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The dual of $(C_b(X), t_\sigma)$: measures and Gulick's conjecture

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Abstract: In [4], D. Gulick introduced a new topology, σ-compact-open topology t_σ on $C_b(X)$, where $C_b(X)$ is the set of all real-valued bounded continuous functions on a Tychonoff space X. This space was denoted $(C_b(X), t_\sigma)$ by him and he studied its dual space. First we give an elegant measure-theoretic characterization of this dual space. Gulick compared this dual space with the dual of $(C_b(X), t_n)$, where t_n is the topology on $C_b(X)$ generated by the supremum norm $\|\cdot\|_\infty$ on $C_b(X)$. Since $t_\sigma \subseteq t_n$, the dual $(C_b(X), t_\sigma)^* \subseteq (C_b(X), t_n)^*$. Gulick has conjectured that for a locally compact Hausdorff space X, if the duals are same, then $t_\sigma = t_n$. By constructing a complicated locally compact Hausdorff space X, we show that the duals are the same but t_σ is strictly weaker than t_n.

Key words: Function space, σ-compact-open topology, Dual, measures, Borel sets.

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1. Introduction

Let X be a completely regular Hausdorff space, that is, a Tychonoff space, and $C_b(X)$ the set of all real-valued bounded continuous functions on X. The classical well-known topology on $C_b(X)$ is t_n, the topology generated by the supremum norm $\|\cdot\|_\infty$, that is,

$$\|f\|_\infty = \sup_{x\in X} |f(x)| \qquad \forall\, f \in C_b(X).$$

It is well-known that $(C_b(X), t_n)$ is a Banach space. We denote the dual space of $(C_b(X), t_n)$ by $\Lambda_\infty(X)$ and the dual (or conjugate) norm on $\Lambda_\infty(X)$

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is given by

$$\|\lambda\|_* = \sup\{|\lambda(f)| : f \in C_b(X), \|f\|_\infty \leq 1\}.$$

In order to define the σ-compact-open topology on $C_b(X)$, we first consider the collection $\sigma(X)$, the set of all non-empty σ-compact subsets of X. Recall that a subset A of X is called σ-compact if it is a countable union of compact sets and the closure of a σ-compact set is called almost σ-compact.

For each $A \in \sigma(X)$, define the semi-norm p_A on $C_b(X)$ as follows:

$$p_A(f) = \sup_{x \in A} |f(x)|, \qquad \forall\, f \in C_b(X).$$

The locally convex topology on $C_b(X)$ generated by the collection of semi-norms $\{p_A : A \in \sigma(X)\}$ is denoted by t_σ. Note that since t_σ is finer than the compact-open topology t_k on $C_b(X)$, t_σ is Hausdorff. Also it is easy to see that $t_\sigma \leq t_n$, that is, t_σ is weaker than t_n. And $t_\sigma = t_n$ if and only if X is almost σ-compact.

Moreover, for t_σ on $C_b(X)$, we can consider the following subbase:

$$\{[A, V] : A \in \sigma(X), V \text{ is a bounded open interval in } \mathbb{R}\},$$

where $[A, V] = \{f \in C_b(X) : \overline{f(A)} \subseteq V\}$. This actually justifies calling t_σ the σ-compact-open topology. Also t_σ can be seen as the topology of uniform convergence on the σ-compact subsets of X as follows: for each $f \in C_b(X)$, the subsets $\{\langle f, A, \epsilon\rangle : A \in \sigma(X), \epsilon > 0\}$ give a neighborhood base at f, where

$$\langle f, A, \epsilon\rangle = \{g \in C_b(X) : |f(x) - g(x)| < \epsilon, \forall\, x \in A\}.$$

For more details on t_σ, see [4, 6, 7].

Since $t_\sigma \leq t_n$, the dual of $(C_b(X), t_\sigma)$ is contained in $\Lambda_\infty(X)$. The dual of $(C_b(X), t_\sigma)$ is denoted by $\Lambda_\sigma(X)$. Here we mention that if t_k is the compact-open topology on $C_b(X)$, then the dual of $(C_b(X), t_k)$ is denoted by $\Lambda_k(X)$. Hence $\Lambda_k(X) \subseteq \Lambda_\sigma(X) \subseteq \Lambda_\infty(X)$.

In Section 2, we develop a nice measure-theoretic characterization of $\Lambda_\sigma(X)$, while in Section 3 we deal with Gulick's conjecture.

2. Measure-theoretic characterization of $\Lambda_\sigma(X)$

In order to have this measure-theoretic characterization, we need to talk about the following concepts and results. X will always denote a Tychonoff space.

A linear functional λ on $C_b(X)$ is called *positive* provided that $\lambda(f) \geq 0$ for each $f \in C_b(X)$ such that $f \geq 0$ and we express this by the notation $\lambda \geq 0$. Now let $\Lambda_j^+(X) = \{\lambda \in \Lambda_j(X) : \lambda \geq 0\}$, where j is k, σ or ∞ and this is called the *positive cone* of $\Lambda_j(X)$.

If λ is a linear functional on $C_b(X)$ and A is a subset of X, then λ is said to be supported on A provided that whenever $f \in C_b(X)$ with $f|_A = 0$, then $\lambda(f) = 0$. Since λ is linear, this is equivalent to saying that whenever $f, g \in C_b(X)$ with $f|_A = g|_A$, then $\lambda(f) = \lambda(g)$.

The next result ([5, Lemma 4.1]) says what kind of linear functionals on $C_b(X)$ can be continuous on $(C_b(X), t_\sigma)$.

THEOREM 2.1. *For each $\lambda \in \Lambda_\sigma(X)$, there exists a σ-compact subset A of X such that λ is supported on A. Conversely, if λ is a positive linear functional on $C_b(X)$, which is supported on some σ-compact subset of X, then $\lambda \in \Lambda_\sigma^+(X)$.*

In fact, it has been established in [5] that $(\Lambda_\sigma(X), \|\cdot\|_*)$ is a Banach lattice.

Now we recall some basic concepts from the measure theory. The σ-algebra generated by the family of all closed subsets of X is denoted by $B_o(X)$. By a measure on $B_o(X)$, we usually mean a (countably additive, real-valued) signed measure (also called charge). A member of $B_o(X)$ is called a Borel set, while a measure μ defined on $B_o(X)$ is called a Borel measure. It can be shown that a Borel measure has finite total variation. For a proof, see [1, Theorem 36.4, pp. 332-334].

A positive measure μ defined on $B_o(X)$ is called closed regular if for each $A \in B_o(X)$, $\mu(A) = \sup\{\mu(C) : C \subseteq A, C \text{ is closed in } X\}$. Now if μ is a (signed) measure on $B_o(X)$, then it is called closed regular provided both μ^+ and μ^- are closed regular. This definition of regularity can be rephrased as follows: a (signed) measure μ on $B_o(X)$ is called closed regular whenever $A \in B_o(X)$ and $\epsilon > 0$, there are closed and open sets C and U in X such that $C \subseteq A \subseteq U$ and $|\mu|(U \setminus C) < \epsilon$. Such a closed regular measure is called compact-regular or tight if the closed set C can be replaced by a compact subset of X. Obviously, a closed regular Borel measure having a compact support is compact regular. Recall that a Borel measure μ has support A, where $A \subseteq X$ and $A \in B_o(X)$, if $|\mu|(X \setminus A) = 0$.

Now we fix some notations.

Let $M(X)$ be the linear space of all closed regular Borel measures defined on $B_o(X)$, $M_t(X) = \{\mu \in M(X)\ \mu \text{ is compact regular}\}$, and

$$M_k(X) = \{\mu \in M(X) : \mu \text{ has a compact support}\}.$$

Note that $M_k(X) \subseteq M_t(X) \subseteq M(X)$. Let $M^+(X)$, $M_t^+(X)$ and $M_k^+(X)$ denote the corresponding subsets of positive measures in $M(X)$, $M_t(X)$ and $M_k(X)$ respectively, that is,

$$M^+(X) = \{\mu \in M(X) \,:\, \mu \geq 0\},$$
$$M_t^+(X) = \{\mu \in M_t(X) \,:\, \mu \geq 0\},$$
$$M_k^+(X) = \{\mu \in M_k(X) : \mu \geq 0\}.$$

For $\mu \in M(X)$, define $\|\mu\| = |\mu|(X)$. It can be easily shown that $\|\cdot\|$ is actually a norm on $M(X)$. This norm $\|\cdot\|$ is called the total variation norm.

Also for a pair of elements μ and ν in $M(X)$, define $\mu \leq \nu$ provided $\mu(A) \leq \nu(A)$, for all $A \in B_o(X)$. Then it can be verified that $\leq$ is an order relation under which $M(X)$ becomes a vector lattice and consequently, the total variation norm becomes a lattice norm. Hence each of $M(X)$, $M_t(X)$ and $M_k(X)$ equipped with the total variation norm $\|\cdot\|$ is a normed vector lattice, while each of $M^+(X)$, $M_t^+(X)$ and $M_k^+(X)$ is a metric space equipped with the metric induced by the total variation norm.

Theorem 2.2. (Riesz Representation Theorem) *If X is compact and Hausdorff, then the map $F : (M(X), \|\cdot\|) \to (\Lambda_\infty(X), \|\cdot\|_*)$ defined by*

$$F(\mu)(f) = \int_X f d\mu$$

is an isometric lattice isomorphism of $M(X)$ onto $\Lambda_\infty(X)$. Under this isomorphism, $M^+(X)$ is identified with $\Lambda_\infty^+(X)$.

Note that if X is compact and Hausdorff, then $M(X) = M_k(X)$ and $\Lambda_\infty(X) = \Lambda_\sigma(X) = \Lambda_k(X)$. By Theorem 2.2, $M_k(\beta X)$ is isometrically isomorphic to $\Lambda_k(\beta X) = \Lambda_\infty(\beta X)$. Actually, $\Lambda_\infty(\beta X)$ can be considered same as $\Lambda_\infty(X)$ by identifying an element λ of $\Lambda_\infty(X)$ with the unique element $\hat{\lambda}$ of $\Lambda_\infty(\beta X)$, where $\hat{\lambda} : (C_b(\beta X), t_n) \to \mathbb{R}$ is defined as follows:

$$\hat{\lambda}(f) = \lambda(f|_X), \qquad \forall\, f \in C_b(\beta X).$$

Conversely, if $\lambda' \in \Lambda_\infty(\beta X)$, define $\lambda : (C_b(X), t_n) \to \mathbb{R}$ as follows:

$$\lambda(f) = \lambda'\big(f^\beta\big), \qquad \forall\, f \in C_b(X),$$

where f^β is the unique continuous extension of f to βX. Clearly, $\lambda' = \hat{\lambda}$. Hence $M_k(\beta X)$ is actually isometrically isomorphic to $\Lambda_\infty(X)$.

An element $\hat{\lambda}$ of $\Lambda_k(\beta X)$ has a compact support. Now, we show that if such a compact support is the closure of a σ-compact subset A of X, then $\hat{\lambda} \in \Lambda_\sigma(X)$. Actually, it gives a useful characterization of $\Lambda_\sigma(X)$.

Let A be a σ-compact subset of X and $f \in C_b(X)$. If f^β is the unique continuous extension of f to βX, then $cl_{\mathbb{R}} f(A) = f^\beta(cl_{\beta X} A)$ and consequently $p_A(f) = p_{\overline{A}}(f^\beta)$ where $\overline{A} = cl_{\beta X} A$. Since βX is compact, every real-valued continuous function on βX is bounded. Note that $t_n = t_\sigma = t_k$ on $C_b(\beta X)$ and consequently $\Lambda_\infty(\beta X) = \Lambda_\sigma(\beta X) = \Lambda_k(\beta X)$.

Given $\lambda \in \Lambda_\sigma(X)$, a $\hat{\lambda}$ in $\Lambda_k(\beta X)$ can be defined as follows: for $h \in C_b(\beta X)$, define $\hat{\lambda} : C_b(\beta X) \to \mathbb{R}$ by $\hat{\lambda}(h) = \lambda(h|_X)$. Since $\lambda \in \Lambda_\sigma(X)$, there exist $M > 0$ and a σ-compact subset A of X such that $|\lambda(g)| \leq M p_A(g)$ for all $g \in C_b(X)$. Now, $|\hat{\lambda}(h)| = |\lambda(h|_X)| \leq M p_A(h|_X) = M p_{\overline{A}}(h)$ and $\hat{\lambda} \in \Lambda_k(\beta X)$.

Conversely, let $\mu \in \Lambda_k(\beta X)$ and suppose μ is supported on $cl_{\beta X} A$ where $A \in \sigma(X)$. Since $|\mu| = \mu^+ + \mu^-$, both μ^+ and μ^- are supported on $cl_{\beta X} A$. Now define $\lambda_1, \lambda_2 : C_b(X) \to \mathbb{R}$ as follows. For each $f \in C_b(X)$, define $\lambda_1(f) = \mu^+(f^\beta)$ and $\lambda_2(f) = \mu^-(f^\beta)$. Both λ_1 and λ_2 are positive linear functionals and supported on A. Hence λ_1 and λ_2 are in $\Lambda_\sigma^+(X)$. Now, let $\lambda = \lambda_1 - \lambda_2$. Then $\lambda \in \Lambda_\sigma(X)$ and for each

$$f \in C_b(X), \qquad \lambda(f) = \lambda_1(f) - \lambda_2(f) = \mu^+(f^\beta) - \mu^-(f^\beta) = \mu(f^\beta).$$

Consequently, $\mu = \hat{\lambda}$.

Also note that if $f \in C_b(X)$, then $\|f\|_\infty \leq 1$ if and only if $\|f^\beta\|_\infty \leq 1$. Consequently, $\|\lambda\|_* = \|\hat{\lambda}\|_*$. Also it is easy to see that if $\lambda_1, \lambda_2 \in \Lambda_\sigma(X)$, then $\lambda_1 = \lambda_2$ if and only if $\hat{\lambda}_1 = \hat{\lambda}_2$. Given $\lambda \in \Lambda_\sigma(X)$, we call $\hat{\lambda}$ the natural expansion of λ to $(C_b(\beta X), t_k)$.

Now denote by $M_\sigma(X)$ the set

$$M_\sigma(X) = \{\mu \in M_k(\beta X) : |\mu|(\beta X - cl_{\beta X} A) = 0 \text{ for some } \sigma\text{-compact } A \subset X\}$$

and $M_\sigma^+(X) = \{\mu \in M_\sigma(X) : \mu \geq 0\}$. Note that

$$M_k(\beta X) = M_t(\beta X) = M_\sigma(\beta X) = M(\beta X).$$

The following result justifies the use of the notation $M_\sigma(X)$ as it identifies $\Lambda_\sigma(X)$ with $M_\sigma(X)$.

THEOREM 2.3. *For any X, the map $F : (M_\sigma(X), \|\cdot\|) \to (\Lambda_\sigma(X), \|\cdot\|_*)$ defined by $F(\mu)(f) = \int_{\beta X} f^\beta d\mu$ is an isometric lattice isomorphism of $M_\sigma(X)$ onto $\Lambda_\sigma(X)$. Under this isomorphism, $M_\sigma^+(X)$ is identified with $\Lambda_\sigma^+(X)$.*

Proof. Let $\mu \in M_\sigma(X)$. So there exists a σ-compact subset A of X such that $|\mu|(\beta X - cl_{\beta X} A) = 0$. Now given $f \in C_b(X)$,

$$|F(\mu)(f)| = \left|\int_{\beta X} f^\beta d\mu\right| \leq \int_{\beta X} |f^\beta| d|\mu| = \int_{cl_{\beta X} A} |f^\beta| d|\mu|$$
$$\leq p_{\overline{A}}(f^\beta)|\mu|(cl_{\beta X} A) = p_A(f)|\mu|(\beta X),$$

where $\overline{A} = cl_{\beta X} A$. Hence $F(\mu) \in \Lambda_\sigma(X)$.

Note that for all $h \in C_b(\beta X)$, $\widehat{F(\mu)}(h) = F(\mu)(h|_X) = \int_{\beta X} h d\mu$ and consequently $|\widehat{F(\mu)}(h)| \leq p_{\overline{A}}(h)|\mu|(cl_{\beta X} A)$. Hence $\widehat{F(\mu)} \in \Lambda_k(\beta X)$. So by Theorem 2.2, $\|\widehat{F(\mu)}\|_* = \|\mu\|$. But $\|\widehat{F(\mu)}\|_* = \|F(\mu)\|_*$.

Now we need to show that F is onto. Here it suffices to show that if $\lambda \in \Lambda_\sigma^+(X)$, then there exists $\mu \in M_\sigma^+(X)$ with $F(\mu) = \lambda$. So let $\lambda \in \Lambda_\sigma^+(X)$ be arbitrarily chosen and let $\hat{\lambda}$ be the natural expansion of λ to $\Lambda_k^+(\beta X)$. By Theorem 2.1, there exists a σ-compact subset A of X such that λ is supported on A. Since $\hat{\lambda} \in \Lambda_k^+(\beta X)$, by Theorem 2.2 there exists a $\mu \in M_k^+(\beta X)$ such that $\hat{\lambda}(h) = \int_{\beta X} h d\mu$ for all $h \in C_b(\beta X)$. So for each $f \in C_b(X)$,

$$\lambda(f) = \hat{\lambda}(f^\beta) = \int_{\beta X} f^\beta d\mu = F(\mu)(f).$$

Hence, $F(\mu) = \lambda$.

Now let B be any compact subset of $\beta X - cl_{\beta X} A$. Because of the normality of βX, there exists a continuous function $h : \beta X \to [0, 1]$ such that $h(x) = 1$, for every $x \in B$ and $h(x) = 0$, for every $x \in cl_{\beta X} A$. Consequently,

$$0 = \lambda(h|_X) = \hat{\lambda}(h) = \int_{\beta X} h d\mu = \int_B h d\mu + \int_{\beta X - B} h d\mu = \mu(B) + \int_{\beta X - B} h d\mu.$$

Hence, $\mu(B) = 0$.

But, since μ is compact-regular, $\mu(\beta X - cl_{\beta X} A) = 0$ and consequently, $\mu \in M_\sigma^+(X)$. We have shown that if $\lambda \in \Lambda_\sigma^+(X)$, then there exists $\mu \in M_\sigma^+(X)$ such that $F(\mu) = \lambda$. Also our definition of F shows that if $\mu \in M_\sigma^+(X)$, then $F(\mu) \in \Lambda_\sigma^+(X)$. Hence by [1, Lemma 38.6], F is an isometric lattice isomorphism. ∎

3. Gulick's Conjecture

Gulick's conjecture arises out of his result stated in [4, Theorem 5.5]. In this result, Gulick has proved that if X is locally compact and paracompact and $\Lambda_\sigma(X) = \Lambda_\infty(X)$, then $t_\sigma = t_n$. In fact, he has proved that X is σ-compact. In [4], by giving a counter-example, Gulick has shown that if X fails to be locally compact, then his aforesaid result may not hold.

But Gulick conjectures that paracompactness of X may be dropped from his result, that is, he believes that for a locally compact space X, $\Lambda_\sigma(X) = \Lambda_\infty(X)$ if and only if $t_\sigma = t_n$ on $C_b(X)$. In this section, we show that Gulick's conjecture is false. In view of Theorem 2.3, it will be so if we can have a space mentioned in the following result.

Theorem 3.1. *There is a locally compact Hausdorff space X which is not almost σ-compact and which satisfies the following condition: there exists a σ-compact subset B of X such that $(\beta X \setminus X) \subseteq cl_{\beta X} B$.*

Before constructing a space as mentioned in Theorem 3.1, we explain why the existence of such a space makes Gulick's conjecture false.

Let X be a locally compact Hausdorff space as mentioned in Theorem 3.1.

1. Since X is not almost σ-compact, t_σ is strictly weaker than t_n, that is, $t_\sigma \neq t_n$ on $C_b(X)$.
2. Since X is locally compact and dense in βX, X is open in βX. Hence, X is a Borel subset of βX. Now for each $\mu \in M_k(\beta X)$, $|\mu|(\beta X) = |\mu|(\beta X \setminus X) + |\mu|(X)$. Since μ is compact regular, there exists a σ-compact subset C of X such that $|\mu|(X) = |\mu|(C)$. Let $D = C \cup B$. So D is a σ-compact subset of X.

   Note $\overline{C} \cup \overline{B} = \overline{C \cup B} = \overline{D}$, where the closures are taken in βX. So,

   $$|\mu|(\beta X) = |\mu|(X) + |\mu|(\beta X \setminus X) = |\mu|(C) + |\mu|(\beta X \setminus X) = |\mu|(C \cup (\beta X \setminus X)) = |\mu|(C \cup \overline{B}) = |\mu|(\overline{D}).$$

   Hence, $|\mu|(\beta X \setminus \overline{D}) = 0$. So $\mu \in M_\sigma(X)$ and one has $M_k(\beta X) = M_\sigma(X)$. Now by using Theorem 2.3, it can be easily shown that $\Lambda_\infty(X) = \Lambda_\sigma(X)$.

Proof of Theorem 3.1. Let $T_1 = [0, \omega_2] \times [0, \omega_1]$ and $T_2 = [0, \omega_1] \times [0, \omega_0]$, where ω_1 and ω_2 are the first and the second uncountable ordinals (respectively), while ω_0 is the first infinite ordinal. Of course, $[0, \omega_2]$, $[0, \omega_1]$ and $[0, \omega_0]$ are given the interval topology.

Let T be the quotient space obtained from $T_1 \oplus T_2$ by identifying each point $\langle \omega_2, \alpha \rangle$ of T_1 with $\langle \alpha, \omega_0 \rangle$ of T_2, where $\alpha \in [0, \omega_1]$. Let $f : T_1 \oplus T_2 \to T$ be the corresponding quotient map. Put $p = f(\langle \omega_2, \omega_1 \rangle) = f(\langle \omega_1, \omega_0 \rangle)$ and let $X = T \setminus \{p\}$. Also, let $T_1^* = T_1 \setminus \{\langle \omega_2, \omega_1 \rangle\}$ and $T_2^* = T_2 \setminus \{\langle \omega_1, \omega_0 \rangle\}$. Let $X_1 = f(T_1^*)$ and $X_2 = f(T_2^*)$. Then $f|_{T_1^*} : T_1^* \to X_1$ and $f|_{T_2^*} : T_2^* \to X_2$ are homeomorphisms.

Claim: $\beta X = T$.

Let $h : X \to [0, 1]$ be a continuous function. Then there exist $\alpha \in [0, \omega_2)$, $\beta \in [0, \omega_1)$ such that $h(x, y) = h(\alpha, \beta)$ for all $x > \alpha$, $y > \beta$ where $\langle x, y \rangle \in f|_{T_1^*}([\alpha, \omega_2) \times [\beta, \omega_1))$, that is h is a constant function on the set $f|_{T_1^*}([\alpha, \omega_2) \times [\beta, \omega_1))$. Clearly, we can determine $h(p) := h(\alpha, \beta)$. Note that $h^{-1}(O(h(p)))$ is an open set in T for every open neighborhood $O(h(p)) \subset [0, 1]$ of the point $h(p)$ because there is $n < \omega_0$ such that $h(y, m) \in O(h(p))$ for all $m > n$ and $y > \beta$ where $\langle y, m \rangle \in f|_{T_2^*}([\beta, \omega_1) \times [n, \omega_0))$. Thus, h is a continuous extension on T. By [2, Corollary 3.6.3], $T = \beta X$.

Claim: X is not almost σ-compact.

If a subset A of X is σ-compact, then $A \cap X_1$ is also σ-compact, because X_1 is closed in X. So if X is almost σ-compact, so is X_1, and hence so is T_1^* (because X_1 is homeomorphic to T_1^*). Therefore, to show that X is not almost σ-compact, it suffices to show that T_1^* is not almost σ-compact.

If a subset A of T_1^* is σ-compact, then A is also σ-compact in T_1; and since $\langle \omega_2, \omega_1 \rangle$ is a P-point in T_1 (recall that a point p in a topological space X is called a P-point [3] if every G_δ-set containing p is a neighborhood of p) and it does not belong to A, there exist $\alpha < \omega_2$ and $\beta < \omega_1$ such that $A \cap ((\alpha, \omega_2] \times (\beta, \omega_1]) = \emptyset$. Thus $\overline{A} \neq T_1^*$.

If we set $A = \{p\}$ and $B = f(\{\langle \omega_1, n \rangle \in T_2 : n < \omega_0\})$, then $\beta X \setminus A = X$ and $A \subseteq cl_{\beta X} B$ hold. ∎

Remark. Actually, X of the space mentioned in Theorem 3.1 is not paracompact. Note T_2^* is not normal, see [8, Example 87]. Hence X_2 is not normal either. Also X_2 is closed in X. Hence X cannot be normal. So X is not paracompact either. Also it follows from [4, Theorem 5.5] that X cannot be paracompact.

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