

Linear Extensions, Almost Isometries, and Diameter Two

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Abstract: Heinrich, Mankiewicz, Sims, and Yost proved that every separable subspace of a Banach space Y is contained in a separable ideal in Y . We improve this result by replacing the term “ideal” with the term “almost isometric ideal”. As a consequence of this we obtain, in terms of subspaces, characterizations of diameter 2 properties, the Daugavet property along with the properties of being an almost square space and an octahedral space.

Key words: Almost isometric ideal; Diameter 2 property; Daugavet property; Octahedral space; Almost square space.

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1. INTRODUCTION

Let Y be a Banach space and X a subspace of Y . Recall that X is an *ideal* in Y if X^\perp , the annihilator of X , is the kernel of a contractive projection on the dual Y^* of Y . A linear operator φ from X^* to Y^* is called a *Hahn-Banach extension operator* if $\varphi(x^*)(x) = x^*(x)$ and $\|\varphi(x^*)\| = \|x^*\|$ for all $x \in X$ and $x^* \in X^*$. We denote by $\mathbf{HB}(X, Y)$ the set of all Hahn-Banach extension operators from X^* to Y^* . We say that X is locally 1-complemented in Y if for every $\varepsilon > 0$ and every finite dimensional subspace E of Y there exists a linear operator $T : E \rightarrow X$ such that $Te = e$ for all $e \in E \cap X$ and $\|T\| \leq 1 + \varepsilon$. The fact that a Banach space is locally 1-complemented in its bidual is commonly referred to as *the Principle of Local Reflexivity (PLR)*.

The following theorem is a collection of known results.

THEOREM 1.1. *Let X be a subspace of a Banach space Y . The following statements are equivalent.*

- (a) X is an ideal in Y .
- (b) There exists $\varphi \in \mathbf{HB}(X, Y)$.

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- (c) Y is locally 1-complemented in X .
- (d) There exists $\varphi \in \mathbf{IB}(X, Y)$ such that for every $\varepsilon > 0$, every finite dimensional subspace E of Y and every finite dimensional subspace F of X^* there exists a linear operator $T : E \rightarrow X$ such that
- (d1) $Te = e$ for all $e \in E \cap X$,
- (d2) $\|Te\| \leq (1 + \varepsilon)\|e\|$ for every $e \in E$, and
- (d3) $\varphi f(e) = f(Te)$ for every $e \in E, f \in F$.

Equivalence of (a), (b), and (c) was independently discovered by Fakhoury [10] and Kalton [21]. Later Oja and Pöldvere [28] showed that these in turn are equivalent to statement (d).

The following result appears for the first time in [31, Proposition 2] by Sims and Yost. Earlier Heinrich and Mankiewicz had proved in [16, Proposition 3.4] a version of Theorem 1.2 for arbitrary subspaces X of Y (but which did not say anything about subspaces in the dual) using some of the deeper results in Model Theory. Sims and Yost's proof, however, does not use Model Theory, but rests instead upon a finite dimensional lemma and a compactness argument due to Lindenstrauss [25]. Using the latter results Sims and Yost gave in [32, Theorem] also a proof of [16, Proposition 3.4].

THEOREM 1.2. *Let Y be a Banach space, X a separable subspace of Y , and W a separable subspace of Y^* . Then there exists a separable subspace Z of Y containing X and $\varphi \in \mathbf{IB}(Z, Y)$ such that $\varphi(Z^*) \supset W$.*

In the language of ideals Theorem 1.2 says that every separable subspace of a Banach space Y is contained in a separable ideal in Y . Thus every non-separable Banach space contains an infinite number of ideals. Looked upon in this way ideals seems to occur quite frequently.

The following stronger form of an ideal was introduced and studied in [5].

DEFINITION 1.3. A subspace X of a Banach space Y is said to be an *almost isometric ideal (ai-ideal)* if for every $\varepsilon > 0$ and every finite dimensional subspace E of Y , there exists a linear operator $T : E \rightarrow X$ which satisfies (d1) in Theorem 1.1 as well as

$$(d2') \quad (1 - \varepsilon)\|e\| \leq \|Te\| \leq (1 + \varepsilon)\|e\| \text{ for } e \in E.$$

In [5] the following was shown.

THEOREM 1.4. *Let X be a subspace of a Banach space Y . The following statements are equivalent.*

- (a) X is an ai-ideal in Y .
- (b) *There exists $\varphi \in \mathbf{HB}(X, Y)$ such that for every $\varepsilon > 0$, every finite dimensional subspace E of Y , and every finite dimensional subspace F of X^* there exists a linear operator $T : E \rightarrow X$ which satisfies (d1) and (d3) in Theorem 1.1 and (d2') in Definition 1.3.*

The φ in Theorem 1.4 is called an *almost isometric Hahn-Banach extension operator* associated with the ai-ideal X in Y . We denote by $\mathbf{HB}_{ai}(X, Y)$ the set of such operators.

The main result of this paper is an improvement of Theorem 1.2 in which the Hahn-Banach extension operator is replaced by an almost isometric one.

THEOREM 1.5. (Main Theorem) *Let Y be a Banach space, X a separable subspace of Y , and W a separable subspace of Y^* . Then there exists a separable subspace Z of Y containing X and $\varphi \in \mathbf{HB}_{ai}(Z, Y)$ such that $\varphi(Z^*) \supset W$.*

So every separable subspace of a Banach space is contained in a separable ai-ideal. Thus ai-ideals seems to occur just as frequently as ideals. Nevertheless, being an ai-ideal is strictly stronger than being an ideal. This can e.g. be seen from Theorem 1.6 below and the two paragraphs that follow. Theorem 1.6 is a combination of [10, Proposition 3.4] and [5, Theorem 4.3].

THEOREM 1.6. *For a Banach space X the following statements are equivalent:*

- (i) X is a Lindenstrauss (resp. Gurarii) space.
- (ii) X is an ideal (resp. ai-ideal) in every superspace.

Recall that a *Lindenstrauss space* is a Banach space with a dual isometric to $L_1(\mu)$ for some positive measure μ . A Banach space X is called a *Gurarii space* if it has the property that whenever $\varepsilon > 0$, E is a finite-dimensional Banach space, $T_E : E \rightarrow X$ is isometric and F is a finite-dimensional Banach space with $E \subset F$, then there exists a linear operator $T_F : F \rightarrow X$ such that

- (i) $T_F(f) = T_E(f)$ for all $f \in E$, and
- (ii) $(1 - \varepsilon)\|f\| \leq \|T_F f\| \leq (1 + \varepsilon)\|f\|$ for all $f \in F$.

The class of Gurariĭ spaces is a subclass of the class of Lindenstrauss spaces [22]. In [15] Gurariĭ showed that this subclass is non-empty as he constructed the first separable Gurariĭ space. Later Lusky [27] proved that all separable Gurariĭ spaces are in fact linearly isometric. Also non-separable Gurariĭ spaces exist (see e.g. [11]). But, no Gurariĭ space is a dual space as e.g. the unit ball of such a space contains no extreme points [3, Proposition 3.3]. The bidual of a Lindenstrauss space is, however, again a Lindenstrauss space [24]. Thus we see that the classes of separable and non-separable Gurariĭ spaces are non-empty proper subclasses of respectively the classes of separable and non-separable Lindenstrauss spaces.

Let us now relate the notion of an ai-ideal to the well established notion of a strict ideal (see e.g. [13], [23], [29], and [1]). We say that X is a *strict ideal* in Y if X is an ideal in Y with an associated $\varphi \in \mathbf{HB}(X, Y)$ whose range is 1-norming for Y , i.e. for every $y \in Y$ we have $\|y\| = \sup\{y^*(y) : y^* \in \varphi(X^*) \cap S_{Y^*}\}$ where S_{Y^*} is the unit sphere of Y^* . Let $\mathbf{HB}_s(X, Y) = \{\varphi \in \mathbf{HB}(X, Y) : \varphi \text{ is strict}\}$. Using the PLR it is straightforward to show that every strict ideal is an ai-ideal. However, the converse is not true (see e.g. [5, Example 1] and [3, Remark 3.2]). We can sum up the last paragraphs by

$$\mathbf{HB}(X, Y) \supset \mathbf{HB}_{ai}(X, Y) \supset \mathbf{HB}_s(X, Y),$$

where the containment may be proper.

The paper is organized as follows: In Section 2 we give a proof of Theorem 1.5. In Section 3 we use this theorem to obtain characterizations of diameter 2 properties, the Daugavet property as well as the properties of being an almost square space and an octahedral space.

We will consider real Banach spaces only (though many of the results are true in the complex case as well). The notation used is mostly standard and is, if considered necessary, explained as the text proceeds.

2. THE MAIN THEOREM

The proof of Theorem 1.5 depends on Lemma 2.1 below. This lemma is a strengthening of [31, Lemma 1]. (A detailed proof of this lemma along with a proof of [31, Proposition 2] are given in [19, III.Lemma 4.2 and III.Lemma 4.3]. For this reason, it will be referred to [19, III.Lemma 4.2 and III.Lemma 4.3] rather than [31, Lemma 1 and Proposition 2] below.) The roots of [19, III.Lemma 4.2] goes back to [24]. Lemma 2.1 differs from [19, III. Lemma 4.2] simply by the fact that the partial conclusion

ii) $\|Tx\| \leq (1 + \varepsilon)\|x\|$ for every $x \in E$,

in [19, III.Lemma 4.2] is replaced by the stronger partial conclusion ii') in Lemma 2.1. The proof of Lemma 2.1 is interestingly enough already contained in the proof of [19, III.Lemma 4.2]. This is perhaps not so easy to spot at first glance. So to make this clearer, we present a complete proof here.

LEMMA 2.1. *Let Y be a Banach space, B a finite dimensional subspace of Y , $k \in \mathbb{N}$, $\varepsilon > 0$, and C a finite subset in Y^* . Then there is a finite dimensional subspace Z containing B such that for every subspace E of Y containing B and satisfying $\dim E/B \leq k$ one can find a linear operator $T : E \rightarrow Z$ such that*

- i) $Ty = y$ for every $y \in B$,
- ii') $(1 - \varepsilon)\|y\| \leq \|Ty\| \leq (1 + \varepsilon)\|y\|$ for every $y \in E$,
- iii) $|f(Ty) - f(y)| \leq \varepsilon\|y\|$ for every $y \in E$ and $f \in C$.

Proof. Choose $\delta > 0$ such that $\delta < \varepsilon$ and $(1 + \delta)^{-1} > 1 - \varepsilon$. Let $C = \{f_1, \dots, f_m\} \subset Y^*$ and P a projection on Y onto B . Put $U = \ker P$, the kernel of P . Then we can write $Y = B \oplus U$. Choose M so large that

$$M > \frac{5k\|I - P\|}{\delta} \quad \text{and} \quad \frac{M + 1}{M - 1} < 1 + \delta.$$

Let $(b_\rho)_{\rho \leq r}$ and $(\lambda_\sigma)_{\sigma \leq s}$ be finite $1/M$ -nets for $\{b \in B : \|b\| \leq M\}$ and $S_{\ell_1(k)}$ respectively ($\ell_1(k)$ denotes the k -dimensional ℓ_1 space). Let B_U be the unit ball of U and define $\phi : (B_U)^k \rightarrow \mathbb{R}^{rs} \times \mathbb{R}^{mk} = \mathbb{R}^{rs+mk}$, by

$$\phi(u_1, \dots, u_k) = \left(\left(\|b_\rho + \sum_{\kappa=1}^k \lambda_{\kappa, \sigma} u_\kappa\| \right)_{\rho \leq r, \sigma \leq s}, (f_\mu(u_\kappa))_{\mu \leq m, \kappa \leq k} \right).$$

Since $\phi(B_U)^k$ is totally bounded, we can find $(u_\nu)_{\nu \leq n} \subset (B_U)^k$ such that $(\phi u_\nu)_{\nu \leq n}$ is a finite $1/M$ -net for $\phi(B_U)^k$ where we may take any norm on \mathbb{R}^{rs+mk} for which the coefficient functionals have norm ≤ 1 . Put

$$Z = B \oplus \text{span}\{u_{\kappa, \nu} : \kappa \leq k, \nu \leq n\}.$$

Now, given a subspace $E \supset B$ with $\dim E/B = k$, there are $u_1, \dots, u_k \in U$ such that $E = B \oplus \text{span}\{u_\kappa : \kappa \leq k\}$. By Auerbach's lemma we can choose $\mathbf{u} = (u_1, \dots, u_k)$ such that

$$\|u_\kappa\| = 1, 1 \leq \kappa \leq k \quad \text{and} \quad \left\| \sum_{\kappa=1}^k \lambda_\kappa u_\kappa \right\| \geq \frac{1}{k} \sum_{\kappa=1}^k |\lambda_\kappa| \quad \text{for all } (\lambda_\kappa) \in \mathbb{R}^k. \quad (1)$$

Indeed, find $u_\kappa^* \in S_{\text{span}\{u_\kappa:1,\dots,k\}^*}$ with $u_\kappa^*(u_j) = 0$ if $\kappa \neq j$ and $\text{sign}(\lambda_\kappa)$ otherwise. Then the norm of $u^* = 1/k \sum_{\kappa=1}^k u_\kappa^*$ is ≤ 1 and we have

$$\left\| \sum_{\kappa=1}^k \lambda_\kappa u_\kappa \right\| \geq u^* \left(\sum_{\kappa=1}^k \lambda_\kappa u_\kappa \right) = \frac{1}{k} \sum_{\kappa=1}^k |\lambda_\kappa|.$$

This means that there is $\nu \leq n$ such that

$$\|\phi u - \phi u_\nu\| < \frac{1}{M},$$

i.e. with $\lambda u = \sum_{\kappa=1}^k \lambda_\kappa u_\kappa$ where $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ we have

$$\left| \|b_\rho + \lambda_\sigma u\| - \|b_\rho + \lambda_\sigma u_\nu\| \right| < \frac{1}{M} \quad \text{for all } \rho \leq r, \sigma \leq s, \quad (2)$$

and

$$\left| f_\mu(u_\kappa) - f_\mu(u_{\kappa,\nu}) \right| < \frac{1}{M} \quad \text{for all } \mu \leq m, \kappa \leq k. \quad (3)$$

Now, define $T : E \rightarrow Z$ by

$$T(b + \lambda u) = b + \lambda u_\nu.$$

Clearly T is the identity on B . To show $(1 - \varepsilon)\|y\| \leq \|Ty\| \leq (1 + \varepsilon)\|y\|$ for all $y \in E$, it suffices to prove

$$(1 - \varepsilon)\|b + \lambda u\| \leq \|b + \lambda u_\nu\| \leq (1 + \varepsilon)\|b + \lambda u\| \quad \text{for } \|\lambda\|_{\ell_1(k)} = 1. \quad (4)$$

To this end assume first $\|b\| \leq M$. Then $\|b - b_\rho\| < \frac{1}{M}$ for some ρ and $\|\lambda - \lambda_\sigma\| < \frac{1}{M}$ for some σ . Thus

$$\begin{aligned} \|b + \lambda u_\nu\| &\leq \|b_\rho + \lambda_\sigma u_\nu\| + \|b - b_\rho\| + \|\lambda u_\nu - \lambda_\sigma u_\nu\| < \|b_\rho + \lambda_\sigma u_\nu\| + \frac{2}{M} \\ &\stackrel{(2)}{<} \|b_\rho + \lambda_\sigma u\| + \frac{3}{M} \leq \|b + \lambda u\| + \|b_\rho - b\| + \|\lambda_\sigma u - \lambda u\| + \frac{3}{M} \\ &< \|b + \lambda u\| + \frac{5}{M}. \end{aligned}$$

Similarly we also get $\|b + \lambda u\| < \|b + \lambda u_\nu\| + \frac{5}{M}$, so we have

$$\|b + \lambda u\| - \frac{5}{M} < \|b + \lambda u_\nu\| < \|b + \lambda u\| + \frac{5}{M}.$$

Also

$$\begin{aligned} \|b + \lambda u\| &\geq \frac{1}{\|I - P\|} \|(I - P)(b + \lambda u)\| = \frac{1}{\|I - P\|} \left\| \sum_{\kappa=1}^k \lambda_{\kappa} u_{\kappa} \right\| \\ &\stackrel{(1)}{\geq} \frac{1}{k\|I - P\|} \sum_{\kappa=1}^k |\lambda_{\kappa}| = \frac{1}{k\|I - P\|} > \frac{5}{\delta M}, \end{aligned}$$

so $\varepsilon \|b + \lambda u\| \geq \frac{5}{M}$ and thus (4) holds for $\|b\| \leq M$.

For $\|b\| > M$ we have

$$\|b\| - 1 \leq \|b + \lambda u_{\nu}\| \leq \|b\| + 1 \quad \text{and} \quad \|b\| - 1 \leq \|b + \lambda u\| \leq \|b\| + 1,$$

so both

$$\frac{\|b + \lambda u\|}{\|b + \lambda u_{\nu}\|} \quad \text{and} \quad \frac{\|b + \lambda u_{\nu}\|}{\|b + \lambda u\|} \quad \text{are} \quad \leq \frac{\|b\| + 1}{\|b\| - 1} < \frac{M + 1}{M - 1} < 1 + \delta,$$

as $y \rightarrow \frac{y+1}{y-1}$ is a positive and decreasing function for $y > 1$. Thus (4) holds also for $\|b\| > M$.

Finally for any $y = b + \sum_{\kappa=1}^k \lambda_{\kappa} u_{\kappa} \in E$ we have

$$\begin{aligned} |f_{\mu}(y) - f_{\mu}(Ty)| &= \left| f_{\mu} \left(\sum_{\kappa=1}^k \lambda_{\kappa} (u_{\kappa} - u_{\kappa, \nu}) \right) \right| \leq \sum_{\kappa=1}^k |\lambda_{\kappa}| |f_{\mu}(u_{\kappa} - u_{\kappa, \nu})| \\ &\stackrel{(3)}{\leq} \frac{1}{M} \sum_{\kappa=1}^k |\lambda_{\kappa}| \stackrel{(1)}{\leq} \frac{k}{M} \left\| \sum_{\kappa=1}^k \lambda_{\kappa} u_{\kappa} \right\| \\ &\leq \frac{k\|I - P\|}{M} \|y\| < \frac{\varepsilon}{5} \|y\|. \end{aligned}$$

■

By the proof of [5, Theorem 1.4] the following holds.

LEMMA 2.2. *Let X be an ideal in Y and let $\varphi \in \mathbf{HB}(X, Y)$. Then the following statements are equivalent.*

- (a) $\varphi \in \mathbf{HB}_{ai}(X, Y)$.
- (b) For every $\delta, \varepsilon > 0$, for every finite dimensional subspace E of Y , and every finite dimensional subspace F of X^* there exists a linear operator $T : E \rightarrow X$ which satisfies

- (d1') $\|Te - e\| \leq \varepsilon\|e\|$ for every $e \in E \cap X$,
- (d2') in Definition 1.3, and
- (d3') $|\varphi f(e) - f(Te)| < \delta\|e\| \cdot \|f\|$ for every $e \in E, f \in F$.

Along with Lemma 2.1 we will use Lemma 2.2 to prove our Main Theorem.

Proof of Theorem 1.5. The first part of the proof is identical to that of [19, III.Lemma 4.3] except at the crucial point where we use Lemma 2.1 in place of [19, III.Lemma 4.2]. The reward for this is that we are able to prove the existence of a Hahn-Banach extension operator φ for which statement (b) in Lemma 2.2 holds.

Let (x_n) be a sequence dense in X and (f_n) a sequence dense in W . Starting with $M_1 = \{0\}$ we inductively define subspaces M_n as follows: Put $B_n = \text{span}(M_n, x_n)$, $C_n = \{f_1, \dots, f_n\}$, and let M_{n+1} be the subspace Z given by Lemma 2.1 when $B = B_n, k = n, \varepsilon = \frac{1}{n}$, and $C = C_n$. Without loss of generality assume $\dim M_{n+1}/B_n \geq n + 1$. Clearly $M = \overline{\cup M_n}$ is separable and contains X . For $n \in \mathbb{N}$ define

$$I_n = \{E \subset Y : B_n \subset E, \dim E/B_n \leq n\}$$

and put

$$I = \cup I_n.$$

Since

$$E \in I_n, F \in I_m \Rightarrow E + F + B_{\dim E + \dim F} \in I_{\dim E + \dim F}, \tag{5}$$

we have that I is a directed set. Moreover, it is clear that every finite dimensional subspace F of Y is contained in some $E \in I$. Just take $E = F + B_{\dim F}$. Then $E \in I_{\dim F}$.

Note that the condition $\dim M_{n+1}/B_n \geq n + 1$ implies $\dim B_{n+1}/B_n \geq n + 1$. This easily gives that for each $E \in I$ there is a unique $n \in \mathbb{N}$ such that $E \in I_n$. So by Lemma 2.1 there exists a linear operator $T_E : E \rightarrow M_{n+1} \subset M$ such that $T_E|_{B_n} = I_{B_n}$, $(1 - \frac{1}{n})\|y\| \leq \|T_E y\| \leq (1 + \frac{1}{n})\|y\|$, and $|f_i(T_E y) - f_i(y)| < \frac{1}{n}\|y\| \cdot \|f_i\|$ for every $y \in E$ and $1 \leq i \leq n$. Extend T_E (nonlinearly) to Y by setting $S_E(y) = T_E(y)$ if $y \in E$ and $S_E(y) = 0$ otherwise. Since $\|S_E(y)\| \leq 2\|y\|$ and regarding $S_E(y)$ as an element in M^{**} we can consider

$$(S_E)_{E \in I} \subset \prod_{y \in Y} B_{M^{**}}(0, 2\|y\|).$$

By Tychonoff's compactness theorem the net $(S_E)_{E \in I}$ has a convergent subnet $(S_{E_F})_{F \in J}$ in the product weak* topology with limit S say. This means that

for every finite number of points $(y_j)_{j \leq K} \subset Y$ we have $S_E y_j \rightarrow S y_j$ with respect to the weak* topology on M^{**} . Using this it is easy to see that the mapping $S : Y \rightarrow M^{**}$ is linear, of norm 1, and the identity on M . Now, if we define $\varphi : M^* \rightarrow Y^*$ by

$$\varphi m^*(y) = m^*(Sy),$$

it is straightforward to check that $\varphi \in \text{IB}(M, Y)$ with $\varphi(M^*) \supset W$.

Finally we check that condition (b) in Lemma 2.2 holds for φ . To this end let H be a finite dimensional subspace of Y and G a finite dimensional subspace of M^* . Let $(g_l)_{l=1}^m$ be a δ -net for S_G and choose n so big that S_{B_n} contains a δ -net $(h_p)_{p=1}^q$ for $S_{H \cap M}$ where $(h_p)_{p=1}^r$, $r \geq q$, is a δ -net for S_H . Then choose $H' \in J$ with $E_{H'} \supset \text{span}(H, B_n)$ such that the linear operator $T_{E_{H'}} : E_{H'} \rightarrow M$ satisfies $|\varphi g_l(h_p) - g_l(T_{E_{H'}} h_p)| < \delta$ for every $l \leq m, p \leq r$. Note that $T_{E_{H'}}|_{B_n} = I_{B_n}$ and $(1 - \frac{1}{N})\|h\| \leq \|T_{E_{H'}} h\| \leq (1 + \frac{1}{N})\|h\|$ for every $h \in E_{H'}$ where $N > n$ is the unique number such that $E_{H'} \in I_N$. Now, for $h \in S_{H \cap M}$ we can find $h_{p'} \in (h_p)_{p=1}^q$ such that $\|h - h_{p'}\| < \delta$. Thus we get

$$\begin{aligned} \|T_{E_{H'}} h - h\| &\leq \|T_{E_{H'}} h - T_{E_{H'}} h_{p'}\| + \|T_{E_{H'}} h_{p'} - h_{p'}\| + \|h_{p'} - h\| \\ &\leq (2 + 1/N)\delta. \end{aligned}$$

For $h \in S_H, g \in S_G$ find $h_{p''} \in (h_p)_{p=1}^r$ and $g_{l'} \in (g_l)_{l=1}^m$ with $\|h - h_{p''}\| < \delta$ and $\|g - g_{l'}\| < \delta$. We get

$$\begin{aligned} |\varphi g(h) - g(T_{E_{H'}} h)| &\leq |\varphi g(h) - \varphi g_{l'}(h)| + |\varphi g_{l'}(h) - \varphi g_{l'}(h_{p''})| \\ &\quad + |\varphi g_{l'}(h_{p''}) - g_{l'}(T_{E_{H'}} h_{p''})| \\ &\quad + |g_{l'}(T_{E_{H'}} h_{p''}) - g_{l'}(T_{E_{H'}} h)| \\ &\quad + |g_{l'}(T_{E_{H'}} h) - g(T_{E_{H'}} h)| \\ &\leq \|g - g_{l'}\| + \|h - h_{p''}\| + \delta \\ &\quad + \|T_{E_{H'}}\|(\|h_{p''} - h\| + \|g_{l'} - g\|) \\ &\leq \delta(5 + 2/N). \end{aligned}$$

Now, as δ can be chosen arbitrary small, the operator $T_{E_{H'}}$ restricted to H will do the work. ■

Remark 2.3. Note that Theorem 1.5 can be by transfinite induction, just as [19, III.Lemma 4.3], be extended to a non-separable version similar to [19, III.Lemma 4.4] but with an almost isometric Hahn-Banach extension operator

in place of a Hahn-Banach extension operator. The proof is like that of [19, III.Lemma 4.4], but uses Theorem 1.5 instead of [19, III.Lemma 4.3], and with a final part similar to the last part of the proof of Theorem 1.5.

As pointed out in Section 1 examples of ideals which are not ai-ideals are plentiful. Similarly examples of ai-ideals which are not strict ideals are also plentiful. Indeed, take any non-separable space Y for which Y^* contains no proper 1-norming subspace (e.g. the case for spaces being M-ideals in their biduals [14] or more generally for strict u-ideals in their biduals [13]). Then for every separable subspace X of Y there exists a separable ai-ideal Z in Y containing X . This ai-ideal cannot be strict. Moreover, this reasoning actually shows that one cannot extend the main theorem replacing “ai-ideal” with “strict ideal”.

3. CHARACTERIZATIONS IN TERMS OF SUBSPACES

Let Y be a Banach space with unit ball B_Y . By a *slice* of B_Y we mean a set $S(y^*, \varepsilon) = \{y \in B_Y : y^*(y) > 1 - \varepsilon\}$ where y^* is in the unit sphere S_{Y^*} of Y^* and $\varepsilon > 0$. A *finite convex combination of slices* of B_Y is a set of the form

$$S = \sum_{i=1}^n \lambda_i S(y_i^*, \varepsilon_i)$$

where $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, $y_i^* \in S_{Y^*}$, and $\varepsilon_i > 0$ for $i = 1, 2, \dots, n$.

The relations between the following three successively stronger properties were investigated in [4]:

DEFINITION 3.1. A Banach space Y has the

- (i) *local diameter 2 property* (LD2P) if every slice of B_Y has diameter 2.
- (ii) *diameter 2 property* (D2P) if every non-empty relatively weakly open subset in B_Y has diameter 2.
- (iii) *strong diameter 2 property* (SD2P) if every finite convex combination of slices of B_Y has diameter 2.

From [9, Theorem 2.4] it is known that LD2P $\not\Rightarrow$ D2P and from [17, Theorem 1] or [6, Theorem 3.2] that D2P $\not\Rightarrow$ SD2P.

Using Theorem 1.5 we obtain in terms of subspaces characterizations of the mentioned diameter 2 properties in the following way. (Recall from [5, p. 404] that diameter 2 properties do not in general pass to ideals.)

PROPOSITION 3.2. *Let Y be a Banach space. Then Y has the SD2P (resp. the D2P, LD2P) if and only if every separable ai-ideal in Y does.*

Proof. Necessity was established in [5, Propositions 3.2, 3.3 and Corollary 3.4] where it was proved that every ai-ideal in Y has the SD2P (resp. the D2P, LD2P) whenever Y has.

First let us prove the sufficiency for the SD2P. To this end let $\varepsilon_k > 0$ for $k = 1, \dots, n$ and $S = \sum_{k=1}^n \lambda_k S_k$ a finite convex combination of slices $S_k = \{y \in B_Y : y_k^*(y) > 1 - \varepsilon_k\}$ of the unit ball of Y . By Theorem 1.5 find a separable ai-ideal X in Y such that $\text{span}(y_k^*)_{k=1}^n \subset \varphi(X^*)$ where $\varphi \in \text{HB}_{ai}(X, Y)$. For $k = 1, \dots, n$ find $x_k^* \in S_{X^*}$ such that $y_k^* = \varphi(x_k^*)$. Define the slices $S'_k = \{x \in B_X : x_k^*(x) > 1 - \varepsilon_k\} = \{x \in B_X : \varphi x_k^*(x) > 1 - \varepsilon_k\}$ and note that $S'_k \subset S_k$. Now the convex combination of slices $S' = \sum_{k=1}^n \lambda_k S'_k$ has diameter 2 by assumption. As $S' \subset S$, we get that S has diameter 2 as well.

For the LD2P the result follows by taking $k = 1$ in the argument above.

For the D2P property let V be a relatively weakly open subset in B_Y . Find $y_0 \in V$ and $y_i^* \in Y^*$ such that $V_\varepsilon = \{y \in B_Y : |y_i^*(y - y_0)| < \varepsilon, i = 1, \dots, n\} \subset V$. By Theorem 1.5 find a separable ai-ideal X in Y which contains y_0 and such that $\text{span}(y_k^*)_{k=1}^n \subset \varphi(X^*)$ where $\varphi \in \text{HB}_{ai}(X, Y)$. Then a similar argument as above will finish the proof. ■

Recall that a Banach space Y has the *Daugavet property* if for every rank one operator $T : Y \rightarrow Y$ the equation $\|I + T\| = 1 + \|T\|$ holds where I is the identity operator on Y . One can show that the Daugavet property is equivalent to the following statement (see [30]): For every $\varepsilon > 0$, every $y_0^* \in S_{Y^*}$, and every $y_0 \in S_Y$ there exists a point y in the slice $S(y_0^*, \varepsilon_0)$ such that $\|y + y_0\| \geq 2 - \varepsilon$.

Using Theorem 1.5 we get a similar characterization of the Daugavet property as for the diameter 2 properties.

PROPOSITION 3.3. *A Banach space Y has the Daugavet property if and only if every separable ai-ideal in Y does.*

Proof. That the Daugavet property is inherited by ai-ideals is proved in [5, Proposition 3.8]. Let $\varepsilon > 0$ and choose positive $\delta < \varepsilon$ such that $(1 - \delta)^2 > 1 - \varepsilon$. Let $y_0^* \in S_{Y^*}$ and $y_0 \in S_Y$. We must show that the slice $S(y_0^*, \varepsilon) = \{y \in B_Y : y_0^*(y) > 1 - \varepsilon\}$, contains y such that $\|y_0 + y\| > 2 - \varepsilon$. To this end, choose $y_1 \in S(y_0^*, \delta)$ and find a separable ai-ideal Z which contains $\text{span}\{y_0, y_1\}$. By construction the slice $S(y_0^*|_Z / \|y_0^*|_Z\|, \delta)$ of B_Z is non-empty

and by assumption there exists $y \in S(y_0^*|_Z/\|y_0^*|_Z\|, \delta)$ with $\|y_0 + y\| > 2 - \delta$. Since $y_0^*(y) = \|y_0^*|_Z\| y_0^*|_Z(y)/\|y_0^*|_Z\| > \|y_0^*|_Z\|(1 - \delta) > (1 - \delta)(1 - \delta) > 1 - \varepsilon$, we are done. ■

In [2] the notion of an almost square Banach space was introduced and studied.

DEFINITION 3.4. A Banach space Y is said to be *almost square* (ASQ) if for every $\varepsilon > 0$ and every finite set $(y_n)_{n=1}^N \subset S_Y$, there exists $y \in S_Y$ such that

$$\|y_n - y\| \leq 1 + \varepsilon \quad \text{for every } 1 \leq n \leq N.$$

The notion of an ASQ space is in some sense, but not quite, dual to the well established notion of an octahedral space (see Definition 3.8 and the paragraph that follows) introduced by Godefroy in [12]. We will discuss octahedral spaces briefly below.

Among the examples of ASQ spaces we find the much studied class of (non-reflexive) M-embedded spaces [2, Corollary 4.3], i.e. spaces Y of the form $Y^{***} = Y^* \oplus_1 Y^\perp$ (cf. [19] for the theory of such spaces). It is not hard to see from Theorem 3.5 below that ASQ spaces contain copies of c_0 . ASQ spaces also possess the SD2P (see [2, Proposition 2.5] and [18, Theorem 2.4]), but not all spaces with the SD2P are ASQ. Take e.g. $C[0, 1]$ which is Lindenstrauss and thus has the SD2P [5, Proposition 4.6]. Moreover, if we let y_1 to be the constant one function and $y_2 = -y_1$ one easily sees that $C[0, 1]$ fails to be ASQ.

The following characterization of ASQ spaces was obtained in [2, Theorem 2.4].

THEOREM 3.5. *Let Y be a Banach space. If Y is ASQ then for every finite dimensional subspace $E \subset Y$ and $\varepsilon > 0$ there exists $y \in S_Y$ such that*

$$(1 - \varepsilon) \max(\|x\|, |\lambda|) \leq \|x + \lambda y\| \leq (1 + \varepsilon) \max(\|x\|, |\lambda|) \quad (6)$$

for all $\lambda \in \mathbb{R}$ and all $x \in E$.

We will use this result and Theorem 1.5 to obtain characterizations of an ASQ space in terms of its subspaces.

THEOREM 3.6. *Let Y be a Banach space. Then the following statements are equivalent.*

- (a) Y is ASQ.
- (b) Every separable ai-ideal in Y is ASQ.
- (c) Every subspace X of Y for which Y/X does not contain a copy of c_0 is ASQ.
- (d) Every subspace of finite codimension in Y is ASQ.

Proof. (a) \Rightarrow (b). This is proved in [2, Lemma 4.1].

(b) \Rightarrow (a). Let $\varepsilon > 0$ and $(y_n)_{n=1}^N \subset S_Y$, and find by Theorem 1.5 a separable ai-ideal Z in Y containing $(y_n)_{n=1}^N$. As Z is ASQ there exists $z \in S_Z$ such that $\|y_n - z\| \leq 1 + \varepsilon$ and we are done.

(c) \Rightarrow (d) is trivial and (d) \Rightarrow (a) is clear as every finite set of points is contained in a subspace of finite codimension in Y .

(a) \Rightarrow (c). Let $(y_n)_{n=1}^N \subset S_Y$, $E = \text{span}(y_n)_{n=1}^N$, and $\varepsilon, \delta > 0$ with $\frac{1+\delta}{1-\delta} + \delta < 1 + \varepsilon$. Choose a sequence $(\delta_k)_{k=0}^\infty$ of positive reals such that $(\delta_k) \downarrow 0$ and

$$\prod_{k=0}^\infty (1 - \delta_k) > 1 - \delta \quad \text{and} \quad \prod_{k=0}^\infty (1 + \delta_k) < 1 + \delta. \tag{7}$$

Using Theorem 3.5 we can find a sequence $(z_k) \subset S_Y$ such that for every $y \in \text{span}(E \cup \{z_1, \dots, z_k\})$ and every $\lambda \in \mathbb{R}$ we have

$$(1 - \delta_k) \max(\|y\|, |\lambda|) \leq \|y + \lambda z_{k+1}\| \leq (1 + \delta_k) \max(\|y\|, |\lambda|). \tag{8}$$

Now, if $z = \sum_{k=1}^K \lambda_k z_k$ we get from (7) and (8) that

$$\begin{aligned} (1 - \delta) \max \{ \|y\|, |\lambda_k|_{k=1}^K \} &< \|y + z\| \\ &< (1 + \delta) \max \{ \|y\|, |\lambda_k|_{k=1}^K \} \end{aligned} \tag{9}$$

for every $y \in E$. It is clear that the space $\overline{\text{span}}(z_k)_{k=1}^\infty$ is isomorphic to c_0 . As Y/X does not contain a copy of c_0 , the quotient map $\pi : Y \rightarrow Y/X$ fails to be bounded below on $\overline{\text{span}}(z_k)_{k=1}^\infty$. From this it follows that there exists a linear combination $\sum_{k=1}^{K_1} \lambda_k z_k$ whose norm is 1 and with $\|\pi(\sum_{k=1}^{K_1} \lambda_k z_k)\| \leq \delta/4$. Thus there is $f \in X$ with $\|f - \sum_{k=1}^{K_1} \lambda_k z_k\| \leq \delta/2$. Putting $g = f/\|f\|$ we get

$\|g - \sum_{k=1}^{K_1} \lambda_k z_k\| \leq \delta$. Hence using (9) we get for $n = 1, \dots, N$

$$\begin{aligned} \|y_n - g\| &\leq \left\| y_n + \sum_{k=1}^{K_1} \lambda_k z_k \right\| + \left\| g - \sum_{k=1}^{K_1} \lambda_k z_k \right\| \\ &< (1 + \delta) \max \{ (\|y_n\|, |\lambda_k|_{k=1}^{K_1}) \} + \delta \\ &\leq \frac{1 + \delta}{1 - \delta} + \delta < 1 + \varepsilon, \end{aligned}$$

which is what we need. ■

Remark 3.7. From Theorem 1.5 and Theorem 3.6 (a) \Rightarrow (b) we get that every separable subspace of an ASQ space Y is contained in a separable subspace Z in Y which is both ASQ and ai-ideal in Y . This improves [2, Proposition 6.5] which says only that Z can be taken to be ASQ.

Let us end the paper with a result similar to Theorem 3.6 for octahedral spaces.

DEFINITION 3.8. A Banach space Y is said to be *octahedral* if for every $\varepsilon > 0$ and every finite set $(y_n)_{n=1}^N \subset S_Y$, there exists $y \in S_Y$ such that

$$\|y_n - y\| \geq 2 - \varepsilon \quad \text{for every } 1 \leq n \leq N.$$

An easy consequence of the PLR is that spaces Y of the form $Y^{**} = Y \oplus_1 X$ where X is a subspace of Y^{**} , i.e. the L-embedded spaces (cf. [19] for the theory of such spaces) are octahedral. It is not so hard to see that octahedral spaces contain ℓ_1 [12]. Quite recently octahedral spaces were studied in [8] and [18]. In [18, Theorem 3.3] it was proved that a space has the SD2P if and only if its dual is octahedral. Characterizations of the D2P and the LD2P in terms of weaker forms of octahedrality in the dual were also established in this paper (see [18, Theorems 2.3 and 2.7]).

In [20] it was proved that spaces with the almost Daugavet property are octahedral and that in the separable case the two properties are equivalent. Recall that a Banach space is said to be an *almost Daugavet space* if there exists a 1-norming subspace X of Y^* such that for every $y_0^* \in S_X$, every $y_0 \in S_Y$, and every $\varepsilon > 0$ there exists a point y in the slice $S(y_0^*, \varepsilon)$ such that $\|y + y_0\| \geq 2 - \varepsilon$.

Lücking in [26, Theorem 2.5] proved that each separable almost Daugavet space satisfies statement (c) in Theorem 3.9 below. For non-separable spaces it is as far as the author knows unknown whether octahedrality implies almost

Daugavet. We obtain, in terms of subspaces, the following characterization of octahedral spaces.

THEOREM 3.9. *Let Y be a Banach space. Then the following statements are equivalent.*

- (a) *Y is octahedral.*
- (b) *Every separable ai-ideal in Y is octahedral.*
- (c) *Every subspace X of Y for which Y/X does not contain a copy of ℓ_1 is octahedral.*
- (d) *Every subspace of finite codimension in Y is octahedral.*

The proof of this result follows along the same lines as the proof Theorem 3.6 using [18, Proposition 2.4] instead of Theorem 3.5 and otherwise adjusting to the ℓ_1 setting. Therefore the proof will be omitted. Theorem 3.9 should be compared with [7, Proposition 2.2 and Theorem 2.6].

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