



Perturbations of bi-continuous cosine families

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Abstract: In this paper we prove a bounded perturbation theorem for bi-continuous cosine families. We also apply our results to adjoints of strongly continuous cosine families.

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INTRODUCTION

The theory of operator cosine families, which provides the mathematical for solving second-order abstract Cauchy problems, has been extensively developed since the pioneering works of Sova [23] and Fattorini [12]. These operator families, satisfying the fundamental identity $2C(t)C(s) = C(t+s) + C(t-s)$ for $t \geq s \geq 0$, are indispensable for studying wave equations, elasticity problems, and other second-order evolution equations in Banach spaces.

The classical theory predominantly concerns strongly continuous cosine families, where the mapping $t \mapsto C(t)x$ is continuous in the norm topology for each x in the Banach space X . However, many physically significant operators, particularly those arising in spaces of bounded functions or measure spaces, fail to generate strongly continuous semigroups or cosine families. This limitation motivated the development of more general frameworks that accommodate weaker continuity requirements while preserving essential qualitative properties.

The concept of bi-continuous semigroups, introduced by Kühnemund [14], provides such a framework by equipping the Banach space with an additional locally convex topology τ that is coarser than the norm topology but satis-



fies appropriate completeness and compatibility conditions. This approach naturally extends to cosine families, with the initial definition proposed by Duan and Sun [8] and later refined by the author [4]. While [8] introduced the basic framework, our previous work [4] established a more comprehensive theory including detailed generator properties, approximation theorems, and examples that demonstrate the necessity of this generalized framework.

Perturbation theory represents a crucial aspect of operator theory, allowing the analysis of modified systems through understanding of their unperturbed counterparts. For strongly continuous cosine families, perturbation theory has been investigated by several authors including Travis and Webb [25], Lutz [17, 18], and others [1, 19, 22, 24, 26]. In the bi-continuous setting, Farkas [11] initiated the study of perturbations for semigroups, with subsequent developments in [2, 3, 6, 9]. An overview on bi-continuous semigroups can be found in the monograph [5]. However, the corresponding theory for bi-continuous cosine families remains largely unexplored.

Our main result, Theorem 2.5, establishes that bounded linear perturbations preserving τ -continuity on norm-bounded sets yield generators of bi-continuous cosine families. This extends classical results for strongly continuous cosine families to the bi-continuous setting and provides explicit construction via Dyson–Phillips series. It should be noted that our work complements and differs from Shaw’s results on w^* -continuous cosine families [21] in several aspects: we employ the axiomatic framework of bi-admissible spaces, obtain explicit perturbation series, and establish connections with the broader theory of bi-continuous semigroups.

The paper is organized as follows: Section 1 recalls the basic definitions and properties of bi-continuous cosine families. Section 2 contains our main bounded perturbation theorem, and Section 3 presents applications to dual cosine families on $L^\infty(\mathbb{R})$ and other function spaces where the classical theory fails.

The examples in Section 3 demonstrate the applicability of our theory to operators that are not accessible through the strongly continuous framework, including perturbed wave operators on $L^\infty(\mathbb{R})$ and translation cosine families on spaces of bounded continuous functions.

1. PRELIMINARIES

As usual, the general idea is to equip the Banach space we are working on with another additional locally convex topology τ which is subjected to

some general assumption. Those assumptions were introduced by Kühnemund [14, Assumption 1] and also appear in a similar fashion in the theory of Saks spaces [7].

ASSUMPTION 1.1. Let $(X, \|\cdot\|)$ be a Banach space and τ a locally convex topology on X such that

- (i) τ is a Hausdorff topology and is coarser than the norm-topology on X , i.e., the identity map $(X, \|\cdot\|) \rightarrow (X, \tau)$ is continuous.
- (ii) τ is sequentially complete on $\|\cdot\|$ -bounded sets, i.e., every $\|\cdot\|$ -bounded τ -Cauchy sequence is τ -convergent.
- (iii) The dual space of (X, τ) is norming for X , i.e.,

$$\|x\| = \sup_{\substack{\varphi \in (X, \tau)' \\ \|\varphi\| \leq 1}} |\varphi(x)|.$$

We call the triple $(X, \|\cdot\|, \tau)$ a *bi-admissible space*.

The definition of bi-continuous cosine families was initially proposed in [8] and later refined by the author [4]. It is inspired by the theory of bi-continuous semigroups [14, Definition 3]. The study of strongly continuous cosine families on Banach spaces goes back to Sova [23, Definition 2.2] and is included as a special case in the definition of bi-continuous cosine families.

DEFINITION 1.2. Let $(X, \|\cdot\|, \tau)$ be a bi-admissible space. We call a family of bounded linear operators $(C(t))_{t \geq 0}$ a *bi-continuous cosine family* if the following holds:

- (i) $2C(t)C(s) = C(t+s) + C(t-s)$ and $C(0) = I$ for all $t \geq s \geq 0$.
- (ii) $(C(t))_{t \geq 0}$ is strongly τ -continuous.
- (iii) There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|C(t)\| \leq Me^{\omega t}$ for each $t \geq 0$.
- (iv) $(C(t))_{t \geq 0}$ is *locally bi-equicontinuous*, i.e., if $(x_n)_{n \in \mathbb{N}}$ is a norm-bounded sequence in X which is τ -convergent to 0, then also $(C(s)x_n)_{n \in \mathbb{N}}$ is τ -convergent to 0 uniformly for $s \in [0, t_0]$ for each fixed $t_0 \geq 0$.

The generator of a bi-continuous cosine family combines the definition for strongly continuous cosine families [23, Definition 2.12] and for bi-continuous semigroups [14, p. 214].

DEFINITION 1.3. Let $(C(t))_{t \geq 0}$ be a bi-continuous cosine family on a bi-admissible space $(X, \|\cdot\|, \tau)$. The *generator* $(A, D(A))$ of $(C(t))_{t \geq 0}$ is the linear operator on X defined by¹

$$Ax := \tau \lim_{t \rightarrow 0+} \frac{2}{t^2} (C(t)x - x),$$

$$D(A) := \left\{ x \in X : \begin{array}{l} \tau \lim_{t \rightarrow 0+} \frac{2}{t^2} (C(t)x - x) \text{ exists in } X \\ \text{and } \sup_{t \in (0,1]} \frac{2}{t^2} \|C(t)x - x\| < \infty \end{array} \right\}.$$

For properties of generators of bi-continuous cosine families we refer to [4, Section 2 and Section 3].

2. A BOUNDED PERTURBATIONS RESULT

In this section we consider bounded perturbations of bi-continuous cosine families following the approach developed by Farkas for bi-continuous semigroups [11]. Our main result extends the classical bounded perturbation theorem for strongly continuous cosine families to the bi-continuous setting.

DEFINITION 2.1. For $t_0 > 0$, we define the space

$$\mathbb{X}_{t_0} := \left\{ F: [0, t_0] \rightarrow \mathcal{L}(X) : \begin{array}{l} F \text{ is } \tau\text{-strongly continuous and} \\ \|\cdot\| \text{-bounded, and } \{F(t) : t \in [0, t_0]\} \\ \text{is bi-equicontinuous} \end{array} \right\}$$

equipped with the norm

$$\|F\|_\infty := \sup_{t \in [0, t_0]} \|F(t)\|, \quad F \in \mathbb{X}_{t_0}.$$

This space is a Banach algebra [11, Lemma 3.2 and Lemma 3.3].

Remark 2.2. In what follows we will make use of integration. The integrals we will work with have to be understood as τ -Riemann integrals. The existence follows from Assumption 1.1, Definition 1.2 as well as [13, Proposition 1.1]. In what follows, we will make use of the τ -Riemann integrals without

¹ Notice that the definition presented here is slightly different from the one given in [4] as one needs the norm boundedness.

mentioning it explicitly. For more information, we also refer to [14, Proposition 4]. This remark can also be found here [5, Remark 9.9]. Scalar-valued integrals arising in norm estimates are understood as Lebesgue integrals. We briefly explain why the relevant integrands are measurable. Since the dual space $(X, \tau)'$ is norming for X , the norm is lower semicontinuous with respect to τ . Consequently, for every $x \in X$, the map $t \mapsto \|C(t)x\|$ is lower semicontinuous, since $t \mapsto C(t)x$ is τ -continuous, so in particular, the map $t \mapsto \|C(t)x\|$ is Borel measurable. By exponential boundedness, it is also bounded on compact intervals. Similarly, the scalar-valued functions occurring in the norm estimates below are measurable and locally bounded, and hence Lebesgue integrable. In particular, the integrals involving $s \mapsto \|C(t-s)\|$ in the proof of Lemma 2.3 below are well-defined as Lebesgue integrals. We shall use these integration conventions without further explicit mention.

Let $(C(t))_{t \geq 0}$ be a bi-continuous cosine family on X and let $B \in \mathcal{L}(X)$ be a bounded linear operator that is τ -continuous on $\|\cdot\|$ -bounded sets. For $t \in [0, t_0]$ and $x \in X$, we define the Volterra operator

$$(\mathbb{V}_{t_0}(F))(t)x := \int_0^t C(t-s) \int_0^s BF(r)x \, dr \, ds, \quad F \in \mathbb{X}_{t_0}.$$

This construction originates from [22, 24]. The integral exists as a τ -Riemann integral due to the properties of $\mathbb{X}_{t_0}$, and $(\mathbb{V}_{t_0}(F))(t)$ is independent of the choice of $t_0 > 0$.

LEMMA 2.3. *Let $t_0 > 0$. Then $\mathbb{V}_{t_0} \in \mathcal{L}(\mathbb{X}_{t_0})$ and $r(\mathbb{V}_{t_0}) = 0$.*

Proof. Let $t_0 > 0$, $x \in X$ and $F \in \mathbb{X}_{t_0}$. Then

$$\begin{aligned} \|(\mathbb{V}_{t_0}(F))(t)x\| &= \sup_{\substack{\varphi \in (X, \tau)' \\ \|\varphi\| \leq 1}} \left| \varphi \left(\int_0^t C(t-s) \int_0^s BF(r)x \, dr \, ds \right) \right| \\ &\leq \sup_{\substack{\varphi \in (X, \tau)' \\ \|\varphi\| \leq 1}} \int_0^t \left| \varphi \left(C(t-s) \int_0^s BF(r)x \, dr \right) \right| \, ds \\ &\leq \int_0^t \|C(t-s)\| \cdot \left\| \int_0^s BF(r)x \, dr \right\| \, ds \\ &= \int_0^t \|C(t-s)\| \cdot \sup_{\substack{\varphi \in (X, \tau)' \\ \|\varphi\| \leq 1}} \left| \varphi \left(\int_0^s BF(r)x \, dr \right) \right| \, ds \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^t \|C(t-s)\| \cdot \sup_{\substack{\varphi \in (X, \tau)' \\ \|\varphi\| \leq 1}} \int_0^s |\varphi(BF(r)x)| \, dr \, ds \\
&\leq \int_0^t \|C(t-s)\| \cdot \int_0^s \|BF(r)x\| \, dr \, ds \\
&\leq \int_0^t \|C(t-s)\| \cdot \int_0^s \|B\| \cdot \|F(r)\| \cdot \|x\| \, dr \, ds \\
&\leq \int_0^t \|C(t-s)\| \cdot \|B\| \cdot \|x\| \cdot s \|F\|_\infty \, ds \\
&\leq \frac{t_0^2}{2} \cdot \|B\| \cdot \|F\|_\infty \cdot \|C|_{[0, t_0]}\|_\infty \cdot \|x\|.
\end{aligned}$$

We note that the function $s \mapsto \|C(t-s)\|$ is continuous on $[0, t]$ since $(C(t))_{t \geq 0}$ is strongly τ -continuous and locally norm-bounded, and the norm is τ -lower semicontinuous by Assumption 1.1(i). Hence the integral exists as a τ -Riemann integral. The inner integral exists similarly. This indeed shows $(\mathbb{V}_{t_0}(F))(t) \in \mathcal{L}(X)$. In the next step, we have to show that the set $\{(\mathbb{V}_{t_0}(F))(t) : t \in [0, t_0]\}$ is bi-equicontinuous. For this, let $(x_n)_{n \in \mathbb{N}}$ be a $\|\cdot\|$ -bounded τ -null sequence. Let $\varepsilon > 0$ be arbitrary. In what follows we denote by $\mathcal{P}$ the set of generating seminorms of the locally convex topology τ , cf. [20, Theorem 1.36 and Theorem 1.37] Then for $p \in \mathcal{P}$ one obtains

$$\begin{aligned}
p((\mathbb{V}_{t_0}(F))(t)x_n) &= p\left(\int_0^t C(t-s) \int_0^s BF(r)x_n \, dr \, ds\right) \\
&\leq \int_0^t p\left(C(t-s) \int_0^s BF(r)x_n \, dr\right) \, ds.
\end{aligned}$$

We now claim that the set $\{C(t-s) \int_0^s BF(r)x \, dr : s \in [0, t]\}$ is also bi-equicontinuous. In fact, let $(y_n)_{n \in \mathbb{N}}$ be any $\|\cdot\|$ -bounded τ -null sequence. Since $F \in \mathbb{X}_{t_0}$ and B is τ -continuous on $\|\cdot\|$ -bounded sets we conclude that $(\int_0^s BF(r)y_n \, dr)_{n \in \mathbb{N}}$ is a $\|\cdot\|$ -bounded τ -null sequence. As $(C(t))_{t \geq 0}$ is a bi-continuous cosine family, we obtain for sufficiently large $n \in \mathbb{N}$ that $p((\mathbb{V}_{t_0}(F))(t)x_n) < t_0\varepsilon$. The $\|\cdot\|$ -boundedness and strong τ -continuity of $\mathbb{V}_{t_0}(F)$ follow immediately. This shows that indeed $\mathbb{V}_{t_0} \in \mathcal{L}(\mathbb{X}_{t_0})$.

Moreover, we want to show that for every $n \in \mathbb{N}$, one has that

$$\|\mathbb{V}_{t_0}^n(F)\|_\infty \leq \frac{(t_0^2 \cdot \|C|_{[0, t_0]}\|_\infty \cdot \|B\|)^n}{(2n)!} \cdot \|F\|_\infty. \quad (2.1)$$

To do so, we indeed show something sharper. In fact, we claim that for every $n \geq 1$, $F \in \mathbb{X}_{t_0}$, and $t \in [0, t_0]$ one has

$$\|(\mathbb{V}_{t_0}^n F)(t)\| \leq \frac{(\|C|_{[0, t_0]}\|_\infty \|B\|)^n t^{2n}}{(2n)!} \|F\|_\infty. \quad (2.2)$$

For $n = 1$, this is exactly the bound derived above. Assume (2.2) holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned} \|(\mathbb{V}_{t_0}^{n+1} F)(t)\| &\leq \|C|_{[0, t_0]}\|_\infty \|B\| \int_0^t \int_0^s \|(\mathbb{V}_{t_0}^n F)(r)\| \, dr \, ds \\ &\leq \|C|_{[0, t_0]}\|_\infty \|B\| \int_0^t \int_0^s \frac{(\|C|_{[0, t_0]}\|_\infty \|B\|)^n r^{2n}}{(2n)!} \|F\|_\infty \, dr \, ds \\ &= \frac{(\|C|_{[0, t_0]}\|_\infty \|B\|)^{n+1}}{(2n)!} \|F\|_\infty \int_0^t \frac{s^{2n+1}}{2n+1} \, ds \\ &= \frac{(\|C|_{[0, t_0]}\|_\infty \|B\|)^{n+1}}{(2n)!} \|F\|_\infty \frac{t^{2n+2}}{(2n+1)(2n+2)} \\ &= \frac{(\|C|_{[0, t_0]}\|_\infty \|B\|)^{n+1} t^{2n+2}}{(2n+2)!} \|F\|_\infty. \end{aligned}$$

This completes the induction. Thus (2.2) holds for all $n \in \mathbb{N}$. Taking the supremum over $t \in [0, t_0]$ yields

$$\|\mathbb{V}_{t_0}^n F\|_\infty \leq \frac{(t_0^2 \|C|_{[0, t_0]}\|_\infty \|B\|)^n}{(2n)!} \|F\|_\infty,$$

showing that (2.1) holds true. In fact, (2.1) shows that for the spectral radius one obtains $r(\mathbb{V}_{t_0}) = 0$. ■

Remark 2.4. Lemma 2.3 implies $1 \in \rho(\mathbb{V}_{t_0})$ with

$$(\mathbf{I} - \mathbb{V}_{t_0})^{-1} = \sum_{n=0}^{\infty} \mathbb{V}_{t_0}^n$$

converging in $\mathcal{L}(\mathbb{X}_{t_0})$. This Neumann series representation is crucial for our perturbation result. Here $\rho(\mathbb{V}_{t_0})$ denotes the resolvent set of the operator $\mathbb{V}_{t_0}$.

We now state our main bounded perturbation theorem. The proof adapts techniques from [11, 22, 24] to the bi-continuous cosine family setting.

THEOREM 2.5. *Let $(C(t))_{t \geq 0}$ be a bi-continuous cosine family on a bi-admissible space $(X, \|\cdot\|, \tau)$ with generator $(A, D(A))$. Suppose $B \in \mathcal{L}(X)$ is τ -continuous on $\|\cdot\|$ -bounded sets. Then $(A + B, D(A))$ generates a bi-continuous cosine family $(\widehat{C}(t))_{t \geq 0}$ given by the Dyson–Phillips series*

$$\widehat{C}(t) = \sum_{n=0}^{\infty} C_n(t), \quad t \geq 0,$$

which converges uniformly in $\|\cdot\|$ on compact intervals. Here, $C_0(t) := C(t)$ and for $n \geq 1$:

$$C_n(t)x := \int_0^t C_{n-1}(t-s) \int_0^s BC(r)x \, dr \, ds, \quad x \in X,$$

where the integrals are τ -Riemann integrals. Moreover, the variation of constants formula holds:

$$\widehat{C}(t)x = C(t)x + \int_0^t C(t-s) \int_0^s B\widehat{C}(r)x \, dr \, ds, \quad x \in X, \quad t \geq 0.$$

Proof. For fixed $t_0 > 0$, define $\widehat{C}_{t_0}(t) := \delta_t((I - \mathbb{V}_{t_0})^{-1}C|_{[0, t_0]})$, where $\delta_t : \mathbb{X}_{t_0} \rightarrow \mathcal{L}(X)$ is evaluation at t . Set $\widehat{C}(0) = I$ and $\widehat{C}(t) = \widehat{C}_{t_0}(t)$ for $t \in [0, t_0]$. This definition is independent of t_0 since $(\mathbb{V}_{t_0}(F))(t)$ does not depend on t_0 for $t \leq t_0$.

We now verify the cosine identity for $0 \leq s \leq t \leq s+t \leq t_0$:

$$2\widehat{C}_{t_0}(t)\widehat{C}_{t_0}(s) = \widehat{C}_{t_0}(t+s) + \widehat{C}_{t_0}(t-s).$$

Expanding both sides using the Neumann series, it suffices to prove for all $n \in \mathbb{N}_0$:

$$\begin{aligned} 2 \sum_{k=0}^n \mathbb{V}_{t_0}^k(C|_{[0, t_0]})(t) \mathbb{V}_{t_0}^{n-k}(C|_{[0, t_0]})(s) \\ = \mathbb{V}_{t_0}^n(C|_{[0, t_0]})(t+s) + \mathbb{V}_{t_0}^n(C|_{[0, t_0]})(t-s). \end{aligned} \tag{2.3}$$

The case $n = 0$ follows from the cosine law for $(C(t))_{t \geq 0}$. Assume (2.3) holds for some $n \in \mathbb{N}$.

Starting from the left-hand side of (2.3) at level $n + 1$, we have

$$\begin{aligned}
& 2 \sum_{k=0}^{n+1} \mathbb{V}_{t_0}^k(C)(t) \mathbb{V}_{t_0}^{n+1-k}(C)(s) \\
&= 2 C(t) \mathbb{V}_{t_0}^{n+1}(C)(s) + 2 \sum_{j=0}^n \mathbb{V}_{t_0}^{j+1}(C)(t) \mathbb{V}_{t_0}^{n-j}(C)(s) \\
&= 2 C(t) \mathbb{V}_{t_0}^{n+1}(C)(s) \\
&\quad + \int_0^t C(t-r) \int_0^r B \left[2 \sum_{j=0}^n \mathbb{V}_{t_0}^j(C)(u) \mathbb{V}_{t_0}^{n-j}(C)(s) \right] du dr.
\end{aligned}$$

By the induction hypothesis (2.3), the bracket equals

$$\mathbb{V}_{t_0}^n(C)(u+s) + \mathbb{V}_{t_0}^n(C)(u-s)$$

whenever $u \geq s$, and equals

$$\mathbb{V}_{t_0}^n(C)(u+s) + \mathbb{V}_{t_0}^n(C)(s-u)$$

whenever $u < s$. Since $C(t)$ is defined for $t \geq 0$, we always have $u + s \geq 0$, and in the second case $s - u \geq 0$. Thus the induction hypothesis applies in both cases.

Applying the cosine law $2C(t)C(s-r) = C(t+s-r) + C(t-s+r)$ to the leading term, we obtain

$$2 \sum_{k=0}^{n+1} \mathbb{V}_{t_0}^k(C)(t) \mathbb{V}_{t_0}^{n+1-k}(C)(s) = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &:= \int_0^s C(t+s-r) \int_0^r B \mathbb{V}_{t_0}^n(C)(u) du dr, \\
I_2 &:= \int_0^s C(t-s+r) \int_0^r B \mathbb{V}_{t_0}^n(C)(u) du dr, \\
I_3 &:= \int_0^t C(t-r) \int_0^r B \mathbb{V}_{t_0}^n(C)(u+s) du dr, \\
I_4 &:= \int_0^t C(t-r) \int_0^r B \mathbb{V}_{t_0}^n(C)(|u-s|) du dr.
\end{aligned}$$

The four integrals combine via a direct change-of-variables calculation – fully analogous to the argument in the proof of [22, Theorem 1] and [24, Theorem 1] for the strongly continuous case, and formally identical in the bi-continuous setting since the calculation is purely algebraic and uses only the definition of $\mathbb{V}_{t_0}$ and the cosine law – to yield

$$I_1 + I_2 + I_3 + I_4 = \mathbb{V}_{t_0}^{n+1}(C)(t+s)x + \mathbb{V}_{t_0}^{n+1}(C)(t-s)x.$$

This proves (2.3) for $n+1$, completing the induction.

The bi-continuity properties of $(\hat{C}(t))_{t \geq 0}$ follow from the construction and the properties of $\mathbb{X}_{t_0}$. The uniform convergence on compact intervals is a consequence of the norm convergence of the Neumann series.

Let us now prove the exponential boundedness of $(\hat{C}(t))_{t \geq 0}$. By the Dyson–Phillips series representation and (2.1) we obtain

$$\|\hat{C}(t)\| \leq \|C(t)\| + \sum_{n=1}^{\infty} \|\mathbb{V}_{t_0}^n(C)(t)\| \leq \|C(t)\| + \sum_{n=1}^{\infty} \frac{(\|C|_{[0,t_0]}\|_{\infty} \|B\|)^n t^{2n}}{(2n)!}.$$

Since $(C(t))_{t \geq 0}$ is exponentially bounded there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that $\|C(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. In particular, one has

$$\|\hat{C}(t)\| \leq Me^{\omega t} + \cosh\left(t\sqrt{\|C|_{[0,t_0]}\|_{\infty} \|B\|}\right) - 1,$$

showing that there exist $\hat{M} \geq 0$ and $\hat{\omega} \in \mathbb{R}$ such that $\|\hat{C}(t)\| \leq \hat{M}e^{\hat{\omega}t}$ for all $t \geq 0$. This proves the exponential boundedness of $(\hat{C}(t))_{t \geq 0}$.

We now prove the norm boundedness condition that appears in the domain of the generator as defined in Definition 1.3. For $x \in D(A)$ we obtain by using the estimate (2.1) that

$$\frac{2}{h^2} \|\hat{C}(h)x - x\| \leq \frac{2}{h^2} \|C(h)x - x\| + \frac{2}{h^2} \sum_{n=1}^{\infty} \|\mathbb{V}_{t_0}^n(C)(h)\| \|x\|.$$

The first summand is bounded for $h \in (0, 1]$ because $x \in D(A)$. For the second summand series (2.1) yields

$$\begin{aligned} \sum_{n=1}^{\infty} \|\mathbb{V}_{t_0}^n(C)(h)\| &\leq \sum_{n=1}^{\infty} \frac{(\|C|_{[0,t_0]}\|_{\infty} \|B\|)^n h^{2n}}{(2n)!} \\ &= \cosh\left(h\sqrt{\|C|_{[0,t_0]}\|_{\infty} \|B\|}\right) - 1, \end{aligned}$$

so

$$\frac{2}{h^2} \sum_{n=1}^{\infty} \|\mathbb{V}_{t_0}^n(C)(h)\| \|x\| \leq \frac{2\|x\|}{h^2} \left(\cosh \left(h \sqrt{\|C|_{[0,t_0]}\|_{\infty} \|B\|} \right) - 1 \right),$$

which converges to $\|C|_{[0,t_0]}\|_{\infty} \|B\| \cdot \|x\|$ as $h \rightarrow 0^+$ and hence is uniformly bounded for $h \in (0, 1]$ as desired.

To show that $(A+B, D(A))$ generates $(\widehat{C}(t))_{t \geq 0}$, we compute for $x \in D(A)$ and $h > 0$:

$$\begin{aligned} \frac{2}{h^2} (\widehat{C}(h)x - x) &= \frac{2}{h^2} (C(h)x - x) + \frac{2}{h^2} \mathbb{V}_{t_0}(C|_{[0,t_0]})(h)x \\ &\quad + \frac{2}{h^2} \sum_{n=2}^{\infty} \mathbb{V}_{t_0}^n(C|_{[0,t_0]})(h)x. \end{aligned}$$

As $h \rightarrow 0+$, the first term converges to Ax in τ since $x \in D(A)$. For the second term:

$$\frac{2}{h^2} \mathbb{V}_{t_0}(C|_{[0,t_0]})(h)x - Bx = \frac{2}{h^2} \int_0^h C(h-s) \int_0^s BC(r)x \, dr \, ds - Bx.$$

By the τ -continuity of B and strong τ -continuity of $(C(t))_{t \geq 0}$, this converges to 0 in τ . The remaining term satisfies

$$\left\| \frac{2}{h^2} \sum_{n=2}^{\infty} \mathbb{V}_{t_0}^n(C|_{[0,t_0]})(h)x \right\| \rightarrow 0$$

as $h \rightarrow 0+$ by estimate (2.1). Therefore,

$$\tau\text{-}\lim_{h \rightarrow 0+} \frac{2}{h^2} (\widehat{C}(h)x - x) = Ax + Bx.$$

Moreover, the uniform boundedness condition in Definition 1.3 follows from the exponential boundedness of $(\widehat{C}(t))_{t \geq 0}$. This completes the proof. ■

Remark 2.6. The τ -continuity of B on norm-bounded sets is essential for Theorem 2.5. To show this, we adapt [11, Example 2.1] to the cosine family setting.

To do so, we consider the Banach space $X = C([0, 1])$ and its dual space $X' = \mathcal{M}([0, 1])$, the space of all complex Borel measures on $[0, 1]$. We equip

the dual space with the weak*-topology. Moreover, we define a bounded linear operator B on X' by

$$(B\mu)(\Omega) := \mu\left(\left(\Omega - \frac{3}{4}\right) \cap [0, 1]\right)$$

for all $\mu \in \mathcal{M}([0, 1])$ and all Borel measurable sets $\Omega \subseteq [0, 1]$. This operator is norm-bounded and satisfies $B^2 = 0$.

One can also show that B is not weak*-continuous on norm-bounded sets. Indeed, consider the sequence of Dirac measures $\delta_{\frac{3}{4} - \frac{1}{n+1}}$ which converges to $\delta_{\frac{3}{4}}$ with respect to the weak*-topology. However, we also have $B\delta_{\frac{3}{4} - \frac{1}{n+1}} = 0$ for all $n \in \mathbb{N}$, but $B\delta_{\frac{3}{4}} = \delta_0$. This indeed shows that B fails to be weak*-continuous.

Now consider the bounded operator B as a perturbation of the zero operator $A = 0$ on X' . The cosine family generated by $A = 0$ is trivially given by $C(t) = I$ for all $t \geq 0$ which is bi-continuous with respect to the weak*-topology.

The perturbed cosine family generated formally by $A + B$ would be

$$C_B(t) := I + \frac{t^2}{2}B, \quad t \geq 0,$$

since $B^2 = 0$. However, this family is not bi-continuous with respect to the weak*-topology. Indeed, applying the same sequence $\delta_{\frac{3}{4} - \frac{1}{n+1}}$ as above yields

$$C_B(t)\delta_{\frac{3}{4} - \frac{1}{n+1}} = \delta_{\frac{3}{4} - \frac{1}{n+1}} \rightarrow \delta_{3/4},$$

where the convergence is with respect to the weak*-topology. However,

$$C_B(t)\delta_{\frac{3}{4}} = \delta_{\frac{3}{4}} + \frac{t^2}{2}\delta_0$$

which is not equal to the limit of $C_B(t)\delta_{\frac{3}{4} - \frac{1}{n+1}}$ for any $t > 0$. Hence $C_B(t)$ is not weak*-continuous on norm-bounded sets for any $t > 0$. Consequently, $C_B(t)$ cannot be a bi-continuous cosine family.

Summarizing, the operator B is norm-bounded but fails to be τ -continuous on norm-bounded sets and the formal perturbation of the trivial cosine family $C(t) = I$ by B does not yield a bi-continuous cosine family. This demonstrates that the τ -continuity assumption on B in Theorem 2.5 cannot be omitted.

3. EXAMPLES: ADJOINTS OF COSINE FAMILIES ON $L^\infty(\mathbb{R})$

Before we start with examples of perturbations of bi-continuous cosine families, we first consider some other examples of bi-continuous cosine families which naturally appear beside the one discussed in [4, Example 1.7 and Example 2.11].

PROPOSITION 3.1. *Let $(C(t))_{t \geq 0}$ be a strongly continuous cosine family on a Banach space X . Then the dual operator family $(C'(t))_{t \geq 0}$ forms a bi-continuous cosine family on X' with respect to the weak*-topology.*

Proof. It is clear that $(C'(t))_{t \geq 0}$ satisfies the cosine family law and that $(C'(t))_{t \geq 0}$ is exponentially bounded. To show strong continuity with respect to the weak*-topology let $x \in X$ and $\varphi \in X'$. Then

$$|(C'(t)\varphi - C'(s)\varphi)(x)| = |\varphi(C(t)x - C(s)x)| \leq \|\varphi\| \cdot \|C(t)x - C(s)x\|,$$

showing that indeed $(C'(t))_{t \geq 0}$ is strongly continuous with respect to the weak*-topology. For the bi-equicontinuity, let $(\varphi_n)_{n \in \mathbb{N}}$ be a $\|\cdot\|$ -bounded null sequence with respect to the weak*-topology. Then for $x \in X$ we have

$$(C'(t)\varphi_n)(x) = \varphi_n(C(t)x) \rightarrow 0,$$

with respect to the weak*-topology uniformly on compact sets due to the compactness of the orbit $\{C(t)x : t \in [0, t_0]\}$. This proves the result. $\blacksquare$

EXAMPLE 3.2. Consider on $X = L^1(\mathbb{R})$ the strongly continuous cosine family $(C(t))_{t \geq 0}$ defined by

$$(C(t)f)(x) := \frac{1}{2}(f(x+t) + f(x-t)), \quad t \geq 0, f \in L^1(\mathbb{R}), x \in \mathbb{R}.$$

Then the dual cosine family $(C'(t))_{t \geq 0}$ on $X' = L^\infty(\mathbb{R})$ is given by

$$(C'(t)f)(x) = \frac{1}{2}(f(x-t) + f(x+t)), \quad t \geq 0, f \in L^\infty(\mathbb{R}), x \in \mathbb{R}. \quad (3.1)$$

By Proposition 3.1, this cosine family on $L^\infty(\mathbb{R})$ is bi-continuous with respect to the weak*-topology on $L^\infty(\mathbb{R})$. To consider cosine families with respect to the weak*-topology on $L^\infty(\mathbb{R})$ makes sense in terms of [15, Corollary 3.3], which is the version for strongly continuous cosine family of the result by Lotz for strongly continuous operator semigroups [16, Theorem 3].

EXAMPLE 3.3. For a locally compact Hausdorff space Ω we consider the space $X = C_b(\Omega)$ equipped with both the supremum norm as well as the compact-open topology τ_{co} . This space is known to satisfy Assumption 1.1. Now, let $q: \Omega \rightarrow \mathbb{C}$ be a continuous function satisfying

$$\sup_{\omega \in \Omega} \operatorname{Re} q(\omega) < \infty,$$

and define

$$(C(t)f)(\omega) := \cosh(tq(\omega))f(\omega), \quad t \geq 0, \quad f \in C_b(\Omega), \quad \omega \in \Omega.$$

It is straight forward to check that $(C(t))_{t \geq 0}$ indeed defines a bi-continuous cosine family. Its generator is given by

$$Af = q^2 f, \quad D(A) = \{f \in C_b(\Omega) : q^2 f \in C_b(\Omega)\}.$$

It is worth mentioning that this example does not arise from duality.

Let us now also give an example for a perturbation. In the view of [10, Proposition 4.2.1] we can state the following result without proof.

PROPOSITION 3.4. *Let $(C(t))_{t \geq 0}$ be a bi-continuous cosine family on X' with respect to the weak*-topology with generator $(A, D(A))$. Then for any $B \in \mathcal{L}(X)$ the operator $(A + B', D(A))$ is the generator of a bi-continuous cosine family.*

Proof. (Sketch) The result follows from [10, Proposition 4.2.1] and Theorem 2.5. The key observation is that for any $B \in \mathcal{L}(X)$, the adjoint operator B' is weak*-continuous on $\|\cdot\|_{X'}$ -bounded sets. Therefore, B' satisfies the conditions of Theorem 2.5, and $(A + B', D(A))$ generates a bi-continuous cosine family. ■

EXAMPLE 3.5. Consider the operator $Af = f''$ on $D(A) = W^{2,\infty}(\mathbb{R})$. By Example 3.2 this operator generates the bi-continuous cosine family on $L^\infty(\mathbb{R})$ given by (3.1). Now, consider the operator $f \mapsto g \cdot f$ where $g \in L^\infty(\mathbb{R})$. This operator is the adjoint operator of the bounded operator $Bf = g \cdot f$ on $L^1(\mathbb{R})$. Indeed, under the duality pairing $\langle h, f \rangle = \int_{\mathbb{R}} h(x)f(x)dx$, the adjoint of multiplication by g on $L^1(\mathbb{R})$ is multiplication by g on $L^\infty(\mathbb{R})$. Hence, by Proposition 3.4 the operator $Cf = f'' + g \cdot f$ with $D(C) = D(A)$ generates a bi-continuous cosine family.

Remark 3.6. We note that Proposition 3.1 is related to the results in [21, Theorem 2.2], and Proposition 3.4 can be compared with [21, Corollary 2.4].

EXAMPLE 3.7. To further illustrate the scope of Theorem 2.5 we investigate nonlocal perturbations that arise naturally in population dynamics, viscoelasticity as well as mathematical biology.

Let $k \in L^1(\mathbb{R})$ and define an operator B on $L^\infty(\mathbb{R})$ by

$$(Bf)(x) := (k * f)(x) = \int_{\mathbb{R}} k(y)f(x-y) dy, \quad f \in L^\infty(\mathbb{R}).$$

In this case the straightforward estimate $\|Bf\|_\infty \leq \|k\|_1 \|f\|_\infty$ yields $\|B\| \leq \|k\|_1$. Moreover, the operator B is weak*-continuous on norm-bounded sets. Indeed, by considering the reflected kernel $\tilde{k}(y) := k(-y)$ and the operator $\tilde{B}$ on $L^1(\mathbb{R})$ given by

$$(\tilde{B}h)(x) := (\tilde{k} * h)(x) = \int_{\mathbb{R}} k(y-x)h(y) dy, \quad h \in L^1(\mathbb{R}),$$

we see that $\tilde{B}$ is bounded on $L^1(\mathbb{R})$ with $\|\tilde{B}\| \leq \|k\|_1$. Under the natural duality pairing $\langle h, f \rangle = \int_{\mathbb{R}} h(x)f(x) dx$ between $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$, we obtain

$$\begin{aligned} \langle \tilde{B}h, f \rangle &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} k(y-x)h(y) dy \right) f(x) dx \\ &= \int_{\mathbb{R}} h(y) \left(\int_{\mathbb{R}} k(y-x)f(x) dx \right) dy \\ &= \int_{\mathbb{R}} h(y)(k * f)(y) dy = \langle h, Bf \rangle, \end{aligned}$$

for all $h \in L^1(\mathbb{R})$ and $f \in L^\infty(\mathbb{R})$. In particular, we have that $B = (\tilde{B})'$. Since every adjoint operator is weak*-continuous on norm-bounded sets, B satisfies the assumptions of Theorem 2.5.

Consequently, for any $k \in L^1(\mathbb{R})$, the operator

$$(A + B)f = f'' + (k * f),$$

with domain

$$D(A + B) = W^{2,\infty}(\mathbb{R}),$$

generates a bi-continuous cosine family $(\hat{C}_k(t))_{t \geq 0}$ on $L^\infty(\mathbb{R})$ with respect to the weak*-topology. This example corresponds to the *nonlocal wave equation*

$$u_{tt}(t, x) = u_{xx}(t, x) + (k * u)(t, x), \quad t \geq 0, x \in \mathbb{R},$$

which describes wave propagation in a medium with long-range interactions. It is worth mentioning that each term in the Dyson–Phillips has a clear physical interpretation. In fact, the n -th term represents n -fold nonlocal scattering of the wave. This construction provides a numerical scheme for approximating wave propagation in nonlocal media.

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