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The Killing algebra of a Minkowskian space

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Abstract: I describe the structure of the Lie algebra of Killing vector fields of the metric derived from a Minkowski norm on $\mathbb{R}^n_\circ$, and discuss some applications.

Key words: Minkowski, Killing.

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1. Introduction

The name Minkowski is associated with (at least) three different kinds of mathematical object: the 4-dimensional spacetime of special relativity; a seminorm used in functional analysis; and a class of vector space structures which play an important role in Finsler geometry. This paper is concerned with the last of these. I have called them Minkowski*an* spaces in the title in the hope of avoiding the otherwise almost inevitable confusion: it is rarely if ever used in the other contexts.

The subject of the paper is the Killing vector fields, or infinitesimal isometries, of the Riemannian metric defined by the norm in a Minkowskian space. As everyone knows, Euclidean space, that is $\mathbb{R}^n$ with the Euclidean metric, defining hyperspheres centred at the origin, is the simplest kind of Minkowskian space. Its Killing fields consist of infinitesimal rotations and infinitesimal translations. Together they form a Lie algebra which is the semi-direct sum of these two components.

Research on Minkowskian spaces in general is bound to be influenced by the presence of this simple and important example. For instance, there is the search for properties of Euclidean space which also hold more generally in Minkowskian spaces: the result that the indicatrices of a Minkowskian space are always totally umbilical hypersurfaces, just like Euclidean hyperspheres, falls into this class. In another direction, there are various results about necessary and sufficient conditions for a Minkowskian space to reduce to Euclidean

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space: one thinks of the theorems of Deicke and Brickell.

The present paper follows this standard pattern. There are two main results. The first, Theorem 1 in Subsection 3.2, shows that the structure of the Lie algebra of Killing vector fields on a Minkowskian space mirrors that of the Euclidean case: it explains how to define infinitesimal rotaions and translations in the general context, and proves that the Killing algebra is the semi-direct sum of these two subalgebras. The second is concerned with conditions on the Killing algebra of a Minkowskian space necessary and sufficient for the space to be Euclidean, with conclusions to be found in Theorems 2 to 6 in Subsection 3.5. Unsurprisingly the result is, with some caveats, that the space must be maximally symmetric, in the sense that the dimension of the Killing algebra is $\frac{1}{2}n(n+1)$ (where n is the dimension of the space).

Each result is embedded in an extensive analysis of the relevant theory of Minkowskian spaces, which itself contains several new results. There are also a couple of examples.

For background reading I suggest Chapter 14 of the standard text on Finsler geometry by Bao, Chern and Shen [1]. I have to say however that many of the main topics of the present paper are scarcely covered in depth there – nor anywhere else so far as I know.

2. Basics

2.1. Definitions and notation. Assuming that the definitions are well known, the main function of this section is actually to establish my notation.

An n-dimensional Minkowskian space is the space $\mathbb{R}^n_\circ = \mathbb{R}^n - \{0\}$ equipped with a homogeneous positive function F such that, at any point of $\mathbb{R}^n_\circ$, with respect to linear coordinates y^i,

$$g_{ij} = \tfrac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} = \frac{\partial^2 E}{\partial y^i \partial y^j}, \qquad E = \tfrac{1}{2}F^2$$

are the components of a positive definite quadratic form. By homogeneous here I mean that for any positive λ, $F(\lambda y) = \lambda F(y)$. Occasionally we will encounter functions F which satisfy the stronger condition $F(\lambda y) = |\lambda|F(y)$ for all nonzero λ: equivalently $F(-y) = F(y)$. Such functions are said to be absolutely homogeneous.

F is often called the Minkowski norm, E the energy, and g is sometimes called the Hessian metric of E. A Minkowski norm which is not absolutely homogeneous may be called positively homogeneous, but I will not bother.

2-dimensional Minkowskian spaces have their own idiosyncratic properties, which I don't intend to deal with, so I shall assume throughout that $n \geq 3$.

I should straightaway make some comments about coordinates. Many calculations below could be made with respect to natural coordinates on $\mathbb{R}^n$: but any linearly related choice will do, without significant alteration to formulae, so I usually, as above, speak of linear coordinates without singling any system out. In fact there are occasions when one wants to use a linear change of coordinates to bring some object into a standard form: for example, if the g_{ij} are constant, to transform them into the components δ_{ij} of the Euclidean metric. Since this is always possible – it is essentially equivalent to finding an orthonormal basis – in order to show that a Minkowski space is Euclidean it is enough to show that the components of its metric with respect to linear coordinates are constants. So far as linear coordinates are concerned, another way of saying much the same thing is to point out that most geometrical objects we have to deal with are linear tensor fields, that is, they transform as tensors of the type indicated by their indices under linear transformations of coordinates.

Some calculations, however, are made using the index-free methods usually found in books and papers about differential geometry nowadays. The objects appearing in such calculations are tensors in the full sense. Representing such objects in terms of linear coordinates produces linear tensors, and the calculations have linear coordinate versions. The catch occurs if one tries to go the other way. Consider the formula for g_{ij} above. Suppose that z^a (with a running from 1 to n) is a system of general, that is nonlinear, local coordinates. We have two choices in how we extend the objects above to the general coordinate system. First we may put

$$g_{ab} = g_{ij} \frac{\partial y^i}{\partial z^a} \frac{\partial y^j}{\partial z^b}.$$

The g_{ab} are the components of a type (0,2) tensor field with respect to the z^a, which coincide with the g_{ij} when the coordinates are chosen to be linear. On the other hand we may consider $\partial E/\partial z^a \partial z^b$. But it is not the case that $g_{ab} = \partial E/\partial z^a \partial z^b$: indeed the right-hand side is not a tensor component.

It will generally be my practice to represent a geometric object by a single kernel letter, and to indicate its components with respect to linear coordinates by using the letters i, j, k and others from the adjacent part of the alphabet as superscripts and subscripts. Thus g is the type (0,2) tensor field whose components with respect to linear coordinates are the g_{ij} defined above. This

means that a Minkowskian space is a Riemannian space over $\mathbb{R}^n_\circ$, whose metric is g. Its Riemannian structure is of a special kind, of course: it is the purpose of this paper to describe some of its special features. What makes the study of Minkowski geometry special is the availability of linear coordinates, homogeneity, and the fact that the metric is the Hessian of the energy. These special factors allow for special techniques, such as the use of partial derivatives with respect to linear coordinates rather than always depending on covariant derivatives; these features will be frequently exploited below.

What I have said here about coordinates corresponds to the usual practice in Finsler geometry. A Finslerian structure on a manifold defines a Minkowskian structure on each fibre of its slit tangent bundle. Linear coordinates on a fibre are those naturally induced by coordinates on the base. A coordinate transformation on the base induces a linear coordinate transformation in the fibre, whose matrix is given by the Jacobian of the base coordinate transformation evaluated at the base point, which is of course constant over the fibre. More general coordinate transformations of the fibres are rarely if ever encountered in Finsler geometry.

Back to Minkowskian spaces and to expressions in linear coordinates. To save space I shall use a comma to denote a derivative.

We can express g alternatively as

$$g_{ij} = FF_{,ij} + F_{,i}F_{,j}.$$

Note that, by the definition of g_{ij}, if we set $C_{ijk} = g_{ij,k}$ then C_{ijk} is completely symmetric in its indices. Moreover, since evidently each component function g_{ij} of g is homogeneous of degree 0,

$$y^k C_{ijk} = y^k g_{ij,k} = 0 = y^k C_{kij} = y^k C_{ikj}.$$

Furthermore, every derivative of C_{ijk} is symmetric in all of its indices. The case that will be most important for us is $C_{ijk,l} = C_{ijl,k}$. Moreover,

$$y^l C_{ijk,l} = y^l C_{ijl,k} = (y^l C_{ijl})_{,k} - C_{ijk} = -C_{ijk},$$

another result that will be very useful later.

It follows from the 2-homogeneity of E that

$$E_{,i} = g_{ij}y^j = y_i = FF_{,i}.$$

Thus in particular the 1-form $y_i dy^i = dE$ is exact.

The map $y^i \mapsto y_j = g_{ij}y^i$ can be regarded simply as a change of coordinates, or from a more sophisticated point of view as a Legendre transformation. Note that

$$\frac{\partial y_i}{\partial y^j} = g_{ik,j}y^k + g_{ij} = C_{ijk}y^k + g_{ij}, \qquad \text{or } \frac{\partial}{\partial y^i} = g_{ij}\frac{\partial}{\partial y_j}.$$

In the Legendre transformation interpretation one regards the y_i as linear coordinates on the space $\mathbb{R}^{n*}_\circ = \mathbb{R}^{n*} - \{0\}$. The Legendre transformation corresponding to the Lagrangian E is

$$y^i \mapsto \frac{\partial E}{\partial y^i} = g_{ij}y^j$$

as before. The corresponding Hamiltonian is

$$H(y_i) = y_i y^i - E^*(y_i) = 2E^*(y_i) - E^*(y_i) = E^*(y_i)$$

where E^* is the function on $\mathbb{R}^{n*}_\circ$ obtained by expressing E in terms of the y_i, that is $E^*(y_i) = E(g^{ij}y_j)$ (where $g^{ik}g_{kj} = \delta^i_j$). I mention this as a point of passing interest, though I make no use of it below.

For any vector field ξ,

$$\xi^i E_{,i} = g_{ij}\xi^i y^j = FF_{,i}.$$

In particular

$$FF_{,i}y^i = F^2 = g_{ij}y^i y^j = 2E.$$

I denote by Δ the dilation field,

$$\Delta = y^i\frac{\partial}{\partial y^i}.$$

It is of course well-defined on $\mathbb{R}^n$, not just $\mathbb{R}^n_\circ$, and its integral curves are the rays in $\mathbb{R}^n$, together with the origin itself (which is a fixed point). The calculations above show that, on $\mathbb{R}^n_\circ$, $F^{-1}\Delta$ is the unit vector field along the rays: it will be denoted by $\hat{\Delta}$. We have

$$\xi(F) = g(\xi, \hat{\Delta}), \qquad \text{equivalently } \xi(E) = g(\xi, \Delta),$$

a formula that will be used frequently. In particular $\xi(E) = 0$ if and only if $g(\xi, \Delta) = 0$: ξ is tangent to the level sets of E if and only if it is orthogonal (with respect to g) to the radial lines in $\mathbb{R}^n_\circ$.

The level sets of F or E are called indicatrices; in particular the set $\{y : F(y) = 1\}$ is called the unit indicatrix, or even just the indicatrix. I denote by $\mathcal{I}$ the unit indicatrix, $\mathcal{I}_r$ the set $\{y : F(y) = r\}$. I usually work on $\mathcal{I}$ when considering indicatrices: corresponding results for $\mathcal{I}_r$ are easy to guess.

The unit indicatrix in Euclidean space is of course the unit hypersphere centred at the origin. It is tempting to think of $\mathcal{I}$ as its equivalent for a general Minkowskian space. Since each ray intersects each indicatrix once and once only, it can be seen that the map from (say) $\mathcal{I}$ to the unit Euclidean hypersphere S by projection along the rays is a diffeomorphism. It follows that $\mathcal{I}$ (indeed, any indicatrix) has the same topological properties as the Euclidean hypersphere: in particular it is compact, connected and simply connected.

One can form a direct product of two Minkowskian spaces by adding the energy functions on the components. Thus if E_1 and E_2 are energy functions on n_1- and n_2-dimensional Minkowskian spaces we take E_1+E_2 as total energy function on (n_1+n_2)-dimensional space. This is not the only way of combining two Minkowskian spaces: for example one could add the Minkowkski norms rather than the energies. The product using the energies is in particular the Riemannian direct product of its components regarded as Riemannian spaces. I shall therefore call it the Riemannian product of two Minkowskian spaces. It is the simplest product to deal with, and will be enough for my purposes. In terms of linear coordinates y^a, $a = 1, 2, \ldots, n_1$ and z^α , $\alpha = 1, 2, \ldots, n_2$, so that $E_1 = E_1(y^a)$, $E_2 = E_2(z^\alpha)$, the metric components on the product satisfy

$$\frac{\partial g_{ab}}{\partial z^\alpha} = 0, \qquad \frac{\partial g_{\alpha\beta}}{\partial y^a} = 0, \qquad g_{a\alpha} = 0.$$

Conversely, if for a given Minkowskian space we can find linear (and I stress linear) coordinates y^a, $a = 1, 2, \ldots, r$ and y^α, $\alpha = r+1, r+2, \ldots, n$ such that the metric components satisfy

$$\frac{\partial g_{ab}}{\partial y^\alpha} = 0, \qquad \frac{\partial g_{\alpha\beta}}{\partial y^a} = 0, \qquad g_{a\alpha} = 0,$$

then the space is the Riemannian product of an r-dimensional and an $(n - r)$-dimensional Minkowskian space whose components have energy functions $\frac{1}{2}g_{ab}y^a y^b$ and $\frac{1}{2}g_{\alpha\beta}y^\alpha y^\beta$, respectively.

2.2. Homogeneity. The test for homogeneity of a tensor is to apply $\mathcal{L}_\Delta$, the Lie derivative with respect to Δ. A tensor T is homogeneous of degree r, or r-homogeneous for short, if

$$\mathcal{L}_\Delta T = rT.$$

T is homogeneous if and only if its component functions are. But be warned: not of the same degree. For example, a vector field is r-homogeneous if and only if its coefficient functions are all homogeneous of degree $r+1$; thus ξ is 0-homogeneous if and only if

$$\Delta\xi^i = \xi^i = y^j \xi^i_{,j}.$$

Again, a covariant tensor of rank k is r-homogeneous if and only if each of its coefficient functions is homogeneous of degree $r-k$. Of course a function f is r-homogeneous if and only if $\Delta f = rf$. In particular, $\Delta F = F$ and $\Delta E = 2E$. Furthermore, $\Delta g_{ij} = y^k C_{ijk} = 0$, so (as a tensor) g is homogeneous of degree 2.

(I should mention here that my convention is not universally accepted. For example, Szilasi, Lovas and Kertész, in their classic text on Finsler geometry and related matters [6], use the homogeneity degree of components to measure the degree of the object. So they say that a vector field ξ is homogeneous of degree 0 if its components satisfy $\Delta\xi^i = 0$, rather than $\Delta\xi^i = \xi^i$.)

Note that for any vector field ξ,

$$\mathcal{L}_\Delta \xi = [\Delta, \xi] = -\mathcal{L}_\xi \Delta.$$

So $\mathcal{L}_\Delta \xi = 0$ if and only if $\mathcal{L}_\xi \Delta = 0$. The first gives us information about the behaviour of ξ under the flow of Δ, that is, along the rays; the second gives us information about the behaviour of Δ under the flow of ξ; together they say that the two flows commute.

If a vector field ξ satisfies $\mathcal{L}_\Delta \xi = 0$ then along any ray $t \mapsto e^t y_0$, for some initial point y_0, its coefficients satisfy $\xi^i(y) = \xi^i(e^t y_0) = e^t \xi(y_0) = e^t \xi_0$. In particular, if $\xi_0 = 0$ then $\xi(y) = 0$ for every point on the ray. Conversely, if we are given some vector ξ_0 at a point y_0 then we can define a vector field ξ along the ray which satisfies $\mathcal{L}_\Delta \xi = 0$ by setting $\xi^i(e^t y_0) = e^t \xi^i_0$, a process quaintly described as dragging ξ_0 along the ray.

A function which is homogeneous of degree 0 and continuous at 0 is constant, while a function which is homogeneous of degree 1 and continuously differentiable at 0 is linear (see [6, Lemma 4.2.5]). No function which is homogeneous of negative degree can be so much as continuous at 0, unless it vanishes completely.

2.3. Connection theory. I denote by ∇ the covariant derivative operator of the Levi-Civita connection of g. Its connection coefficients (Christoffel symbols) are given in linear coordinates by

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{il} \left(\frac{\partial g_{jl}}{\partial y^j} + \frac{\partial g_{jl}}{\partial y^i} - \frac{\partial g_{jk}}{\partial y^l} \right) = \tfrac{1}{2} g^{il} C_{jkl}.$$

That is to say

$$\nabla_\xi \eta = \left(\xi^j \frac{\partial \eta^i}{\partial y^j} + \tfrac{1}{2} g^{il} C_{jkl} \xi^j \eta^k \right) \frac{\partial}{\partial y^i}.$$

Another way of expressing this formula is to say that $\frac{1}{2}C^i_{jk}$ is the difference tensor between ∇ and the covariant differentiation operator D of the standard flat connection on $\mathbb{R}^n$, given in terms of linear coordinates by

$$D_\xi \eta = \xi^j \frac{\partial \eta^i}{\partial y^j} \frac{\partial}{\partial y^i}.$$

Thus in geometrical terms $\frac{1}{2}C^i_{jk}$ has a dual role: it can be regarded as a connection coefficient or as a component of a tensor; the expressions of these two geometrical objects in linear coordinates coincide.

I point out that $(\nabla_\Delta \eta)^i = \Delta \eta^i$, so that $\nabla_\Delta \eta = 0$ if and only if $\Delta \eta^i = 0$, that is, $\nabla_\Delta \eta = 0$ if and only if η is homogeneous of degree -1.

On the other hand, $\nabla_\xi \Delta = \xi$, which can be expressed in the typographically entertaining form $\nabla \Delta = \mathrm{id}$. Notice also that $\nabla \Delta = D\Delta$ (D is defined above): so this is a universal expression, independent of the choice of Minkowski norm.

The curvature tensor is

$$R_{ijkl} = \tfrac{1}{4} g^{mn} (C_{ilm} C_{jkn} - C_{ikm} C_{jln}) = \tfrac{1}{4} (C^m_{il} C_{mjk} - C^m_{ik} C_{mjl}).$$

It is obvious from this formula by inspection that R_{ijkl} enjoys the well-known algebraic properties of a Riemannian curvature tensor. The components of the curvature tensor are homogeneous of degree -2.

Note in particular that contracting R_{ijkl} with y^i on any index gives zero. Moreover, if the components of the metric are constant, so that the C_{ijk} all vanish, then the space is flat, as one would expect.

The sectional curvature of a two-plane is given by $R_{ijkl}\eta^i \zeta^j \eta^k \zeta^l$ where η and ζ form a basis for the two-plane, assumed (without loss of generality) to be unit vectors orthogonal to each other. It follows from the remark above that any two-plane containing a radial vector has zero sectional curvature. To put it another way, if we set $\eta = Y + \lambda\Delta$, $\zeta = Z + \lambda\Delta$ in the expression above for the sectional curvature, where Y and Z are the components of η and ζ orthogonal to Δ, then $R_{ijkl}\eta^i \zeta^j \eta^k \zeta^l = R_{ijkl} Y^i Z^j Y^k Z^l$. In particular, if at $p \in \mathcal{I}$ the sectional curvature of all two-planes tangent to $\mathcal{I}$ vanishes then the sectional curvature of all two-planes vanishes.

I now briefly consider some aspects of the geometry of the indicatrix $\mathcal{I}_r = \{y : F(y) = r\}$, considered as a hypersurface (embedded submanifold of

codimension 1) of the Minkowskian Riemannian space. Following [1] I denote by $\dot{g}$ the metric on $\mathcal{I}_r$ defined by restriction of the ambient metric g, and by $\dot{R}$ the derived curvature. The relation between $\dot{R}$ and R is obtained from the Gauss curvature equations, and is

$$\dot{R}_{ijkl} = R_{ijkl} + r^{-2}(\dot{g}_{ik}\dot{g}_{jl} - \dot{g}_{il}\dot{g}_{jk}).$$

Note that $\dot{R}$, though its components may be expressed in terms of linear coordinates on $\mathbb{R}^n_\circ$, is an object on $\mathcal{I}_r$. In other words, its vector arguments must always be tangent to $\mathcal{I}_r$, or orthogonal to Δ; and similarly for the terms involving $\dot{g}$.

(One can avoid the awkwardness of having to restrict the arguments in what appears to be a tensor equation as follows. First, consider the tensor $h_{ij} = g_{ij} - F_{,i}F_{,j}$: it agrees with g_{ij} when both of its arguments are orthogonal to Δ, and vanishes when either of them is a multiple of Δ. It is therefore the projection of g orthogonal to Δ, and its restriction to $\mathcal{I}_r$ is $\dot{g}$. h is often called the angular metric. It can be written $h_{ij} = FF_{,ij}$. I define a modified covariant derivative $\tilde{\nabla}$ on $\mathbb{R}^n_\circ$ by

$$\tilde{\nabla}_\xi \eta = \nabla_\xi \eta + F^{-2}h(\xi, \eta)\Delta,$$

where as usual ∇ is the Levi-Civita covariant derivative of g. $\tilde{\nabla}$ is evidently a symmetric connection: its restriction to $\mathcal{I}_r$ is the Levi-Civita covariant derivative of $\dot{g}$, and it agrees with ∇ when either of its arguments is a multiple of Δ. Its connection coefficients are

$$\tilde{\Gamma}^i_{jk} = \tfrac{1}{2}g^{il}C_{jkl} + F^{-1}F_{,jk}y^i.$$

Conveniently, its curvature turns out to be

$$\tilde{R}_{ijkl} = R_{ijkl} + F^{-2}(h_{ik}h_{jl} - h_{il}h_{jk}).$$

This is a proper tensor equation. On restriction to $\mathcal{I}_r$, $\tilde{R}$ becomes $\dot{R}$, since it is just the curvature tensor of the Levi-Civita covariant derivative of $\dot{g}$, and h becomes $\dot{g}$. Contracted with Δ on any index $\tilde{R}$ returns 0. The formula above for $\dot{R}$ is effectively equivalent to this expression for $\tilde{R}$, though the latter is fully tensorial. This calculation unexpectedly provides a way of deriving the Gauss equations directly.)

Consider the sectional curvatures of two-planes tangent to $\mathcal{I}$, as calculated with respect to $\dot{g}$ and g. I call the first an intrinsic sectional curvature and

denote it by $\dot{\kappa}(P)$ (where P represents the two-plane), and the second an extrinsic sectional curvature and denote it by $\kappa(P)$. Then using an orthonormal basis for P one sees that

$$\dot{\kappa}(P) = \kappa(P) + 1.$$

If $\mathcal{I}$ has constant intrinsic curvature 1 (that is, if all of its intrinsic sectional curvatures are 1) then all extrinsic sectional curvatures of two-planes tangent to $\mathcal{I}$ are zero. It follows that the sectional curvature of any two-plane in the ambient space is zero. The sectional curvatures determine the curvature tensor, and if all sectional curvatures vanish then $R_{ijkl} = 0$. Thus if $\mathcal{I}$ is of constant intrinsic curvature 1 then the Minkowskian space, as a Riemannian space, is flat. According to Brickell's Theorem, provided that $n \geq 3$ and that F is absolutely homogeneous (that is, it satisfies $F(-y) = F(y)$), the Minkowskian space is Euclidean. The more general case is unresolved.

(See [1, Chapter 14] for a more detailed exposition of some of the results used here, including a more conventional derivation of the Gauss equations, and the fact that sectional curvatures determine the curvature tensor.)

The shape operator S of the hypersurface $\mathcal{I}_r \subset \mathbb{R}^n_\circ$, relative to the unit normal $\hat{\Delta}$, is given by

$$S(X) = -\nabla_X \hat{\Delta},$$

where X is tangent to $\mathcal{I}_r$: it is a type $(1,1)$ tensor field on $\mathcal{I}_r$. Now for any X we have (since $X(F) = 0$)

$$F\nabla_X \hat{\Delta} = \nabla_X \Delta = D_X \Delta = X.$$

That is to say, S is a multiple of the identity I:

$$S = -(1/F)I = -(1/r)I.$$

A hypersurface whose shape operator is a multiple of the identity is said to be totally umbilical. The prime example of a totally umbilical hypersurface is the Euclidean hypersphere. Indicatrices share with Euclidean hyperspheres the property of being totally umbilical hypersurfaces. The indicatrix $\mathcal{I}_r$ has mean curvature $-1/r$ (negative since it curves away from the normal). Remarkably, this holds for any Minkowskian space whatsoever. Don't forget, though, that the definition of the shape operator implicitly involves the ambient metric, which of course differs from Minkowskian space to Minkowskian space. Each indicatrix is umbilical with mean curvature -1, just like the unit hypersphere in Euclidean space: but whereas the curvature of the hypersphere is calculated with respect to the Euclidean background that of the indicatrix

of a different Minkowskian space refers to a different background. The shape operator of the indicatrix of a general Minkowskian space relative to the Euclidean background is quite a different, and rather more complicated, thing: for a start, its unit normal won't be $\hat{\Delta}$. So let us not get too excited by the statement that all indicatrices are umbilical.

In the course of their discussion of intrinsic and extrinsic curvatures, Bao et al. ([1, 14.5]) repeat 'an interesting observation by Xinyue Chen'. He is reported as pointing out that the indicatrices $\mathcal{I}_r$ 'do not look "round" intrinsically', but extrinsically they 'do look "round"'. The use of the word 'round' implies of course comparison with Euclidean spheres. The sphere of radius r has constant intrinsic sectional curvature $1/r$, and constant mean curvature $-1/r$ (relative to the outward normal). For most hypersurfaces of Riemannian manifolds, however, these two kinds of curvature are unrelated. The indicatrix of a Minkowskian space need not have constant intrinsic sectional curvature, so in general does not look round intrinsically. It is however always totally umbilical with mean curvature $-1/r$, so does look round extrinsically – but with the caveat discussed above.

However, special cases are of interest. If the indicatrix does have constant intrinsic curvature 1, then the ambient space is flat, as I pointed out above. What happens if it has constant intrinsic curvature different from 1 is not so clear. If on the other hand an indicatrix should turn out to be totally umbilical with respect to Euclidean ambient space then it would indeed be round, since a connected totally umbilical hypersurface with non-zero mean curvature in a Euclidean space is a standard hypersphere.

3. Killing fields

3.1. Basic properties. A vector field ξ on $\mathbb{R}^n_\circ$ is a Killing field, Killing vector, or infinitesimal isometry, of g if

$$\mathcal{L}_\xi g = 0.$$

The vector field

$$\xi = \xi^i \frac{\partial}{\partial y^i}$$

is a Killing field if and only its components ξ^i satisfy Killing's equation

$$\xi^k \frac{\partial g_{ij}}{\partial y^k} + g_{ik}\frac{\partial \xi^k}{\partial y^j} + g_{jk}\frac{\partial \xi^k}{\partial y^i} = 0.$$

Killing's equation may be written more succinctly as

$$\xi^k C_{ijk} + g_{ik}\xi^k_{,j} + g_{jk}\xi^k_{,i} = 0.$$

Alternatively, with $\xi_i = g_{ik}\xi^k$,

$$\xi_{i,j} + \xi_{j,i} - \xi_k C^k_{ij} = 0.$$

Equivalently

$$\xi_{i;j} + \xi_{j;i} = 0,$$

where the semi-colon indicates a covariant derivative. In short, the 2-covector $\xi_{i;j}$ is skew symmetric. This is a well-known alternative formulation of Killing's equation which holds (mutatis mutandis) for all Riemannian spaces and in general coordinates.

Please note carefully that the definition above is not the same as – or better not the obvious translation of – that of an infinitesimal isometry of a Finsler space according to the standard definition of an isometry of a Finsler space given by Deng and Hou [3]. The latter would amount to defining a Killing field as a vector field that satisfies $\xi(F) = 0$. As will appear later, this difference has important consequences.

To continue briefly in the general Riemannian context, let me remind the reader that, as a consequence of the invariance of g, a Killing field leaves invariant most objects derived from g: for example, the Levi-Civita connection, the curvature, and the metric volume form. (The metric volume form defined by a Riemannian structure is

$$\Omega = \sqrt{\det g}\, dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n.$$

It is to be expected, and can easily be proved, that for any Killing field ξ, $\mathcal{L}_\xi \Omega = 0$.)

Though these things hold true for Riemannian spaces in general, in Riemannian spaces of Minkowskian type there are some unexpected departures from the rule that a Killing field leaves invariant objects derived from g.

Back to the Minkowskian case. On carrying out the Legendre transformation $y^i \mapsto y_j = g_{ij}y^i$, using the formula $\partial/\partial y^i = g_{ij}\partial/\partial y_j$, Killing's equation converts into

$$\xi_k \frac{\partial g^{ij}}{\partial y_k} + g^{ik}\frac{\partial \xi_k}{\partial y_j} + g^{jk}\frac{\partial \xi_k}{\partial y_i} = 0, \qquad \xi_j = g_{jk}\xi^k.$$

This shows that $\xi_k \partial/\partial y_k$ is a Killing field of the metric derived from E^* on $\mathbb{R}^{n*}_\circ$ if and only if $\xi^k \partial/\partial y^k$ is a Killing field of the metric derived from E on $\mathbb{R}^n_\circ$.

From Killing's equation

$$\xi^l C_{ijl} + g_{il}\xi^l_{,j} + g_{jl}\xi^l_{,i} = 0,$$

differentiating with respect to y^k yields

$$g_{jl}\xi^l_{,ik} + g_{il}\xi^l_{,jk} + C_{jkl}\xi^l_{,i} + C_{ikl}\xi^l_{,j} + C_{ijl}\xi^l_{,k} + C_{ijk,l}\xi^l = 0.$$

The terms involving C are collectively symmetric in i, j and k, the terms involving g only in i and j. Interchanging j and k gives $g_{jl}\xi^l_{,ik} = g_{kl}\xi^l_{,ij}$, whence

$$2g_{kl}\xi^l_{,ij} + C_{jkl}\xi^l_{,i} + C_{ikl}\xi^l_{,j} + C_{ijl}\xi^l_{,k} + C_{ijk,l}\xi^l = 0,$$

or equivalently

$$\xi^k_{,ij} = -\tfrac{1}{2}g^{km}(C_{jml}\xi^l_{,i} + C_{iml}\xi^l_{,j} + C_{ijl}\xi^l_{,m} + C_{ijm,l}\xi^l).$$

One important point to note about this expression is that the right-hand side is homogeneous linear in ξ^i and $\xi^i_{,j}$, because it allows one to use known facts about systems of homogeneous linear ordinary differential equations to prove results on the uniqueness of Killing fields. Using standard arguments one can show that a Killing field is uniquely determined by the values of ξ^i and $\xi^i_{,j}$ at any chosen point p.

I point out in passing the interesting and perhaps unexpected fact that for any Killing field ξ, $\xi^k_{,ij} y^i y^j = 0$.

In the expression

$$g_{kl}\xi^l_{,ij} + C_{jkl}\xi^l_{,i} + C_{ikl}\xi^l_{,j} + C_{ijl}\xi^l_{,k} + C_{ijk,l}\xi^l = 0.$$

the terms not involving $\xi^l_{,ij}$ actually form $(\mathcal{L}_\xi C)_{ijk}$:

$$(\mathcal{L}_\xi C)_{ijk} = \xi^l C_{ijk,l} + C_{jkl}\xi^l_{,i} + C_{ikl}\xi^l_{,j} + C_{ijl}\xi^l_{,k}.$$

Thus for a Killing field ξ, $\mathcal{L}_\xi C = 0$ if and only if $\xi^k_{,ij} = 0$, that is, if and only ξ^k is an affine function of linear coordinates.

(This is a result about the components of the tensor C, which as I pointed out earlier are not the same as the connection coefficients of the Levi-Civita connection of g, despite the fact that the two effectively agree when expressed

in terms of linear coordinates. The condition for ξ to preserve the connection is in origin quite different from the one for the tensor: in fact it turns out to be the formula for $\xi^k_{,ij}$ given above, which we know holds for Killing fields. So this result does not contradict the fact that, in any Riemannian space, infinitesimal isometries are connection-preserving. With hindsight, indeed, the result is not unexpected: transforms of connection coefficients typically involve second derivatives, after all, while transforms of tensors involve only first derivatives. Since as a tensor C is the difference tensor of the Levi-Civita connection and the flat connection, for a transformation to preserve the Levi-Civita connection and the difference tensor it must also preserve the flat connection, that is, be affine with respect to linear coordinates.)

I denote the vector space of Killing fields by $\mathfrak{k}$. Recall that for $\xi \in \mathfrak{k}$, $\xi_{i;j} + \xi_{j;i} = 0$. Denote the vector space of skew-symmetric bilinear forms on $\mathbb{R}^n$ by $\mathbb{S}_n$. By uniqueness the map $\mathfrak{k} \to \mathbb{R}^n \oplus \mathbb{S}_n$ by $\xi \mapsto (\xi^i(p), \xi_{i;j}(p))$ is injective, so $\dim \mathfrak{k} \leq n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$.

The Lie bracket of any two Killing fields is again a Killing field, and since, as we now know, $\mathfrak{k}$ is finite-dimensional it is a Lie algebra.

(The result just derived is standard, but not all of the methods used to obtain it.)

This seems as good a point as any to make a few remarks about skew symmetry. One reason why skew symmetry is of interest is that $\mathfrak{so}(n)$, the Lie algebra of the orthogonal group in n dimensions, is the Lie algebra of $n \times n$ skew-symmetric matrices with the commutator as bracket: and when it acts on $\mathbb{R}^n$, $\mathfrak{so}(n)$ is a key ingredient of the Killing algebra of Euclidean space – it is the Lie algebra of infinitesimal rotations.

The argument above about uniqueness depended on the fact that for a Killing field ξ, the $\xi_{i;j}$ are components of a skew-symmetric bilinear form. However, in most applications I prefer to deal with the type $(1,1)$ linear tensor whose components are $\xi^i_{,j}$. Suppose that $\xi(p) = 0$ at some point p. Then, at p, $g_{ik}\xi^k_{,j} + g_{jk}\xi^k_{,i} = 0$. This says that $\xi^i_{,j}(p)$ is skew-symmetric with respect to $g_{ij}(p)$, in the sense that I shall now explain. In order to avoid continually having to repeat the phrase 'at p', let me consider the $\xi^i_{,j}(p)$ as the components of a matrix R on a vector space V equipped with a scalar product g. Then R is skew-symmetric with respect to g if and only if $g(R(u), v) + g(u, R(v)) = 0$ for all $u, v \in V$. Now the conventional way of defining skew symmetry of a matrix is to say that a matrix M is skew-symmetric if $M(u) \cdot v + u \cdot M(v) = 0$. If R is skew-symmetric with respect to g and we denote by M the matrix whose entries are the components of R with respect to an orthonormal basis for g

then $M(u) \cdot v + u \cdot M(v) = 0$, and conversely. To put it another way, we can find a non-singular linear transformation L of V such that $g(L(u), L(v)) = u \cdot v$ (that is, we can convert g to the standard scalar product by a linear change of coordinates, determined in fact by the components of an orthonormal basis with respect to the original basis). Then $M = L^{-1}RL$. So a matrix R which is skew-symmetric with respect to g is simply a matrix which is skew-symmetric in the usual sense, subject to a change of basis. What works for conventionally skew-symmetric matrices works equally well for those skew-symmetric with respect to some other scalar product. I shall tend therefore to drop the qualifier 'with respect to g' from the phrase 'skew-symmetric with respect to g'.

Consider next a linear vector field ξ, with $\xi^i = R^i_j y^j$, and suppose that $\xi(p) = 0$. Since ξ is linear, $\mathcal{L}_\xi C = 0$, so that at p

$$C_{jkl}R^l_i + C_{ikl}R^l_j + C_{ijl}R^l_k = 0,$$

given that $\xi^l = 0$. Again, let me write this as

$$C\big(R(u), v, w\big) + C\big(u, R(v), w\big) + C\big(u, v, R(w)\big) = 0, \qquad \text{for all } u, v, w \in V$$

where C is to be thought of as a completely symmetric trilinear form. The most basic kind of skew-symmetric matrix is one which satisfies $R(u) = v$, $R(v) = -u$, for an orthonormal pair of vectors u and v, and $R(w) = 0$ for w orthogonal to u and v. (It is easy to check that R is skew-symmetric. In fact the set of such matrices with u and v taken in pairs from an orthonormal basis of V forms a basis for the vector space of all skew-symmetric matrices.) For this R we have

$$\begin{aligned}
C\big(R(u), u, u\big) + C\big(u, R(u), u\big) + C\big(u, u, R(u)\big) = 0 &\Rightarrow C(u, u, v) = 0,\\
C\big(R(u), u, v\big) + C\big(u, R(u), v\big) + C\big(u, u, R(v)\big) = 0 &\Rightarrow C(u, u, u) = 2C(u, v, v),\\
C\big(R(u), v, v\big) + C\big(u, R(v), v\big) + C\big(u, v, R(v)\big) = 0 &\Rightarrow C(v, v, v) = 2C(u, u, v),\\
C\big(R(v), v, v\big) + C\big(v, R(v), v\big) + C\big(v, v, R(v)\big) = 0 &\Rightarrow C(u, v, v) = 0.
\end{aligned}$$

That is to say, C vanishes on the two-dimensional subspace of V spanned by u and v. This result will make a brief reappearance later.

3.2. Homogeneity of Killing fields. It may be thought that since Killing fields preserve g, and g is directly derived from E and vice versa, that Killing fields preserve E. However things are more complicated than that: and homogeneity plays a significant part in explaining why.

For a Killing field ξ we have from Killing's equation, by contracting with y^j,

$$\begin{aligned} 0 &= g_{ik}y^j\xi^k_{,j} + g_{jk}y^j\xi^k_{,i} \\ &= g_{ik}\Delta\xi^k + \big(g_{jk}y^j\xi^k\big)_{,i} - C_{ijk}y^j\xi^k - g_{ik}\xi^k \\ &= g_{ik}\big(\Delta\xi^k - \xi^k\big) + \big(\xi(E)\big)_{,i}. \end{aligned}$$

So if $\xi(E) = 0$ then ξ is homogeneous of degree 0. If ξ is a 0-homogeneous Killing field then $\xi(E)$ is constant on $\mathbb{R}^n_\circ$. Now when ξ is 0-homogeneous,

$$\Delta\big(\xi(E)\big) = \xi\big(\Delta(E)\big) = 2\xi(E).$$

But $\xi(E)$ is constant, so the left-hand side is zero, and therefore $\xi(E) = 0$. In summary a Killing field preserves E, or equivalently F, if and only if it is 0-homogeneous. Equally a Killing field is tangent to indicatrices if and only if it is 0-homogeneous.

I next consider the converse question: when is a vector field ξ which satisfies $\xi(E) = 0 = g_{ij}y^i\xi^j$ a Killing field? On differentiating that equation twice we find that

$$\begin{aligned} g_{ij}\xi^j + g_{jk}y^j\xi^k_{,i} &= 0, \\ C_{ijk}\xi^k + g_{ik}\xi^k_{,j} + g_{jk}\xi^k_{,i} + g_{kl}y^l\xi^k_{,ij} &= 0. \end{aligned}$$

This doesn't reveal a clear answer in general: but if $\xi^k_{,ij} = 0$ then ξ is a Killing field. Bearing in mind that ξ must be 0-homogeneous, that amounts to ξ being linear in terms of linear coordinates, that is, $\xi^i = R^i_j y^j$ with constant R^i_j. So a linear vector field which is tangent to indicatrices is a Killing field. Indeed, by homogeneity it is sufficient for ξ to be tangent to any single indicatrix, say $\mathcal{I}$.

Let us go back to the consideration of homogeneous Killing fields. Suppose that ξ is an r-homogeneous Killing field with $r \neq -1$. Then

$$\begin{aligned} 0 = \mathcal{L}_\xi g(\Delta,\Delta) &= 2\xi(E) - 2g(\mathcal{L}_\xi\Delta,\Delta) \\ &= 2\xi(E) + 2g(\mathcal{L}_\Delta\xi,\Delta) = 2(1+r)\xi(E). \end{aligned}$$

But we know that if $\xi(E) = 0$ then ξ must be 0-homogeneous. So the only homogeneous Killing fields are of degree 0 or degree -1.

Note that

$$\mathcal{L}_\Delta(\mathcal{L}_\xi g) = \mathcal{L}_\xi(\mathcal{L}_\Delta g) + \mathcal{L}_{\mathcal{L}_\Delta\xi}g = 2\mathcal{L}_\xi g + \mathcal{L}_{\mathcal{L}_\Delta\xi}g,$$

so that if $\mathcal{L}_\xi g = 0$ then $\mathcal{L}_{\mathcal{L}_\Delta \xi} g = 0$. That is to say, if ξ is a Killing field so is $\mathcal{L}_\Delta \xi$. Of course if ξ is a homogeneous Killing field this tells us nothing new. But suppose that ξ is a Killing field which is a linear combination of homogeneous vector fields, say (dealing with the case of most interest)

$$\xi = \lambda\rho + \mu\tau,$$

with ρ a 0-homogeneous vector field and τ one of degree -1, where λ and μ are non-zero constants. (We don't really need λ and μ in fact, because each can be absorbed into the corresponding vector field.) Then

$$\mathcal{L}_\Delta \xi = -\mu\tau,$$

so τ is a Killing field, and therefore ρ, as a linear combination of Killing fields, is also one. That is, in this particular case, if a Killing field is a linear combination of homogeneous vector fields (of different degrees) then each homogeneous component is a Killing vector field.

The bracket of two Killing fields is again a Killing field. For any two vector fields ξ and η,

$$\mathcal{L}_\Delta[\xi, \eta] = [\mathcal{L}_\Delta \xi, \eta] + [\xi, \mathcal{L}_\Delta \eta].$$

So the bracket of two 0-homogeneous Killing fields is again a 0-homogeneous Killing field. Next, if ξ is 0-homogeneous and η is -1-homogeneous then $[\xi, \eta]$ is -1-homogeneous. Finally, if ξ and η are both -1-homogeneous then $[\xi, \eta]$ is, in principle, -2-homogeneous. But there are no non-zero -2-homogeneous Killing fields. Thus, in this case, $[\xi, \eta] = 0$.

I call the space of 0-homogeneous Killing fields $\mathcal{R}$, the space of -1 homogeneous Killing fields $\mathcal{T}$, and their vector space direct sum $\mathcal{K}$. We know that $\mathcal{R}$ is a subalgebra of $\mathcal{K}$ while $\mathcal{T}$ is an Abelian ideal in $\mathcal{K}$, which as a Lie algebra is therefore the semi-direct sum of $\mathcal{R}$ and $\mathcal{T}$.

The next task is to investigate the relationship between $\mathcal{K}$ and $\mathfrak{k}$. The question is, given an arbitrary $\xi \in \mathfrak{k}$, are there vector fields $\rho \in \mathcal{R}$ and $\tau \in \mathcal{T}$ such that $\xi = \rho + \tau$? Here the fact that for any Killing field ξ, $\xi^k_{,ij} y^i y^j = 0$, proved earlier, becomes significant. This equation may be rewritten in terms of $\Delta\xi^k$:

$$0 = \xi^k_{,ij} y^i y^j = (\xi^k_{,ij} y^i) y^j = (\xi^k_{,i} y^i)_{,j} y^j - \xi^k_{,j} y^j = \Delta^2 \xi^k - \Delta\xi^k.$$

That is to say, $\Delta\xi^i$ is of homogeneity degree 1 and $\xi^k - \Delta\xi^k$ is of homogeneity degree 0. So if we take $\rho^k = \Delta\xi^k$ then $\rho \in \mathcal{R}$, and if $\tau^k = \xi^k - \Delta\xi^k$ then

$\tau \in \mathcal{T}$: and $\rho + \tau = \xi$. Thus $\mathcal{K} = \mathfrak{k}$. (This essentially repeats a calculation just above, but with the extra information about $\Delta\xi^i$.)

We need ways of identifying elements of $\mathcal{R}$ and $\mathcal{T}$ more readily.

So far as $\mathcal{R}$ is concerned, we may use the fact that it consists of Killing fields which are tangent to indicatrices. I shall show that if $\xi \in \mathcal{R}$ then its restriction $\dot{\xi}$ to the indicatrix $\mathcal{I}$, or indeed any indicatrix, is a Killing field of $\dot{g}$, the restriction of g to $\mathcal{I}$. Let X, Y be any vector fields on $\mathbb{R}^n_\circ$ which are tangent to $\mathcal{I}$. Then

$$\begin{aligned} 0 = \mathcal{L}_\xi g(X,Y) &= \xi\big(g(X,Y)\big) - g\big([\xi,X],Y\big) - g\big(X,[\xi,Y]\big) \\ &= \dot{\xi}\big(\dot{g}(X,Y)\big) - \dot{g}\big([\dot{\xi},X],Y\big) - \dot{g}\big(X,[\dot{\xi},Y]\big) = \mathcal{L}_{\dot{\xi}}\dot{g}(X,Y), \end{aligned}$$

since $[\xi, X]$ is tangent to $\mathcal{I}$, and equal to $[\dot{\xi}, X]$ there. Conversely, suppose that we have a vector field $\dot{\xi}$ on $\mathcal{I}$ which is a Killing field of $\dot{g}$, the restriction of g to $\mathcal{I}$. By dragging $\dot{\xi}$ along rays we generate a vector field ξ on $\mathbb{R}^n_\circ$ which satisfies $\mathcal{L}_\Delta \xi = 0$; and likewise for X and Y. Then (remembering that $\mathcal{L}_\Delta g = 2g$)

$$\begin{aligned} \Delta\big(\mathcal{L}_\xi g(X,Y)\big) &= \Delta\Big(\xi\big(g(X,Y)\big) - g\big([\xi,X],Y\big) - g\big(X,[\xi,Y]\big)\Big) \\ &= \xi\big(\Delta(g(X,Y)\big) - 2g\big([\xi,X],Y\big) - 2g\big(X,[\xi,Y]\big) \\ &= 2\Big(\xi\big(g(X,Y)\big) - g\big([\xi,X],Y\big) - g\big(X,[\xi,Y]\big)\Big) \\ &= 2\mathcal{L}_\xi g(X,Y). \end{aligned}$$

That is to say, $\mathcal{L}_\xi g(X,Y)$ satisfies a homogeneous linear ordinary differential equation along each ray. But it vanishes on $\mathcal{I}$: it follows that $\mathcal{L}_\xi g(X,Y) = 0$. To show (as I intend) that $\mathcal{L}_\xi g = 0$ we must consider $\mathcal{L}_\xi g(X,\Delta)$ (where $g(X,\Delta) = 0$) and $\mathcal{L}_\xi g(\Delta,\Delta)$. For the first,

$$\mathcal{L}_\xi g(X,\Delta) = \xi\big(g(X,\Delta)\big) - g\big([\xi,X],\Delta\big) - g\big(X,[\xi,\Delta]\big) = 0$$

because $[\xi, X]$ is tangent to indicatrices, and so orthogonal to Δ, when both ξ and X are. Finally, and familiarly,

$$\mathcal{L}_\xi g(\Delta,\Delta) = 2\xi(E) = 0.$$

For $\xi \in \mathcal{T}$, on the other hand, we know that, since $\Delta\xi^i = 0$, $(\xi(E))_{,i} = g_{ik}\xi^k$. But then

$$\begin{aligned} \big(\xi(E)\big)_{,ij} &= g_{ik}\xi^k_{,j} + C_{ijk}\xi^k \\ &= g_{jk}\xi^k_{,i} + C_{ijk}\xi^k, \end{aligned}$$

since of course the left-hand side is symmetric in i and j. It follows that

$$g_{ik}\xi^k_{,j} = g_{jk}\xi^k_{,i} = -\tfrac{1}{2}C_{ijk}\xi^k.$$

Equally, we may write

$$\xi_{i;j} = \xi_{j;i} = 0.$$

Conversely, if ξ satisfies these conditions then, firstly, $\xi_{i;j} + \xi_{j;i} = 0$ so ξ is a Killing field; and secondly

$$g_{jk}\Delta\xi^k = g_{jk}\xi^k_{,i}y^i = -\tfrac{1}{2}C_{ijk}y^i\xi^k = 0,$$

so $\Delta\xi^k = 0$ and $\xi \in \mathcal{T}$. That is to say, $\mathcal{T}$ is the space of covariant constant, or as it is more usually called, parallel, vector fields of the Levi-Civita connection.

One important consequence of this is that no nonzero element of $\mathcal{T}$ can vanish at any point: for if $\xi \in \mathcal{T}$ and $\xi^i(p) = 0$ then $\xi^i_{,j}(p) = 0$, so $\xi = 0$.

Furthermore any $\xi \in \mathcal{T}$ has constant length, that is, $g(\xi, \xi)$ is constant; the integral curves of ξ are affinely parametrised geodesics; and $R^i_{jkl}\xi^j = 0$ where R is the curvature tensor. These in fact are all properties of a parallel vector field on any Riemannian manifold, as is demonstrated in the appendix on parallel vector fields.

To conclude:

THEOREM 1. *The Lie algebra $\mathfrak{k}$ of the Killing fields of the metric g of any Minkowskian space, if it is not empty, is the semi-direct sum of two subalgebras $\mathcal{R}$ and $\mathcal{T}$, where $\mathcal{T}$ is an Abelian ideal in $\mathfrak{k}$. The subalgebra $\mathcal{R}$ consists of Killing fields which are homogenous of degree 0; it can be identified with the Lie algebra of Killing fields of the metric induced by g on the indicatrix $\mathcal{I}$. The ideal $\mathcal{T}$ consists of Killing fields which are homogenous of degree -1; equivalently, it is the Abelian Lie algebra of vector fields which are parallel with respect to the Levi-Civita connection of g.*

In the case of Euclidean space $\mathcal{R}$ consists of the infinitesimal generators of rotations, $\mathcal{T}$ of translations (whence the notation).

Of course, we have no reason to believe that $\mathcal{I}$ has any nonzero Killing fields; nor that there are any nonzero parallel vector fields. If there are neither than the Minkowskian space is no longer of any interest so far as this enquiry is concerned. I proceed on the assumption that there are some nonzero Killing fields of one kind or the other.

3.3. Further properties of $\mathcal{T}$. I want first to consider how the value of E changes as one moves along an integral curve of an element of $\mathcal{T}$. (In the case of $\mathcal{R}$ the answer is obvious of course: it doesn't.)

Take $\tau \in \mathcal{T}$ with integral curve $t \mapsto \sigma(t)$, where $\sigma(0) = y_0$ say. Set $E(t) = E(\sigma(t))$. Then

$$\dot{E} = E_{,i}\dot{\sigma}^i = E_{,i}\tau^i,$$
$$\ddot{E} = E_{,i}\ddot{\sigma}^i + E_{,jk}\dot{\sigma}^j\dot{\sigma}^k = E_{,i}\ddot{\sigma}^i + g_{jk}\dot{\sigma}^j\dot{\sigma} = E_{,i}\ddot{\sigma}^i + g_{jk}\tau^j\tau^k.$$

But τ, being parallel, has constant length, say λ (and of course $\lambda > 0$). So $\ddot{E} = E_{,i}\ddot{\sigma}^i + \lambda$. Now σ, being an integral curve of a parallel vector field, is a geodesic: that is,

$$\ddot{\sigma}^k + \tfrac{1}{2}C^k_{ij}\dot{\sigma}^i\dot{\sigma}^j = 0.$$

Since $E_{,k} = g_{kl}y^l$ we obtain

$$E_{,k}\ddot{\sigma}^k = -\tfrac{1}{2}g_{kl}y^lC^k_{ij}\dot{\sigma}^i\dot{\sigma}^j = -\tfrac{1}{2}y^lC_{lij}\dot{\sigma}^i\dot{\sigma}^j = 0.$$

It follows that $\ddot{E} = \lambda$, that is,

$$E(t) = \tfrac{1}{2}\lambda t^2 + \dot{E}(0)t + E(y_0).$$

I shall be interested below in the case in which τ is tangent to the indicatrix through y_0, that is, $\dot{E}(0) = E_{,k}(y_0)\tau^k(y_0) = 0$. For such τ

$$E(t) = \tfrac{1}{2}\lambda t^2 + E(0), \qquad \text{so } E(t) > E(0).$$

The important point to note is that an integral curve of an element of $\mathcal{T}$, if once tangent to an indicatrix, cannot meet the same indicatrix anywhere else.

For a given point p, consider the linear map $\mathfrak{k} \to T_p\mathbb{R}^n_\circ$ by $\xi \in \mathfrak{k} \mapsto \xi(p)$. Its image is a subspace of $T_p\mathbb{R}^n_\circ$ and its kernel is just the space of Killing fields that vanish at p. Let me denote the image space at p by $\mathfrak{k}_p$. Then $\mathfrak{k}$ defines a distribution on $\mathbb{R}^n_\circ$, whose constituent subspaces are the $\mathfrak{k}_p$. Since $\mathfrak{k}$ is closed under bracket (it is a Lie algebra after all) this distribution is involutive and therefore integrable by Frobenius' Theorem: the leaves of the resultant foliation are the orbits of the integrated action of $\mathfrak{k}$. I denote by $\mathfrak{L}_p$ the leaf of the foliation through p: it is a submanifold of $\mathbb{R}^n_\circ$ whose tangent space at p is $\mathfrak{k}_p$. The ideal $\mathcal{T}$ of $\mathfrak{k}$ is also integrable, and its leaves are contained in those of $\mathfrak{k}$. I denote its leaf at p by $\bar{\mathfrak{L}}_p$.

Of course if $p' \in \mathfrak{L}_p$ then $\mathfrak{L}_{p'} = \mathfrak{L}_p$. It is a property of leaves of a foliation that two leaves are either completely distinct or they coincide.

I shall show that, provided of course that $\mathcal{T} \neq \{0\}$, at each point p, $T_p\mathfrak{L}_p = \mathfrak{k}_p = T_p\bar{\mathfrak{L}}_p = \mathcal{T}_p$.

Consider first $\mathcal{T}$. There is one leaf of the distribution which would pass through the origin if it could. On any other leaf $\bar{\mathfrak{L}}_p$ there is a point at which it is tangent to an indicatrix: without loss of generality (that is, by choosing another leaf if necessary) we may take the indicatrix to be $\mathcal{I}$. It is a point at which F takes its minimum value on the leaf. Such a point is unique on the leaf, by the argument that began this subsection. This will be my choice of basepoint: I'll call it q, for clarity.

Now $\mathcal{R}$ consists of Killing fields which are tangent to indicatrices. That is to say, $\mathcal{R}_q$ is tangent to $\mathcal{I}$ at q, as well as $\mathcal{T}_q$. But $\mathfrak{k}_q = \mathcal{T}_q \oplus \mathcal{R}_q$. It follows that if $\bar{\mathfrak{L}}_q$ is tangent to $\mathcal{I}$ at q then so is $\mathfrak{L}_q$. Suppose that there were some $\xi \in \mathcal{R}$ with $\xi(q) \neq 0$. Let $t \mapsto \rho_t$ be its flow, so that $t \mapsto \rho_t q$ is the integral curve of ξ through q. The integral curve of any Killing field lies in the leaf of $\mathfrak{k}$ through its initial point. It follows that ρ_t maps $\mathfrak{L}_q$ to itself. Moreover, since $[\mathcal{R}, \mathcal{T}] \subset \mathcal{T}$, ρ_t maps $\bar{\mathfrak{L}}_q$ to itself. But $\rho_t q \in \mathcal{I}$. That is to say, $\mathfrak{L}_q$ would be tangent to $\mathcal{I}$ at $\rho_t q$, and $\bar{\mathfrak{L}}_q$ likewise. But again, the argument that began this subsection shows that this cannot hold.

It follows that for all $\xi \in \mathcal{R}$ we must have $\xi(q) = 0$. Now $\mathfrak{k}$ projects surjectively onto $T_q\mathfrak{L}_q$. Since $\mathcal{R}_q = \{0\}$, and $\mathcal{T}$ has no zeros, it follows that the kernel of the projection is precisely $\mathcal{R}_q$, and so $\mathcal{T}_q = T_q\mathfrak{L}_q$. In particular, $\dim \mathcal{T} = \dim \mathfrak{L}_q = \dim \mathfrak{L}_p$ for every p. Now $\mathcal{T}$ consists of Killing fields, that is, $\mathcal{T} \subseteq \mathfrak{k}$, whence for every p, $\mathcal{T}_p \subseteq T_p\mathfrak{L}_p$. But their dimensions are the same, so in fact $\mathcal{T}_p = T_p\mathfrak{L}_p$. In other words, the leaves of the integrable distribution defined by $\mathfrak{k}$ coincide with the leaves of the integrable distribution defined by $\mathcal{T}$. Moreover, elements of $\mathcal{T}$ never vanish, and they commute, so we can find coordinates z^a, $a = 1, 2, \ldots, \dim \mathcal{T}$, such that $\mathcal{T}$ is spanned by the corresponding coordinate fields. However, such coordinates are not necessarily linear.

In the discussion above I have made no assumptions about the behaviour of elements of $\mathcal{T}$ at the origin of $\mathbb{R}^n$. The components of elements of $\mathcal{T}$ are 0-homogeneous, and as I have pointed out earlier, a function which is homogeneous of degree 0 and continuous at 0 is constant. If all elements of $\mathcal{T}$ have components which are continuous at the origin then the situation regarding linearity is much improved, as I show in the next section.

3.4. A SIMPLE EXAMPLE. There is a straightforward situation in which a Minkowskian space has a non-vanishing Killing algebra: when it is a Rie-

mannian product of Minkowskian spaces one component of which is an Euclidean space. Suppose the second component admits no Killing fields. Then the Killing algebra of the product is the familiar Lie algebra of infinitesimal isometries of the Euclidean component, acting in the fibres of the projection onto the other component. In particular $\mathcal{T}$ consists of constant vector fields (vector fields whose components are constant with respect to linear coordinates).

I now consider the converse.

One obvious subspace of $\mathcal{T}$ consists of constant vector fields ξ which satisfy $\xi^k C_{ijk} = 0$. I shall suppose that in fact all elements of $\mathcal{T}$ take this form. This occurs if and only if all elements of $\mathcal{T}$ have components which are continuous at the origin.

In this case the leaves of the distribution spanned by $\mathfrak{k}$, equivalently by $\mathcal{T}$, consist of parallel affine subspaces of $\mathbb{R}^n$, including one which contains the origin (which is of course a vector subspace of $\mathbb{R}^n$).

We can choose linear coordinates y^a, $a = 1, 2, \ldots, d$, $d = \dim \mathcal{T}$, and y^α, $\alpha = d+1, d+2, \ldots, n$, so that the leaves of $\mathcal{T}$ are given by $y^\alpha = c^\alpha$ for constants c^α, and the y^a are coordinates on each leaf (in particular on the leaf $y^\alpha = 0$, the one through the origin). In such coordinates the condition that determines $\mathcal{T}$, which is that it consists of constant vector fields that satisfy $\xi^k C_{ijk} = 0$, comes down to

$$\frac{\partial g_{ij}}{\partial y^a} = 0 = C_{ija}, \qquad a = 1, 2, \ldots, d.$$

Since

$$C_{ijk} = \frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{jk}}{\partial y^i} = \frac{\partial g_{ki}}{\partial y^j},$$

the condition $C_{ija} = 0$ amounts to the facts that g_{ai} is constant, that is, g_{ab} and $g_{a\alpha}$ are both constant for all a and α. Moreover, $g_{\alpha\beta}$ is independent of y^a, in other words is a function of y^γ alone. If we make the coordinate transformation $\hat{y}^a = y^a + g^{ab} g_{b\alpha} y^\alpha$, $\hat{y}^\alpha = y^\alpha$ (where $g^{ac} g_{cb} = \delta^a_b$), which is linear because the coefficients g^{ab} and $g_{a\alpha}$ are constants, we find that $\hat{g}_{ab} = g_{ab}$, $\hat{g}_{\alpha\beta} = g_{\alpha\beta}$, and $\hat{g}_{a\alpha} = 0$. This coordinate transformation amounts simply to a change of origin of the coordinates in each leaf. I assume below that this coordinate transformation has already been made.

Each Killing field ξ is tangent to every leaf, that is, is a linear combination of the vector fields $\partial/\partial y^a$, or again satisfies $\xi^\alpha = 0$, $\alpha = d+1, d+2, \ldots, n$. Recall that for any Killing field

$$\xi^k_{,ij} = -\tfrac{1}{2}g^{km}\big(C_{jml}\xi^l_{,i} + C_{iml}\xi^l_{,j} + (C_{ijl}\xi^l)_{,m}\big).$$

Since $\xi^\alpha = 0$ this reduces to

$$\xi^a_{,ij} = -\tfrac{1}{2}g^{am}\big(C_{jmb}\xi^b_{,i} + C_{imb}\xi^b_{,j} + (C_{ijb}\xi^b)_{,m}\big).$$

But $C_{ija} = 0$ for every $a = 1, 2, \ldots, d$, so $\xi^a_{,ij} = 0$, which is to say that ξ^a is an affine function of the y^i:

$$\xi^a = R^a_i y^i + T^a \qquad \text{for some constants } R^a_i \text{ and } T^a.$$

Killing's equation for ξ gives $g_{ac}R^c_b + g_{bc}R^c_b = 0$ and $g_{ab}R^b_\alpha = 0$. From the latter, $R^b_\alpha = 0$. So the Killing fields take the form

$$\xi^a = R^a_b y^b + T^a \qquad \text{for some constants } R^a_b \text{ and } T^a,$$

where R is skew-symmetric. This is of course the familiar expression from the Euclidean case, acting in the leaves of the distribution determined by $\mathcal{T}$. The Minkowskian space is therefore a Riemannian product of Minkowskian spaces, one component of which is Euclidean, the other being undetermined. Assuming that the original Killing algebra is the full Killing algebra of the Minkowskian space the second component can have no Killing fields of its own.

3.5. The maximally symmetric case. I consider now the case of an n-dimensional Minkowskian space for which $\dim \mathfrak{k} = \frac{1}{2}n(n+1)$, so that the Minkowskian space is maximally symmetric. It is well known that a maximally symmetric Riemannian space is a space of constant curvature: its curvature tensor takes the form

$$R_{ijkl} = \kappa(g_{il}g_{jk} - g_{ik}g_{jl})$$

for constant κ. If the manifold is simply connected and geodesically complete then it is one of the three standard space forms: Euclidean if $\kappa = 0$, spherical if κ is positive, hyperbolic if κ is negative. Unfortunately $\mathbb{R}^n_\circ$ is not complete, so we cannot apply this result directly. However, $\mathcal{I}$ is. Moreover, in our context we have to bear linearity in mind.

However, let us consider the relation above for the whole Minkowskian space (not just for $\mathcal{I}$). It is a tensor equation (and κ is a scalar) so holds in the same form in linear coordinates. Clearly, if $\kappa = 0$ then the space is flat.

I show that κ must be 0. Let X be any vector field orthogonal to Δ, that is, tangent to indicatrices. Then

$$R_{ijkl}y^iX^jX^ky^l = \kappa(g_{il}y^iy^l)(g_{jk}X^jX^k).$$

The left-hand side is zero, since $R_{ijkl}y^l = 0$ (recall that the sectional curvature of any two-plane containing Δ vanishes). On the other hand, $g_{il}y^iy^l \neq 0$, $g_{jk}X^jX^k \neq 0$. So $\kappa = 0$. So $R_{ijkl} = 0$: the space is flat. However, this is not in general enough to ensure that the space is Euclidean: to establish that it is, we may appeal to Brickell's theorem, but to do so we have to assume that the norm F is absolutely homogeneous. Thus:

THEOREM 2. *A maximally symmetric Minkowskian space whose norm is absolutely homogeneous is Euclidean.*

It is easy to see that $\dim\mathcal{T}$ must be n, in which case the dimension of its complement $\mathcal{R}$ is $\frac{1}{2}n(n-1)$. Then $\mathcal{R}$ consists of Killing fields of $\mathcal{I}$, whose dimension is $n-1$, and so it has the maximum possible dimension for the Lie algebra of such Killing fields.

First, consider $\mathcal{T}$. Its dimension is n. Every element τ of $\mathcal{T}$ is parallel, and therefore satisfies $R^i_{jkl}\tau^j = 0$ (see the appendix on parallel vector fields for the straightforward proof). Since for any p we can choose $\tau(p)$ to be any element of $T_p\mathbb{R}^n_\circ$ we must have $R^i_{jkl} = 0$: the Riemannian space is flat. Thus:

THEOREM 3. *A Minkowskian space whose norm is absolutely homogeneous and for which $\mathcal{T}$ has maximal dimension is Euclidean.*

Secondly, suppose that $\dim\mathcal{R} = \frac{1}{2}n(n-1)$. Then the indicatrix is a space of positive constant curvature. If the curvature is 1, then provided its norm is absolutely homogeneous the space is Euclidean, as I pointed out in Subsection 2.3. However, there seems to be no good reason why the curvature should be 1. It might be thought that carrying out a rescaling, that is, making a homothetic transformation, would rescale the curvature: but bearing in mind that this operation would also change the indicatrix, in fact the curvature would be unchanged. Likewise, changing to a new indicatrix has no effect. The problem is that the formula $\dot\kappa = \kappa+1$, derived from the Gauss equations, holds only for the indicatrix $\mathcal{I}$: the corresponding equation for $\mathcal{I}_r$ is $\dot\kappa = \kappa + r^{-2}$. So what you win on the swings by changing the indicatrix you lose on the roundabouts. As I discussed earlier (in Subsectionn 2.3) the problem arises because one has to consider not just the intrinsic geometry of the indicatrix but also its relation to the ambient space.

So I shall have to add this condition on the curvature as an extra hypothesis.

THEOREM 4. *A Minkowskian space whose norm is absolutely homogeneous, for which $\mathcal{R}$ has maximal dimension, and for which the sectional curvature of $\mathcal{I}$ is 1 is Euclidean.*

I should like to make an observation about the problem that arises in this result. It strikes me that what is lacking is a means of bridging the gap between general geometrical results of the form 'a compact Riemannian manifold with such-and-such a property is isometric to a sphere' and our special case 'if the indicatrix has this property then, as a hypersurface in $\mathbb{R}^n_\circ$ it *is* an Euclidean sphere' (or perhaps better 'it coincides with an Euclidean sphere'). I think of the required result as a counterpart to Brickell's Theorem: 'a Minkowskian space whose indicatrix is isometric to a sphere and ... is Euclidean' (dots to be filled in), compared with 'a Minkowskian space whose curvature vanishes and whose norm is absolutely homogeneous is Euclidean'.

It is worth pausing here to briefly discuss a possible alternative proof of Theorem 3, assuming ab initio that the norm is absolutely homogeneous.

I denote the map $\mathbb{R}^n_\circ \to \mathbb{R}^n_\circ : y \mapsto -y$ by μ: it is simply reflection in the origin. Absolute homogeneity amounts to $\mu^* F = F$. It follows from this that μ is an isometry of g, and therefore that the Killing algebra is invariant under μ. Moreover, μ preserves the semi-direct sum decomposition: $\mu_* \mathcal{T} = \mathcal{T}$, $\mu_* \mathcal{R} = \mathcal{R}$, since evidently μ preserves (positive) homogeneity degree. By direct calculation one finds that for any vector field ζ, $(\mu_* \zeta)^i(y) = -\zeta^i(-y)$. For $\tau \in \mathcal{T}$, consider the vector field $\frac{1}{2}(\tau - \mu_* \tau)$, which also belongs to $\mathcal{T}$. Its components are $\frac{1}{2}(\tau^i(y) + \tau^i(-y)) = T^i(y)$ say: and $T^i(-y) = T^i(y)$. Then $T^i(ty_0) = T^i(y_0) = T^i(-y_0) = T^i(-ty_0)$ for any fixed $y_0 \in \mathbb{R}^n_\circ$, since T^i is 0-homogeneous. If we set $T^i(0)$ to be the common value of $\lim_{t\to 0} T^i(ty_0) = \lim_{t\to 0} T^i(-ty_0)$ then $T^i(y)$ is constant on the line $t \mapsto ty_0$. It appears that the value of $T^i(0)$ might depend on the choice of y_0: but it follows from the continuity of T^i on $\mathbb{R}^n_\circ$ that this cannot be the case. So $T^i(0)$ is independent of y_0, which means that T^i is constant. Then $T^k C_{ijk} = 0$.

Notice that μ_* is an involution, and so its eigenvalues are ± 1. For affine Killing fields $\mathcal{T}$ is the -1 eigenspace of μ_*, $\mathcal{R}$ the $+1$ eigenspace. If the same is true in general – which seems very plausible – then the argument above applies directly to every $\tau \in \mathcal{T}$. However, it is not clear to me whether this holds in general. Of course $\mu_* T = \mu_*(\frac{1}{2}(\tau - \mu_* \tau)) = -T$.

The bugbear in all of this is the problem of establishing the linearity of

coordinates. Of course assuming absolute homogeneity resolves the problem immediately. Otherwise, there are several, admittedly somewhat artificial, ways of forcing linearity. One I have used already, in Subsection 3.4, namely assuming that all elements of $\mathcal{T}$ are continuous at 0. Indeed, the argument following from that assumption can be modified to apply to the present situation, to show that the Minkowskian space is Euclidean. Another is the observation that for a Killing field ξ, $\mathcal{L}_\xi C = 0$ if and only if $\xi^k_{,ij} = 0$.

Be that as it may, let us consider the case in which ξ is affine, say $\xi^i = T^i + R^i_j y^j$ where T and R are constant. Killling's equation becomes

$$(T^k + R^k_l y^l)C_{ijk} + g_{ik}R^k_j + g_{jk}R^k_i = 0.$$

Separating out the terms of homogeneity degree -1 and 0 we obtain

$$T^k C_{ijk} = 0, \quad R^k_l y^l C_{ijk} + g_{ik}R^k_j + g_{jk}R^k_i = 0.$$

In the maximally symmetric case we see that $C_{ijk} = 0$, so g_{ij} is constant in linear coordinates; and $g_{ik}R^k_j + g_{jk}R^k_i = 0$. We obtain this rather straightforward result.

THEOREM 5. *If a Minkowskian space is maximally symmetric and its Killing algebra consists of affine vector fields then it is Euclidean.*

It is noteworthy, if not entirely unexpected, that of the two subalgebras it is $\mathcal{T}$ provides the most direct approach. That being so, there is an obvious challenge, to invoke only the properties of $\mathcal{R}$. So I now consider the linear case for $\mathcal{R}$. Somewhat surprisingly, perhaps, one doesn't have to explicitly mention $\mathcal{R}$, since linear vector fields ξ that satisfy $\xi(E) = 0$ are Killing, as I showed in Subsection 3.2. I shall prove the following theorem.

THEOREM 6. *If in an n-dimensional Minkowskian space the space of linear vector fields ξ that satisfy $\xi(E) = 0$ is of dimension $\frac{1}{2}n(n-1)$ then the Minkowskian space is Euclidean.*

I shall suggest two different ways of proving this: the first picks up a point made earlier, the second introduces an idea that I will use later. Despite the remarks made earlier, both proofs depend on the fact that $\dim \mathcal{R} = \frac{1}{2}n(n-1)$.

Proof. The first proof uses the following fact derived in Subsection 3.1. For a linear vector field $\xi \in \mathcal{R}$, with $\xi^i = R^i_j y^j$, such that $\xi(p) = 0$, where R is given by $R(u) = v$, $R(v) = -u$ for an orthonormal pair of vectors u and v, and

$R(w) = 0$ for w orthogonal to u and v, C_p vanishes on the two-dimensional subspace spanned by u and v. In particular, $C_p(u,u,u) = C_p(v,v,v) = 0$.

I apply this result to all $\xi \in \mathcal{R}$ such that $\xi(p) = 0$ (with $\dim \mathcal{R} = \frac{1}{2}n(n-1)$). The map $\mathcal{R} \to T_p\mathcal{I}$ by $\xi \mapsto \xi(p)$ is surjective, and its kernel, which is of dimension $\frac{1}{2}n(n-1) - (n-1) = \frac{1}{2}(n-1)(n-2)$, consists of all $(n-1)\times(n-1)$ skew-symmetric matrices, acting in $T_p\mathcal{I}$. So the result above holds for any orthonormal pair of vectors. It follows that $C_p(u,u,u) = 0$ for any u. Since C_p is completely symmetric, this is enough to ensure that $C_p = 0$. This holds for every p, so $C = 0$. ∎

Here is my second proof.

Proof. I pointed out in Subsection 3.1 that the metric volume form Ω on $\mathbb{R}^n_\circ$ defined by the Minkowskian Riemannian structure satisfies $\mathcal{L}_\xi\Omega = 0$ for any Killing field ξ. I shall actually be interested in the induced volume form ω on $\mathcal{I}$, which is given by

$$\omega = \iota_\Delta\Omega.$$

Then

$$\mathcal{L}_\xi\omega = \iota_{[\xi,\Delta]}\Omega + \iota_\Delta(\mathcal{L}_\xi\Omega) = \iota_{[\xi,\Delta]}\Omega.$$

For $\xi \in \mathcal{R}$, $[\xi,\Delta] = -\mathcal{L}_\Delta\xi = 0$. So for such ξ, $\mathcal{L}_\xi\omega = 0$.

Consider the averaged metric $\tilde{g}$ on $\mathbb{R}^n_\circ$ defined by

$$\tilde{g}_{ij} = \int_{\mathcal{I}} g_{ij}\,\omega.$$

(In the integral g_{ij} is to be treated simply as a function on $\mathcal{I}$. Strictly speaking, to obtain an average one should divide by the volume of $\mathcal{I}$: but this is just a positive constant, and so can safely be ignored.) $\tilde{g}$ is clearly a type $(0,2)$ linear tensor, which defines a positive-definite symmetric bilinear form whose components with respect to linear coordinates are constant: that is, it is an Euclidean metric on $\mathbb{R}^n$.

For any diffeomorphism ϕ of $\mathcal{I}$,

$$\int_{\mathcal{I}} \phi^*(g_{ij}\,\omega) = \int_{\phi(\mathcal{I})} g_{ij}\,\omega = \int_{\mathcal{I}} g_{ij}\,\omega.$$

Using this formula with ϕ the flow of any vector field ξ tangent to $\mathcal{I}$, by differentiation we obtain

$$\int_{\mathcal{I}} \mathcal{L}_\xi(g_{ij}\,\omega) = 0.$$

Suppose that $\xi \in \mathcal{R}$:

$$0 = \int_{\mathcal{I}} \mathcal{L}_\xi(g_{ij}\,\omega) = \int_{\mathcal{I}} \xi(g_{ij})\omega = \int_{\mathcal{I}} \xi^k C_{ijk}\,\omega = -\int_{\mathcal{I}} (g_{ik}\xi^k_{,j} + g_{jk}\xi^k_{,i})\,\omega.$$

Finally, suppose that $\xi \in \mathcal{R}$ is linear, say $\xi^i = R^i_j y^j$ with R^i_j constant:

$$0 = \int_{\mathcal{I}} (g_{ik}R^k_j + g_{jk}R^k_i)\,\omega = \tilde{g}_{ik}R^k_j + \tilde{g}_{jk}R^k_i.$$

That is to say, R is skew-symmetric with respect to the constant metric $\tilde{g}$. So the linear vector field ξ is a Killing field of an Euclidean metric on $\mathbb{R}^n_\circ$ as well as the original Minkowskian one. In the case when its dimension is $\frac{1}{2}n(n-1)$, therefore, $\mathcal{R}$ coincides with $\mathfrak{so}(n)$.

We can identify $\mathcal{I}$ with an Euclidean hypersphere of suitable radius as follows. Consider the function $\chi(y) = \tilde{g}_{ij}y^i y^j$ on $\mathcal{I}$. With ξ tangent to $\mathcal{I}$ where $\xi^i = R^i_j y^j$ we have

$$\xi(\chi) = 2\tilde{g}_{ij}R^i_k y^j y^k = 0$$

since R is skew-symmetric. That is to say, χ is constant on $\mathcal{I}$. But $\chi(p)$ is just the Euclidean distance of $p \in \mathcal{I}$ from the origin, up to a linear coordinate transformation. That is, all points of $\mathcal{I}$ are at the same Euclidean distance from the origin, in other words $\mathcal{I}$ is an Euclidean hypersphere. ∎

This second proof is the infinitesimal version of an argument I used in [2]. What makes this style of argument so appealing is that it establishes that for any Minkowskian space, the subalgebra of the Killing algebra consisting of the linear vector fields tangent to $\mathcal{I}$ can be identified with $\mathfrak{so}(k)$ where k is its dimension. Moreover, it bridges the gap between general results and results specific to Minkowskian spaces that I mentioned earlier (immediately after the proof of Theorem 4).

3.6. A somewhat more sophisticated example. The analysis of the simple example in Subsection 3.4 ended with an unspoken question: what happens if the undetermined component has Killing fields of its own. I shall now discuss a case which, while certainly not answering this question completely, may throw some light on it – as well as being interesting in its own right.

It concerns essentially what in the Finslerian context would be called an (α_1, α_2) space. It is assumed that we are given a pair of complementary orthogonal subspaces of $\mathbb{R}^n$ of dimensions k and $n-k$, with $1 < k < n-1$.

In other words, we write $\mathbb{R}^n$ as $\mathbb{R}^k \oplus \mathbb{R}^{n-k}$, with linear coordinates y^a, $a = 1, 2, \ldots, k$ on the first component, z^α, $\alpha = 1, 2, \ldots, n-k$ on the second. Each of the two subspaces is to be regarded as an Euclidean space, by restriction of the standard Euclidean metric to it. We may assume that the coordinates are chosen so that the two Euclidean metrics take the form δ_{ab} and $\delta_{\alpha\beta}$. Suppose that we are given a function f of two variables, say $f(u, v)$, which is smooth on an open subset of $\mathbb{R}^2 - \{0\}$ containing the quadrant $u > 0, v > 0$, is positive, and is homogeneous of degree 1. Define a function E on $\mathbb{R}^n_\circ$ by

$$E(y^i) = E(y^a, y^\alpha) = f\big(\tfrac{1}{2}\delta_{ab}y^a y^b, \tfrac{1}{2}\delta_{\alpha\beta}z^\alpha z^\beta\big).$$

Then provided it is strictly convex, which requires that f satisfies some complicated conditions that I won't go into, E is the energy function of a Minkowskian space. (In fact the Minkowski norm $\sqrt{2E}$ is absolutely homogeneous.) I denote by g the metric defined by E in the usual way. It is not necessarily the case that the component subspaces $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ are orthogonal with respect to g (even though they are with respect to the Euclidean metric). In other words, the cross terms $g_{a\alpha}$ of the metric need not vanish: and unlike on previous occasions, we cannot linearly transform them away because in order to preserve the underlying structure we may not use transformations that mix the y^a and the z^α.

The discussion is concerned with the Lie algebra of linear vector fields on $\mathbb{R}^n_\circ$ which are tangent to $\mathcal{I}$, the indicatrix of g. It is a subalgebra of $\mathcal{R}$ for this space: shall denote it by $\mathcal{LR}$.

It is easy to see that if μ is a Killing field of the Euclidean structure on $\mathbb{R}^k$ then it is also a Killing field of g: in fact μ is a linear vector field, which satisfies $\mu(E) = 0$ because $\mu(\delta_{ab}y^a y^b) = 0$. So $\mu \in \mathcal{LR}$. The same holds for the Killing fields of the Euclidean structure on $\mathbb{R}^{n-k}$. By the averaged metric trick,

$$\mathfrak{so}(k) \oplus \mathfrak{so}(n-k) \subseteq \mathcal{LR} \subseteq \mathfrak{so}(n).$$

Moreover, $\mathcal{LR} = \mathfrak{so}(n)$ if and only if the space is Euclidean. Now $\mathfrak{so}(k) \oplus \mathfrak{so}(n-k)$ is maximal as a subalgebra of $\mathfrak{so}(n)$: that is to say, there is no proper subalgebra of $\mathfrak{so}(n)$ which properly contains $\mathfrak{so}(k) \oplus \mathfrak{so}(n-k)$. So if $\mathcal{LR} \neq \mathfrak{so}(k) \oplus \mathfrak{so}(n-k)$ – in other words, if there is an element of $\mathcal{LR}$ which is not a linear combination of the individual Killing fields of its component parts – then $\mathcal{LR} = \mathfrak{so}(n)$, and the space is Euclidean. Conversely, if the space is not Euclidean $\mathcal{LR} = \mathfrak{so}(k) \oplus \mathfrak{so}(n-k)$.

It is easy to see that the metric components $g_{a\alpha}$ of E are given by

$$g_{a\alpha} = f_{,uv}\delta_{ab}y^b\delta_{\alpha\beta}z^\beta.$$

For this to vanish we must have $f_{,uv} = 0$, that is, $f(u,v) = p(u) + q(v)$ for some functions p and q of a single variable, each homogeneous fo degree 1. But then $p(u) = up(1)$, $q(v) = vq(1)$, so that $f(u,v) = au + bv$ for constants a and b, and the space is clearly Euclidean. If $f_{,uv} \neq 0$, so that $g_{a\alpha} \neq 0$, the two Euclidean subspaces $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ are not orthogonal with respect to g, and $\mathcal{LR} = \mathfrak{so}(k) \oplus \mathfrak{so}(n-k)$.

(For more information about (α_1, α_2) spaces consult Deng and Xu [4], the paper in which the concept, in the Finslerian setting, was first introduced. The conditions for E to be strictly convex are given there. The result above is essentially the Killing algebra version of Lemma 3.4 of the paper of Deng and Xu. It is a particular case of a situation discussed at length by Xu and Matveev in [7].)

4. Concluding remarks

One could regard the results of this investigation as reassuring, since none of them is very unexpected. But equally, for the same reason, one could regard the results of this investigation as boring. For the interesting features one has to look a little deeper.

One interesting feature is the fact that the expression of $\mathfrak{k}$ as the vector space direct sum of its differently homogeneous parts, $\mathcal{T}$ and $\mathcal{R}$, directly reveals its Lie algebraic semi-direct sum structure, which is highly satisfactory. The identification of $\mathcal{T}$ as parallel vector fields and $\mathcal{R}$ as Killing fields of the induced metric on $\mathcal{I}$ is also illuminating. Of course the results obtained here are dependent on the special properties of the Minkowskian nature of the space.

Another interesting, and challenging, feature is what a difference assuming that elements of $\mathcal{R}$ are linear makes. You have to assume less and you get more: every linear vector field on $\mathcal{I}$ is a Killing field; and it can be established directly that if the dimension of $\mathcal{R}$ is $\frac{1}{2}n(n-1)$ the Minkowskian space is Euclidean without it being necessary to assume that its norm is absolutely homogeneous.

Optimists believe that Brickell's Theorem holds without the extra proviso of absolute homogeneity: but nothing in this enquiry suggests that this is likely to be true. It strikes me, indeed, that it would be worth investing effort in

studying absolutely homogeneous Minkowskian spaces as objects of interest in their own right.

A. Parallel vector fields

This appendix is about the properties of parallel vector fields on Riemannian spaces in general, that is, it doesn't appeal at all to facts about Minkowskian spaces. I shall use $\mathcal{P}$ denote the vector space of parallel vector fields, on a Riemannian manifold M with metric g. A vector field U is parallel if and only if $\nabla U = 0$, that is, $\nabla_v U = 0$ for all vectors v (where ∇ is the covariant differentiation operator of the Levi-Civita connection). $\mathcal{P}$ is a (finite-dimensional) vector space. At each point p, $\mathcal{P}_p$ is a subspace of T_pM, of dimension equal to the dimension of $\mathcal{P}$, and this assignment to each p of a subspace of T_pM is a distribution.

For $U, V \in \mathcal{P}$, $[U, V] = 0$, since $[U, V] = \nabla_U V - \nabla_V U$. It follows that the distribution generated from $\mathcal{P}$ is involutive.

For $U \in \mathcal{P}$ and any vector field Z on M, $[U, Z] = \mathcal{L}_U Z = \nabla_U Z$.

For $U \in \mathcal{P}$, $\mathcal{L}_U g = 0$, since for all vector fields X, Y

$$\begin{aligned}\mathcal{L}_U g(X,Y) &= U\big(g(X,Y)\big) - g(\mathcal{L}_U X, Y) - g(X, \mathcal{L}_U Y)\\ &= U\big(g(X,Y)\big) - g(\nabla_U X, Y) - g(X, \nabla_U Y)\\ &= \nabla_U g(X,Y) = 0.\end{aligned}$$

For all vector fields Z and for $V, W \in \mathcal{P}$, $Z(g(V,W)) = 0$, since

$$Z\big(g(V,W)\big) = g(\nabla_Z V, W) + g(V, \nabla_Z W) = 0.$$

In particular elements of $\mathcal{P}$ have constant length.

The integral curves of $U \in \mathcal{P}$ are affinely parametrised geodesics, since if $\nabla U = 0$ then in particular $\nabla_U U = 0$.

For $U \in \mathcal{P}$ and for any vector fields X, Y, $R(X,Y)U = 0$ where R is the curvature tensor, since

$$R(X,Y)U = \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U = 0.$$

I denote by $\mathcal{P}^\perp$ the distribution on M consisting of vector fields orthogonal to $\mathcal{P}$.

For $Y \in \mathcal{P}^\perp$ and any X, $\nabla_X Y \in \mathcal{P}^\perp$, since for $U \in \mathcal{P}$

$$0 = X\big(g(U,Y)\big) = g(\nabla_X U, Y) + g(U, \nabla_X Y) = g(U, \nabla_X Y).$$

The distribution $\mathcal{P}^\perp$ is involutive: this follows immediately from the previous result.

Since $\mathcal{P}$ and $\mathcal{P}^\perp$ are complementary and both involutive there are local coordinates on M such that pointwise

$$\mathcal{P} \text{ is the span of } \frac{\partial}{\partial x^a}, \quad a = 1, 2, \ldots, r$$

$$\mathcal{P}^\perp \text{ is the span of } \frac{\partial}{\partial x^\alpha}, \quad \alpha = r+1, r+2, \ldots, n.$$

Then $g_{a\alpha} = 0$ by definition, and g_{ab} is constant because $Z(g(V,W)) = 0$ for all vector fields Z and for $V, W \in \mathcal{P}$. From the fact that $\nabla_X Y \in \mathcal{P}^\perp$ when $Y \in \mathcal{P}^\perp$ we have

$$\nabla_{\partial_i} \partial_\alpha = \Gamma^j_{i\alpha} \partial_j \in \mathcal{P}^\perp, \qquad i, j \in \{a, \alpha\};$$

in particular

$$\Gamma^a_{\alpha\beta} = 0 = -\tfrac{1}{2} g^{ab} \frac{\partial g_{\alpha\beta}}{\partial x^b}.$$

That is to say, $g_{\alpha\beta}$ is independent of the x^a.

It follows that locally M is the Riemannian product of a flat space of dimension r with local coordinates $\{x^a\}$ and constant metric g_{ab}, and a Riemannian space of dimension $n - r$ with local coordinates $\{x^\alpha\}$ and metric $g_{\alpha\beta}(x^\gamma)$. The parallel vector fields are the constant vector fields on the first component (the vector fields which have constant coefficients with respect to the coordinate vector fields ∂_a).

It is instructive to investigate the full Killing algebra of M, though its relationship with that of a Minkowskian Riemannian space is not entirely clear (the special coordinates invoked above need not of course be linear in the Minkowskian case). Killing's equation reads

$$\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik} \frac{\partial \xi^k}{\partial x^j} + g_{jk} \frac{\partial \xi^k}{\partial x^i} = 0.$$

Remembering that the g_{ab} are constants, and $g_{a\alpha} = 0$, we have the following cases:

$$ij = ab \qquad g_{ac} \frac{\partial \xi^c}{\partial x^b} + g_{bc} \frac{\partial \xi^c}{\partial x^a} = 0;$$

$$ij = a\alpha \qquad g_{ab} \frac{\partial \xi^b}{\partial x^\alpha} + g_{\alpha\beta} \frac{\partial \xi^\beta}{\partial x^a} = 0;$$

$$ij = \alpha\beta \qquad \xi^\gamma \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + g_{\alpha\gamma} \frac{\partial \xi^\gamma}{\partial x^\beta} + g_{\beta\gamma} \frac{\partial \xi^\gamma}{\partial x^\alpha} = 0.$$

It is clear that the parallel fields, for which ξ^a is constant and $\xi^\alpha = 0$, satisfy these equations. Imposing the equivalent of the condition that $\mathcal{T}$ is an ideal in $\mathfrak{k}$ simplifies things somewhat: by that condition I mean in effect that

$$\left[\frac{\partial}{\partial x^a}, \xi^i \frac{\partial}{\partial x^i}\right] \in \left\langle \frac{\partial}{\partial x^a} \right\rangle,$$

that is, that

$$\frac{\partial \xi^\alpha}{\partial x^a} = 0.$$

From the case $ij = a\alpha$ we see that

$$\frac{\partial \xi^a}{\partial x^\alpha} = 0.$$

So the equations break into separate non-interacting parts: the first says that $\partial \xi^a / \partial x^b$ is a component of a skew-symmetric matrix; the last that $\xi^\alpha \partial / \partial x^\alpha$ satisfies Killing's equation for $g_{\alpha\beta}$. If the Riemannian space is derived from a Minkowski norm, and the coordinates are linear, we get back a slightly enhanced version of the simple example in Subsection 3.4.

For further information about parallel vector fields and 1-forms see [5].

Disclaimer. AI has not been used in the preparation of this paper, otherwise than in the form of summaries of the results of Google searches.

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