **104** (1) (1993), 13 – 59.

[6] P. Harmand, D. Werner, W. Werner, "M-Ideals in Banach Spaces and Banach Algebras", Lecture Notes in Mathematics 1547, Springer-Verlag, Berlin, 1993.

[7] Z. Hu, B. L. Lin, RNP and CPCP in Lebesgue-Bochner function spaces, *Illinois J. Math.* **37** (2) (1993), 329 – 347.

[8] Ü. Kahre, L. Kirikal, E. Oja, On M-ideals of compact operators in Lorentz sequence spaces, *J. Math. Anal. Appl.* **259** (2) (2001), 439 – 452.

[9] H. E. Lacey, "The Isometric Theory of Classical Banach Spaces", Die Grundlehren der mathematischen Wissenschaften 208, Springer-Verlag, New York-Heidelberg, 1974.

[10] B. L. Lin, P. K. Lin, S. L. Troyanski, Characterization of denting points, *Proc. Amer. Math. Soc.* **102** (3) (1988), 526 – 528.

[11] T. S. S. R. K. Rao, On ideals in Banach spaces, *Rocky Mountain J. Math.* **31** (2) (2001), 595 – 609.

[12] H. P. Rosenthal, On the structure of non-dentable closed bounded convex sets, *Adv. in Math.* **70** (1) (1988), 1 – 58.

[13] W. Schachermayer, The Radon Nikodym property and the Krein-Milman property are equivalent for strongly regular sets, *Trans. Amer. Math. Soc.* **303** (2) (1987), 673 – 687.