



Partial Poisson-Lie groups and groupoids. Application to Von Neumann algebras

FERNAND PELLETIER [†] , PATRICK CABAU ^{*,@} *El camino, 31520 Ramonville-Saint-Agne, France**patrickcabau@gmail.com*Received December 5, 2025
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Abstract: This paper develops a comprehensive framework for Poisson geometry in the infinite-dimensional convenient setting, as introduced by Frölicher, Kriegl and Michor, with a particular focus on Poisson-Lie groups, Lie groupoids, and applications to operator algebras. Building upon the foundational work of Weinstein on Poisson groupoids, Mackenzie and Xu on Lie bialgebroids and Poisson groupoids, Tumpach on Banach-Lie groupoids associated with operator algebras, and Odziejewicz and collaborators on groupoids arising from W^* -algebras, we extend several classical finite-dimensional constructions to the convenient framework. After reviewing the fundamental properties of partial Poisson manifolds and establishing several complementary results, we introduce and study convenient partial Poisson-Lie groups. We then extend key arguments of Weinstein on Poisson-Lie groupoids to the infinite-dimensional context, identifying the results that remain valid beyond the finite-dimensional setting. Further developments include the extension of fundamental constructions from Lie groupoid theory, notably the canonical identification between the conormal bundle of the unit space and the dual of the associated Lie algebroid established by Mackenzie and Xu. Building on these foundations, we construct sub-Poisson structures on cotangent bundles of Lie groupoids and investigate their behavior under direct limits of finite-dimensional Poisson groupoids. The theory is illustrated through explicit examples. As an application, we consider groupoids associated with von Neumann algebras and endow them with natural Poisson-Lie structures. An appendix collects the background material on W^* -algebras required for the development of the theory.

Key words: partial Poisson structure, convenient Lie algebroid, convenient partial Lie groupoid, convenient partial Poisson-Lie group, direct limit, von Neumann algebra.

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À la mémoire de mon ami Fernand Pelletier

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[@] Corresponding author

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1. INTRODUCTION

In finite dimension, the theory of Poisson groupoids was introduced by Weinstein in [40] as a natural extension of symplectic groupoids, which had previously been introduced in [10] and, independently, in [42]. Both Poisson-Lie groups and symplectic groupoids play a fundamental rôle in the integration theory of Poisson manifolds (see, e.g., [10, 19]). Weinstein further observed that, for a Poisson groupoid G , the dual vector bundle $A^*(G)$ of its Lie algebroid $A(G)$ can be endowed with a canonical Lie algebroid structure.

This infinitesimal aspect was further developed by Mackenzie and Xu in [27]. In that work, the authors emphasized the close analogy between the Lie theory of Poisson-Lie groups and that of Poisson groupoids. Recall that, for connected and simply connected Lie groups, Poisson-Lie group structures are in one-to-one correspondence with Lie bialgebra structures on their Lie algebras. Extending this correspondence to the groupoid setting, Mackenzie and Xu introduced a natural infinitesimal invariant associated with a Poisson groupoid G , namely a Lie bialgebroid structure on its Lie algebroid $A(G)$, and investigated its fundamental properties.

Motivated by the correspondence between Poisson and infinitesimal structures in finite dimensions, several extensions of Poisson geometry to infinite-dimensional settings have been proposed. A Banach-space version of a Poisson manifold was first introduced in [33] and further developed by Odziejewicz and coauthors. In parallel, the authors of the present paper introduced in [35] the notion of a *partial convenient Poisson manifold*, providing a generalization of Poisson geometry to the convenient setting. Furthermore, A. B. Tumpach defined in [38] the notion of a *Banach Poisson-Lie group*. A detailed comparison of these approaches may be found in [16], which also contains an extensive bibliography on the topic.

In [34], Odziejewicz and Sliżewska associated a Banach Lie groupoid with W^* -algebras. The theory of Banach-Lie groupoids was subsequently developed in [5], where connections were established with earlier studies of infinite-dimensional manifolds arising from Banach algebras. In [32], Odziejewicz, Jakimowicz, and Sliżewska introduced the notion of a *sub-Poisson groupoid* in the setting of W^* -algebras and provided a variety of examples illustrating this framework.

The first aim of this paper is to propose a framework in which one can define notions of Banach and, more generally, convenient partial Poisson and sub-Poisson groupoids, as a generalization of Weinstein's finite-dimensional construction. This approach allows us to recover the results of [32].

Secondly, we show how the results of [27] can be extended to this infinite-dimensional setting.

We end this paper with the notion of von Neumann algebra and endow an associated object with a Lie-Poisson structure.

This paper is self-contained and is organized as follows:

- Section 2 summarizes all the essential properties of a partial Poisson manifold which are well known in the convenient setting (cf. [8, Chapter 7]) and some complements which are needed in the other sections.
- Section 3 is devoted to the study of convenient partial Poisson-Lie groups.
- Section 4 shows how Weinstein's arguments in [40] on finite-dimensional Poisson-Lie groupoids extend to the infinite-dimensional setting, and which results remain valid.
- Section 5 is devoted to the setting in which some results of [27], in particular the canonical identification between the conormal bundle of the unit space and the dual of the Lie algebroid of a Lie groupoid, can be extended to the convenient setting.
- Section 6 contains essentially the construction of sub-Poisson structures on the cotangent bundle of a Lie groupoid.
- In section 7, the previous results are extended to an ascending sequence of finite dimensional Poisson groupoids and is illustrated by an example.
- In Section 8, we recall the definition of a von Neumann algebra, introduce the associated groupoid, and endow it with a Poisson-Lie structure.
- Finally, in an Appendix, we summarize the main properties needed to understand the results on groupoids associated with W^* -algebras presented in this paper.

2. CONVENIENT PARTIAL POISSON MANIFOLDS

We begin by reviewing the basic framework of convenient calculus developed by Frölicher, Kriegl, and Michor in [14] and [21].

We also collect the main properties of partial Poisson manifolds in this infinite-dimensional setting that will be used in the subsequent subsections. Several of these results are available in the book [8], some are refinements or extensions of known results, and the rest are original contributions, such as §2.8.

2.1. CONVENIENT CALCULUS. A *convenient vector space* $\mathbb{E}$ is a locally convex topological vector space (l.c.t.v.s) such that a curve $c : \mathbb{R} \rightarrow \mathbb{E}$ is *smooth* if and only if $\lambda \circ c$ is smooth for all continuous linear functionals λ on $\mathbb{E}$ [21, Theorem 2.14 (4)]. We then get a second topology on $\mathbb{E}$ which is the final topology relatively to the set of all smooth curves and called the *c^∞ -topology* [21, Definition 2.12]. This last topology may be different from the l.c.t.v.s topology and, for this topology, $\mathbb{E}$ cannot be a topological vector space. However for Fréchet (and so Banach) spaces, both topologies coincide.

A map $f : \mathbb{E} \rightarrow \mathbb{R}$ is *smooth* if and only if $f \circ c : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth map for any smooth curve c in $\mathbb{E}$ [21, Definition 3.11]. Any composition of smooth mappings is also smooth.

The convenient calculus provides an appropriate extension of differential calculus to such spaces because, for any c^∞ -open set U of a convenient space $\mathbb{E}$ and any convenient space $\mathbb{F}$, we have the following properties:

- the space $C^\infty(U, \mathbb{F})$ of smooth maps may also be endowed with a structure of convenient space;
- the differential operator $d : C^\infty(U, \mathbb{F}) \rightarrow C^\infty(U, L(\mathbb{E}, \mathbb{F}))$ defined by:

$$df(x).v = \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

exists for every $f \in C^\infty(U, \mathbb{F})$. Moreover, d is a linear and smooth map [21, 3.18]. Here $L(\mathbb{E}, \mathbb{F})$ denotes the space of bounded (equivalently smooth) linear mappings¹ from $\mathbb{E}$ to $\mathbb{F}$;

¹Let $\mathbb{E}$ and $\mathbb{F}$ be convenient vector spaces, and let $T : \mathbb{E} \rightarrow \mathbb{F}$ be a linear map. Then the following conditions are equivalent:

- (a) T is bounded, i.e. it maps bounded subsets of $\mathbb{E}$ to bounded subsets of $\mathbb{F}$;
- (b) T is C^∞ -smooth;

- the chain rule holds [21, 3.18].

The notion of *convenient manifold* modelled on a convenient vector space is then defined *via* an equivalent class of smooth atlases [21, 27.1].

On a convenient vector space $\mathbb{E}$, two kinds of tangent spaces at a point a of this space can be defined [21, 28.1]: *the kinematic tangent space* and *the operational tangent space*. We are only interested in the first one² $T_a\mathbb{E}$ consisting of all derivatives $c'(0)$ at 0 of smooth curves $c : \mathbb{R} \rightarrow \mathbb{E}$ with $c(0) = a$.

The *kinematic tangent bundle* TM of a convenient manifold M [21, 29.9], is the quotient space for the equivalence relation:

$$(\alpha, x, v) \sim (\beta, y, w) \iff \begin{cases} x = y \\ (d\phi_{\alpha\beta})_{\phi_\beta(x)} w = v \end{cases}$$

defined on the disjoint union $\bigcup_{\alpha \in A} \{\alpha\} \times U_\alpha \times \mathbb{M}$ where $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ is a smooth atlas of M .

A *kinematic vector field* is a smooth section of the bundle TM [21, 32.1].

The *kinematic cotangent bundle* $T'M$ is defined as the union of the convenient vector spaces $T'_x M$ of all bounded linear functionals on the convenient vector space $T_x M$ endowed with an equivalence class of atlases [21, 33.1].

A *kinematic 1-form* is a smooth section of the bundle $T'M$ [21, 33.2].

2.2. RELATIONS IN LINEAR POISSON AND SYMPLECTIC CONVENIENT SPACES

2.2.1. PARTIAL LINEAR POISSON CONVENIENT SPACES. Let $\mathbb{E}$ be a convenient space. Denote by $\mathbb{E}'$ its kinematic dual and by $\langle ., . \rangle$ the duality bracket.

Let $\mathbb{E}^b$ be a vector subspace of $\mathbb{E}'$ endowed with its own convenient structure where the inclusion map $\mathbb{E}^b \hookrightarrow \mathbb{E}'$ is bounded. The duality pairing $\langle ., . \rangle$ on $\mathbb{E}' \times \mathbb{E}$ then restricts to a bounded bilinear form on $\mathbb{E}^b \times \mathbb{E}$, which is non degenerate in the first variable. By abuse of notation, this restricted pairing will also be denoted by $\langle ., . \rangle$ whenever no confusion can arise.

A subspace $\mathbb{Y}$ of any convenient space $\mathbb{X}$ will be call *closed* if it is c^∞ -closed according to the terminology used in [21].

(c) T is c^∞ -continuous, i.e. continuous with respect to the c^∞ -topologies on $\mathbb{E}$ and $\mathbb{F}$.

²The second one $D_a\mathbb{E}$ corresponds to a family of bounded derivations.

More generally, we systematically adopt notations and terminology used in [21].

DEFINITION 2.1. A *partial linear Poisson structure* on $\mathbb{E}$ consists in the following data:

- a vector subspace $\mathbb{E}^b$ of $\mathbb{E}'$ endowed with its own convenient structure, and such that the inclusion map $\mathbb{E}^b \hookrightarrow \mathbb{E}'$ is bounded;
- a bounded linear map $P : \mathbb{E}^b \rightarrow \mathbb{E}$ such that:

$$\langle \omega_1, P\omega_2 \rangle = - \langle \omega_2, P\omega_1 \rangle$$

for all ω_1 and ω_2 in $\mathbb{E}^b$.

We then say that $(\mathbb{E}^b, \mathbb{E}, P)$ is a *partial linear Poisson space*.

If P is a partial linear Poisson structure on $\mathbb{E}$, so is $-P$. When no confusion is possible, if P is a fixed partial linear Poisson structure on $\mathbb{E}$, we simply denote by $\mathbb{E}^-$ the space $\mathbb{E}$ provided with the partial linear Poisson structure $-P$.

Moreover, if P is a convenient isomorphism, then the 2-form $\Omega_P \in \bigwedge^2 \mathbb{E}'$ defined by

$$\Omega_P(u, v) = \langle P^{-1}(u), v \rangle$$

is a weak symplectic form on $\mathbb{E}$ whenever it is closed.

When $\mathbb{E}^b = \mathbb{E}'$ then Ω_P is a linear strong symplectic form. In general, when $\mathbb{E}^b \neq \mathbb{E}'$, Ω_P is a linear symplectic form on $\mathbb{E}$. Conversely, if Ω is a weak symplectic linear form on $\mathbb{E}$ and let $\mathbb{E}^b$ be the range of the associated linear operator $\Omega^b : \mathbb{E} \rightarrow \mathbb{E}'$. Then $P = (\Omega^b)^{-1} : \mathbb{E}^b \rightarrow \mathbb{E}$ is a partial Poisson structure which is a convenient isomorphism. This justifies the following terminology:

DEFINITION 2.2. Let $P : \mathbb{E}^b \rightarrow \mathbb{E}$ be a partial Poisson structure.

If P is a convenient isomorphism, we say that P is a *partial linear symplectic structure* on $\mathbb{E}$. When $\mathbb{E}^b = \mathbb{E}'$, we will simply say that P is *linear symplectic*.

If P is a partial linear symplectic structure, then $-P$ is also of the same type. When no confusion can arise, we simply write $\mathbb{E}^-$ for this opposite structure.

DEFINITION 2.3. For any subspace $\mathbb{F}$ of a convenient space $\mathbb{E}$, the *annihilator* of $\mathbb{F}$ is the subspace

$$\mathbb{F}^a := \{ \alpha \in \mathbb{E}' : \forall u \in \mathbb{F}, \langle \alpha, u \rangle = 0 \}.$$

DEFINITION 2.4. Let $(\mathbb{E}^b, \mathbb{E}, P)$ be a partial linear Poisson space.

1. Given a vector subspace $\mathbb{A} \subset \mathbb{E}^b$, the vector space:

$$\mathbb{A}^{\perp_P} := \{\omega \in \mathbb{E}^b : \forall \alpha \in \mathbb{A}, \langle \omega, P\alpha \rangle = 0\}$$

is called the *orthogonal of $\mathbb{A}$ relative to P* .

The space $\mathbb{A}$ is called *isotropic* if $\mathbb{A} \subset \mathbb{A}^{\perp_P}$.

2. If $\mathbb{F}$ is a convenient subspace of $\mathbb{E}$ (that is $\mathbb{F}$ is c^∞ -closed) we set:

$$\mathbb{F}^0 := \{\alpha \in \mathbb{E}^b : \forall u \in \mathbb{F}, \langle \alpha, u \rangle = 0\}.$$

The *orthogonal $\mathbb{F}^{\perp_P}$* of a convenient subspace $\mathbb{F}$ of $\mathbb{E}$ is the vector space

$$\mathbb{F}^{\perp_P} = P(\mathbb{F}^0).$$

The space $\mathbb{F}$ is called *coisotropic* if $\mathbb{F}^0$ is isotropic in $\mathbb{E}^b$.

LEMMA 2.5. *We have the following properties:*

1. *For every vector subspace $\mathbb{A}$ of $\mathbb{E}^b$, one has $\ker P \subset \mathbb{A}^{\perp_P}$.*
2. *If $\mathbb{A}$ is a vector subspace of $\mathbb{E}^b$, then $\mathbb{A}^{\perp_P}$ is always c^∞ -closed in $\mathbb{E}^b$.*
3. *If $\overline{\mathbb{F}}$ is the c^∞ -closure of a subspace $\mathbb{F}$ of $\mathbb{E}$, then $\mathbb{F}^0 = (\overline{\mathbb{F}})^0$.*
4. *If $\mathbb{F}$ is a subspace of $\mathbb{E}$, then both $(\mathbb{F}^0)^{\perp_P}$ and $\mathbb{F}^0$ contain $\ker P$.*
5. *$\mathbb{F}$ is coisotropic if and only if*

$$\forall (\omega_1, \omega_2) \in (\mathbb{F}^0)^2, \quad \langle \omega_1, P\omega_2 \rangle = 0.$$

6. *If $\mathbb{F}_1 \subset \mathbb{F}_2$ and $\mathbb{F}_1$ is coisotropic, so is $\mathbb{F}_2$.*

PROPOSITION 2.6. *Let $(\mathbb{E}^b, \mathbb{E}, P)$ be a partial linear Poisson space such that P is an isomorphism. Consider the symplectic form Ω_P on $\mathbb{E}$ defined by:*

$$\Omega_P(u, v) = \langle P^{-1}(u), v \rangle.$$

If $\mathbb{F}$ is any c^∞ -closed subspace of $\mathbb{E}$, we have:

$$\mathbb{F}^{\perp_P} = \{u \in \mathbb{E} : \forall v \in \mathbb{F}, \Omega_P(u, v) = 0\}.$$

Moreover, we have: $(\mathbb{F}^{\perp_P})^{\perp_P} = \mathbb{F}$ if $\mathbb{F}$ is closed.

Now, following the classical terminology (cf. [41]), we have:

DEFINITION 2.7. Let $(\mathbb{E}^b, \mathbb{E}, P)$ be a partial linear symplectic space.

A convenient subspace $\mathbb{F}$ of $\mathbb{E}$ is called *Lagrangian* if $P(\mathbb{F}^0) = \mathbb{F}$.

We say that a Lagrangian space $\mathbb{F}$ is a *split Lagrangian* if $\mathbb{F}$ supplemented in $\mathbb{E}$ by a Lagrangian space.

Note that in a partial symplectic structure, any Lagrangian space is also coisotropic. However, any Lagrangian space is not supplemented in general. Even more, the reader will find in [20] an example of a linear symplectic Banach space in which no Lagrangian space admits a supplemented space.

PROPOSITION 2.8. Let $(\mathbb{E}^b, \mathbb{E}, P)$ be a partial linear Poisson space. If $\mathbb{F}$ is a convenient subspace of $\mathbb{E}$, we have the following properties:

1. $P((\mathbb{F}^0)^{\perp_P}) = \mathbb{F} \cap P(\mathbb{E}^b)$ and so $\mathbb{F}$ is coisotropic if and only if $\mathbb{F}^{\perp_P} \subset \mathbb{F}$.
2. The following properties are equivalent:
 - (i) $\mathbb{F}$ is coisotropic;
 - (ii) $\mathbb{F} \cap P(\mathbb{E}^b)$ is coisotropic in $\mathbb{E}$.

2.2.2. PARTIAL LINEAR POISSON MORPHISMS

DEFINITION 2.9. For $i \in \{1, 2\}$, let $(\mathbb{E}_i^b, \mathbb{E}_i, P_i)$ be a partial linear Poisson space.

A bounded linear map $\phi : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ is called a *partial linear Poisson morphism* if one has:

(LpPMorph1) $\phi^*(\mathbb{E}_2^b) \subset \mathbb{E}_1^b$

(LpPMorph2) $P_2 = \phi \circ P_1 \circ \phi^*$.

The bounded linear map ϕ is called a *partial linear anti-Poisson morphism* if the adjoint ϕ^* of ϕ satisfies **(LpPMorph1)** and the following property:

(LpAPMorph) $P_2 = -\phi \circ P_1 \circ \phi^*$.

Moreover, if each P_i is partial symplectic and a Poisson morphism (resp. anti-Poisson morphism), ϕ is then called a *partial symplectic morphism* (resp. partial anti-symplectic morphism).

The product $\mathbb{E}_1 \times \mathbb{E}_2$ can be provided with two partial linear partial Poisson structures:

(**LpP**+) $P_{\mathbb{E}_1 \times \mathbb{E}_2^+} : \mathbb{E}_1^b \times \mathbb{E}_2^b \rightarrow \mathbb{E}_1 \times \mathbb{E}_2$ given by $P_{\mathbb{E}_1 \times \mathbb{E}_2^+} = (P_1, P_2)$

(**LpP**-) $P_{\mathbb{E}_1 \times \mathbb{E}_2^-} : \mathbb{E}_1^b \times \mathbb{E}_2^b \rightarrow \mathbb{E}_1 \times \mathbb{E}_2$ given by $P_{\mathbb{E}_1 \times \mathbb{E}_2^-} = (P_1, -P_2)$.

Note that if each P_i is partial (resp. strong) symplectic, so are $P_{\mathbb{E}_1 \times \mathbb{E}_2^+}$ and $P_{\mathbb{E}_1 \times \mathbb{E}_2^-}$. For simplicity, we denote by $\mathbb{E}_1^+ \times \mathbb{E}_2^+$ and $\mathbb{E}_1^+ \times \mathbb{E}_2^-$ this first and second structure respectively.

Recalling that the graph ³ of ϕ is

$$G_\phi = \{ (u, \phi(u)) : u \in \mathbb{E}_1 \}$$

we then have the following characterization of a partial linear Poisson morphism:

PROPOSITION 2.10. *A bounded linear map $\phi : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ such that $\phi^* (\mathbb{E}_2^b) \subset \mathbb{E}_1^b$ is a partial linear Poisson morphism (resp. partial linear anti-Poisson morphism) if and only if its graph is coisotropic in $\mathbb{E}_1 \times \mathbb{E}_1^-$ (resp. $\mathbb{E}_1^- \times \mathbb{E}_2$).*

Moreover, if P_1 and P_2 are partial symplectic, ϕ is a partial symplectic or partial anti-symplectic map, respectively, if and only if its graph is Lagrangian in $\mathbb{E}_1 \times \mathbb{E}_1^-$ or in $\mathbb{E}_1^- \times \mathbb{E}_2$ respectively.

Proof. Note that since ϕ is bounded, G_ϕ is closed (cf. [3, 1.1])

$$\{ (\phi^*(\alpha), -\alpha) : \alpha \in \mathbb{E}_2^b \}. \quad (2.1)$$

is c^∞ closed (cf. [3, 1.1]).

For the linear partial Poisson structure $P_{\mathbb{E}_1 \times \mathbb{E}_2^-}$, we have

$$P_{\mathbb{E}_1 \times \mathbb{E}_2^-} ((G_\phi)^0) = \{ (P_1(\phi^*(\alpha)), P_2(\alpha)) : \alpha \in \mathbb{E}_2^b \}.$$

Now, if ϕ is a linear Poisson morphism, we have $P_2 = \phi \circ P_1 \circ \phi^*$ and so

$$P_{\mathbb{E}_1 \times \mathbb{E}_2^-} ((G_\phi)^0) = \{ (P_1 \circ \phi^*(\alpha), \phi(P_1 \circ \phi^*(\alpha))) : \alpha \in \mathbb{E}_2^b \}.$$

which is contained in G_ϕ .

Conversely, if $P_{\mathbb{E}_1 \times \mathbb{E}_2^-} ((G_\phi)^0)$ is contained in G_ϕ , this implies that $P_2 = \phi \circ P_1 \circ \phi^*$.

³This notation is classical. However, usually, in finite dimension in the classical papers on symplectic groupoids or Poisson groupoids (cf. [17, 10, 1, 41] among many others), the graph of such a map is denoted $G_\phi = \{ (\phi(u), u) : u \in \mathbb{E}_1 \}$.

Assume that P_1 and P_2 are partial symplectic if G_ϕ is Lagrangian, then this implies that ϕ is a Poisson morphism.

Conversely, assume that ϕ is partial symplectic, this means that P_1 and P_2 are isomorphisms, so $P_{\mathbb{E}_1 \times \mathbb{E}_2^-}$ is an isomorphism from $\mathbb{E}_1^\flat \times \mathbb{E}_2^\flat$ to $\mathbb{E}_1 \times \mathbb{E}_2$. Thus it remains to show that:

$$P_{\mathbb{E}_1 \times \mathbb{E}_2^-}^{-1}(G_\phi) \subset (G_\phi)^0.$$

But we have:

$$\begin{aligned} P_{\mathbb{E}_1 \times \mathbb{E}_2^-}^{-1}(G_\phi) &= \{(P_1^{-1}(u), -P_2^{-1} \circ \phi(u)) : u \in \mathbb{E}_1\} \\ &= \{(\phi^*(P_2^{-1} \circ \phi(u)), -P_2^{-1} \circ \phi(u)) : u \in \mathbb{E}_1\}. \end{aligned}$$

which is included in $(G_\phi)^0$ from (2.1). ■

2.2.3. LINEAR POISSON (RESP. SYMPLECTIC) RELATIONS. Following [17, 10, 1, 41], we introduce the following notions:

DEFINITION 2.11. Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be convenient vector spaces.

1. Any closed subspace $R \subset \mathbb{E}_1 \times \mathbb{E}_2$ will be called a *linear relation*.

Given a linear relation $R \subset \mathbb{E}_1 \times \mathbb{E}_2$, we will say that R is a linear relation from $\mathbb{E}_1$ to $\mathbb{E}_2$ and we will write $R : \mathbb{E}_1 \rightarrow \mathbb{E}_2$. If $\mathbb{E}_1 = \mathbb{E}_2 = \mathbb{E}$, we simply say that R is a *relation* in $\mathbb{E}$.

2. Let $R : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ and $S : \mathbb{E}_2 \rightarrow \mathbb{E}_3$ be two relations. The composition $S \circ R$ is the relation $S \circ R : \mathbb{E}_1 \rightarrow \mathbb{E}_3$ given by

$$S \circ R = \{(x, z) \in \mathbb{E}_1 \times \mathbb{E}_3 : \exists y \in \mathbb{E}_2, (x, y) \in R \text{ and } (y, z) \in S\}.$$

In particular, the graph G_ϕ of a bounded linear map $\phi : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ (which is always a convenient subspace of $\mathbb{E}_1 \times \mathbb{E}_2$) is a linear relation.

We will reserve the symbol $\rightarrow$ for a graph G_ϕ and the symbol $\rightarrow$ will be used for a general relation. Of course, if $\phi : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ and $\psi : \mathbb{E}_2 \rightarrow \mathbb{E}_3$ are two bounded linear maps, the composition $G_\psi \circ G_\phi$ is $G_{\psi \circ \phi}$.

Following the terminology introduced by Weinstein in [41], and using Proposition 2.10, we introduce:

DEFINITION 2.12. 1. A relation $R : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ is called a *Poisson relation*, if R is coisotropic according to the partial Poisson structure $P_{\mathbb{E}_1 \times \mathbb{E}_2^-}$.

2. If for $i \in \{1, 2\}$, $(\mathbb{E}_i^b, \mathbb{E}_i, P_i)$ is partial symplectic, a relation $R : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ is called a *symplectic relation* if R is Lagrangian in $P_{\mathbb{E}_1 \times \mathbb{E}_2^-}$.

As in [41], we begin by the following result which is the version of Proposition 1.3.2 in our context and whose proof uses the same type of arguments.

THEOREM 2.13. *For $i \in \{1, 2\}$, let $(\mathbb{E}_i^b, \mathbb{E}_i, P_i)$ be a partial linear Poisson space and R (resp. C) a coisotropic space of $\mathbb{E}_1 \times \mathbb{E}_2$ (resp. $\mathbb{E}_2$). Then, one has:*

$$R(C) = \{y \in \mathbb{E}_2 : \exists x \in C, (x, y) \in R\}.$$

is a coisotropic space in $\mathbb{E}_2$.

Moreover, if each P_1 and P_2 is partial symplectic and if R and C are Lagrangian, so is $R(C)$.

COROLLARY 2.14. *For $i \in \{1, 2, 3\}$, let $P_i : \mathbb{E}_i^b \rightarrow \mathbb{E}_i$ be a partial linear Poisson structure on $\mathbb{E}_i$. Given two Poisson relations $S : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ and $T : \mathbb{E}_2 \rightarrow \mathbb{E}_3$, $T \circ S$ is a Poisson relation from $\mathbb{E}_1 \rightarrow \mathbb{E}_3$. Moreover, if each P_i is partial symplectic and S and T are Lagrangian, so is $T \circ S$.*

Proof. See proof of [41, Theorem 1.3.4]. ■

2.3. BASIC PROPERTIES OF PARTIAL POISSON MANIFOLDS. All of the following properties can also be found in the book [8].

2.3.1. PARTIAL POISSON MANIFOLDS. Let M be a convenient manifold modelled on the convenient space $\mathbb{M}$. We denote by $p_M : TM \rightarrow M$ its kinematic tangent bundle and by $p'_M : T'M \rightarrow M$ its kinematic cotangent bundle.

Consider a vector subbundle $p^b : T^b M \rightarrow M$ of $p'_M : T'M \rightarrow M$ such that $p^b : T^b M \rightarrow M$ is a convenient bundle and the canonical inclusion $\iota : T^b M \hookrightarrow T'M$ is a convenient bundle morphism. Such a bundle $p^b : T^b M \rightarrow M$ will be called a *weak subbundle* of $p'_M : T'M \rightarrow M$ and its typical fibre is denoted $\mathbb{M}^b$.

We denote by $\langle \cdot, \cdot \rangle$ the canonical bilinear pairing between $T'M$ and TM . Since the inclusion $\iota : T^b M \hookrightarrow T'M$ is an injective bundle morphism, we have an induced bilinear pairing $\langle \cdot, \cdot \rangle^b$ on $T^b M \times TM$ defined by:

$$(\alpha, u) \longmapsto \langle \alpha, u \rangle^b = \langle \iota \circ \alpha, u \rangle$$

which is continuous and non degenerate on the first variable. In what follows, and when no confusion can arise, this pairing will also be denoted by $\langle ., . \rangle$.

According to [21], Definition 48.5, we introduce, for any open set U in M , the set $\mathfrak{A}(U)$ of smooth functions $f \in C^\infty(U)$ such that, for any $k \in \mathbb{N}^*$ and any x in U , the derivative $d^k f(x) \in L_{\text{sym}}^k(T_x M, \mathbb{R})$ satisfies:

$$\forall (u_2, \dots, u_k) \in (T_x M)^{k-1}, \quad d_x^k f(., u_2, \dots, u_k) \in T_x^\flat M. \quad (2.2)$$

Remark 2.15. In particular, the previous condition implies that, for any bounded linear functional λ on $T_x^\flat M$, the map $u \mapsto \lambda(d_x^k f(u, u_2, \dots, u_k))$ is bounded on bounded sets of $T_x M$.

The sets $\{\mathfrak{A}(U) \subset C^\infty(U), U \text{ open in } M\}$ define a sub-sheaf $\mathfrak{A}_M$ of the sheaf $C_M^\infty = \{C^\infty(U), U \text{ open set in } M\}$ (cf. [8]). Of course, if $T^\flat M = T'M$, then $\mathfrak{A}_M = C_M^\infty$.

We then have the following properties of $\mathfrak{A}(U)$ proved in [8]:

PROPOSITION 2.16. *Fix any open set U in M .*

1. *The set $\mathfrak{A}(U)$ is a subalgebra of $C^\infty(U)$.*
2. *For each $k \in \mathbb{N}$, any local vector fields $X_1, \dots, X_k$ on U and any smooth map $f \in \mathfrak{A}(U)$, the map*

$$x \longmapsto d^k f(X_1, \dots, X_k)(x)$$

belongs to $\mathfrak{A}(U)$.

Then $\{\mathfrak{A}(U) \subset C^\infty(U), U \text{ open set in } M\}$ defines a sub-sheaf $\mathfrak{A}_M$ of the sheaf $C_M^\infty = \{C^\infty(U), U \text{ open in } M\}$ (cf. [8]). Of course, if $T^\flat M = T'M$, then $\mathfrak{A}_M = C_M^\infty$.

A morphism $P : T^\flat M \rightarrow TM$ is called *skew-symmetric* if it satisfies the relation:

$$\langle \xi, P(\eta) \rangle = - \langle \eta, P(\xi) \rangle$$

for all sections ξ and η of $T^\flat M$.

Given such a morphism P , for each open set U in M , on $\mathfrak{A}(U)$ we define:

$$\{f, g\}_U = - \langle df, P(dg) \rangle = \langle dg, P(df) \rangle. \quad (2.3)$$

In these conditions, the relation (2.3) defines a skew-symmetric bilinear map

$$\{., .\}_U : \mathfrak{A}(U) \times \mathfrak{A}(U) \longrightarrow \mathfrak{A}(U)$$

which satisfies the Leibniz identity

$$\{f, gh\}_U = g\{f, h\}_U + h\{f, g\}_U$$

(cf. [35, 2.1]).

In fact, the family of the bilinear maps $\{.,.\}_U$ defines a sheaf of brackets which will be denoted by $\{.,.\}_P$

DEFINITION 2.17. Let $p^b : T^b M \rightarrow M$ be a weak subbundle of $p'_M : T' M \rightarrow M$.

1. Any skew-symmetric morphism $P : T^b M \rightarrow TM$ is called an *almost Poisson anchor* and $\{.,.\}_P$ an *almost Poisson bracket*.
2. Given an almost Poisson anchor $P : T^b M \rightarrow TM$, we say that P is a *Poisson anchor* if the associated bracket $\{.,.\}_P$ satisfies the Jacobi identity:

$$\{f, \{g, h\}_P\}_P + \{g, \{h, f\}_P\}_P + \{h, \{f, g\}_P\}_P = 0.$$

In this case, $(T^b M, TM, P, \{.,.\}_P)$ is called a *partial Poisson structure* on M .

As classically, given a partial Poisson structure $(T^b M, TM, P, \{.,.\}_P)$, any function $H \in \mathfrak{A}(U)$ is called a *Hamiltonian* and the associated vector field $X_H = P(dH)$ is called a *Hamiltonian vector field*.

We then have $\{f, g\} = X_f(g)$ and also $[X_f, X_g] = X_{\{f, g\}}$ (see [31]), which is equivalent to:

$$P(d\{f, g\}) = [P(df), P(dg)]. \quad (2.4)$$

If $(T^b M, TM, P, \{.,.\}_P)$ is a partial Poisson structure on M , then, for any open set U , the space $\mathfrak{A}(U)$ provided with the Poisson bracket $\{.,.\}_P$ has a structure of Lie algebra, called classically a *Lie-Poisson algebra* and, according to (2.4), the Poisson anchor P gives rise to a Lie algebra morphism from $\mathfrak{A}(U)$ to the Lie algebra $\mathfrak{X}(U)$ of smooth vector fields on U .

2.3.2. POISSON BRACKET ASSOCIATED WITH A CONVENIENT LIE ALGEBROID. We recall some results of [8, Chapter 7] and, as in § 2.3.1, we consider the set $\mathfrak{A}(U)$.

Let $\pi : \mathcal{A} \rightarrow M$ be a convenient vector bundle and $\pi' : \mathcal{A}' \rightarrow M$ its associated dual convenient bundle. We consider a closed subbundle $q^b : T^b \mathcal{A}' \rightarrow \mathcal{A}'$ of $q_{\mathcal{A}'} : T' \mathcal{A}' \rightarrow \mathcal{A}'$.

DEFINITION 2.18. Let U be an open set in M and denote by $\mathcal{A}_U$ (resp. $\mathcal{A}'_U$) the restriction of $\mathcal{A}$ (resp. $\mathcal{A}'$) to U . A function $\Phi \in C^\infty(\mathcal{A}'_U)$ is called *linear*, if the restriction of Φ to each fibre $\mathcal{A}'_x$ is a linear map into $\mathbb{R}$ ⁴.

Let $\mathfrak{a}$ be a section of $\mathcal{A}_U$ and denote by $\Phi_{\mathfrak{a}} : \mathcal{A}' \rightarrow \mathbb{R}$ the smooth map defined by:

$$\Phi_{\mathfrak{a}}(\sigma) = \langle \sigma, \mathfrak{a} \circ \pi_*(\sigma) \rangle. \quad (2.5)$$

It is clear that $\Phi_{\mathfrak{a}}$ is linear.

We denote by $\mathfrak{A}(\mathcal{A}'_U)$ the set of smooth functions on $\mathcal{A}'_U$ which satisfy (2.2).

According to [8, 6.3.3], we have:

$$T^\flat_\sigma \mathcal{A}' = \left\{ (\sigma, \eta, A) \in T'_\sigma \mathcal{A}' : \begin{array}{l} \eta = d(f \circ \pi'), \ A = \mathfrak{a} \circ \pi', \\ \mathfrak{a} \in \Gamma(\mathcal{A}_U), \ f \in C^\infty(U) \end{array} \right\} \quad (2.6)$$

for any small enough open set U around x .

Each function of the form $\Phi_{\mathfrak{a}}$ and each map $f \circ \pi'$ are such that $d\Phi_{\mathfrak{a}}$ and $df \circ \pi'$ belong to the set $\Gamma(T^\flat \mathcal{A}'_U)$ of sections of $(T^\flat \mathcal{A}'_U)$, for any section $\mathfrak{a}$ of $\Gamma(\mathcal{A}'_U)$ and any function f of $C^\infty(U)$.

Let $C^b(\mathcal{A}'_U)$ be the set of smooth maps on $\mathcal{A}'_U$ whose restriction to each fibre $\mathcal{A}'_x$ is constant. Such a function φ naturally defines a function $\bar{\varphi}$ on U such that $\varphi = \bar{\varphi} \circ \pi'$.

Therefore, we can identify the subalgebra $\{f \circ \pi' : f \in C^\infty(U)\}$ of $C^b(\mathcal{A}'_U)$ with $C^\infty(U)$.

Now, it is clear that the set $C^b(\mathcal{A}'_U)$ of functions $F \in C^\infty(\mathcal{A}'_U)$, whose differential belongs to $\Gamma(T^\flat \mathcal{A}'_U)$, has a structure of algebra. Note that from (2.2), if F belongs to $\mathfrak{A}(\mathcal{A}'_U)$, this clearly implies that dF must be a section of $T^\flat \mathcal{A}'$. Since dF is a smooth section of $T' \mathcal{A}'$ and that $T^\flat \mathcal{A}'$ is closed convenient subbundle of $T' \mathcal{A}'$, it follows that dF is a smooth section of $T^\flat \mathcal{A}'_U$ and so $\mathfrak{A}(\mathcal{A}'_U)$ is contained in $C^b(\mathcal{A}'_U)$. In fact, we have (cf. [8, Proposition 7.2]):

PROPOSITION 2.19. *Let U be an open set which is a chart domain. The algebra $\mathfrak{A}(\mathcal{A}'_U)$ is equal to $C^b(\mathcal{A}'_U)$. This algebra is generated by functions of type $f \circ \pi_*$ and $\Phi_{\mathfrak{a}}$, for any $f \in C^\infty(U)$ and any $\mathfrak{a} \in \Gamma(\mathcal{A}_U)$.*

We introduce the following notion:

⁴Since Φ is smooth, its restriction to $\mathcal{A}'_x$ is linear and smooth, so it is bounded (cf. [21]).

DEFINITION 2.20. Consider a partial Poisson structure $(T^b\mathcal{A}', \mathcal{A}', P, \{.,.\}_P)$ on $\mathcal{A}'$. We say that P is a linear Poisson anchor, or that the Poisson bracket $\{.,.\}_P$ is linear if, for any open set U and any linear functions Φ_1 and Φ_2 in $\mathfrak{A}(\mathcal{A}'_U)$, the Poisson bracket $\{\Phi_1, \Phi_2\}_P$ is linear.

The following result extends to the convenient setting the classical equivalence between algebroid structures on a finite-dimensional anchored vector bundle and linear Poisson structures on its dual (cf. [8, Theorem 7.1]).

THEOREM 2.21. *Let $P : T^b\mathcal{A}' \rightarrow T\mathcal{A}'$ be a linear Poisson anchor on $\mathcal{A}'$. Then there exists a unique structure of convenient Lie algebroid $(\mathcal{A}, \pi, M, \rho, [.,.]_P)$ characterized in the following way, for any open set U in M :*

$$\forall (\mathbf{a}_1, \mathbf{a}_2) \in \Gamma(\mathcal{A}_U)^2, \quad \Phi_{[\mathbf{a}_1, \mathbf{a}_2]_P} = \{\Phi_{\mathbf{a}_1}, \Phi_{\mathbf{a}_2}\}_P \quad (2.7)$$

$$\forall f \in C^\infty(U), \forall \mathbf{a} \in \Gamma(\mathcal{A}_U), \quad \{\Phi_{\mathbf{a}}, f \circ \pi_*\}_P = df(\rho(\mathbf{a})) \circ \pi_* \quad (2.8)$$

where $\rho(\mathbf{a}) = T\pi'(P(\Phi_{\mathbf{a}}))$.

Conversely, a structure of convenient Lie algebroid $(\mathcal{A}, \pi, M, \rho, [.,.]_\rho)$ defines a unique linear Poisson anchor $P : T^b\mathcal{A}' \rightarrow T\mathcal{A}'$ which gives rise to a linear Poisson bracket $\{.,.\}_\rho$ on $\mathfrak{A}(\mathcal{A}'_U)$, for any open set U in M , which is characterized by relations (2.7) and (2.8). Moreover, the unique structure of convenient Lie algebroid $(\mathcal{A}, \pi, M, \rho, [.,.]_P)$ associated with P is the original one.

From Proposition 2.19 and Theorem 2.21, we obtain:

COROLLARY 2.22. *Let $(\mathcal{A}, \pi, M, \rho, [.,.]_P)$ be a convenient Lie algebroid and $\{.,.\}_P$ the Poisson Lie bracket associated with this structure. Then $(M, C^b(\mathcal{A}'), \{.,.\}_P)$ is a partial Poisson manifold.*

2.3.3. CHARACTERISTIC DISTRIBUTION AND FOLIATION. Given a partial Poisson structure (T^bM, TM, P) , the family of vector spaces

$$D := \{D_x = P(T_x^bM) \subset T_xM : x \in M\}$$

is a distribution on M called the *characteristic distribution*.

In finite dimension, when $T^bM = T'M$, it is well known that the characteristic distribution gives rise to a Stefan-Sussman foliation. Moreover, each leaf carries a natural structure of immersed symplectic manifold.

In the convenient setting, this property does not hold in general, and may even fail in the Banach setting (see Remarks 2.24). Nevertheless, the following theorem provides sufficient conditions under which an analogous result remains valid (cf. [35]).

THEOREM 2.23. *Let $(T^\flat M, TM, P)$ be a partial Banach Poisson manifold such that the kernel of P is supplemented in each fibre $T_x^\flat M$ of $T^\flat M$ and $P(T^\flat M)$ is a closed distribution. Then we have the following:*

1. $D = P(T^\flat M)$ is integrable and the foliation defined by D is an almost symplectic foliation, i.e. on each leaf N of this foliation, we have a unique non degenerate smooth skew symmetric bilinear form ω_N on TN such that for any u and v in $T_x N$, we have

$$\omega_N(u, v) = \langle \alpha, P(\beta) \rangle$$

for any $\alpha, \beta \in T_x^\flat M$ with $P(\alpha) = u$ and $P(\beta) = v$.

2. For each maximal leaf N , let ω_N denote the natural weak symplectic form on N . Given an open subset V of N , denote by $\mathcal{E}_{\omega_N}(V)$ the subset of functions $f \in C^\infty(V)$ such that df belongs to the range $\omega_N^\flat(TV)$. Then the restriction f_N to $U \cap N$ of any $f \in \mathcal{E}_{\omega_N}(U)$ belongs to $\mathcal{E}_{\omega_N}(U \cap N)$ and we have, for any f and g in $\mathcal{E}_{\omega_N}(U)$,

$$\{f|_N, g|_N\}_{P|_N} = \{f_N, g_N\}_{\omega_N}.$$

Remarks 2.24. (1) In general, in the convenient setting, the characteristic distribution need not be integrable. Even in the Banach setting, if D is not closed, we think that the distribution D is not integrable, but, unfortunately, we do not have any example.

(2) Theorem 2.23 provides sufficient conditions for the integrability of the characteristic distribution and, on each leaf N of the associated foliation, the 2-form ω_N is not, *a priori*, closed (as stated in [35]). But, in fact, ω_N is effectively closed. Indeed, according to Lemma 2.25, for any $x \in M$, there exists an open set U around x such that any $\sigma \in T_x^\flat M$ can be written $\sigma = d_x f$ for some local smooth function $f : U \rightarrow \mathbb{R}$. Thus, for $i \in \{1, 2, 3\}$, each $X_i \in T_x N$ can be written $X_i = P(d_x f_i)$ for some f_i locally defined around x . Each Hamiltonian vector field is tangent to the characteristic foliation and the tangent space of a leaf N at x is generated by the value at x of the set of Hamiltonian vector fields $\{X_f, f \in \mathcal{E}(U)\}$. Using Cartan's formulae

for the exterior derivative, the expression $d\omega_N(X_{f_0}, X_{f_1}, X_{f_2})$ involves two occurrences of the Jacobi identity for the triple (f_0, f_1, f_2) . Since the Poisson bracket $\{, \}_P$ satisfies the Jacobi identity, it follows that ω_N is closed.

(3) When $(T^\flat M, TM, P)$ is a partial symplectic structure, even in the convenient setting, the characteristic distribution coincides with TM . Consequently, the characteristic foliation reduces to a single leaf, namely M itself. Thus, the argument in (2) applies, and it follows that M is a weak symplectic manifold.

2.3.4. PARTIAL LIE ALGEBROIDS AND PARTIAL POISSON MANIFOLDS. We refer to [8, 3.18.4] for the notion of convenient partial Lie algebroid and we adopt the same notation as in that reference. We begin with the following fundamental result (cf. [8, Lemma 7.3]).

LEMMA 2.25. *Let $(T^\flat M, M, P, \{.,.\}_P)$ be a partial Poisson structure.*

For each $x \in M$, there exists an open neighbourhood U of x such that the vector space

$$\mathfrak{P}_x(U) = \{d_x f : f \in \mathfrak{A}(U)\}$$

is equal to the fibre $T_x^\flat M$.

We then have the following result (cf. [8, Proposition 7.3 and Corollary 7.2]).

PROPOSITION 2.26. *Let $(T^\flat M, M, P, \{.,.\}_P)$ be a partial Poisson structure. We denote by $\mathfrak{P}_M$ the sheaf of $\mathfrak{A}(U)$ -modules generated by the set $\{df : f \in \mathfrak{A}(U)\}$. Then the following properties hold:*

1. *We can define a sheaf of almost brackets $[\cdot, \cdot]_P$ on the sheaf $\mathfrak{P}_M$ by:*

$$[\alpha, \beta]_P = L_{P(\alpha)}\beta - L_{P(\beta)}\alpha - d\langle \alpha, P(\beta) \rangle \quad (2.9)$$

for any open set U in M and any sections α and β in $\mathfrak{P}(U)$ where L_X is the Lie derivative.

Moreover, $[\cdot, \cdot]_P$ satisfies:

$$\forall (f, g) \in (\mathfrak{A}(U))^2, \quad [df, dg]_P = d\{f, g\}_P. \quad (2.10)$$

2. *$(\mathfrak{P}_M, [\cdot, \cdot]_P)$ is a sheaf of Poisson-Lie algebras.
In particular, $(T^\flat M, p_M^\flat, M, P, \mathfrak{P}_M)$ is a strong partial convenient Lie*

algebroid and $(T^b M, M, P, [\cdot, \cdot]_P)$ is an pre-Lie algebroid that is P satisfies $P([\alpha, \beta]_P) = [P(\alpha), P(\beta)]$ for all local sections α, β de $T^b M$. Moreover, if P is an injective morphism, then $(T^b M, p_M^b, M, P, \mathfrak{P}_M)$ is a Lie algebroid.

Remark 2.27. In fact, Assertion (1) of Proposition 2.26, is also valid if we only assume that P is an almost partial Poisson anchor. The proof of this assertion only uses the fact that P is, in particular, an almost partial Poisson anchor (cf. Proof of [8, Proposition 7.3]). Note that, in this case, we only have a well defined almost Poisson bracket $\{f, g\}_P = \langle dg, Pdf \rangle$ on each algebra $\mathfrak{A}(U)$ for any open set U in M .

2.3.5. JACOBI IDENTITY AND SCHOUTEN CONDITION. It is well known that, in finite dimensions, if $P : T^*M \rightarrow TM$ is an almost Poisson anchor, then it induces a skew symmetric contravariant tensor Λ of type $(2, 0)$ on T^*M defined by

$$\Lambda(\alpha, \beta) = \langle \beta, P\alpha \rangle. \quad (2.11)$$

Then P defines a Poisson structure on M if and only if the Schouten bracket $[\Lambda, \Lambda]$ vanishes identically on M (cf. [25]). In fact, the property $[\Lambda, \Lambda] = 0$ is equivalent to the Jacobi identity of the bracket $\{\cdot, \cdot\}_P$ naturally associated with P .

Now consider an almost Poisson anchor $P : T^b M \rightarrow TM$.

On the one hand, as in finite dimension, one can associate to P a skew symmetric contravariant tensor Λ of type $(2, 0)$ on $T^b M$ defined by

$$\Lambda(\alpha, \beta) = \langle \beta, P\alpha \rangle. \quad (2.12)$$

On the other hand, on any open set U in M , and for any σ_1 and σ_2 in $\mathfrak{P}(U)$, we have the map $[P, P] : \mathfrak{P}(U) \times \mathfrak{P}(U) \rightarrow \mathfrak{X}(U)$ given by:

$$[P, P](\sigma_1, \sigma_2) = P([\sigma_1, \sigma_2]_P) - [P\sigma_1, P\sigma_2]. \quad (2.13)$$

Then $[P, P]$ is a tensor of type $(2, 1)$ with takes values in TM .

On the other hand, as in finite dimension, to P is associated a skew symmetric tensor Λ on $T^b M$ of type $(2, 0)$ also defined by (2.12). Thus we can consider the tensor $[\Lambda, \Lambda]$ of type $(3, 0)$ on $T^b M$ defined by:

$$[\Lambda, \Lambda](\sigma_1, \sigma_2, \sigma_3) = \langle \sigma_3, [P, P](\sigma_2, \sigma_1) \rangle \quad (2.14)$$

for any $\sigma_i \in \mathfrak{P}(U)$ where $i \in \{1, 2, 3\}$.

DEFINITION 2.28. Let $P : T^b M \rightarrow TM$ be an almost Poisson anchor and Λ the associated contravariant tensor of type $(3, 0)$. Then $2[\Lambda, \Lambda]$ is called the *Schouten bracket* of Λ .

Note that this Definition is justified in [8, 7.2.2]. Now, as in finite dimension, we have (cf. [8, Theorem 7.2]):

THEOREM 2.29. *Let $P : T^b M \rightarrow TM$ be an almost Poisson anchor and Λ the associated contravariant tensor of type $(3, 0)$. Then P defines a partial Poisson structure on M if and only if $[\Lambda, \Lambda] = 0$.*

2.4. PARTIAL POISSON BIALGEBROID AND ITS ASSOCIATED COHOMOLOGY. In this section, we consider a partial Poisson manifold $(M, \mathfrak{A}_M, \{\cdot, \cdot\}_P)$. We have seen that $(T^b M, p_M^b, M, P, \mathfrak{P}_M)$ is a strong partial Banach Lie algebroid and $(T^b M, p_M^b, M, P, [\cdot, \cdot]_P)$ is an almost Lie algebroid (cf. Proposition 2.26, (2)).

2.4.1. PARTIAL BIALGEBROID. According to [8, 3.18.6.4], let d_P be the associated exterior differential associated with the strong partial Lie algebroid $(T^b M, p_M^b, M, P, \mathfrak{P}_M)$. Then, if X is a vector field defined on an open set U in M , according to [8, Example 7.4], X can be considered as an element of $\bigwedge^1 \Gamma^*(T^b M)$, so we have:

$$d_P X(\alpha, \beta) = L_\alpha^P \langle \beta, X \rangle - L_\beta^P \langle \alpha, X \rangle - \langle [\alpha, \beta]_P, X \rangle. \quad (2.15)$$

Therefore, as for finite dimensional Poisson manifolds (see [25]), we have (cf. [8, Proposition 7.8]):

PROPOSITION 2.30. *Let Λ be the Poisson tensor associated with P . Then with the previous notations we have:*

- (1) $d_P X = -L_X \Lambda$;
- (2) $d_P [X, Y] = L_X d_P Y - L_Y d_P X$.

Remark 2.31. In finite dimension, recall that a Lie bi-algebroid is a pair of Lie algebroids $(\mathcal{A}, \pi, M, \rho, [\cdot, \cdot]_{\mathcal{A}})$ and $(\mathcal{A}', \pi', M, \rho', [\cdot, \cdot]_{\mathcal{A}'})$ where $\pi' : \mathcal{A}' \rightarrow M$ is the dual bundle of $\pi : \mathcal{A} \rightarrow M$ with the following compatibility condition:

$$d_{\rho'} [X, Y]_{\mathcal{A}} = L_X d_{\rho'} Y - L_Y d_{\rho'} X,$$

for all sections X and Y of $\mathcal{A}$ where d_ρ (resp. $d_{\rho'}$) denotes the exterior differential associated with the Lie algebroid $(\mathcal{A}, \pi, M, \rho, [\cdot, \cdot]_\mathcal{A})$ (resp. $(\mathcal{A}', \pi', M, \rho', [\cdot, \cdot]_{\mathcal{A}'})$) (cf. [27]).

In particular, if M is provided with a Poisson structure defined by a bivector Λ and a Poisson anchor $P : T^*M \rightarrow M$, then the pair $(TM, p_{TM}, M, \text{Id}, [\cdot, \cdot])$ and $(T^*M, p_{T^*M}, M, P, [\cdot, \cdot]_P)$ is a Lie bi-algebroid (cf. [27]). According to our notations, in this finite dimensional context, the compatibility relation is exactly Proposition 2.30 (2).

Unfortunately, in our infinite dimensional context, even if $T^\flat M = T'M$, the Poisson structure, in general, does not provide $T'M$ with a Lie algebroid structure.

Taking Proposition 2.30 into account, we propose the following generalization of the notion of Lie bialgebroid in finite dimension:

DEFINITION 2.32. Let $(E, \pi, M, \rho, [\cdot, \cdot]_E)$ be a Banach Lie algebroid. Let $\pi' : E' \rightarrow M$ be the dual bundle of E . Assume that there exists a weak Banach subbundle $\pi^\flat : E^\flat \rightarrow M$ with the following properties:

- $E^\flat \subset E'$, $\pi^\flat = \pi'|_{E^\flat}$ and the canonical inclusion $\iota : E^\flat \hookrightarrow E'$ is a bundle morphism;
- there exists a pre-Lie algebroid structure $(E^\flat, \pi^\flat, M, \rho^\flat, [\cdot, \cdot]_{E^\flat})^5$ on $E^\flat$ and also a strong Lie algebroid structure on $E^\flat$.

We denote by d_ρ (resp. $d_{\rho^\flat}$) the exterior differential on E (resp. $E^\flat$) associated with these structures. We say that the pair $(E, E^\flat)$ is a partial bialgebroid if, for any open set U in M , we have:

$$d_{\rho^\flat}[X, Y]_E = L_X^{\rho^\flat} d_{\rho^\flat} Y - L_Y^{\rho^\flat} d_{\rho^\flat} X,$$

for all sections X and Y of E_U , considered as 1-forms on $E^\flat$.

EXAMPLES 2.33. (a) Any finite dimensional bialgebroid (E, E') is a partial bialgebroid.

(b) From Proposition 2.30, to any partial Poisson manifold $(M, \mathfrak{A}_M, \{\cdot, \cdot\}_P)$ is associated a partial bialgebroid $(TM, T^\flat M)$ (see also Theorem 3.22).

⁵cf. Proposition 2.26.

Proof of Proposition 2.30. Fix some open set U in M . All sections α and β of T^*M and vector fields X and Y considered in this proof are assumed to be defined on U .

(1) Using the expression of the Lie derivative of a 2-form, we obtain:

$$\begin{aligned} L_X \Lambda(\alpha, \beta) &= - \langle \beta, [P(\alpha), X] \rangle \\ &\quad + \langle \alpha, [P(\beta), X] \rangle - X(\langle \beta, P(\alpha) \rangle). \end{aligned} \quad (2.16)$$

On the other hand, using the exterior derivative of a 2-form on a partial strong convenient Lie algebroid (cf. [8, 3.18.6.3]), we have:

$$d_P X(\alpha, \beta) = P(\alpha)(\langle \beta, X \rangle) - P(\beta)(\langle \alpha, X \rangle) - \langle [\alpha, \beta]_P, X \rangle. \quad (2.17)$$

Using the value of $[\alpha, \beta]_P$, we get:

$$d_P X(\alpha, \beta) = \langle \beta, [P(\alpha), X] \rangle - \langle \alpha, [P(\beta), X] \rangle + X(\langle \beta, P(\alpha) \rangle)$$

which ends the proof of (1).

(2) From the value of a Lie derivative and (1), we have:

$$\begin{aligned} L_X d_P Y(\alpha, \beta) &= X(d_P Y(\alpha, \beta)) - d_P Y(L_X \alpha, \beta) - d_P Y(\alpha, L_X \beta) \\ &= X(Y(\langle \beta, P(\alpha) \rangle)) - X(\langle L_Y \alpha, P(\beta) \rangle) \\ &\quad + X(\langle L_Y \beta, P(\alpha) \rangle) - Y(\langle L_X \beta, P(\alpha) \rangle) \\ &\quad + \langle L_X L_Y \alpha, P(\beta) \rangle - \langle L_Y \beta, P(L_X \alpha) \rangle \\ &\quad - Y(\langle \beta, P(L_X \alpha) \rangle) + \langle L_Y \alpha, P(L_X \beta) \rangle \\ &\quad - \langle L_X L_Y \beta, P(\alpha) \rangle. \end{aligned}$$

By permutation of X and Y , we obtain the corresponding expression of $L_Y d_P X$. By a direct calculation, we obtain easily that

$$\begin{aligned} L_X d_P Y - L_Y d_P X &= [X, Y](\langle \alpha, P(\beta) \rangle) \\ &\quad - \langle L_{[X, Y]} \alpha, P(\beta) \rangle + \langle L_{[X, Y]} \beta, P(\alpha) \rangle. \end{aligned} \quad (2.18)$$

This expression coincides with $d_P[X, Y](\alpha, \beta)$. ■

2.4.2. CONVENIENT LINEAR PARTIAL POISSON-LIE SPACES. The concept of Banach Poisson-Lie space was introduced in [33] and generalized in [38] or in [31, Example 2.10].

In this section, we propose a generalization of this concept to the convenient setting.

DEFINITION 2.34. Let $\mathfrak{g}$ be a convenient space. Consider a convenient subspace $\mathfrak{g}^b$ of $\mathfrak{g}'$ such that the inclusion is bounded. Let $P : T^b\mathfrak{g} := \mathfrak{g} \times \mathfrak{g}^b \rightarrow T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$ be an injective almost Poisson anchor.

We say that $(\mathfrak{g}, \mathfrak{g}^b, P)$ is a *convenient partial Poisson-Lie space* if $\mathfrak{g}^b$ is provided with a Lie bracket $[\cdot, \cdot]^b$ such that, for all $x \in \mathfrak{g}$,

$$P([\alpha, \beta]^b)(x) = [P\alpha, P\beta](x) \quad (2.19)$$

where α and β are considered as functions in $\mathfrak{A}(\mathfrak{g}) := \{f \in C^\infty(\mathfrak{g}) : d_x f \in \{x\} \times \mathfrak{g}^b\}$.

We then have the following characterization:

PROPOSITION 2.35. *Consider a convenient subspace $\mathfrak{g}^b$ of $\mathfrak{g}'$ such that the inclusion is bounded and let $P : T^b\mathfrak{g} := \mathfrak{g} \times \mathfrak{g}^b \rightarrow T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$ be an almost Poisson anchor. Then $(\mathfrak{g}, \mathfrak{g}^b, P)$ is a partial Poisson-Lie space if and only if P is a Poisson-Lie anchor.*

Moreover, in this case, $(T\mathfrak{g}, T^b\mathfrak{g})$ is a partial Lie bialgebroid.

Proof. Consider a convenient subspace $\mathfrak{g}^b$ of $\mathfrak{g}'$ such that the inclusion is bounded and let $P : T^b\mathfrak{g} \rightarrow T\mathfrak{g}$ be an almost Lie-Poisson anchor. We have an almost Lie bracket on the set $\mathfrak{A}(\mathfrak{g}) := \{f \in C^\infty(\mathfrak{g}) : d_x f \in \{x\} \times \mathfrak{g}^b\}$ defined by $\{f, g\}_P = \langle dg, Pdf \rangle$. Then if Λ is the skew symmetric contravariant tensor of type $(2, 0)$ on $T^b\mathfrak{g}$ associated with P as defined in (2.12), the bracket $\{\cdot, \cdot\}_P$ satisfies the Jacobi identity if and only if the Schouten bracket vanishes: $[\Lambda, \Lambda] = 0$ (cf. Theorem 2.29). But, according to (2.13) and (2.14), this condition is equivalent to:

$$P(d\{f, g\}) = [Pdf, Pdg] \quad (2.20)$$

for all f and g in $\mathfrak{A}(\mathfrak{g})$. In particular, this condition implies that:

$$P([\alpha, \beta]) = [P\alpha, P\beta] \quad (2.21)$$

for all sections α, β of $T^b\mathfrak{g}$. Since $\mathfrak{g}^b$ can be identified with the set

$$\{f_\alpha \in \mathfrak{A}(\mathfrak{g}) : \forall \alpha \in \mathfrak{g}^b, \forall x \in \mathfrak{g}, f_\alpha(x) = \langle \alpha, x \rangle\}$$

then we have a Lie bracket on $\mathfrak{g}^b$ defined by

$$[\alpha, \beta]^b = d\{f_\alpha, f_\beta\}. \quad (2.22)$$

Thus $\mathfrak{g}^b$ is endowed with a Lie algebra structure and the condition (2.22) is satisfied.

Conversely, assume that the assumptions of Definition 2.34 are satisfied. Since P is injective, it follows that the almost Lie bracket on $\mathfrak{g}^b$ defined by (2.21) coincides with the given Lie bracket on $\mathfrak{g}^b$. Thus the relation (2.20) is satisfied and so the Schouten bracket of P is zero. This concludes the proof of the first part.

Now under the assumption that $(\mathfrak{g}, \mathfrak{g}^b, P)$ is a partial Lie-Poisson space the last part is a direct consequence of Proposition 2.30. ■

EXAMPLE 2.36. Assume that $\mathfrak{g}^b = \mathfrak{g}'$. According to [33], a Lie algebra $\mathfrak{g}$ is a Poisson-Lie algebra if the set $C^\infty(\mathfrak{g})$ is provided with a Lie-Poisson algebra $\{.,.\}$ such that considering its dual $\mathfrak{g}'$ as a subset of $C^\infty(\mathfrak{g})$, then $\{.,.\}$ induces a Lie bracket on $\mathfrak{g}'$. In fact, we have a natural bounded map $P : \mathfrak{g} \times \mathfrak{g}' \rightarrow \mathfrak{g} \times \mathfrak{g}$ such that $\{f, g\} = \langle dg, P(df) \rangle$ which is given by $P\alpha(x) = -\text{ad}_\alpha^* x$ where, as classically, $\text{ad}_\alpha^* x = -\alpha \circ \text{ad}_x$ (cf. next Example) which satisfies the Jacobi identity (cf. [33]).

EXAMPLE 2.37. Assume that $(\mathfrak{g}, [.,.])$ is a convenient Lie algebra. Then, to each $x \in \mathfrak{g}$, we have a bounded linear map $\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}$ given by $\text{ad}_x(y) = [x, y]$. This gives rise to a bounded ⁶ coadjoint action of $\mathfrak{g}$ onto $\mathfrak{g}'$ in an evident way. We also have a bounded coadjoint action

$$\text{ad}^* : \mathfrak{g} \times \mathfrak{g}' \longrightarrow \mathfrak{g}'$$

given by: $\text{ad}_x^* \alpha = -\alpha \circ \text{ad}_x$ for all $x \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}'$.

Consider a convenient subspace $\mathfrak{g}^b$ of $\mathfrak{g}'$ whose inclusion is bounded. If $\mathfrak{g}^{b'}$ denote the dual of $\mathfrak{g}^b$, we have the inclusions: $\mathfrak{g} \subset \mathfrak{g}'' \subset \mathfrak{g}^{b'}$. Assume that the restriction of the previous coadjoint action to $\mathfrak{g} \times \mathfrak{g}^b$ takes values in $\mathfrak{g}^b$ and is bounded⁷. When there is no ambiguity, this action will still be denoted ad^* .

When $\mathfrak{g}^b$ separates points of $\mathfrak{g}$, for any $x \in \mathfrak{g}$, then the map $P_x(\alpha(x)) = -\text{ad}_{\alpha(x)}(x)$ is a well defined vector field on $\mathfrak{g}$ which satisfies the assumption of Definition 2.34. Then, in the Banach setting, from [38, Theorem 3.14], $(\mathfrak{g}, \mathfrak{g}^b, P)$ is a partial Lie-Poisson space in the sense of Definition 2.34. Thus this definition can be considered as an adaptation of [38, Definition 3.12].

In [38], the reader can find other more specific and very interesting examples in the Banach setting.

⁶Note that this action is continuous for the associated bornological topology of $\mathbb{E}$.

⁷This condition is automatically satisfied if $\mathfrak{g}^b$ is closed in $\mathfrak{g}'$.

2.5. POISSON (RESP. SYMPLECTIC) RELATIONS. By analogy with the linear setting, in the context of convenient manifolds, a *relation* $R : M_1 \rightarrow M_2$ is a closed weak submanifold of $M_1 \times M_2$. Any map $\phi : M_1 \rightarrow M_2$ will be also considered as the relation given by its graph $G_\phi = \{(x, \phi(x)), x \in M_1\}$. If $S : M_2 \rightarrow M_3$ is also a relation, then the relation $S \circ R : M_1 \rightarrow M_3$ is again defined by

$$S \circ R = \{(x, z) \in M_1 \times M_3 : (x, y) \in R, (y, z) \in S \text{ with } y \in M_2\}. \quad (2.23)$$

The inverse of R is the relation $R^{-1} : M_2 \rightarrow M_1$ given by

$$R^{-1} = \{(y, x) \in M_2 \times M_1 : (x, y) \in R\}.$$

As in [41], we introduce:

DEFINITION 2.38. For $i \in \{1, 2\}$, let $(T^\flat M_i, TM_i, P_i, \{ , \}_{P_i})$ be a Poisson structure (resp. symplectic structure). A relation $R : M_1 \rightarrow M_2$ is a Poisson relation, if R is a closed weak submanifold of $M_1 \times M_2$ which is coisotropic for the partial Poisson structure (resp. Lagrangian for the partial symplectic structure) $M_1 \times M_2^-$.

Unfortunately, the composition of two Poisson relations is not in general a Poisson relation and even not a closed submanifold without some adequate assumptions. In fact, if $R : M_1 \rightarrow M_2$ and $S : M_2 \rightarrow M_3$ are Poisson relations, then the product $R \times S$ is a closed submanifold of $M_1 \times M_2 \times M_2 \times M_3$ and if $p_{1,3}$ is the projection of $M_1 \times M_2 \times M_2 \times M_3$ onto $M_1 \times M_3$ then according to (2.23), we have:

$$S \circ R = p_{1,3}(R \times S \cap (M_1 \times \Delta_2 \times M_3))$$

where Δ_2 is the diagonal of $M_2 \times M_2$.

This is why is introduced the following conditions under which $S \circ R$ will be a weak closed submanifold in $M_1 \times M_3$ (cf. [41, Definition 1.3.7]):

DEFINITION 2.39. (1) A pair (N_1, N_2) of closed submanifolds of a convenient manifold M is called a *clean pair* if $N_1 \cap N_2$ is a closed submanifold of M such that $T(N_1 \cap N_2) = TN_1 \cap TN_2$ ⁸.

(2) Let $R : M_1 \rightarrow M_2$ and $S : M_2 \rightarrow M_3$ be relations. The pair of relations (R, S) is called a *clean pair* if:

⁸This notion was firstly introduced by R. Bott in his book [6].

- (i) If Δ_2 denotes the diagonal in $M_2 \times M_2$, the intersection of the closed submanifolds $R \times S$ and $D = M_1 \times \Delta_2 \times M_3$ is a clean pair in $M_1 \times M_2 \times M_2 \times M_3$;
- (ii) $S \circ R$ is a closed submanifold of $M_1 \times M_3$ and the restriction to $(R \times S) \cap D$ of the projection of $p_{1,3} : M_1 \times M_2 \times M_2 \times M_3 \rightarrow M_1 \times M_3$ is a submersion on $S \circ R$.

Remark 2.40. In [41], A. Weinstein introduced the notion of clean pair and very clean pair for the finite dimensional context. Definition 2.39 corresponds to Weinstein's notion of very clean pair. But since we will only need the assumptions of this Definition we simply use the terminology "clean pair" instead of "very clean pair".

Note that from [8, Remark 1.26], if a pair (R, S) of relations is such that $R \times S$ and D are transverse, then the condition (2) (i) of Definition 2.39 is satisfied, but, in general, the condition (2) (ii) is not satisfied. The following Lemma gives examples of clean pair of relations:

LEMMA 2.41. *Consider a smooth $\phi : M_1 \rightarrow M_2$. If $p_i : M_2 \times M_3 \rightarrow M_i$ for $i \in \{2, 3\}$ is the canonical projection, let $R : M_2 \rightarrow M_3$ be a relation such that the restriction of p_3 to R is a diffeomorphism onto M_3 and $\phi_1(M_1) \subset p_2(R)$. If we identify ϕ with the relation from M_1 to M_2 given by its graph, then (ϕ, R) is a clean pair. In particular, if ψ is a submersion from M_3 to M_2 then (ψ^{-1}, ϕ) is a clean pair.*

Proof. We consider the relation product

$$\phi \times R = \{(x, \phi(x), y_2, y_3) : x \in M_1, (y_2, y_3) \in R\}.$$

From the assumptions, $\phi \times S$ is a closed submanifold of $M_1 \times M_2 \times M_2 \times M_3$ modelled on $\mathbb{M}_1 \times \mathbb{M}_3$. If Δ_2 is the diagonal of $M_2 \times M_2$ and $D = M_1 \times \Delta_2 \times M_3$, then we have

$$\begin{aligned} (\phi \times R) \cap D &= \{(x, \phi(x), \phi(x), y) : (\phi(x), y) \in R\} \\ &= \{(x, \phi(x), \phi(x), y) : (x, y) \in M_1 \times M_3\}. \end{aligned}$$

From the assumptions, it follows that $(\phi \times R) \cap D$ is a closed submanifold in $M_1 \times M_2 \times M_2 \times M_3$ modelled on $\mathbb{M}_1 \times \mathbb{M}_3$.

Now, for $m = (x, \phi(x), \phi(x), y) \in (\phi \times R) \cap D$, we have

$$\begin{aligned} T_m((\phi \times R) \cap D) & \quad (2.24) \\ &= \{(u, v, -v, w) \in T_x M_1 \times T_{\phi(x)} M_2 \times T_{\phi(x)} M_2 \times T_y M_2 : v = T_x \phi(u)\} \\ &= T_m(\phi \times R) \cap T_m D. \end{aligned}$$

Thus condition (2) (i) of Definition 2.39 is satisfied.

On the other hand the relation $(R \circ \phi)$ is

$$\{(x, y) \in M_1 \times M_3 : (\phi(x), y) \in R\}$$

which, from assumption, is equal to $M_1 \times M_3$.

Moreover, for any $m = (x, \phi(x), \phi(x), y) \in (\phi \times R) \cap D$, according to (2.24), we obtain:

$$T_m p_{1,3}(T_m((\phi \times R) \cap D)) = T_x M_1 \times T_y M_3 = T_{(x,y)}(R \circ \phi).$$

Now, as $p_{1,3}^{-1}(x, y) = \{(x, \phi(x), \phi(x), y)\}$ which is reduced to a point in $(\phi \times R) \cap D$, this trivially implies that the condition (2) (ii) of Definition 2.39 is satisfied. Finally, if $\psi : M_3 \rightarrow M_2$ is a submersion, $R = \psi^{-1}$ satisfies clearly the assumptions required for R in Lemma 2.41. This implies the last Assertion. ■

Now, for a clean pair of Poisson relations, we have:

THEOREM 2.42. *Let $R : M_1 \rightarrow M_2$ and $S : M_2 \rightarrow M_3$ be Poisson (resp. symplectic) relations such that the pair (R, S) is clean. Then $S \circ R : M_1 \rightarrow M_3$ is a Poisson (resp. symplectic) relation.*

Proof. We only consider the case of Poisson relation. The symplectic case uses point by point the same type of argument.

Since the pair (R, S) is clean, it follows that for $(x, y) \in R$ and $(y, z) \in S$, the tangent space of $R \times S \cap D$ at $m = (x, y, y, z)$ is the intersection of $T_m(R \times S)$ and $T_m D$. From Definition 2.39, assumption (ii), $T_m p_{1,3}$ is a surjective map from $T_m(R \times S) \cap T_m D$ onto $T_{(x,z)} S \circ R$. But $T_{(x,z)}(S \circ R)$ is exactly the composition $(T_{(x,y)} S) \circ (T_{(y,z)} R)$ of the linear Poisson relations $T_{(x,y)} R : T_x M_1 \rightarrow T_y M_2$ and $T_{(y,z)} S : T_y M_2 \rightarrow T_z M_3$. Thus, from Definition 2.12 and Corollary 2.14, it follows that $S \circ R$ is a Poisson relation. ■

Let $R : M_1 \rightarrow M_2$ be a Poisson (resp. symplectic) relation between two partial Poisson (resp. partial symplectic) manifolds. Given any coisotropic (resp. Lagrangian) submanifold C in M_1 , following [41, (2.1)], if $\{*\}$ is the trivial Poisson manifold reduced to a point with the zero Poisson structure, then C can be considered as a Poisson (resp. symplectic) relation $C : \{*\} \rightarrow M_1$; since C is coisotropic (resp. Lagrangian), C can be identified with the graph of the relation $C : \{*\} \rightarrow M_1$. Thus we introduce:

DEFINITION 2.43. Let $R : M_2 \rightarrow M_1$ be a relation and C a closed submanifold of M_2 . We will say that (R, C) is a clean pair if the pair of relations (R, C) is a clean pair.

From now on, if (R, C) is a clean pair, we identify $R(C)$ with $R \circ C$. In this way, Theorem 2.42 has the following Corollary:

COROLLARY 2.44. Let $R : M_1 \rightarrow M_2$ be a Poisson (resp. symplectic) relation between two partial Poisson (resp. partial symplectic) manifolds and a coisotropic (resp. Lagrangian) weak closed submanifold C in M_1 . If (R, C) is clean, then $R(C)$ is a coisotropic (resp. Lagrangian) submanifold of M_2 .

2.6. COISOTROPIC SUBMANIFOLDS. Fix a partial convenient Poisson manifold $(T^b M, TM, P, \{ , \}_P)$. In this way, for each $x \in M$, $P_x : T_x^b M \rightarrow T_x M$ is a Poisson anchor.

- DEFINITION 2.45.
1. An ideal $\mathcal{I}$ in a Poisson Lie algebra $\mathcal{E}$ whose bracket is $\{ , \}$ is called *coisotropic* if $\mathcal{I}$ is closed under the bracket $\{ , \}$.
 2. A closed submanifold N of M^9 is called a *coisotropic submanifold* of M , relative to P if $T_x N \subset T_x M$ is a coisotropic subspace of $T_x M$ relatively to P_x .

We have the following characterization of a coisotropic weak closed submanifold.

PROPOSITION 2.46. Let N be a closed convenient submanifold of M . Then the following properties are equivalent:

- (a) N is coisotropic.

⁹ N is a closed submanifold of M if it is a convenient manifold, the inclusion i of N into M is smooth and $T_x i$ is injective with closed range for any $x \in N$.

- (b) For any open set U in M , the set $\mathcal{I}_N(U)$ of smooth functions of $\mathcal{E}(U)$ which vanish on $U \cap N$ is coisotropic in $\mathcal{E}(U)$.
- (c) For any open set U and any $h \in \mathcal{I}_N(U)$, the Hamiltonian field X_h is tangent to N .

Proof. First, it is clear that the set $\mathcal{I}_N(U)$ is an ideal in $\mathcal{E}(U)$. For $f \in \mathcal{I}_N(U)$ $f|_{U \cap N} \equiv 0$ then for each $x \in U$, $d_x f$ belongs to $(T_x N)^0 \in T'_x M$. Now from Lemma 2.25 it follows that $(T_x N)^0$ is generated by all the forms $\{d_x f, f \in \mathcal{I}_N(U)\}$ for all $x \in U$. According to Lemma 2.5, 5, the definition of $\{, \}_p$ and (2.3), $N \cap U$ is coisotropic if and only if $\{f, g\}_P = 0$ on $N \cap U$, which gives the equivalence between (a) and (b).

The equivalence between (a) and (c) is a consequence of Proposition 2.8 applied at each point $x \in U$ according to the definition of X_h and Lemma 2.25. ■

2.7. POISSON MORPHISMS AND POISSON MAPS

DEFINITION 2.47. Let $(T^\flat M_1, TM_1, P_1, \{\cdot, \cdot\}_{P_1})$ and $(T^\flat M_2, TM_2, P_2, \{\cdot, \cdot\}_{P_2})$ be two convenient partial Poisson structures.

A smooth map $\phi : M_1 \rightarrow M_2$ is called a *Poisson morphism* if:

- (pPMorph1) $\forall x \in M_1, (T_x \phi)^* (T_{\phi(x)}^\flat M_2) \subset T_x^\flat M_1$
where $(T_x \phi)^*$ is the adjoint of the tangent map $T_x \phi$;
- (pPMorph2) $T_x \phi : T_x M_1 \rightarrow T_{\phi(x)} M_2$ is a linear Poisson morphism from $(T_x^\flat M_1, T_x M_1, (P_1)_x)$ to $(T_{\phi(x)}^\flat M_2, T_{\phi(x)} M_2, (P_2)_{\phi(x)})$.

If, moreover, each P_i is partial symplectic, a Poisson morphism is called a *symplectic morphism*.

The notions of *anti-Poisson morphism* and *anti-symplectic morphism* are defined analogously.

Note that the tangent map $T_x \phi : T_x M_1 \rightarrow T_{\phi(x)} M_2$ is a linear Poisson morphism from $(T_x^\flat M_1, T_x M_1, (P_1)_x)$ to $(T_{\phi(x)}^\flat M_2, T_{\phi(x)} M_2, (P_2)_{\phi(x)})$ means precisely that it is bounded and we have:

$$(P_2)_{\phi(x)} = T_x \phi \circ (P_1)_x \circ (T_x \phi)^*. \quad (2.25)$$

As in the context of partial linear Poisson structure, consider for $i \in \{1, 2\}$, a Poisson anchor $P_i : T^\flat M_i \rightarrow TM_i$ on a Banach manifold M_i . Then we can provide the product $M_1 \times M_2$ with two Poisson anchors:

1. $P_{M_1 \times M_2} : T^\flat M_1 \times T^\flat M_2 \rightarrow TM_1 \times TM_2$ given by $P_{M_1 \times M_2} = (P_1, P_2)$;
2. $P_{M_1 \times M_2^-} : T^\flat M_1 \times T^\flat M_2 \rightarrow TM_1 \times TM_2$ given by $P_{M_1 \times M_2^-} = (P_1, -P_2)$ which will be simply denoted $M_1 \times M_2^-$.

Moreover, if each $(T^\flat M_i, TM_i, P_i, \{ \cdot, \cdot \}_{P_i})$ is a partial Poisson manifold on M_i , we obtain in this way two partial Poisson structures on $M_1 \times M_2$ associated with these previous Poisson anchors. Moreover, if each $(T^\flat M_i, TM_i, P_i, \{ \cdot, \cdot \}_{P_i})$ is partial symplectic, so are the associated partial Poisson structures on $M_1 \times M_2$.

By application of Proposition 2.10, the graph of a smooth map $\phi : M_1 \times M_2$ is a closed submanifold of $M_1 \times M_2$ and from the definition of a coisotropic submanifold, we obtain:

PROPOSITION 2.48. *Let $(T^\flat M_1, TM_1, P_1, \{ \cdot, \cdot \}_{P_1})$ and $(T^\flat M_2, TM_2, P_2, \{ \cdot, \cdot \}_{P_2})$ be two partial Poisson structures and consider a smooth map $\phi : M_1 \rightarrow M_2$ such that $(T_x \phi)^*(T_{\phi(x)}^\flat M_2) \subset T_x^\flat M_1$ for all $x \in M_1$.*

ϕ is a Poisson morphism (resp. anti-Poisson morphism) if and only if the graph of ϕ is a coisotropic submanifold of $M_1 \times M_2^-$ (resp. $M_1 \times M_2$).

Moreover, if each partial Poisson structure is partial symplectic or partial anti-symplectic, then such a map ϕ is partial symplectic (resp. partial anti-symplectic) if and only if the graph of ϕ is Lagrangian in $M_1 \times M_2^-$ (resp. $M_1 \times M_2$).

The notion of Poisson map is defined as follows:

DEFINITION 2.49. For $i \in \{1, 2\}$, let

$$\{(\mathcal{E}_i(U_i), \{ \cdot, \cdot \}_{U_i})\}, \text{ for any open subset } U_i \text{ in } M_i\}$$

be a Poisson-Lie algebras sheaf on M_i .

A map $\phi : M_1 \rightarrow M_2$ is a *Poisson map* if, for any open set U_i in M_i such that $\phi(U_1) \subset U_2$, the induced map $\phi^* : \mathcal{C}^\infty(U_2) \rightarrow \mathcal{C}^\infty(U_1)$, defined by $\phi^*(f) := f \circ \phi$, is such that

$$(\mathbf{pPMap1}) \quad \phi^*(\mathcal{E}(U_2)) \subset \mathcal{E}(U_1);$$

$$(\mathbf{pPMap2}) \quad \forall (f, g) \in \mathcal{E}_2(U_2)^2, \{ \phi^*(f), \phi^*(g) \}_{P_1} = \phi^*(\{f, g\}_{P_2}).$$

PROPOSITION 2.50. *Let $(T^\flat M_1, TM_1, P_1, \{ \cdot, \cdot \}_{P_1})$ and $(T^\flat M_2, TM_2, P_2, \{ \cdot, \cdot \}_{P_2})$ two convenient partial Poisson structures and let ϕ be a convenient Poisson map.*

Then, for any $x \in M_1$, we have $(T_x^*\phi)(T_{\phi(x)}^\flat M_2) \subset T_x^\flat M_1$ and the restriction of $T_x^*\phi$ to $T_{\phi(x)}^\flat M_2$ is a bounded map from $T_{\phi(x)}^\flat M_2$ to $T_x^\flat M_1$ respectively endowed with their own topology of convenient spaces.

Proof. Fix some $x \in M_1$ and consider an open set U_2 around $\phi(x)$ in M_2 such that $\{df_2, f_2 \in \mathcal{E}_2\}$ generates $T_{\phi(x)}^\flat M_2$. Let U_1 be an open set around x in M_1 such that $\phi^*(\mathcal{E}(U_2)) \subset \mathcal{E}(U_1)$. Thus from the definition and properties of $\mathcal{E}(U_1)$, this implies that, for any $f_2 \in \mathcal{E}(U_2)$, the map $T_{\phi(x)}^*\phi(df_2)$ belongs to $T_x^\flat M_1$ provided with its own convenient structure. This implies that, for any bounded linear functional λ on $T_x^\flat M_1$, $\lambda(T_{\phi(x)}^*\phi(df_2))$ is bounded for any f_2 such that $d_{\phi(x)}f_2$ belongs to a bounded set of $T_{\phi(x)}^\flat M_2$. Since $T_{\phi(x)}^\flat M_2$ is generated by $\{d_{\phi(x)}f_2, f_2 \in \mathcal{E}_2\}$, it follows that $T_{\phi(x)}^*\phi$ is a bounded linear map from $T_{\phi(x)}^\flat M_2$ to $T_x^\flat M_1$ endowed with their own topologies of convenient spaces. ■

Remark 2.51. Definition 2.49 of a Poisson map is quite different from the definition given in [38]. From Proposition 2.50, the condition in the first assumption is in Tumpach's Definition but the second one is not imposed and cannot be satisfied. On the other hand, the definition of a Poisson morphism (cf. Definition 2.47) does not require that the restriction of $T_x^*\phi$ to $T_{\phi(x)}^\flat M_2$ is a bounded map from $T_{\phi(x)}^\flat M_2$ to $T_x^\flat M_1$ endowed with their own topology of convenient spaces. Note that both definitions of Poisson morphism in [8] and Definition 2.47 are coherent.

The results established in the following lemma will be used in the proof of Theorem 2.53.

LEMMA 2.52. *Let ϕ be a smooth map from a c^∞ -open set U around 0 of a convenient space E_1 into a convenient space E_2 .*

Assume that there exists a convenient space F_2 contained in E_2' with bounded inclusion and a closed convenient subspace F_1 of E_1' . If the restriction of $(T_x\phi)^$ to F_2 is a bounded linear map from F_2 to F_1 for any $x \in U$ any $u_2, \dots, u_n \in E_1$, then $((T_x^n\phi)(\cdot, u_2, \dots, u_n))^*$ is a bounded linear map from F_2 to F_1 , for integer n and $x \in U$.*

Proof. Fix some $x \in U$. By assumption the result is true for $n = 1$.

Assume that the result is true for all $1 \leq i < n$. Then according to the definition of $d_x^n \phi$ (cf. [21, 5.11]), we have

$$T_x^n \phi(u, u_2, \dots, u_n) = \lim_{t \rightarrow 0} \left(\frac{1}{t} (T_{x+tu}^{n-1} \phi(u_2, \dots, u_n) - T_x^{n-1} \phi(u_2, \dots, u_n)) \right).$$

On the other hand, for any $\alpha \in F_2$, we obtain

$$(T_x^n \phi(\cdot, u_2, \dots, u_n))^*(\alpha) = \alpha \circ (T_x^n \phi(\cdot, u_2, \dots, u_n)).$$

Now the map

$$t \longmapsto \alpha \circ T_{x+tu}^{n-1} \phi(\cdot, u_2, \dots, u_n) \quad (2.26)$$

is a smooth map from E'_2 to E'_1 .

Thus we have

$$\frac{d}{dt} \Big|_{t=0} (\alpha \circ (T_{x+tu}^{n-1} \phi(\cdot, u_2, \dots, u_n))) = \alpha \circ (T_x^n \phi(\cdot, u, u_2, \dots, u_n)).$$

But the inclusion of F_2 in E_2 is bounded, F_1 is closed in E'_1 and the map (2.26) takes values in F_1 from the inductive assumption. This implies that $\alpha \circ (T_x^n \phi(\cdot, u, u_2, \dots, u_n))$ belongs to F_1 for any $\alpha \in F_2$. But the map $\alpha \mapsto \alpha \circ (T_x^n \phi(\cdot, u, u_2, \dots, u_n))$ is a bounded map from F_2 to E'_1 which takes values in F_1 . Since F_1 is closed it follows that this map is a linear bounded map from F_2 to F_1 . ■

The following result gives relations between convenient Poisson morphisms and convenient Poisson maps (cf. [8, Theorem 7.53 and Remark 7.54]):

THEOREM 2.53. *Consider two partial Poisson structures $(T^\flat M_i, TM_i, P_i, \{ \cdot, \cdot \}_{P_i})$, for $i \in \{1, 2\}$, such that $T^\flat M_1$ is closed in $T' M_1$ and let $\phi : M_1 \rightarrow M_2$ be a smooth map.*

- (1) *If ϕ is a convenient Poisson morphism, then ϕ is also a convenient Poisson map.*
- (2) *Assume that the typical fibre $\mathbb{M}_2^\flat$ of $T^\flat M_2$ is dense in $\mathbb{M}'$ (typical fiber of $T' M_2$), for the weak* topology. Then if ϕ is a convenient Poisson map then ϕ is also a convenient Poisson morphism.
In particular, if P_2 is partial symplectic or if $T^\flat M_2 = T' M_2$ the previous converse is true.*

Sketch of the proof (1) Suppose that property **(pPMorph1)** is satisfied. Since the problem is local, and according to compatible charts as previously, we may assume that, for $i \in \{1, 2\}$, U_i is a c^∞ open set in $\mathbb{M}_i$ such that $TM_i = U_i \times \mathbb{M}_i$, $T^\flat M_i = U_i \times \mathbb{F}_i$ and ϕ is a smooth map from $U_1 \rightarrow U_2$ where

- $T_x \phi(\{x\} \times \mathbb{M}_1) \subset \{\phi(x)\} \times \mathbb{M}_2$;
- $T_x^* \phi(\{\phi(x)\} \times \mathbb{F}_2) \subset \{x\} \times \mathbb{F}_1$.

Moreover, by assumption, $\mathbb{F}_1$ is closed in $\mathbb{M}'_1$.

Since $\phi^*(f) = f \circ \phi$, we have:

$$d_x(f \circ \phi) = d_{\phi(x)} f \circ T_x \phi = T_x^* \phi(d_{\phi(x)} f).$$

Since f belongs to $\mathfrak{A}(U_2)$, the map $d_x f$ belongs to $T_y^\flat M_2$. Thus the previous relation implies that $d_x(f \circ \phi)$ belongs to $T_x^\flat M_1$ and is bounded since $\mathbb{F}_1$ is closed in $\mathbb{M}'_1$. As in the proof of Proposition 2.16, 2., by induction, we can show that:

$$d_x^k(f \circ \phi)(u_1, \dots, u_k)$$

is a summation of terms of type:

$$d_{\phi(x)}^l f \left(T_x \phi(u^{\sigma^1}), T_x^2 \phi(u^{\sigma^2}), \dots, T_x^i \phi(u^{\sigma^i}), \dots, T_x^h \phi(u^{\sigma^h}) \right) \quad (2.27)$$

where:

- the set $\{\sigma^1, \dots, \sigma^h\}$ is a partition of the set $\{1, \dots, k\}$;
- for σ^i , one has:
 - each σ^i is a disjoint union of ν_i strictly increasing sequences of length i , that is $\sigma_j^i := \left(\binom{s_j^i}{1}, \dots, \binom{s_j^i}{i} \right)$ if $\nu_i \geq 1$;
 - otherwise σ^i is an empty set;
- for all $1 \leq l \leq k$ and $h + l = k$, we have:

$$\begin{cases} \nu_1 + 2\nu_2 + \dots + i\nu_i + \dots + h\nu_h = k, \\ \nu_1 + \dots + \nu_i + \dots + \nu_h = l. \end{cases}$$

- If $\nu_i \geq 1$ and $\sigma_j^i = \left(\binom{s_j^i}{1}, \dots, \binom{s_j^i}{i} \right)$, for $j \in \{1, \dots, \nu_i\}$, we set:

$$\begin{aligned} u_{\sigma_j^i} &= \left(u_{(s_j^i)_1}, \dots, u_{(s_j^i)_i} \right), \\ T_x^i \phi(u^{\sigma^i}) &= \left(T_x^i \phi(u_{\sigma_1^i}), \dots, T_x^i \phi(u_{\sigma_{\nu_i}^i}) \right). \end{aligned}$$

Note that the term (2.27) is completely defined by such a partition $\{\sigma^1, \dots, \sigma^h\}$ and the summation in the expression of $d_x^k(f \circ \phi)(u_1, \dots, u_k)$ is for all such partitions.

According to Lemma 2.52, we fix such a partition $\{\sigma^1, \dots, \sigma^h\}$. Then 1 belongs to one and only one $\sigma_j^{i_0} := ((s_j^{i_0})_1, \dots, (s_j^{i_0})_{i_0})$ and since this sequence is strictly increasing, we $(s_j^{i_0})_1 = 1$. The corresponding term is

$$T_x^{i_0} \phi(u_{\sigma_j^{i_0}}) = T_x^l \phi(u_{(s_j^{i_0})_1}, \dots, u_{(s_j^{i_0})_{i_0}}).$$

After deleting $u_{(s_j^l)_1}$ in this term we obtain the bounded linear map

$$\widehat{\phi}_1 : u_1 \mapsto T_x^{i_0} \phi(u_1, u_{(s_j^{i_0})_2}, \dots, u_{(s_j^{i_0})_{i_0}}).$$

By application of Lemma 2.52, the restriction of $(\widehat{\phi}_1)^*$ to $\mathbb{F}_2$ is a bounded linear map into $\mathbb{F}_1$.

Associated with this partition, we consider again the corresponding term (2.27):

$$d_{\phi(x)}^l f(T_x \phi(u^{\sigma^1}), T_x^2 \phi(u^{\sigma^2}), \dots, T_x^i \phi(u^{\sigma^i}), \dots, T_x^h \phi(u^{\sigma^h})). \quad (2.28)$$

In this expression, all vectors

$$T_x \phi(u^{\sigma^1}), T_x^2 \phi(u^{\sigma^2}), \dots, T_x^i \phi(u^{\sigma^i}), \dots, T_x^h \phi(u^{\sigma^h})$$

belong to $T_{\phi(x)} M_1 \equiv \mathbb{F}_2$, which we denote by $v_1, \dots, v_h$. Among these vectors, we have one and only one which is equal to $\widehat{\phi}_1(u_1)$ and we may assume that $v_1 = \widehat{\phi}_1(u_1)$.

Finally, after having deleted u_1 in (2.28), we obtain a term of type

$$\widehat{\phi}_1^* \left(d_{\phi(x)}^l f(., v_2, \dots, v_h) \right)$$

which belongs to $\mathbb{F}_1$ from the construction of $(\phi_1)^*$. But such a property is true for any term of the decomposition of $d_x^k(f \circ \phi)(., u_2, \dots, u_k)$ for any $k \in \mathbb{N}$ and any $u_2, \dots, u_k$ in $T_x M_1$. This implies that $d_x^k(f \circ \phi)(., u_2, \dots, u_k)$ belongs to $T_x M_1$ for any $k \in \mathbb{N}$ and any $u_2, \dots, u_k$ in $T_x M_1$ and so the proof is completed, according to Proposition 2.16.

(2) By the bipolar theorem (cf. [36, Theorem 3.7 and Theorem 3.12]) the assumption of 2. is equivalent to the assumption:

$$(\mathbb{M}^\flat)^{\text{ann}} \cap \mathbb{M} = \{0\}.$$

This, in turn, is equivalent to the property that $\mathbb{M}^\flat$ separates points in $\mathbb{M}$.

Fix some point $x \in M_1$ and let U be an open set around $\phi(x)$ in M_2 such that set $\{d_{\phi(x)}f, f \in \mathcal{E}(U)\}$ is equal to $T_{\phi(x)}^\flat M_2$ (cf. Lemma 2.25). Since $d_x(\phi_*f) = (T_x^*\phi)(df)$, and $(T_x^*\phi)(T_{\phi(x)}^\flat M_2) \subset T_x^\flat M_1$, the set $\{d_x(f \circ \phi), f \in \mathcal{E}(U)\}$ generates $T_x^\flat M_1$. From the definition of the Poisson bracket from a Poisson anchor, we have:

$$\langle d_x(g \circ \phi), T_x\phi \circ P_1 \circ (T_{\phi(x)}^*\phi)(df_x(f \circ \phi)) \rangle = \langle d_{\phi(x)}g, P_2(d_{\phi(x)}f) \rangle \quad (2.29)$$

for all $f, g \in \mathcal{E}(U)$.

Then, for any fixed $f \in \mathcal{E}(U)$ in relation (2.29), we obtain:

$$\langle d_{\phi(x)}g, T_x\phi \circ P_1 \circ (T_{\phi(x)}^*\phi)(d_{\phi(x)}f) - P_2d_{\phi(x)}f \rangle = 0$$

for any $g \in \mathcal{E}(U)$. Since the set $\{d_{\phi(x)}g, g \in \mathcal{E}(U)\}$ generates $T_{\phi(x)}^\flat M_2$, it follows that $T_x\phi \circ P_1 \circ (T_{\phi(x)}^*\phi)(d_{\phi(x)}f) - P_2d_{\phi(x)}f$ belongs to $(T_{\phi(x)}^\flat M_2)^a \cap T_y M_2$ but by assumption, as we have seen previously, we have:

$$(T_y^\flat M_2)^a \cap T_y M_2 = \{0\}$$

for all $y \in \phi(M_1)$. This completes the proof.

Now, if P_2 is partial symplectic, then P_2 is an isomorphism from $T^\flat M_2$ onto TM_2 . This necessary implies that $(T_y^\flat M_2)^a \cap T_y M_2 = \{0\}$ for all $y \in M_2$. The same is also clear if $T^\flat M_2 = T' M_2$. ■

Remarks 2.54. (a) In finite dimension, the notions of Poisson morphism and Poisson map coincide. However, in the context of Theorem 2.53 (2), in general, $\mathbb{M}_2^\flat \neq \mathbb{M}_2'$ and not dense in $\mathbb{M}_2'$ for its weak* topology and so the assertions (1) and (2) are not equivalent.

For instance, it is always the case if $T^\flat M_1 = T' M_1$, $\mathbb{M}_2$ is reflexive, $\mathbb{M}_2^\flat \neq \mathbb{M}_2'$ and not dense in $\mathbb{M}_2'$ for its weak* topology, and ϕ is a surjective submersion.

(b) If a smooth map $\phi : M_1 \rightarrow M_2$ is a convenient Poisson morphism from $(T^\flat M_1, TM_1, P_1, \{, \}_{P_1})$ to $(T^\flat M_2, TM_2, P_2, \{, \}_{P_2})$, ϕ can be also a Poisson map even if $T^\flat M_1$ is not a closed subbundle of $T' M_1$ (cf. Theorem 2.55).

2.8. PARTIAL POISSON STRUCTURE ON THE SOURCE OF A SUBMERSION. Given a surjective submersion on a partial Poisson manifold, we look for sufficient conditions on the structure of the source for which this submersion is a Poisson map. More precisely, we have:

THEOREM 2.55. *Consider a partial Poisson structure $(T^\flat M_1, TM_1, P_1, \{\cdot, \cdot\}_{P_1})$ and $\phi : M_1 \rightarrow M_2$ a surjective submersion. Then we have*

1. M_2 can be provided with a (unique) Lie-Poisson algebras sheaf

$$\{(\mathfrak{A}(U_2), \{\cdot, \cdot\}_{U_2}); U_2 \text{ open set in } M_2\}$$

such that ϕ is a Poisson map if and only if the relation $\phi^{-1} \circ \phi$ is a Poisson relation from M_1 to M_1 .

2. Assume that M_1 is connected, and assume that $(\ker T\phi)^0 = (\ker T\phi)^a \cap T^\flat M_1$ is a closed convenient subbundle of $T^\flat M_1$ ($(\ker T\phi)^a$ is the annihilator of $\ker T\phi$). Then there exists a unique partial Poisson structure $(T^\flat M_2, TM_2, P_2, \{\cdot, \cdot\}_{P_2})$ such that ϕ is Poisson morphism if and only if the relation $\phi^{-1} \circ \phi$ is a Poisson relation from M_1 to M_1 . In this case ϕ is also a Poisson map.
3. If $(T^\flat M_1, TM_1, P_1, \{\cdot, \cdot\}_{P_1})$ is partial symplectic or if we have $T^\flat M_1 = T'M_1$, then there exists a unique partial Poisson structure $(T^\flat M_2, TM_2, P_2, \{\cdot, \cdot\}_{P_2})$ such that ϕ is a Poisson morphism if and only if the relation $\phi^{-1} \circ \phi$ is a Poisson relation from M_1 to M_1 and so, in this case, ϕ is also a Poisson map.

Remark 2.56. Under the same assumptions of Theorem 2.55, we obtain the same conclusion in 1., 2. and 3. when ϕ is an anti-Poisson morphism if and only if the graph of $\phi^{-1} \circ (-\phi)$ is a Poisson relation.

Proof. 1. Fix some open set U_2 in M_2 . If $U_1 = \phi^{-1}(U_2)$, we consider the set

$$\mathcal{E}(U_2) = \{g \in C^\infty(M_2) : \phi^*(g) \in \mathcal{E}(U_1)\}.$$

Clearly $\mathcal{E}(U_2)$ has a structure of algebra. First, we show that $\mathcal{E}(U_2)$ can be provided with a Lie-Poisson structure.

The Poisson bracket on $M_1 \times M_1^-$ is given by:

$$\{h_1, h_2\}_{(P_1, P_1^-)} = - \langle d_1 h_1, P_1(d_1 h_2) \rangle + \langle d_2 h_1, P_1(d_2 h_2) \rangle$$

where d_1h and d_2h are the partial differential of h relative to the first factor and the second factor respectively.

Now, in $M_1 \times M_1^-$, the graph of the relation $\phi^{-1} \circ \phi : M_1 \rightarrow M_1$ is the set

$$G = \{(x, y) \in M_1 \times M_1^- : \phi(x) = \phi(y)\}.$$

Assume that G is a Poisson relation, which means that G is coisotropic in $M_1 \times M_1^-$. By Proposition 2.46, the ideal $\mathcal{I}_g(U_1 \times U_1)$ is stable under the Poisson bracket on $M_1 \times M_1^-$ in restriction to $U_1 \times U_1$.

For $i \in \{1, 2\}$, to the function $f_i \in \mathcal{E}(U_2)$ we associate $g_i(x, y) = f_i(\phi(x)) - f_i(\phi(y))$ on $U_1 \times U_1^-$. Thus g_i vanishes on $(U_1 \times U_1) \cap G$ and the bracket the $\{g_1, g_2\}_{(P_1, P_1^-)}$ also vanishes on $(U_1 \times U_1) \cap G$ and so

$$\{g_1, g_2\}_{(P_1, P_1^-)}(x, y) = \{f_1 \circ \phi, f_2 \circ \phi\}_{(P_1, P_1^-)}(x) - \{f_1 \circ \phi, f_2 \circ \phi\}_{(P_1, P_1^-)}(y) = 0$$

for all x, y such that $\phi(x) = \phi(y)$. Thus $\{f_1 \circ \phi, f_2 \circ \phi\}_{(P_1, P_1^-)}$ is constant on each fibre of ϕ . It follows that $\{f_1 \circ \phi, f_2 \circ \phi\}_{(P_1, P_1^-)}$ belongs to $\mathfrak{A}(U_2)$ and only depends only on f_1 and f_2 . Since ϕ is a submersion, in this way, we define a Poisson bracket on $\mathfrak{A}(U_2)$.

For the converse assume that ϕ is a partial Poisson map. We first show that the graph of ϕ is coisotropic. First, note that since ϕ is smooth, for any $x \in M$, the linear map $T_x\phi : T_xM_1 \rightarrow T_xM_2$ must be bounded. Thus, if G_ϕ denotes the graph of ϕ , its tangent space $T_{(x, \phi(x))}G_\phi = \{(u, T_x\phi(u)) : u \in T_xM_1\}$ is a closed subspace of $T_xM_1 \times T_{\phi(x)}M_2$ and so G_ϕ is a closed submanifold of $M_1 \times M_2$. We must show that $T_{(x, \phi(x))}G_\phi$ is a coisotropic subspace of $T_xM_1 \times T_{\phi(x)}M_2$ provided with the linear partial Poisson structure $(T_x^\flat M \times T_{\phi(x)}^\flat M_2, T_xM_1 \times T_{\phi(x)}M_2, (P_1)_x \times (-P_2)_{\phi(x)})$.

Indeed, we have an open set U around $x \in M_1$ such that $T_x^\flat M_1$ (resp. $T_{\phi(x)}^\flat M_2$) is generated by $\{d_x f_1 : f_1 \in \mathfrak{A}(U_1)\}$ (resp. $\{d_{\phi(x)} f_2 : f_2 \in \mathfrak{A}(U_2)\}$) where $\phi(U_1) \subset U_2$. Therefore $(T_{(x, \phi(x))}G_\phi)^0 = \{(d_x(f_2 \circ \phi), d_{\phi(x)} f_2) : f_2 \in \mathfrak{A}(U_2)\}$. Since ϕ is a Poisson map, it follows that:

$$\begin{aligned} & \langle d_x(f_2 \circ \phi), P_1 d_x(g_2 \circ \phi) \rangle = - \langle d_{\phi(x)} f_2, P_2(d_{\phi(x)} g_2) \rangle \\ & = \{f_2 \circ \phi, g_2 \circ \phi\}_{P_1} - \{f_2, g_2\}_{P_2} \circ \phi = 0. \end{aligned}$$

Thus, from Lemma 2.5, 5., $T_{(x, \phi(x))}G_\phi$ is coisotropic.

But, (ϕ^{-1}, ϕ) is clean pair from Lemma 2.41, and so from Theorem 2.42 the graph of $\phi \circ \phi^{-1}$ is coisotropic.

2. At the one hand, since M_1 is connected and ϕ is a submersion, we have a convenient space $\mathbb{F}$ such that $\mathbb{M}_1$ is isomorphic to $\mathbb{M}_2 \times \mathbb{F}$. In particular, $\ker T\phi$ is a subbundle of TM with typical fibre $\mathbb{F}$. Let $(\ker T\phi)^a$ be the subbundle of $T'M_1$ which is the annihilator of $\ker T\phi$. Of course, $\ker T\phi$ is a the typical of a convenient bundle isomorphic to $\mathbb{M}'_2$. We consider the pull-back $\phi^*(T'M_2)$ over M_1 . Then $T^*\phi$ can be considered as a bundle morphism from $\phi^*(T'M_2)$ to $T'M_1$. Since ϕ is a submersion, for any $x \in M_1$, we must have $\ker T_x^*\phi = \{0\}$ and $T_x^*\phi(T'_{\phi(x)}M_2) = (\ker T_x\phi)^a$. As $T^*\phi$ is a bundle morphism, this implies that $T^*\phi$ is an isomorphism from $\phi^*(T'M_2)$ to $(\ker T\phi)^a$ and so we can identify $(\ker T\phi)^a$ and $\phi^*(T'M_2)$. Thus we have the following commutative diagram:

$$\begin{array}{ccccc} (\ker T\phi)^a & \xrightarrow{T^*\phi} & \phi^*(T'M_2) & \xrightarrow{\widehat{\phi^*}} & T'M_2 \\ & \searrow & \downarrow & & \downarrow \\ & & M_1 & \xrightarrow{\phi} & M_2 \end{array} \quad (2.30)$$

where $\widehat{\phi^*}$ is a smooth convenient bundle morphism such that the restriction $\widehat{\phi^*}_x$ to the fibre $(\phi^*(T'M_2))_x$ is an isomorphism onto $T'_{\phi(x)}M_2$.

We denote ι_0 the inclusion of $(\ker T\phi)^0$ into $(\ker T\phi)^a$.

LEMMA 2.57. (i) For any $y \in M_2$, the vector space $T_y^\flat M_2 = \widehat{\phi^*} \circ \iota_0((\ker T\phi)_x^0)$ if $\phi(x) = y$ does not depend on such a choice of x and has a structure of convenient space isomorphic to $\mathbb{M}_2^\flat$.

(ii) The range $T^\flat M_2 = \widehat{\phi^*} \circ \iota_0((\ker T\phi)^0)$ has a structure of convenient weak subbundle of $T'M_2$. In particular, $(\ker T\phi)^0$ is the pull-back of $T^\flat M_2$ over ϕ .

Proof. (i) Note that by construction, for each $x \in M_1$, there exists an open set U_2 in M_2 around $\phi(x)$ such that $(\ker T\phi)_x^0$ is generated by $\{d_x(f_2 \circ \phi), f_2 \in \mathfrak{A}(U_2)\}$. Indeed the fibre $(\ker T\phi)_x^a$, is generated by the set $\{d_x(f_2 \circ \phi), f_2 \in C^\infty(U_2)\}$. In particular, we can remark that

$$\forall f_2 \in C^\infty(U_2), \widehat{\phi^*}(d_x(f_2 \circ \phi)) = d_{\phi(x)}f_2.$$

But, since the fibre $T_x^\flat M_1$ is generated by $\{d_x f_1, f_1 \in \mathfrak{A}(U_1)\}$, their intersection is generated by $\{d_x(f_2 \circ \phi), f_2 \in \mathfrak{A}(U_2)\}$.

Fix some $y \in M_2$ and choose an open set U_2 in M_2 around y such that $T_y^\flat M_2$ is generated by $\{d_y f_2, f_2 \in C^\infty(U_2)\}$ and consider $U_1 = \phi^{-1}(U_2)$.

According to the previous Remark, for each $x \in \phi^{-1}(y)$ the fibre $(\ker T\phi)_x^0$ is generated by $\{d_x(f_2 \circ \phi), f_2 \in \mathfrak{A}(U_2)\}$.

Consider two points $x_1 \neq x_2$ such that $y = \phi(x_1) = \phi(x_2)$. As ϕ is a submersion, the convenient space $\mathbb{M}_1$ is isomorphic to $\mathbb{M}_2 \times \mathbb{F}$ and, for $i \in \{1, 2\}$, we have charts (V_i, Φ_i) around x_i in M_1 such that:

- (1) $\Phi_i(x_i) = (0, 0) \in \mathbb{M}_2 \times \mathbb{F}$;
- (2) $\Phi_i(V_i) = \overline{W} \times \overline{V} \subset \mathbb{M}_2 \times \mathbb{F}$ and $\Psi(W) = \overline{W} \subset \mathbb{M}_2$;
- (3) $\Psi \circ \phi \circ \Phi_i^{-1} = q$ where q is the first projection from $\mathbb{M}_2 \times \mathbb{F}$ on $\mathbb{M}_2$.

It follows that $T\Phi_i$ is a bundle isomorphism from $TM_1|_{V_i}$ to $\overline{W} \times \overline{V} \times \mathbb{M}_2 \times \mathbb{F}$ and $T\Psi$ is an isomorphism from $TM_2|_W$ to $\overline{W} \times \mathbb{M}_2$ and so $\ker T\phi|_{V_i}$ is isomorphic to $\overline{W} \times \overline{V} \times \mathbb{F}$. Now from (2) and (3), according to diagram (2.30), this easily implies that we have an isomorphism $\Theta_i = T^*\Phi_i^{-1}|_{V_i}$ from $(\ker T\phi)|_{V_i}^a$ to $\overline{W} \times \overline{V} \times \mathbb{M}_2'$ over Φ_i . Therefore, if we set $\theta = \Phi_2^{-1} \circ \Phi_1$, then $\Theta = \Theta_2^{-1} \circ \Theta_1$ is a convenient isomorphism from $(\ker T\phi)|_{V_1}^a$ to $(\ker T\phi)|_{V_2}^a$ such that, in restriction to $(\ker T\phi)|_{V_1}^a$, we have

$$(4) \quad \widehat{\phi^*} \circ \Theta = \widehat{\phi^*}.$$

Since the restriction of $\widehat{\phi^*}$ to each fibre is an isomorphism, according to (1), we have

$$(5) \quad \Theta(d_{x_1}(f \circ \phi)) = d_{x_2}(f \circ \phi) \text{ which implies point (i).}$$

(ii) From point (i), the set $T^b M_2 = \bigcup_{y \in M_2} T_y^b M_2$ is well defined and let

$\iota_2 : T^b M_2 \rightarrow T' M_2$ be the inclusion.

Now, since by assumption, $(\ker T\phi)^0$ is a closed convenient subbundle of $T^b M_1$ and the inclusion of $T^b M_1$ in $T' M_1$ is a smooth convenient morphism, the inclusion of $(\ker T\phi)^0$ into $(\ker T\phi)^a$ is also a smooth convenient morphism. This gives rise to an injective bounded map from each fibre $(\ker T\phi)_x^0$ into $(\ker T\phi)_x^a$. Thus we may consider that the typical fibre $\mathbb{M}_2^b$ of $(\ker T\phi)^0$ is contained in the typical fibre $\mathbb{M}_2^a$ of $(\ker T\phi)^a$ and the inclusion $\overline{\iota_0}$ is an injective bounded map.

Using the local context introduced in (2) and (3), without loss of generality, we can assume that V_i is simply connected and so $(\ker T\phi)^0|_{V_i}$ is trivial. Thus

we have a trivialization $\Phi_i^b : (\ker T\phi)^0|_{V_i} \rightarrow \overline{W} \times \overline{V} \times \mathbb{M}_2^b$;

$$\begin{array}{ccccc}
 (\ker T\phi)^0|_{V_i} & \xrightarrow{\Phi_i^b} & \overline{W} \times \overline{V} \times \mathbb{M}_2^b & & \\
 \downarrow & \searrow \iota_0 & \downarrow & \searrow \overline{\iota_0} & \\
 & (\ker T\phi)^a|_{V_i} & \xrightarrow{T^*\Phi_i^{-1}} & \overline{W} \times \overline{V} \times \mathbb{M}_2' & \\
 & \downarrow \Phi_i & \downarrow & \downarrow & \\
 V_i & \xrightarrow{\text{Id}} & \overline{W} \times \overline{V}_i & \xrightarrow{\text{Id}} & \overline{W} \times \overline{V}_i \\
 & \searrow & \downarrow \Phi_i & \searrow & \\
 & & V_i & \xrightarrow{\Phi_i} & \overline{W} \times \overline{V}_i
 \end{array} \tag{2.31}$$

$$\begin{array}{ccccc}
 (\ker T\phi)^a|_{V_i} & \xrightarrow{T^*\Phi_i^{-1}} & \overline{W} \times \overline{V} \times \mathbb{M}_2' & & \\
 \downarrow & \searrow \widehat{\phi}^* & \downarrow & \searrow q \times \text{Id} & \\
 & T^b M_{2|W} & \xrightarrow{T^*\Phi_i^{-1}} & \overline{W} \times \mathbb{M}_2' & \\
 & \downarrow \Phi_i & \downarrow & \downarrow & \\
 V_i & \xrightarrow{\phi} & \overline{W} \times \overline{V}_i & \xrightarrow{q} & \overline{W} \\
 & \searrow & \downarrow \Psi & \searrow & \\
 & & W & \xrightarrow{\Psi} & \overline{W}
 \end{array} \tag{2.32}$$

Since $\widehat{\phi}^* \circ \iota_0$ is an injective smooth convenient bundle morphism over $\phi|_{V_i}$, it follows that $\widehat{\phi}^* \circ \iota_0((\ker T\phi)^0|_{V_i}) = T^b M_{2|W}$ and, according to (4), this range does not depend on the choice of such trivializations Φ_i^b . Thus, we can provide $T^b M_{2|W}$ with a structure of trivial convenient bundle over W . Indeed, let $W_i(x) = \Phi_i^{-1}(\overline{W} \times \{0\})$ the closed supplemented submanifold of V_i . Then the restriction of ϕ to $W_i(x)$ is a diffeomorphism onto W and so $\overline{\iota_0} \circ \Phi_i^b \circ (\widehat{\phi}^* \circ \iota_0)^{-1} : T^b M_{2|W} \rightarrow \overline{W} \times \mathbb{M}_2^b \subset \overline{W} \times \mathbb{M}_2'$ is then a trivialization over $q \circ \Phi_i \circ (\phi|_{W_i(x)})^{-1} : W \rightarrow \overline{W}$.

Therefore, by merging Diagrams (2.31) and (2.32), and using composition

of $T^*\Psi^{-1}$ with ι_2 , we obtain the following commutative diagram:

$$\begin{array}{ccccc}
 (\ker T\phi)_{|V_i}^0 & \xrightarrow{\Phi_i^b} & \overline{W} \times \overline{V} \times \mathbb{M}_2^b & & (2.33) \\
 \downarrow & \searrow \widehat{\phi^* \circ \iota_0} & \downarrow & \searrow q \times \overline{\iota_0} & \\
 & T^b M_{2|W} & \xrightarrow{T^*\Psi^{-1} \circ \iota_2} & \overline{W} \times \mathbb{M}_2^b & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 V_i & \xrightarrow{\Phi_i} & \overline{W} \times \overline{V}_i & & \\
 \downarrow \phi & & \downarrow q & & \\
 W & \xrightarrow{\Psi} & \overline{W} & &
 \end{array}$$

In particular, we have

$$\begin{cases} \Psi = q \circ \Phi_i \circ (\phi_{|W(x)})^{-1} & \text{on } W \\ T^*\Psi^{-1} \circ \iota_2 = (q \times \overline{\iota_0}) \circ \Phi_i^b \circ (\widehat{\phi^* \circ \iota_0})^{-1} & \text{on } T^b M_{2|W} \end{cases} \quad (2.34)$$

By merging Diagrams (2.31) and (2.32), and using composition with ι_2 , we see that $\widehat{\phi^* \circ \iota_0}$ is a smooth morphism bundle from $(\ker T\phi)_{|V_i}^0$ to $T^b M_{2|W}$ over ϕ such that

$$T^*\Psi^{-1} \circ (\widehat{\phi^* \circ \iota_0}) = (q \times Id \circ \overline{\iota_0}) \circ \Phi_i^b.$$

According to this relation, since $T^b M_{2|W} = \widehat{\phi^* \circ \iota_0}((\ker T\phi)_{|V_i}^0)$ in $T^b M_2$ and $T^*\Psi^{-1}$ is a smooth isomorphism, it follows that the following diagram is commutative:

$$\begin{array}{ccccc}
 (\ker T\phi)_{|V_i}^0 & \xrightarrow{\Phi_i^b} & \overline{W} \times \overline{V} \times \mathbb{M}_2^b & & (2.35) \\
 \downarrow & \searrow \widehat{\phi^* \circ \iota_0} & \downarrow & \searrow q \times \overline{\iota_0} & \\
 & T^b M_{2|W} & \xrightarrow{T^*\Psi^{-1}} & \overline{W} \times \mathbb{M}_2^b & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 V_i & \xrightarrow{\Phi_i} & \overline{W} \times \overline{V}_i & & \\
 \downarrow \phi & & \downarrow q & & \\
 W & \xrightarrow{\Psi} & \overline{W} & &
 \end{array}$$

and so that the restriction of $T^*\Psi^{-1}$ to $T^b M_{2|W}$ is a trivialization.

In fact, by merging diagrams (2.33) for $i \in \{1, 2\}$, we obtain:

$$\begin{array}{ccccccc}
 & & (\ker T\phi)|_{V_1}^0 & & & & (2.36) \\
 & \swarrow \Theta^b & \downarrow & \searrow \widehat{\phi}^* \circ \iota_0 & & & \\
 (\ker T\phi)|_{V_2}^0 & \xrightarrow{\widehat{\phi}^* \circ \iota_0} & T^b M_{2|W} & \xrightarrow{T^*\Psi^{-1} \circ \iota_2} & \overline{W} \times \mathbb{M}_2^b & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 & & V_1 & & & & \\
 \downarrow & \swarrow \theta & & \searrow \phi & & & \\
 V_2 & \xrightarrow{\phi} & W & \xrightarrow{\Psi} & \overline{W} & &
 \end{array}$$

where $\Theta^b = (\Phi_2^b)^{-1} \circ \Phi_1^b$ is a smooth isomorphism over $\theta = \Phi_2^{-1} \circ \Phi_1$. It follows that such a trivialization of $T^b M_{2|W}$ does not depend on the choice of the trivialization of $(\ker T\phi)^0$ around any point x in the fibre over $y \in W$.

Note that we have $\Theta^b = \Theta \circ \iota_0$.

Consider two charts (W'_i, Ψ'_i) around some $y \in M_2$ for $i \in \{1, 2\}$. Fix some $x \in M_1$ such that $\phi(x) = y$. By same arguments as in (1), (2) and (3), we can find a chart (V'_i, Φ'_i) around x in M_1 such that, after shrinking W_i if necessary, we have:

- (1') $\Psi'_i(y) = \Psi'(y) = 0 \in \mathbb{M}_2$ and $\Phi'_i(x) = (0, 0) \in \mathbb{M}_2 \times \mathbb{F}$;
- (2') $\Psi_i(W'_i) = \overline{W}'_i \subset \mathbb{M}_2$ and $\Phi'_i(V'_i) = \overline{W}'_i \times \overline{V}'_i$;
- (3') $\Psi'_i \circ \phi \circ (\Phi'_i)^{-1} = q$ where q is the first projection of $\mathbb{M}_2 \times \mathbb{F}$ on $\mathbb{M}_2$.

We again assume that V'_i and W'_i are simply connected. Since $(\ker T\phi)^0$ is a convenient bundle, it follows that we have the trivialization

$$(\Phi'_i)^b : (\ker T\phi)|_{V'_i}^0 \longrightarrow \overline{W}'_i \times \overline{V}'_i \times \mathbb{M}_2^b.$$

Then the composition $(\Phi'_2)^b \circ [(\Phi'_1)^b]^{-1}$ in restriction to $(\overline{W}'_1 \cap \overline{W}'_2) \times (\overline{V}'_1 \cap \overline{V}'_2) \times \mathbb{M}_2^b$ is a diffeomorphism of type

$$(\bar{x}, \bar{z}, \bar{\alpha}) \longmapsto (\bar{x}, \Pi_{\bar{x}}(\bar{z}), A_{\bar{x}}(\bar{\alpha})),$$

where $\bar{x} \mapsto \Pi_{\bar{x}}$ (resp. $\bar{x} \mapsto A_{\bar{x}}$) is a field over $\overline{W}'_1 \cap \overline{W}'_2$ of local diffeomorphisms of $\mathbb{F}$ over $\overline{V}'_1 \cap \overline{V}'_2$ (resp. automorphisms of $\mathbb{M}_2^b$). As in the upper situation, we have an analog commutative diagrams as (2.33) and by superposition of these diagrams and using the transition map (5') we obtain a transition map

for $T^\flat M_2|_W$ and $T^\flat M_2|_{W'}$ which show that $T^\flat M_2$ is a convenient subbundle of $T'M_2$. Now, by construction, we have $(\ker T\phi)_x^0 := \{(x, \widehat{\phi}_x^*{}^{-1}(\alpha)), \alpha \in T^\flat M_2\}$ which ends the proof of point (ii). ■

End of the proof of Theorem 2.55.2. First, from the proof of point (i) of Lemma 2.57, the fibre $T^\flat M_2 = \mathfrak{P}_y(U_2)$ for all $y \in M_2$. Now, since $(\ker T\phi)^0$ is the pull-back of $T'M_2$, this implies that $T_x^* \phi = \widehat{\phi}_x^*{}^{-1}$. According to notations in the proof of Lemma 2.57 we have $\theta = \Phi_2^{-1} \circ \Phi_1$, so $T\theta = T\Phi_2^{-1} \circ T\Phi_1$ and so $T^* \theta = T^* \Phi_1 \circ T^* \Phi_2^{-1} = \Theta^{-1}$. From diagram (2.36), (4) and since $\Theta^\flat = \Theta \circ \iota_0$, it follows that for any $\alpha \in T_y^\flat M_2$, if $y = \phi(x_1) = \phi(x_2)$ then $T_{x_1}^* \phi(\alpha) = \Theta^\flat \circ T_{x_2}^* \phi(\alpha) = \Theta^{-1}$. Therefore, for any $\alpha \in T_y^\flat M$, if P_1^0 denotes the restriction of P_1 to $(\ker T\phi)^0$ we have

$$(P_1^0)_{x_2} \circ T_{x_1}^* \phi = (P_1^0)_{x_1} \circ \Theta^\flat \circ T_{x_1}^* \phi = (P_1^0)_{x_1} \circ T^* \theta^{-1} \circ T_{x_1}^* \phi.$$

But $T_{x_2} \phi = T\theta \circ T_{x_1} \phi$. Thus, for any $\alpha \in T_y M_2$, we have

$$T_{x_2} \phi \circ (P_1^0)_{x_2} \circ T_{x_1}^* \phi(\alpha) = (T_{x_1} \phi \circ T\theta) \circ (P_1^0)_{x_1} \circ T_{x_1}^* \phi(\alpha).$$

By setting $(P_2)_y = T\phi \circ P_1 \circ \widehat{\phi}_x^*{}^{-1} \circ \iota_0$, we get a partial Poisson structure $(T^\flat M_2, TM_2, P_2, \{\cdot, \cdot\}_{P_2})$.

3. If $(T^\flat M_1, TM_1, P_1, \{\cdot, \cdot\}_{P_1})$ is partial symplectic, this means that P is an isomorphism and so we have a smooth non-degenerate 2-form Ω_P on TM_1 defined by $\Omega_P(u, v) := \langle P^{-1}(u), v \rangle$. Since Ω is non degenerate, for all $x \in M_1$, the orthogonal $(\ker T_x \phi)^\perp = \{u \in T_x M_1 : \forall v \in \ker T_x \phi, \Omega_P(u, v) = 0\}$ defines a convenient subbundle $(\ker T\phi)^\perp$ of TM_1 and so $(\ker T\phi)^0 = P^{-1}((\ker T\phi)^\perp)$ is a convenient subbundle of $T^\flat M_1$. Then we can apply 2.

If $T^\flat M_1 = T'M_1$, then $(\ker T\phi)^0 = (\ker T\phi)^a$ is a convenient subbundle of $T^\flat M_1$ and so we also can apply 2. ■

2.9. DUAL PAIRS. We end this section by a generalization of the notion of dual pairs introduced in [9].

DEFINITION 2.58. Consider a partial symplectic manifold $(T^\flat M, TM, P)$ and, for $i \in \{1, 2\}$, a Poisson map ϕ_i from M to the partial Poisson manifold $(T^\flat M_i, TM_i, P_i)$. We say that (ϕ_1, ϕ_2) is a *dual pair* if, for any $x \in M$, we have:

$$\ker T_x \phi_1 = (\ker T_x \phi_2)^{\perp P} \quad \text{and} \quad \ker T_x \phi_2 = (\ker T_x \phi_1)^{\perp P}.$$

PROPOSITION 2.59. *Consider a partial symplectic manifold $(T^b M, TM, P, \{ , \}_P)$. For $i \in \{1, 2\}$, let ϕ_i be a Poisson map from $(T^b M, TM, P, \{ , \}_P)$ to the partial Poisson manifold $(T^b M_i, TM_i, P_i, \{ , \}_{P_i})$ such that (ϕ_1, ϕ_2) is a dual pair. Then, for each open set U_i in M_i and any f_i in the Poisson-Lie algebra $(\mathfrak{A}(U_i), \{ , \}_{U_i})$ we have $\{\phi_1^* f_1, \phi_2^* f_2\}_P = 0$ on $\phi_1^{-1}(U_1) \cap \phi_2^{-1}(U_2)$.*

Proof. In the one hand, under our assumptions, from the proof of Theorem 2.55.2, we have $T^b M_i = \bigcup_{y \in M_i} \mathfrak{P}_y(U_i)$ where $\mathfrak{P}_y(U_i) = \{d_y f : y \in \mathfrak{A}_i(U_i)\}$. On the other hand, the characteristic foliation is reduced to M , so by Remark 2.24, (2) and (3), we have a weak symplectic form ω such that $\omega(u, v) = \langle P^{-1}(u), v \rangle$ for all u and v in $T_x M$ and all $x \in M$. On the other hand, for any $\alpha \in T_x^b M$, we have an open set U around x in M and a smooth map $f : U \rightarrow \mathbb{R}$ such that $\alpha = d_x f$ and since $P : T^b M \rightarrow TM$ is an isomorphism, we have $\omega_x(u, v) = \omega_x(X_{f_u}, X_{f_v}) = d_x \{f_u, f_v\}_P$ if $P^{-1}(u) = d_x f_u$ and $P^{-1}(v) = d_x f_v$.

Thus, according to Proposition 2.6, we have

$$(\ker T_x \phi_i)^{\perp P} = \{P(\{d_x f\}) : f \in \mathfrak{A}(U), \forall v \in \ker T_x \phi_i, \langle d_x f, v \rangle = 0\}.$$

But since ϕ_i is a Poisson map, this implies that if f_i belongs to $\mathfrak{A}(U_i)$ then $\phi_i^* f_i$ belongs to $\mathcal{E}_i(\phi_i^{-1}(U_i))$ and then each $d_x(\phi_i^* f_i)$ belongs to $(\ker T_x \phi_j)^0$ for $i \neq j$ and $x \in \phi_1^{-1}(U_1) \cap \phi_2^{-1}(U_2)$ and then if (ϕ_1, ϕ_2) is a dual pair, it follows that $\{\phi_1^* f_1, \phi_2^* f_2\}_P = 0$ on $\phi_1^{-1}(U_1) \cap \phi_2^{-1}(U_2)$. ■

Remark 2.60. The conclusion of Proposition 2.59 is also true when $(\phi_1, -\phi_2)$ or $(-\phi_1, \phi_2)$ is a dual pair.

3. CONVENIENT PARTIAL POISSON LIE GROUPS

This section begins with a survey of definitions and basic properties of convenient Lie groups, as presented in [8], together with adaptations of results on Banach Lie groups from [38] (see also [16]). We then introduce the notion of a convenient partial Poisson-Lie group in the convenient setting *via* a bi-invariant Poisson structure, and investigate its properties and its relation with bialgebras.

3.1. CONVENIENT LIE GROUPS

DEFINITION 3.1. A *convenient Lie group* $\mathbf{G}$ is a convenient manifold endowed with a group structure for which the multiplication $\mathbf{m} : \mathbf{G} \times \mathbf{G} \rightarrow \mathbf{G}$ and the inversion $\mathbf{i} : \mathbf{G} \rightarrow \mathbf{G}$ are smooth mappings.

The unit element of the group $\mathbf{G}$ will be denoted e and the inverse of an element g by g^{-1} .

For $g \in \mathbf{G}$, $L_g : x \mapsto g.x$ is the *left translation* and $R_g : x \mapsto x.g$ is the *right translation*.

The tangent bundle to $\mathbf{G}$ will be the kinematic tangent bundle $T\mathbf{G}$ and the vector fields, smooth sections of this bundle, are the kinematic vector fields whose set is denoted $\mathfrak{X}(\mathbf{G})$.

DEFINITION 3.2. A vector field X on $\mathbf{G}$ is said to be left invariant if

$$\forall g \in \mathbf{G}, \quad (L_g)_* X = X$$

where $(L_g)_* X = TL_g \circ X \circ L_{g^{-1}}$.

Since $(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y]$, the set of all left invariant vector fields on $\mathbf{G}$ forms a Lie subalgebra of $\mathfrak{X}(\mathbf{G})$ denoted $\mathfrak{g}$.

DEFINITION 3.3. $\mathfrak{g}$ is called the Lie algebra of the Lie group $\mathbf{G}$.

$\mathfrak{g}$ is in bijective correspondence with the tangent space $T_e\mathbf{G}$ via the linear isomorphism

$$\begin{aligned} \mathfrak{g} &\longrightarrow T_e\mathbf{G} \\ X &\longmapsto X_e. \end{aligned}$$

So $T_e\mathbf{G}$ becomes a Lie algebra for the bracket still denoted $[\cdot, \cdot]$ and defined by

$$[X_e, Y_e] = [X, Y]_e.$$

PROPOSITION 3.4. Let $\varphi : \mathbf{G} \rightarrow \mathbf{H}$ be a smooth homomorphism of Lie groups. Then $T_e\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra homomorphism.

DEFINITION 3.5. Let $(\mathbf{G}, \cdot)$ be a Lie group with Lie algebra $\mathfrak{g}$. A 1-parameter subgroup of $\mathbf{G}$ is a smooth Lie group homomorphism $\alpha : (\mathbb{R}, +) \rightarrow (\mathbf{G}, \cdot)$.

So a 1-parameter subgroup of $\mathbf{G}$ is a smooth curve α in $\mathbf{G}$ such that

$$\begin{cases} \forall (s, t) \in \mathbb{R}^2, \alpha(s+t) = \alpha(s) \cdot \alpha(t), \\ \alpha(0) = e. \end{cases}$$

PROPOSITION 3.6. *Let $\alpha : \mathbb{R} \rightarrow \mathbf{G}$ be a smooth curve where $\alpha(0) = e$ and let $X \in \mathfrak{g}$. We denote $\hat{X}$ the left invariant vector field such that $\hat{X}_e = X$. Then the following assertions are equivalent:*

1. α is a 1-parameter subgroup of $\mathbf{G}$ and $X = \left. \frac{\partial}{\partial t} \right|_{t=0} \alpha(t)$;
2. α is an integral curve of the left invariant vector field $\hat{X}$;
3. $\text{Fl}^{\hat{X}}(t, x) = x \cdot \alpha(t)$ is the unique global flow of the vector field $\hat{X}$.

DEFINITION 3.7. Let $\mathbf{G}$ be a Lie group with Lie algebra $\mathfrak{g}$. We say that $\mathbf{G}$ admits an exponential map if there exists a smooth mapping $\exp : \mathfrak{g} \rightarrow \mathbf{G}$ such that $t \mapsto \exp(tX)$ is the unique 1-parameter subgroup with tangent vector X at 0.

As in the Banach framework, we have the following properties:

(ExpC1) $\exp(0) = e$;

(ExpC2) $T_0 \exp = \text{Id}_{\mathfrak{g}}$

where **(ExpC2)** is a consequence of $\left. \frac{d}{dt} \right|_{t=0} \exp(tX) = X$.

Unfortunately, for a kinematic vector field, integral curves need not exist locally¹⁰, so the existence of such a map is not guaranteed.

Let $\mathbf{G}$ be a convenient Lie group with Lie algebra $\mathfrak{g}$.

For any $g \in \mathbf{G}$, we define the *conjugation* or *inner automorphism*

$$\begin{aligned} \text{conj}_g : \mathbf{G} &\longrightarrow \mathbf{G} \\ x &\longmapsto g \cdot x \cdot g^{-1}. \end{aligned}$$

DEFINITION 3.8. The adjoint representation is the mapping

$$\begin{aligned} \text{Ad} : \mathbf{G} &\longrightarrow \text{GL}(\mathfrak{g}) \\ g &\longmapsto T_e(\text{conj}_g). \end{aligned}$$

The adjoint representation of the Lie algebra $\mathfrak{g}$ is the map $\text{ad} = T_e \text{Ad}$.

We then have:

$$\forall (X, Y) \in \mathfrak{g}^2, \quad \text{ad}(X)Y = [X, Y]. \quad (3.1)$$

¹⁰cf. [21, 32.12]

3.2. COADJOINT ACTION ON A SUBSPACE OF A DUAL. For this subsection, we refer to [38, 2.5]. But for the sake of completeness, we recall this classical context.

Let G be a convenient Lie group and $\Phi : G \rightarrow \text{Aff}(\mathbb{E})$ a representation into the affine group of a vector space $\mathbb{E}$. As $\text{Aff}(\mathbb{E})$ is a semi-direct product of the linear group $GL(V)$ and the translation group on $\mathbb{E}$ identified with $\mathbb{E}$, Φ can be written as a pair (ϕ, Θ) where $\phi : G \rightarrow GL(\mathbb{E})$ and $\Theta : G \rightarrow \mathbb{E}$. Since Φ is a group morphism it follows that ϕ is a group morphism and Θ satisfies

$$\Theta(gh) = \Theta(g) + \phi(g)(\Theta(h)) \quad (3.2)$$

for all g and h in G . In this case, Θ is called **1-cocycle on G relative to ϕ** .

The derivative $T_e\Phi$ is a Lie algebra morphism from $\mathfrak{g}$ to the Lie algebra $\mathfrak{aff}(\mathbb{E})$ of $\text{Aff}(\mathbb{E})$ which can be written as a pair $(T_e\phi, T_e\Theta)$ where $T_e\phi$ and $T_e\Theta$ gives rise to a Lie algebra morphism $d\phi$ from $\mathfrak{g}$ to $\mathfrak{gl}(\mathbb{E})$ and $d\Theta$ from $\mathfrak{g}$ to $\mathbb{E}$ respectively which satisfy the relation

$$d\Theta([x, y]) = d\phi(x)(d\Theta(y)) - d\phi(y)(d\Theta(x)) \quad (3.3)$$

for all x and y in $\mathfrak{g}$. In this case, $d\Theta$ is called **1-cocycle on $\mathfrak{g}$ relative to $d\phi$** .

In particular, consider a Lie group G and a bi-invariant subbunbdle T^bG of TG . If we denote by $\mathfrak{g}^b$ the fibre T_e^bG , then $\mathfrak{g}^b$ is invariant by the coadjoint action G on $T'G$ (formally, same proof as in [38, Proposition 5.6]).

PROPOSITION 3.9. *Under the previous assumptions, a map $\lambda : \mathfrak{g} \rightarrow \bigwedge^2 \mathfrak{g}^b$ is called a 1-cocycle relative to the coadjoint action of $\mathfrak{g}$ onto $\mathfrak{g}^b$ if, we have the following properties:*

- (1) *there exists $\mathfrak{p} : \mathfrak{g} \times \mathfrak{g}^b \rightarrow \mathfrak{g}$ such that $\lambda(x)(\alpha, \beta) = \langle \alpha, \mathfrak{p}(x)(\beta) \rangle$, for all $x \in \mathfrak{g}$ and α, β in $\mathfrak{g}^b$;*
- (2) *for all x, y in $\mathfrak{g}$ and α, β in $\mathfrak{g}^b$, we have:*

$$\lambda([x, y]) = \text{ad}_x^{(2,0)}(\lambda(y)) - \text{ad}_y^{(2,0)}(\lambda(x)). \quad (3.4)$$

Then, in the Banach setting, from [38, Theorem 3.14], $(\mathfrak{g}, \mathfrak{g}^b, P)$ is a partial Lie-Poisson space in the sense of Definition 2.34. Thus this Definition can be considered as an adaptation of [38, Definition 3.12].

3.3. PARTIAL BIALGEBRAS. The notion of Lie bialgebra in finite dimension¹¹ appears as a particular case of Lie bialgebroid (see [27] for instance). A generalization to the Banach setting can be found in [38]. In the next paragraph, we will propose the following natural adaptation to the convenient framework, which we call a *partial bialgebra*.

DEFINITION 3.10. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a convenient Lie algebra and $\mathfrak{g}^b$ be a convenient subspace of $\mathfrak{g}'$ where the inclusion $\mathfrak{g}^b \rightarrow \mathfrak{g}'$ is bounded. We will say that $(\mathfrak{g}, \mathfrak{g}^b)$ is a *partial Lie bialgebra* if:

- (1) the restriction of the coadjoint action to $\mathfrak{g} \times \mathfrak{g}^b$ takes values in $\mathfrak{g}^b$ and is bounded;
- (2) there exists a Lie algebra bracket $[\cdot, \cdot]^b$ on $\mathfrak{g}^b$ such that the dual map of $[\cdot, \cdot]^b : \mathfrak{g}^b \times \mathfrak{g}^b \rightarrow \mathfrak{g}^b$ restricts to a 1-cocycle $\lambda : \mathfrak{g} \rightarrow \bigwedge^2 \mathfrak{g}^{b'}$ relative to the coadjoint action of $\mathfrak{g}$ onto $\mathfrak{g}^b$.

Remark 3.11. According to [38, Remark 7.3], by formal same arguments adapted to the convenient setting, the 1-cocycle condition can be written:

$$\begin{aligned} \langle [x, y], [\alpha, \beta]^b \rangle &= \langle y, [\text{ad}_x^* \alpha, \beta]^b \rangle + \langle y, [\alpha, \text{ad}_x^* \beta]^b \rangle \\ &\quad - \langle x, [\text{ad}_y^* \alpha, \beta]^b \rangle - \langle x, [\alpha, \text{ad}_y^* \beta]^b \rangle. \end{aligned} \quad (3.5)$$

3.4. CONVENIENT PARTIAL (RESP. SYMPLECTIC) POISSON-LIE GROUPS. We begin by fixing some notations.

As in the Banach setting used in [38], on a convenient Lie group G the left (resp. right) translation L_g (resp. R_g) gives rise to a left (resp. right) smooth action of G on $T'G$ given by $(L_g)^*(\alpha) = \alpha \circ TL_g$ (resp. $R_g^*(\alpha) = \alpha \circ TR_g$) and also a smooth action denoted L_g^{**} (resp. R_g^{**}) on the bidual $T''G$.

The smooth adjoint action $\text{Ad}_g = L_g \circ R_{g^{-1}}$ of G on its Lie algebra $\mathfrak{g}$ induces on the dual $\mathfrak{g}'$ a smooth coadjoint action $\text{Ad}_g^* = L_g^* \circ R_{g^{-1}}^*$ and a smooth action $\text{Ad}_g^{**} = L_g^{**} \circ R_{g^{-1}}^{**}$.

Motivated by the framework of Banach Poisson-Lie groups developed in [38], we introduce the notion of convenient partial Poisson-Lie group, defined via a bi-invariant partial Poisson structure.

DEFINITION 3.12. Let G be a convenient Lie group. A partial Poisson (resp. symplectic) structure $(T^b G, TG, P, \{, \}_P)$ is called a *bi-invariant partial Poisson* (resp. *symplectic*) *structure* if the following conditions hold:

¹¹cf. [23].

(BiT) The subbundle $T^\flat \mathbf{G}$ is bi-invariant:

$$L_g^*(T_{gh}^\flat \mathbf{G}) = T_h^\flat \mathbf{G} \quad \text{and} \quad R_g^*(T_{hg}^\flat \mathbf{G}) = T_h^\flat \mathbf{G}$$

for all $g, h \in \mathbf{G}$.

(BiP) P is bi-invariant:

$$P_h \circ L_g^* = TL_g^{-1} \circ P_{gh} \quad \text{and} \quad P_h \circ R_g^* = TR_g^{-1} \circ P_{hg}$$

for all $g, h \in \mathbf{G}$.

We then say that $(T^\flat \mathbf{G}, T\mathbf{G}, P, \{ , \}_P)$ defines a *convenient partial Poisson* (resp. *symplectic*) *Lie group*.

Remark 3.13. When $T^\flat \mathbf{G}$ is bi-invariant, if we set $\mathfrak{g}^\flat := T_e^\flat \mathbf{G}$, then the map $(TL)^* : T^\flat \mathbf{G} \rightarrow G \times \mathfrak{g}^\flat$ (resp. $(R)^* : T^\flat \mathbf{G} \rightarrow G \times \mathfrak{g}^\flat$) given by $TL^*(g, \alpha) = (g, L_g^*(\alpha))$ (resp. $TR^*(g, \alpha) = (g, R_g^*(\alpha))$) is a convenient bundle isomorphism

As we have already seen, to P is associated a partial Poisson tensor Λ defined by $\Lambda(\alpha, \beta) = \langle \alpha, P(\beta) \rangle$. Therefore, if P is bi-invariant, we have:

$$\Lambda_{gh}(\alpha, \beta) = \Lambda_h(L_g^* \alpha, L_g^* \beta) + \Lambda_g(R_h^* \alpha, R_h^* \beta). \quad (3.6)$$

NOTATIONS 3.14. We will denote by $T^{\flat'} M$ the convenient dual bundle of $T^\flat M$. Since the restriction to $T_h \mathbf{G} \subset T_h'' \mathbf{G}$ of the adjoint $L^{**}g$ to $T_h \mathbf{G}$ is TL_g . However for the sake of clarity, as in [38], for any $\Lambda \in \bigwedge^k T^{\flat'} \mathbf{G}$ we set:

$$(L_g^{**} \Lambda)_{gh}(\alpha_1, \dots, \alpha_k) := \Lambda_h(L_g^* \alpha_1, \dots, L_g^* \alpha_k)$$

and for analogue reason we also set:

$$(R_g^{**} \Lambda)_{hg}(\alpha_1, \dots, \alpha_k) := \Lambda_h(R_g^* \alpha_1, \dots, R_g^* \alpha_k).$$

Since we have: $Ad_g = TR_{g^{-1}} \circ TL_g$ (resp. $Ad_{g^{-1}} = TL_{g^{-1}} \circ TR_g$), we set:

$$Ad_g^{**} \Lambda_e = R_{g^{-1}}^{**} \circ L_g^{**} \Lambda_e \quad (\text{resp. } Ad_{g^{-1}}^{**} \Lambda_e = L_{g^{-1}}^{**} \circ R_g^{**} \Lambda_e).$$

In this way, the relation (3.6) can be written:

$$\Lambda_{gh} = L_g^{**} \Lambda_h + R_h^{**} \Lambda_g. \quad (3.7)$$

The following result generalizes some fundamental results on Banach Poisson-Lie groups proved in [38]:

PROPOSITION 3.15. *Let $\mathbf{G}$ be a convenient Lie group and let $(T^\flat \mathbf{G}, T\mathbf{G}, P, \{.,.\}_P)$ be a partial Poisson structure on $\mathbf{G}$.*

The following assertions are equivalent:

- (1) *$(T^\flat \mathbf{G}, T\mathbf{G}, P, \{.,.\}_P)$ is a convenient partial Poisson (resp. symplectic) Lie group.*
- (2) *The multiplication $\mathbf{m} : \mathbf{G} \times \mathbf{G} \rightarrow \mathbf{G}$ is a Poisson (resp. partial symplectic) morphism when $\mathbf{G} \times \mathbf{G}$ is equipped with the partial Poisson (resp. symplectic) product structure.*
- (3) *With respect to the Poisson structure $\mathcal{G} \times \mathcal{G} \times \mathcal{G}^-$ on $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$, the graph $G_{\mathbf{m}}$ of the multiplication is coisotropic (resp. Lagrangian).*
- (4) *The subbundle $T^\flat \mathbf{G}$ of $T'\mathbf{G}$ is bi-invariant and the associated partial Poisson tensor Λ satisfies, for all $g, h \in \mathbf{G}$,*

$$\Lambda_{gh} = L_g^{**} \Lambda_h \quad \text{and} \quad \Lambda_{hg} = R_g^{**} \Lambda_h. \quad (3.8)$$

Remark 3.16. Note that Proposition 3.15, (2) recovers Definition 5.3 of a Banach Poisson-Lie group in [38]. This is consistent with our Definition 3.12 in the Banach case.

Proof. We only look for the partial Poisson situation. For the partial symplectic framework, the same argument works too.

(1) $\Leftrightarrow$ (2)¹² On the one hand, for the partial Poisson structure on $\mathbf{G} \times \mathbf{G}$, we have $T_{gh}^\flat \mathbf{G} \times \mathbf{G} = T_g^\flat \mathbf{G} \times T_h^\flat \mathbf{G}$. If $(X_g, X_h) \in T_{gh}^\flat (\mathbf{G} \times \mathbf{G})$, then we have:

$$T_{(g,h)} \mathbf{m}(X_g, X_h) = TL_g(X_h) + TR_h(X_g). \quad (3.9)$$

The bi-invariance of $T^\flat \mathbf{G}$ means that, for any $\alpha \in T_h^\flat \mathbf{G}$ and $\alpha \circ T_g R_h \in T_{gh}^\flat \mathbf{G}$ then $\alpha \circ T_h L_g$ belongs to $T_g^\flat \mathbf{G}$. It follows that the relation (3.9), for all $g, h \in \mathbf{G}$ and $X_g, X_h \in T_{gh}^\flat \mathbf{G} \times \mathbf{G}$, is equivalent to **(BiT)**. On the other hand, let (ϕ, ψ) be a smooth function around $(g, h) \in \mathbf{G} \times \mathbf{G}$. We denote by X^ϕ and X^ψ the Hamiltonian vector fields associated with ϕ and ψ respectively. Then, for the partial product Poisson $P \times P$, the Hamiltonian field of (ϕ, ψ) is (X^ϕ, X^ψ) . Then according to (3.9), we must have

$$T_{(g,h)} \mathbf{m}(X^\phi, X^\psi) = T_h L_g(X^\phi) + T_g R_h(X^\psi).$$

¹²This proof is an adaptation of the proof of [38, Proposition 5.4].

Moreover, by definition of the Poisson product structure on $\mathbf{G} \times \mathbf{G}$, it follows that

$$\begin{aligned} (P \times P)(d\phi, d\psi)(g, h) &= X_{(g,h)}^{(\phi,\psi)} = T_g R_h(X^\phi) + T_h L_g(X^\psi) \\ &= T_g R_h(P_g(d\phi)) + T_h L_g(P_h(d\psi)). \end{aligned} \quad (3.10)$$

The map $\mathbf{m}$ is a Poisson morphism if and only if

$$T_{(g,h)}\mathbf{m}\left(X^{(\phi,\psi)}\right) = (P \times P)(d\phi, d\psi)(g, h)$$

which is equivalent to the identity above. This completes the proof of the equivalence.

The equivalence (2) $\Leftrightarrow$ (3) is a consequence of Proposition 2.48.

For the equivalence (1) $\Leftrightarrow$ (4), it suffices to prove the equivalence between condition **(BiP)** and relation (3.8), which follows directly from the definition of Λ in terms of P , together with (3.7). $\blacksquare$

3.5. MULTIPLICATIVE ADMISSIBLE 2-FORM AND LIE BIALGEBRA. In this subsection, we fix a bi-invariant convenient subbundle $T^\flat\mathbf{G}$ of $T'\mathbf{G}$ and we set $\mathfrak{g}^\flat = T_e^\flat\mathbf{G} \subset \mathfrak{g}'$. The convenient dual of $\mathfrak{g}^\flat$ will be denoted $\mathfrak{g}^{\flat'}$.

3.5.1. MULTIPLICATIVE ADMISSIBLE 2-FORMS. According to [8, Definition 7.12], for any integer $k \geq 2$, an *admissible section* $\Lambda \in \bigwedge^k T^{\flat'}\mathbf{G}$ is a section for which there exists a skew-symmetric morphism $P : (T^\flat\mathbf{G})^{k-1} \rightarrow T\mathbf{G}$ such that:

$$\Lambda(\alpha_1, \dots, \alpha_k) = \alpha_1(P(\alpha_2, \dots, \alpha_k))$$

for all $(\alpha_1, \alpha_2, \dots, \alpha_k) \in (T^{\flat'}\mathbf{G})^k$.

For the sake of simplicity, such a section will be called an *admissible k -form*. Note that the partial Poisson tensor Λ associated with a partial Poisson anchor $P : T^\flat\mathbf{G} \rightarrow T\mathbf{G}$ is a 2-admissible form.

According to Property (3.8), as in finite dimension for the notion of multiplicative tensor (cf. for example [13, 26, 28, 23]), we introduce:

DEFINITION 3.17. An admissible k -form Λ is called *multiplicative* if it satisfies:

$$\Lambda_{gh} = L_g^{**}\Lambda_h + R_h^{**}\Lambda_g \quad (3.11)$$

for all $(g, h) \in \mathbf{G}^2$.

Remarks 3.18. (a) If Λ is multiplicative, for $h = g = e$, from the relation (3.11) we obtain $\Lambda_e = 2\Lambda_e$ and so $\Lambda_e = 0$.

(b) If $\Lambda_e = 0$, for any $v \in T_e \mathbf{G}$, let X be a vector field on $\mathbf{G}$ such that $X(e) = v$. Since we have (cf. [8, Chapter 7])

$$\begin{aligned} L_X \Lambda(\alpha_1, \dots, \alpha_k)(e) &= d_e (\Lambda(\alpha_1, \dots, \alpha_k)) (X) \\ &\quad - \sum_{i=1}^k \Lambda_e(\alpha_1, \dots, L_X \alpha_i, \dots, \alpha_k) \end{aligned} \quad (3.12)$$

it follows that $\mathcal{L}_X \Lambda(\alpha_1, \dots, \alpha_k)(e) = d_e (\Lambda(\alpha_1, \dots, \alpha_k)) (X)$ and hence only depends on the 1-jet of Λ at point e and the value $X(e)$. Thus, as in finite dimension, the differential $d\Lambda_e$ is intrinsically defined.

As in finite dimension, (cf. [13]), we have the following properties:

PROPOSITION 3.19. *Given any admissible k -form Λ , we associate to Λ two bounded fields*

$$\Lambda_L : \mathbf{G} \longrightarrow \bigwedge^k \mathfrak{g}^{b'} \quad \text{and} \quad \Lambda_R : \mathbf{G} \longrightarrow \bigwedge^k \mathfrak{g}^{b'}$$

defined, for all $(g, h) \in \mathbf{G}^2$, by $(\Lambda_L)_g = L_{g^{-1}}^{**} \Lambda_g$ and $(\Lambda_R)_g = R_{g^{-1}}^{**} \Lambda_g$. We then have the following equivalences:

- (i) Λ is multiplicative.
- (ii) Λ_L satisfies, for all $(g, h) \in \mathbf{G}^2$:

$$(\Lambda_L)_{gh} = (\Lambda_L)_h + Ad_{h^{-1}}^{**} (\Lambda_L)_g.$$

- (iii) Λ_R satisfies, for all $(g, h) \in \mathbf{G}^2$:

$$(\Lambda_R)_{gh} = Ad_g^{**} (\Lambda_R)_h + (\Lambda_R)_g.$$

Remark 3.20. In finite dimension, if $\mathbf{G}$ is connected, the following properties:

- (iv) $\Lambda_e = 0$ and $\mathcal{L}_X \Lambda$ is left invariant whenever X is a left invariant vector field.
- (v) $\Lambda_e = 0$ and $\mathcal{L}_X \Lambda$ is right invariant whenever X is a right invariant vector field.

are equivalent to properties (i), (ii) and (iii) (cf. [13, 22]).

Properties (iv) or (v) are equivalent to each Properties (i), (ii) and (iii) for a connected convenient Lie group admitting an exponential map and generated by the range of the exponential map. This is, in particular, the case for connected and simply connected Banach-Lie groups. Unfortunately, in the general case for convenient Lie groups, such properties are not true since the previous assumption is not satisfied.

Proof. According to Notations 3.14, for all $(g, h) \in \mathbf{G}^2$, we have

$$\begin{aligned} (\Lambda_L)_{gh} - (\Lambda_L)_h - \text{Ad}_{h^{-1}}^{**}(\Lambda_L)_g &= L_{(gh)^{-1}}^{**}\Lambda_{gh} - L_{h^{-1}}^{**}\Lambda_h - L_{h^{-1}}^{**} \circ R_h^{**} \circ L_{g^{-1}}^{**}\Lambda_g \\ &= L_{(gh)^{-1}}^{**}(\Lambda_{gh} - L_g^{**}\Lambda_h - R_h^{**}\Lambda_g) \end{aligned}$$

which implies the equivalence between (i) and (ii).

A formally comparable calculus shows the equivalence between (i) and (iii). ■

3.5.2. 1-COCYCLES AND LIE BIALGEBRAS. For this paragraph, we refer to [38, 2.5 and 4.1]. For the sake of completeness, we adapt these results to the convenient context.

As in §3.2, let $\mathbf{G}$ be a convenient Lie group and $\Phi : \mathbf{G} \rightarrow \text{Aff}(\mathbb{E})$ a representation into the affine group of a vector space $\mathbb{E}$. In this situation, Φ can be written as a pair (ϕ, Θ) where $\phi : \mathbf{G} \rightarrow GL(\mathbb{E})$ and $\Theta : \mathbf{G} \rightarrow \mathbb{E}$.

EXAMPLE 3.21. Consider a Lie group $\mathbf{G}$ and a bi-invariant subbundle $T^b\mathbf{G}$ of $T\mathbf{G}$. If we denote by $\mathfrak{g}^b$ the fibre $T_e^b\mathbf{G}$. Then $\mathfrak{g}^b$ is invariant by the coadjoint action $\mathbf{G}$ on $T'\mathbf{G}$ (formally, same proof as in [38, Proposition 5.6]). Let Λ be an admissible multiplicative 2-form and $\Lambda_R : \mathbf{G} \rightarrow \bigwedge^2 \mathfrak{g}^b$ given by

$$(\Lambda_R)_g = R_{g^{-1}}^{**}\Lambda_g$$

for all $g \in \mathbf{G}$.

Then Λ_R is a 1-cocycle on $\mathbf{G}$ with respect to the coadjoint action Ad^{**} on the vector space $\bigwedge^2 \mathfrak{g}^b$ that is

$$(\Lambda_R)_{gh} = (\Lambda_R)_g + \text{Ad}^{**}(g)(\Lambda_R)_h \quad (3.13)$$

for all g, h in $\mathbf{G}$ (Formally again, the proof is the same as for [38, Proposition 5.7, Assertion (2)]). Therefore, we have

$$T_e\Lambda_R([x, y]) = \text{ad}_x^{**}(T_e\Lambda_R(y)) - \text{ad}_y^{**}(T_e\Lambda_R(x)) \quad (3.14)$$

which means exactly:

$$\begin{aligned} T_e \Lambda_R([x, y])(\alpha, \beta) &= T_e \Lambda_R(y)(\text{ad}_x^* \alpha, \beta) + T_e \Lambda_R(y)(\alpha, \text{ad}_x^* \beta) \\ &\quad - T_e \Lambda_R(x)(\text{ad}_y^* \alpha, \beta) - T_e \Lambda_R(x)(\alpha, \text{ad}_y^* \beta). \end{aligned} \quad (3.15)$$

3.5.3. LIE BIALGEBRA ASSOCIATED WITH A MULTIPLICATIVE 2-FORM.

The following result gives a link between admissible multiplicative 2-forms on a Lie group $\mathbf{G}$ and 1-cocycles:

THEOREM 3.22. *Let Λ be an admissible multiplicative 2-form and the associated skew-symmetric morphism $P : T^b \mathbf{G} \rightarrow T\mathbf{G}$. Assume that P_e is injective. Then $[\alpha, \beta]_P = d_e \Lambda(\alpha, \beta)$ defines an almost bracket on $\mathfrak{g}^b$ which is a Lie bracket if and only if $(T^b \mathbf{G}, T\mathbf{G}, P, \{., .\}_P)$ is a convenient partial Poisson-Lie group. In this case, $(T\mathbf{G}, T^b \mathbf{G})$ is a partial Poisson bialgebroid and $d_e \Lambda$ is a 1-cocycle on $\mathfrak{g}$ relative to the ad^{**} action on $\bigwedge^2 \mathfrak{g}^b$.*

Proof. First, recall that the bi-invariance of $T^b \mathbf{G}$ implies that $T^b \mathbf{G} \equiv \mathbf{G} \times \mathfrak{g}^b$ via right or left translation on $\mathbf{G}$.

On the one hand, from the right invariance of $T^b \mathbf{G}$, the map $P_R : \mathbf{G} \times \mathfrak{g}^b \rightarrow T\mathbf{G}$ where $(P_R)_g := P \circ R_{g^{-1}}$ is well defined.

On the other hand, by the definition of Λ_R from Λ (cf. Proposition 3.19), we have

$$(\Lambda_R)_g(\alpha, \beta) = \langle \alpha, (P_R)_g \beta \rangle$$

for all $\alpha, \beta \in \mathfrak{g}^b$.

We will need the following result:

LEMMA 3.23. *Under the assumptions of Theorem 3.22, for any $g \in \mathbf{G}$, we have: $P_e \equiv 0$ and*

$$d(\Lambda(\alpha, \beta))(X_e) = T_e \Lambda_R(\alpha, \beta)(X_e) = \langle \alpha, T_e P_R(X_e)(\beta) \rangle$$

for any local sections α, β of $T_e^b \mathbf{G}$ around e and any $X_e \in T_e \mathbf{G}$.

Proof of Lemma 3.23. Fix some neighbourhood V of g in $\mathbf{G}$. For all $h \in V$ we have $\Lambda_h(\alpha, \beta) = (\Lambda_R)(R_{h^{-1}}^* \alpha, R_{h^{-1}}^* \beta) = \langle R_{h^{-1}}^* \alpha, (P_R)_h(\beta) \rangle$.

Since Λ_R is a map from $\mathbf{G}$ to $\bigwedge^2 \mathfrak{g}^b$, it follows that we have:

$$\begin{aligned} d(\Lambda(\alpha, \beta))(X_g) &= \langle T_g \tilde{\alpha}(X_g), (P_R)_g(\beta) \rangle + \langle \alpha, T_g P_R(X_g)(\beta) \rangle \\ &\quad + \langle \alpha, P_R(T_g \tilde{\beta}(X_g)) \rangle \end{aligned} \quad (3.16)$$

where $\tilde{\alpha}(h) = R_{h^{-1}}\alpha(h)$ and $\tilde{\beta}(h) = R_{h^{-1}}\beta(h)$, for all $h \in V$.

But we have $\Lambda_e(\alpha, \beta) = \langle \alpha, P_e\beta \rangle = 0$, for all α and β in $\mathfrak{g}^b$. According to section 2.2.1, this implies that $(\mathfrak{g}^b)^{\perp_{P_e}} = \{0\}$ and so $P_e(\beta)$ belongs to $P_e(\mathfrak{g}^b)^{\perp_{P_e}} = 0$. Since P_e is injective, it follows that $P_e(\beta) = 0$ for all $\beta \in \mathfrak{g}^b$. Thus from (3.16), since we have $(P_R)_e = P_e$ we obtain:

$$d(\Lambda(\alpha, \beta))(X_g) = \langle \alpha, T_g P_R(X_g)(\beta) \rangle. \quad (3.17)$$

In particular, for $g = e$, we have $(P_R)_e = P_e = 0$ and so we get:

$$d(\Lambda(\alpha, \beta))(X_e) = \langle \alpha, T_e P_R(X_e)(\beta) \rangle. \quad (3.18)$$

But we can note that since $(\Lambda_R)_e = \Lambda_e = 0$, the second member of (3.18) is nothing but that $T_e \Lambda_R(\alpha, \beta)(X_e)$, which ends the proof of Lemma 3.23. ■

End of proof of Theorem 3.22. Let $\mathfrak{A}(\mathbf{G}) := \{\phi \in C^\infty(\mathbf{G}), d_g \phi \in T_g^b \mathbf{G}\}$. Then $\mathfrak{g}^b$ can be identified with the set

$$\bigcup_{\alpha \in \mathfrak{g}^b} \{\phi_\alpha \in \mathfrak{A}(\mathbf{G}) : \phi_\alpha(e) = 0 \text{ and } \forall g \in \mathbf{G}, d_g \phi_\alpha = R_{g^{-1}}^* \alpha\}.$$

Since P is an almost Poisson anchor, we have an almost Lie Poisson bracket $\{\phi, \psi\}_P$ on $\mathfrak{A}(\mathbf{G})$ given by

$$\{\phi, \psi\}_P = \langle d\psi, P(d\phi) \rangle$$

which is a Poisson bracket if and only if P is a Poisson anchor (cf. Proposition 2.26 and Remark 2.27).

Now, from Remark 2.27, $\{\phi, \psi\}_P$ induces an almost Lie bracket $[\alpha, \beta]_P = d\{\phi_\alpha, \phi_\beta\}$ on $\mathfrak{g}^b$ which satisfies the Jacobi identity.

From the definition of Λ_R as a map from G to $\bigwedge^2 \mathfrak{g}^b$ it follows that $T_e \Lambda$ is map from $\mathfrak{g}$ to $\bigwedge^2 \mathfrak{g}^b$. Therefore, the almost bracket on $\mathfrak{g}$ satisfies the relations:

$$[\alpha, \beta]_P = d\{\phi_\alpha, \phi_\beta\} = d_g \Lambda(R_{g^{-1}}^* \alpha, R_{g^{-1}}^* \beta) = T_e \Lambda_R(\alpha, \beta).$$

Lemma 3.23 implies that $[\alpha, \beta]_P = d_e \Lambda(\alpha, \beta)$.

Now, if P is a Poisson anchor, this implies that the bracket on $\mathfrak{A}$ satisfies the Jacobi identity and from Proposition 2.26 the almost bracket $[\cdot, \cdot]_P$ also satisfies the Jacobi identity since P is injective by assumption. Therefore, the almost Lie bracket induced on $\mathfrak{g}$ also satisfies the Jacobi identity.

Conversely, assume that the almost bracket on $\mathfrak{g}$ satisfies the Jacobi identity.

First note that from Remark 2.27, we have an almost Lie bracket $[\cdot, \cdot]_P$ on $T^\flat \mathfrak{g}$ which satisfies:

$$[d\phi, d\psi]_P = d\Lambda(d\phi, d\psi).$$

For any function ϕ_1, ϕ_2, ϕ_3 in $\mathfrak{A}$, we set:

$$J_P(\phi_1, \phi_2, \phi_3) = \{\phi_1\{\phi_2, \phi_1\}_P\}_P + \{\phi_2\{\phi_3, \phi_1\}_P\}_P + \{\phi_3\{\phi_1, \phi_2\}_P\}_P.$$

Note that, for any $g \in \mathbf{G}$ and any ϕ_i there exists $\alpha_i \in \mathfrak{g}^\flat$ such that $d_g \phi_i = R_g^* \alpha_i$ for $i \in \{1, 2, 3\}$.

With all these properties we have sucessively:

$$\begin{aligned} d_g \{\phi_2, \phi_1\}_P &= d_g \Lambda(d\phi_2, d\phi_3) = d_g \Lambda(R_g^* \alpha_2, R_g^* \alpha_3) \\ &= d_e \Lambda_R(\alpha_2, \alpha_3) = [\alpha_2, \alpha_3]_P. \end{aligned}$$

So we obtain

$$d_g (\{\phi_1\{\phi_2, \phi_1\}_P\}_P) = d_e \Lambda_R(\alpha_1, [\alpha_2, \alpha_3]_P) = [\alpha_1[\alpha_2, \alpha_3]_P]_P.$$

It follows that $d_g J_P(\phi_1, \phi_2, \phi_3)$ is the Jacobi expression of the bracket on $\mathfrak{g}^\flat$ for $(\alpha_1, \alpha_2, \alpha_3)$. Therefore

$$d_g J_P(\phi_1, \phi_2, \phi_3) = 0$$

for all $g \in \mathbf{G}$ and ϕ_1, ϕ_2, ϕ_3 in $\mathfrak{A}$. By the way, J_P must be constant and this value does not depend on the choice (ϕ_1, ϕ_2, ϕ_3) in particular of their values at g . Thus J_P must be identically zero, which ends the announced equivalence.

Now if P is a Poisson anchor, from Proposition 2.30, $(T\mathbf{G}, T^\flat \mathbf{G})$ is a partial Poisson partial bialgebroid.

It remains to show that $d_e \Lambda$ is a 1-cocycle on $\mathfrak{g}$ relative to the ad^{**} action on $\bigwedge^2 \mathfrak{g}^\flat$.

From Lemma 3.23, we have

$$[\alpha, \beta]_P = d_e \Lambda(\alpha, \beta) = T_e \Lambda_R(\alpha, \beta).$$

The last assertion is then a direct consequence of Example 3.21. $\blacksquare$

3.6. DIRECT LIMITS OF (PARTIAL) BANACH POISSON-LIE GROUPS.

Every Banach Poisson-Lie group in the sense of [38] is, in particular, a convenient partial Poisson-Lie group. In [38], the author presents many interesting examples of Banach Poisson-Lie groups which therefore are also partial convenient Poisson-Lie groups. We do not reproduce the full list of examples here; interested readers are referred to that paper.

In this subsection, direct limits of ascending sequences of (partial) Banach Poisson-Lie groups provide examples of convenient Poisson-Lie groups that are not Banach Poisson-Lie groups.

3.6.1. DIRECT LIMITS OF SOME ASCENDING SEQUENCES OF BANACH (PARTIAL) POISSON-LIE GROUPS. First, we recall some results concerning direct limits of ascending sequences of Banach-Poisson manifolds and Banach-Lie groups. For further details, see [8, 7.6 and 5.8].

Let $(M_i, \mathfrak{A}(M_i), \{., .\}_{P_i})_{i \in \mathbb{N}}$ be a sequence of partial Banach-Poisson manifolds where $p_i^b : T^b M_i \rightarrow M_i$ is a Banach subbundle of the cotangent bundle $p_{M_i}^* : T^* M_i \rightarrow M_i$ and $P_i : T^b M_i \rightarrow TM_i$ is the associated Poisson anchor. We denote by $\mathbb{M}_i$ the Banach space on which M_i is modelled, and by $\mathbb{F}_i$ the model of the typical fibre of the subbundle $p_i^b : T^b M_i \rightarrow M_i$. We assume that $\mathbb{F}_i$ is a Banach subspace of the dual $\mathbb{M}_i^*$ of $\mathbb{M}_i$. For each $i \in \mathbb{N}$, we denote by $\varepsilon_i^{i+1} : M_i \hookrightarrow M_{i+1}$ the inclusion map.

DEFINITION 3.24. The sequence $(M_i, \mathfrak{A}(M_i), \{., .\}_{P_i})_{i \in \mathbb{N}}$ is called a *direct sequence of partial Banach-Poisson manifolds* if

(DSpPM1) $(M_i)_{i \in \mathbb{N}}$ is an ascending sequence of smooth Banach manifolds, where M_i is modelled on the Banach space $\mathbb{M}_i$ such that $\mathbb{M}_i$ is a supplemented Banach subspace of $\mathbb{M}_{i+1}$ and such that M_i is a closed submanifold of M_{i+1} ;

(DSpPM2) for each $i \in \mathbb{N}$, the map

$$\varepsilon_i^{i+1} : \left(T^b M_i, TM_i, P_i, \{., .\}_{P_i} \right) \longrightarrow \left(T^b M_{i+1}, TM_{i+1}, P_{i+1}, \{., .\}_{P_{i+1}} \right)$$

is a Poisson map;

(DSpPM3) around each $x \in M$, there exists an ascending sequence of charts

$(U_i, \phi_i)_{i \in \mathbb{N}}$ such that we have the following commutative diagram:

$$\begin{array}{ccc} T^\flat U_i & \xleftarrow{T^* \varepsilon_i^{i+1}} & T^\flat U_{i+1} \\ \downarrow & & \downarrow \\ U_i & \xrightarrow{\varepsilon_i^{i+1}} & U_{i+1} \end{array}$$

Remarks 3.25. (a) Since $(M_i, \varepsilon_i^{i+1})_{i \in \mathbb{N}}$ is an ascending sequence of Banach manifolds, we may assume that $\mathbb{M}_i$ is a Banach subspace of $\mathbb{M}_{i+1}$. By abuse of notation, we continue to denote by ε_i^{i+1} the natural inclusion of $\mathbb{M}_i$ in $\mathbb{M}_{i+1}$ and $(\varepsilon_i^{i+1})^* : \mathbb{M}_{i+1}^* \rightarrow \mathbb{M}_i^*$ the adjoint operator. Note that $(\varepsilon_i^{i+1})^*$ is surjective. Since $\mathbb{F}_i$ is the typical fibre of $T^\flat M_i$, we must have $(\varepsilon_i^{i+1})^*(\mathbb{F}_{i+1}) = \mathbb{F}_i$ and each $\mathbb{F}_i$ is a Banach subspace of $\mathbb{M}_i'$.

(b) In fact, **(DSpPM2)** implies $P_{i+1}|_{T^\flat M_i} = P_i$.

First, we have the following proposition:

PROPOSITION 3.26. *Let $(M_i, \mathfrak{A}(M_i), \{\cdot, \cdot\}_{P_i})_{i \in \mathbb{N}}$ be a direct sequence of partial Banach-Poisson manifolds and let M be the direct limit of the sequence $(M_i)_{i \in \mathbb{N}}$. We then have:*

1. $p_M : TM \rightarrow M$ is a convenient bundle, which is the kinematic bundle of M .
2. $p'_M : T'M \rightarrow M$ and $p^\flat : T^\flat M \rightarrow M$ are locally trivial convenient bundles over M . Moreover $p'_M : T'M \rightarrow M$ is the kinematic cotangent bundle of M .
3. There exists a canonical bundle morphism $P : T^\flat M \rightarrow TM$ whose typical fibre is $\varprojlim \mathbb{F}_i$ where $\mathbb{F}_i$ denotes the typical fibre of $T^\flat M_i$. This morphism is characterized, for $x_i \in M_i$, by:

$$P(x_i, \xi) = P(x_i, \varprojlim_{k \geq i} (\xi_k)) = \varinjlim_{i \leq k} (P_k(x_i, \xi_k)). \quad (3.19)$$

Moreover, P is skew-symmetric, with respect to the canonical dual pairing between $T'M$ and TM when restricted to $T^\flat M \times TM$.

Finally, we have:

THEOREM 3.27. *Let $(M_i, \mathfrak{A}(M_i), \{\cdot, \cdot\}_{P_i})_{i \in \mathbb{N}}$ be a direct sequence of partial Poisson Banach manifolds and let $M = \varinjlim (M_i)$.*

There exists a weak subbundle $p^\flat : T^\flat M \rightarrow M$ of $p'_M : T'M \rightarrow M$ and a skew-symmetric morphism $P : T^\flat M \rightarrow TM$ such that $(M, \mathfrak{A}_P(M), \{\cdot, \cdot\}_P)$ is a partial Poisson manifold with the following characterization:

If $\varepsilon_i : M_i \rightarrow M$ denotes the canonical injection, then ε_i is a Poisson morphism from $(M_i, \mathfrak{A}(M_i), \{\cdot, \cdot\}_{P_i})$ to $(M, \mathfrak{A}(M), \{\cdot, \cdot\}_P)$ for all $i \in \mathbb{N}$. Moreover, we have:

$$\mathfrak{A}_P(M) = \varinjlim \mathfrak{A}(M_i), \quad \{\cdot, \cdot\}_P = \varinjlim \{\cdot, \cdot\}_{P_i}.$$

Therefore $(T^\flat M, M, P, \{\cdot, \cdot\}_P)$ is not *a priori* a partial Poisson manifold since its Poisson bracket is defined only on the sheaf of subalgebras $\mathfrak{A}_P(M) = \varinjlim \mathfrak{A}(M_i)$ of sheaf $\mathcal{A}_M$ of smooth functions associated with the convenient bundle $T^\flat M$. However:

LEMMA 3.28. *Let $(U_i, \Phi_i)_{i \in \mathbb{N}}$ be an ascending sequence of charts. Set $U = \varinjlim U_i$, $\Phi = \varinjlim \Phi_i$. Then, for any smooth function $f : U \rightarrow M$, if $f_i = f|_{U_i}$, we have $f = \varinjlim f_i$.*

Proof. Recall that if M_i is modelled on the Banach space $\mathbb{M}_i$, we may assume that $\mathbb{M}_i$ is a supplemented subspace of $\mathbb{M}_{i+1}$ and let $\mathbb{M} = \varinjlim \mathbb{M}_i$ which is a convenient space. Without loss of generality, we may assume that U_i is a c^∞ -open set of $\mathbb{M}_i$. Then by [8, Lemma 5.2], we obtain $f = \varinjlim f_i$. ■

It follows that $(T^\flat M, M, P, \{\cdot, \cdot\}_P)$ is really a partial Poisson manifold.

We will apply these results in the following context of Banach (partial) Poisson-Lie groups.

Note that the direct limit of an ascending sequence of Banach manifolds can be viewed as a non-Hausdorff manifold. Nevertheless, an ascending sequence of finite dimensional manifolds is always a Hausdorff manifold ([15]).

However, for an ascending sequence

$$G_1 \subset \cdots \subset G_i \subset G_{i+1} \subset \cdots$$

where G_i is a Banach-Lie subgroup of G_{i+1} , the associated sequence of Lie algebras

$$\mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_i \subset \mathfrak{g}_{i+1} \subset \cdots$$

is a strong ascending sequence of closed Banach spaces. Hence, the direct limit $\mathfrak{g} = \varinjlim \mathfrak{g}_i$ is a Hausdorff locally convex space and therefore a Hausdorff convenient space. Since $C^\omega_{\mathbb{K}}$ -differentiability implies convenient smoothness, by [11, Theorem 4.11] we obtain:

THEOREM 3.29. *Consider an ascending sequence $G_1 \subset \dots \subset G_i \subset G_{i+1} \subset \dots$ of Banach-Lie groups such that G_i is a Banach-Lie subgroup of G_{i+1} . Assume that, for each $i \in \mathbb{N}$, there exists a norm $\|\cdot\|_i$ on $\mathfrak{g}_i$ such that $\|[X, Y]\|_i \leq \|X\|_i \|Y\|_i$ and such that the norm of the morphism $\epsilon_i^{i+1} : \mathfrak{g}_i \rightarrow \mathfrak{g}_{i+1}$ canonically associated with the inclusion ϵ_i^{i+1} of G_i in G_{i+1} is at most 1.*

Then $G = \varinjlim G_i$ is a convenient Lie group whose Lie algebra is $\mathfrak{g}$. Moreover, $\exp_G := \varinjlim \exp_{G_i}$ is a convenient local diffeomorphism on a c^∞ -open set of 0.

Note that this result is particularly valid for sequences of finite dimensional Lie groups (cf. [15]).

In a more general context, we have the following result (cf. [8, 5.8]):

COROLLARY 3.30. *Let $(\mathbb{E}_n)_{n \in \mathbb{N}}$ be an ascending sequence of Banach spaces such that $\mathbb{E}_n$ is a supplemented Banach subspace of $\mathbb{E}_{n+1}$.*

Then $\mathbb{E} = \bigcup_{n \in \mathbb{N}} \mathbb{E}_n$ is an LB-space and $\mathcal{L}(\mathbb{E}) = \bigcup_{n \in \mathbb{N}} \mathcal{L}(\mathbb{E}_n)$ is also an LB-space, where $\mathcal{L}(\mathbb{E}_n)$ denotes the Banach space of continuous linear operators of $\mathbb{E}_n$. Moreover, $\mathcal{GL}(\mathbb{E}) = \bigcup_{n \in \mathbb{N}} \mathcal{GL}(\mathbb{E}_n)$ has a structure of convenient Lie group modelled on $\mathcal{L}(\mathbb{E})$, where $\mathcal{GL}(\mathbb{E}_n)$ denotes the Banach-Lie group of linear continuous automorphisms of $\mathbb{E}_n$.

3.6.2. ASCENDING SEQUENCES OF BANACH (PARTIAL) POISSON-LIE GROUPS. We have seen that any Banach Poisson-Lie group is in particular a Banach partial Poisson-Lie group, so we only consider sequences of Banach partial Poisson-Lie groups. Thus naturally, an ascending sequence of Banach (partial) Poisson-Lie groups $(T^\flat G_i, G_i, P_i)_{i \in \mathbb{N}}$ is an ascending sequence of (partial) Banach-Poisson manifolds.

THEOREM 3.31. *Consider an ascending sequence of Banach (partial) Poisson Lie groups $(T^\flat G_i, G_i, P_i)_{i \in \mathbb{N}}$. Assume that the ascending sequence of groups $(G_i)_{i \in \mathbb{N}}$ satisfies the assumption of Theorem 3.29.*

Then the direct limit $G = \varinjlim G_i$ can be endowed with a (partial) Poisson-Lie group structure.

Proof. It remains to show that the partial Poisson manifold $(T^\flat G, G, P)$ is a (partial) Poisson Lie group.

Assume that $g = \varinjlim g_i$. By using classical results on direct limits and maps and their differentials, we have the following relations and results:

- the left (resp. right) translation L_g (resp. R_g) is $L_g = \varinjlim L_{g_i}$ (resp. $R_g = \varinjlim R_{g_i}$) and an analogous result holds for Ad_g on $\mathbf{G}$;
- since any $x \in \mathfrak{g}$ can be written $x = \varinjlim x_i$ with $x_i \in \mathfrak{g}_i$, one has:

$$\text{ad}_x = \varinjlim \text{ad}_{x_i} \quad \text{and} \quad \text{ad}_x^* = \varprojlim \text{ad}_{x_i}^*$$

- if we set $\mathfrak{g}^\flat = T_e^\flat \mathbf{G}$, we have $\mathfrak{g}^\flat = \varprojlim \mathfrak{g}_i^\flat$ (cf. [8, p. 351]);
- the bi-invariance of each $T^\flat \mathbf{G}_i$ implies the bi-invariance of $T^\flat \mathbf{G}$;
- from Remark 3.25 (b), the bi-invariance of each P_i implies the bi-invariance of P .

This completes the proof. ■

3.6.3. SOME EXAMPLES OF CONVENIENT PARTIAL POISSON-LIE GROUPS

EXAMPLE 3.32. Structure of Poisson-Lie group on $\mathbf{U}(n)$.

We first recall how to build a structure of Poisson-Lie group on the unitary group $\mathbf{U}(n) := \mathbf{U}(n, \mathbb{C})$. Let $\mathfrak{u}(n)$ (cf. [38])¹³ be the Lie algebra of the Lie group $\mathbf{U}(n)$, i.e. the space of skew-symmetric matrices and let $\mathfrak{b}(n)$ be the Lie algebra of the Borel group $\mathbf{B}(n) := \mathbf{B}(n, \mathbb{C})$, i.e. the space of upper triangular matrices with real coefficient on the diagonal. Then the Lie algebra of the group $\mathbf{GL}(n, \mathbb{C})$ is $\mathfrak{gl}(n)$, the space of all complex $n \times n$ matrices and the Iwasawa decomposition gives rise to the decomposition:

$$\mathfrak{gl}(n) = \mathfrak{u}(n) \oplus \mathfrak{b}(n).$$

Consider the non degenerate symmetric bilinear continuous map $\langle \cdot, \cdot \rangle$ given by

$$\langle A, B \rangle = \Im \text{Tr}(AB),$$

where $\Im \text{Tr}(AB)$ denotes the imaginary part of the trace of the matrix AB . Then $(\mathfrak{u}(n), \mathfrak{b}(n))$ is a bialgebra, to which is associated a canonical Poisson Lie structure on $\mathbf{U}(n)$. More precisely, in this finite dimensional situation, $\mathfrak{b}(n)$ is isomorphic to the Lie algebra $\mathfrak{u}(n)^*$. From the decomposition $\mathfrak{gl}(n) = \mathfrak{u}(n) \oplus \mathfrak{b}(n)$ we denote by p_1 (resp. p_2) the canonical projection of $\mathfrak{gl}(n)$ on $\mathfrak{u}(n)$

¹³This situation is a particular case of a more general context exposed in [23] or [26] for instance.

(resp. $\mathfrak{b}(n)$) the associated multiplicative Poisson tensor $\Lambda_R^n : \mathbf{U}(n) \rightarrow \bigwedge \mathfrak{b}(n)$ ¹⁴ is characterized by (cf. [26]):

$$(\Lambda_R^n)_g(\alpha_1, \alpha_2) := \Im \text{Tr} \left(p_2(\text{Ad}_{g^{-1}} \alpha_1) p_1(\text{Ad}_{g^{-1}} \alpha_2) \right). \quad (3.20)$$

The corresponding multiplicative 2-form on $\mathbf{U}(n)$ is¹⁵

$$(\Lambda^n)_g(\alpha_1, \alpha_2) = R^{**}(\Lambda_R^n)_g(\alpha_1, \alpha_2)$$

and the Poisson anchor $P_n : T^*\mathbf{U}(n) \rightarrow T\mathbf{U}(n)$ is the map

$$(g, \alpha) \mapsto (\Lambda^n)_g(., \alpha).$$

EXAMPLE 3.33. A convenient Poisson-Lie group.

We have a canonical embedding from $\mathbf{U}(n)$ into $\mathbf{U}(n+1)$ and associated, canonical embedding from $\mathfrak{u}(n)$ (resp. $\mathfrak{b}(n)$) in $\mathfrak{u}(n+1)$ (resp. $\mathfrak{b}(n+1)$) (cf. proof of Proposition A.4).

Consider the Poisson anchor $P_n : T^*\mathbf{U}(n) \rightarrow T\mathbf{U}(n)$ (resp. $P_{n+1} : T^*\mathbf{U}(n+1) \rightarrow T\mathbf{U}(n+1)$) as previously defined. It is clear for the previous construction that $(T^*\mathbf{U}(n), \mathbf{U}(n), P_n)_{n \in \mathbb{N}}$ is an ascending sequence of finite dimensional Poisson-Lie groups whose direct limit is $\mathbf{U}(\infty)$ and which is thereby equipped with a convenient Poisson-Lie group structure $(T^*\mathbf{U}(\infty), \mathbf{U}(\infty), P_\infty)$ that is not a Banach Poisson-Lie group. Note that if f and g are smooth maps on $\mathbf{U}(\infty)$ and if $x \in \mathbf{U}(\infty)$, then x belongs to some $\mathbf{U}(n)$ and $\{f, g\}_{P_\infty}(x) = \{f, g\}_{P_n}(x)$ which does not depend on the integer n such that g belongs to $\mathbf{U}(n)$.

EXAMPLE 3.34. A partial Banach Poisson-Lie group which is not a Banach Poisson-Lie group.

We consider the context of [39].

Let $\mathcal{H}$ be a separable complex Hilbert space provided with a Hilbert basis $(|i\rangle)_{i \in \mathbb{Z}}$. We consider the group $\mathbf{U}(\mathcal{H})$ of skew-hermitian bounded operators and we denote by $\mathfrak{u}(\mathcal{H})$ its Lie algebra and $\mathfrak{u}_1(\mathcal{H})$ the Lie subalgebra of trace class of elements of $\mathfrak{u}(\mathcal{H})$. We also consider the Lie algebra of algebra of upper triangular trace-class operators :

$$\mathfrak{b}_1^+(\mathcal{H}) := \{\alpha \in \mathfrak{L}_1(\mathcal{H})^{16} : \alpha|i\rangle \in \text{span}\{|j\rangle, j \leq i\} \text{ and } \forall i \in \mathbb{Z}, \langle i|\alpha|i\rangle \in \mathbb{R}\}$$

¹⁴ $\mathfrak{b}(n)$ is identified with $\mathfrak{u}(n)^*$.

¹⁵For notations, see Example 3.21.

¹⁶cf. Appendix A.

Then the bilinear map:

$$\begin{aligned} \mathfrak{u}(\mathcal{H}) \times \mathfrak{b}_1^+(\mathcal{H}) &\longrightarrow \mathbb{R} \\ (A, B) &\longmapsto \Im \mathrm{Tr}(AB) \end{aligned}$$

is a duality pairing between $\mathfrak{u}(\mathcal{H})$ and $\mathfrak{b}_1^+(\mathcal{H})$ (cf. [39, Proposition 10]). By the way, it can be considered as a subspace of the dual $\mathfrak{u}^*(\mathcal{H})$.

On the other hand, let $\Phi : \mathbf{L}_1(\mathcal{H}) \rightarrow \mathfrak{u}^*(\mathcal{H})$ be the map defined by $\Phi(a)(x) = \Im \mathrm{Tr}(ax)$. Then the kernel on Φ is $\mathfrak{u}_1(\mathcal{H})$ and we obtain a continuous injective linear map from $\mathbf{L}_1(\mathcal{H})/\mathfrak{u}_1(\mathcal{H})$ into $\mathfrak{u}^*(\mathcal{H})$ which is the predual $\mathfrak{u}_*(\mathcal{H})$ of $\mathfrak{u}(\mathcal{H})$ is preserved by the coadjoint action of $\mathbf{U}(\mathcal{H})$. Moreover, $\Phi(\mathfrak{b}_1^+(\mathcal{H}))$ is a proper dense subspace of $\mathbf{L}_1(\mathcal{H})/\mathfrak{u}_1(\mathcal{H})$ (cf. [39, Proposition 13]).

The Lie-Poisson structure on $\mathbf{U}(\mathcal{H})$ is built from the decomposition:

$$\mathbf{L}_2(\mathcal{H}) = \mathfrak{u}_2(\mathcal{H}) \oplus \mathfrak{b}_2^+(\mathcal{H})$$

where $\mathfrak{u}_2(\mathcal{H}) = \mathfrak{u}(\mathcal{H}) \cap \mathbf{L}_2(\mathcal{H})$ and $\mathfrak{b}_2^+(\mathcal{H})$ is the Lie algebra of Hilbert-Schmidt upper triangular operators with real coefficients on the diagonal. Then we have a Banach Poisson-Lie structure on $\mathbf{U}(\mathcal{H})$ defined in the following way (cf. [39, Theorem 14]):

$$T^b \mathbf{U}(\mathcal{H}) = \bigcup_{g \in \mathbf{U}(\mathcal{H})} R_{g^{-1}} \mathfrak{u}_*(\mathcal{H})$$

which is the predual bundle of $T\mathbf{U}(\mathcal{H})$.

The Poisson structure is defined by $\Lambda_R : \mathbf{U}(\mathcal{H}) \rightarrow \bigwedge^2 \mathfrak{u}_*(\mathcal{H})$ ¹⁷ given by:

$$(\Lambda_R^{\mathcal{H}})_g([\alpha_1], [\alpha_2]) = \Im \mathrm{Tr} (p_2(\mathrm{Ad}_{g^{-1}} \alpha_1) p_1(\mathrm{Ad}_g(p_2(\alpha_2)))) \quad (3.21)$$

where $[\alpha_l] = \Phi(\alpha_l) \in \mathbf{L}_1(\mathcal{H})/\mathfrak{u}_1(\mathcal{H}) \equiv \mathfrak{u}_*(\mathcal{H})$ for $l \in \{1, 2\}$. We denote by $P_{\mathcal{H}}$ the associated Poisson anchor.

Now, considering $\mathcal{H}$ provided with a fixed Hilbert basis $(|i\rangle)_{i \in \mathbb{Z}}$, for each $n \in \mathbb{N}$, we have a canonical embedding of $\mathbf{U}(n)$ in $\mathbf{U}(\mathcal{H})$ which is a closed Banach-Lie subgroup. By the way, this gives rise to an embedding of Banach spaces from $\mathfrak{b}(n)$ into $\mathfrak{b}_2^+(\mathcal{H})$. Note that $\Phi(\alpha)$ belongs to $\mathfrak{u}^*(n)$ and so the class $[\alpha]$ is reduced to α . Moreover, in the one hand, according to the relations (9) and (10) in [39], $\mathfrak{u}(n)$ and $\mathfrak{b}(n)$ are invariant by conjugation by a unitary operator. On the other hand, according to (3.20), (3.21) and the previous identifications, Λ_R^n is the restriction of $\Lambda_R^{\mathcal{H}}$ to $\mathfrak{b}(n)$.

¹⁷cf. Proposition 3.19.

We denote by $P_{\mathcal{H}} : T^*\mathcal{U}(\mathcal{H}) \rightarrow T\mathcal{U}(\mathcal{H})$ the Poisson anchor associated with $\Lambda_{\mathcal{H}}^{\mathcal{H}}$. It follows that P_n is precisely the restriction of $P_{\mathcal{H}}$ to $T^*\mathcal{U}(n)$. Hence $(T^*\mathcal{U}(n), \mathcal{U}(\mathcal{H}), P_{\mathcal{H}}|_{T^*\mathcal{U}(n)})$ is a Banach partial Poisson-Lie group which is not a Banach Poisson-Lie group.

Remarks 3.35. (a) As in finite dimension, we can define the notion of convenient partial Poisson-Lie subgroup as follow.

Let $(T^b\mathbf{G}, \mathbf{G}, P)$ be a Poisson-Lie group. A closed supplemented subgroup $\mathbf{H}$ of $\mathbf{G}$ will be called a *convenient partial Poisson-Lie subgroup* of $\mathbf{G}$ if there exists a convenient partial Poisson Lie group $(T^b\mathbf{H}, \mathbf{H}, Q)$ on $\mathbf{H}$ such that $T^b\mathbf{H}$ is a closed supplemented subbundle of $T^b\mathbf{G}$ whose typical fibre $\mathfrak{h}^b$ is a Lie subalgebra of $\mathfrak{g}^b$. By the way, $(T^*\mathcal{U}(n), \mathcal{U}(n), P_{\mathcal{H}})$ is a Banach Lie-Poisson subgroup of $(T^*\mathcal{U}(\mathcal{H}), \mathcal{U}(\mathcal{H}), P_{\mathcal{H}})$.

(b) From Example 3.34, it follows easily that $\mathcal{U}(\infty)$ can be considered as a subset of $\mathcal{U}(\mathcal{H})$. The inclusion is a continuous smooth injective map and $\mathcal{U}(\infty)$ is dense in $\mathcal{U}(\mathcal{H})$. It is a convenient weak submanifold of $\mathcal{U}(\mathcal{H})$, but it is not a convenient partial Poisson Lie subgroup of $\mathcal{U}(\mathcal{H})$.

4. CONVENIENT POISSON-LIE GROUPOIDS

In this section, we recall the definition of a topological groupoid and some generalizations of this notion, introduced in [5], to convenient Lie groupoids and their associated Lie algebroids, with particular emphasis on the Banach setting. In the final subsection, we introduce the notions of partial Poisson-Lie groupoids and symplectic Lie groupoids, and review their main properties.

4.1. TOPOLOGICAL GROUPOIDS

DEFINITION 4.1. A *topological groupoid* $\mathcal{G} \rightrightarrows M$ is a pair $(\mathcal{G}, M)$ of topological spaces such that $\mathcal{G}$ need not be Hausdorff but M is Hausdorff, endowed with the following structure maps:

(TGr1) two surjective open continuous maps $\mathbf{s} : \mathcal{G} \rightarrow M$ and $\mathbf{t} : \mathcal{G} \rightarrow M$ called the *source* and *the target* maps, respectively;¹⁸

(TGr2) a continuous map $\mathbf{m} : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$, where $\mathcal{G}^{(2)} := \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid \mathbf{s}(g) = \mathbf{t}(h)\}$ is equipped with the induced topology from the product topology on $\mathcal{G} \times \mathcal{G}$. This map called *the multiplication* is denoted by

¹⁸Each element $g \in \mathcal{G}$ can be seen as an arrow $g : \mathbf{s}(g) \rightarrow \mathbf{t}(g)$ from $\mathbf{s}(g)$ to $\mathbf{t}(g)$.

$\mathbf{m}(g, h) = gh$ and satisfies the associativity property: the product $(gh)k$ is defined if and only if $g(hk)$ is defined and in this case:

$$(gh)k = g(hk);$$

(TGr3) a continuous embedding $\mathbf{1} : M \rightarrow \mathcal{G}$ called *the identity section* such that $g\mathbf{1}_x = g$ for all $g \in \mathbf{s}^{-1}(x)$, and $\mathbf{1}_x g = g$ for all $g \in \mathbf{t}^{-1}(x)$. In particular, $\mathbf{s} \circ \mathbf{1} = \text{id}_M = \mathbf{t} \circ \mathbf{1}$;

(TGr4) a homeomorphism $\mathbf{i} : \mathcal{G} \rightarrow \mathcal{G}$, denoted $\mathbf{i}(g) = g^{-1}$ called *the inversion*, such that $gg^{-1} = \mathbf{1}_{\mathbf{t}(g)}$, $g^{-1}g = \mathbf{1}_{\mathbf{s}(g)}$. In particular, $\mathbf{s} \circ \mathbf{i} = \mathbf{t}$, $\mathbf{t} \circ \mathbf{i} = \mathbf{s}$.

The space M is called the *base* of the groupoid, and $\mathcal{G}$ is called its *total space*.

For any $(x, y) \in M^2$, we denote $\mathcal{G}(x, -) := \mathbf{s}^{-1}(x)$, $\mathcal{G}(-, y) := \mathbf{t}^{-1}(y)$, and

$$\mathcal{G}(x, y) := \{g \in \mathcal{G} : \mathbf{s}(g) = x, \mathbf{t}(g) = y\} = \mathcal{G}(x, -) \cap \mathcal{G}(-, y).$$

The *isotropy group* at $x \in M$ is the set

$$\mathcal{G}(x) = \{g \in \mathcal{G} : \mathbf{s}(g) = x = \mathbf{t}(g)\} = \mathcal{G}(x, x) \subseteq \mathcal{G}$$

and the *orbit* of $x \in M$ is the set

$$\mathcal{G}.x = \{\mathbf{t}(g) : g \in \mathbf{s}^{-1}(x)\} = \mathbf{t}(\mathcal{G}(x, -)) \subseteq M.$$

For any $g \in \mathcal{G}$, if $\mathbf{s}(g) = x$ and $\mathbf{t}(g) = y$, we define its corresponding left translation

$$L_g : \mathcal{G}(-, x) \longrightarrow \mathcal{G}(-, y), \quad h \longmapsto gh,$$

and similarly its right translation

$$R_g : \mathcal{G}(x, -) \longrightarrow \mathcal{G}(y, -), \quad h \longmapsto hg.$$

Each of these maps L_g and R_g is a homeomorphism.

The topological groupoid $\mathcal{G} \rightrightarrows M$ is called *s-connected* if its *s*-fibres $\mathcal{G}(x, -)$ are connected for all $x \in M$, and *s-simply connected* if all its *s*-fibres are connected and simply connected.

The topological groupoid is called *transitive* if the map $(\mathbf{s}, \mathbf{t}) : \mathcal{G} \rightarrow M \times M$ is surjective.

A *topological morphism between topological groupoids* $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ is given by a pair of continuous maps $\Phi : \mathcal{G} \rightarrow \mathcal{H}$ and $\phi : M \rightarrow N$ which are compatible with the structure maps, that is:

- $\forall g \in \mathcal{G}, \phi(\mathbf{s}(g)) = \mathbf{s}(\Phi(g)), \phi(\mathbf{t}(g)) = \mathbf{t}(\Phi(g))$ and $\Phi(g^{-1}) = \Phi(g)^{-1}$;
- $\forall (g, g') \in \mathcal{G}^{(2)}, \Phi(gg') = \Phi(g)\Phi(g')$;
- $\forall x \in M, \Phi(\mathbf{1}_x) = \mathbf{1}_{\phi(x)}$.

Note that ϕ is uniquely determined by Φ and therefore a morphism between the groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ is given only by a continuous map $\Phi : \mathcal{G} \rightarrow \mathcal{H}$ which satisfies the above compatibility conditions.

A *subgroupoid* of $\mathcal{G} \rightrightarrows M$ is a groupoid $\mathcal{H} \rightrightarrows N$ such that $\mathcal{H} \subset \mathcal{G}$ and the inclusion $\iota : \mathcal{H} \rightarrow \mathcal{G}$ is a topological morphism of groupoids. A subgroupoid $\mathcal{H} \rightrightarrows N$ of $\mathcal{G} \rightrightarrows M$ is called a *wide subgroupoid* if $N = M$.

4.2. CONVENIENT LIE GROUPOIDS. As in [5], the definition of a n.n.H. convenient Lie groupoid requires analogues of the following basic facts for Banach-Lie algebroids, whose proofs require suitable adaptations of the classical arguments.

PROPOSITION 4.2. *Let $\mathcal{G} \rightrightarrows M$ be a topological groupoid satisfying the following conditions:*

- (i) $\mathcal{G}$ is a n.n.H. convenient manifold and M is a Hausdorff convenient manifold;
- (ii) the source map $\mathbf{s} : \mathcal{G} \rightarrow M$ is a submersion;
- (iii) the inversion map $\mathbf{i} : \mathcal{G} \rightarrow \mathcal{G}$ is a smooth diffeomorphism.

Then the target map $\mathbf{t}$ is also a submersion. Consequently, for every $x \in M$, the fibres $\mathcal{G}(x, -)$ and $\mathcal{G}(-, x)$ are n.n.H. convenient submanifolds of $\mathcal{G}$ and $\mathcal{G}(x, -)$ is Hausdorff if and only if $\mathcal{G}(-, x)$ is Hausdorff. Moreover

1. the topological space $\mathcal{G}^{(2)}$ is a n.n.H. convenient submanifold of $\mathcal{G} \times \mathcal{G}$;
2. the identity section $\mathbf{1} : M \rightarrow \mathcal{G}$ is smooth and $\mathbf{1}(M)$ is a convenient submanifold of $\mathcal{G}$.

Furthermore, if $\mathcal{G}$ is a Hausdorff manifold, then, for every $x \in M$, the fibres $\mathcal{G}(x, -)$ and $\mathcal{G}(-, x)$ are Hausdorff submanifolds of $\mathcal{G}$ and $\mathcal{G}^{(2)}$ is a Hausdorff submanifold of $\mathcal{G} \times \mathcal{G}$.

Proof. First, since from (ii), $\mathbf{s}$ is a submersion (cf. [8, Definition 1.13]) and, from (iii), $\mathbf{i}$ is a diffeomorphism, it follows that $\mathbf{t} = \mathbf{s} \circ \mathbf{i}$, is a submersion (cf. [8, Proposition 1.15]). Therefore, by [8, Theorem 1.14], the fibres $\mathcal{G}(x, -)$

and $\mathcal{G}(-, x)$ are split n.n.H. convenient immersed submanifolds of $\mathcal{G}$, for every $x \in M$. Now assume that $\mathcal{G}(x, -)$ is Hausdorff. Since $\mathcal{G}(-, x) = \mathbf{i}(\mathcal{G}(x, -))$ and since $\mathbf{i}$ is a diffeomorphism of $\mathcal{G}$, it follows that $\mathcal{G}(-, x)$ is also Hausdorff. The converse follows by the same argument.

Consider the map $(\mathbf{s}, \mathbf{t}) : \mathcal{G} \times \mathcal{G} \rightarrow M \times M$. If $\mathcal{D}$ denotes the diagonal in $M \times M$, we have $\mathcal{G}^{(2)} = (\mathbf{s}, \mathbf{t})^{-1}(\mathcal{D})$. Therefore, again from [8, Theorem 1.14], $\mathcal{G}^{(2)}$ is a n.n.H. convenient submanifold of $\mathcal{G} \times \mathcal{G}$ since $\mathbf{s}$ and $\mathbf{t}$ are submersions. Moreover, if $\mathcal{G}$ is Hausdorff so are $\mathcal{G}(x, -)$ and $\mathcal{G}(-, x)$ for all $x \in M$ and the same is true for $\mathcal{G}^{(2)}$.

The proof of the last statement is formally the same as in [5, Proposition 3.1]. ■

DEFINITION 4.3. A *n.n.H. convenient Lie groupoid* is a topological groupoid $\mathcal{G} \rightrightarrows M$ satisfying the following conditions:

- (**CLGr1**) $\mathcal{G}$ is a n.n.H. convenient manifold and M is a Hausdorff convenient manifold;
- (**CLGr2**) the source map $\mathbf{s} : \mathcal{G} \rightarrow M$ is a submersion;
- (**CLGr3**) the inversion map $\mathbf{i} : \mathcal{G} \rightarrow \mathcal{G}$ is smooth;
- (**CLGr4**) the multiplication $\mathbf{m} : \mathcal{G}^2 \rightarrow \mathcal{G}$ is smooth.

A convenient Lie groupoid $\mathcal{G} \rightrightarrows M$ is called *split* if, for every $(x, y) \in M^2$, the set $\mathcal{G}(x, y)$ is a submanifold of $\mathcal{G}$.

A convenient Lie morphism between the convenient Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ is a topological morphism $\Phi : \mathcal{G} \rightarrow \mathcal{H}$ which is a smooth map.

When $\mathcal{G}$ is modelled on a Banach space (resp. Fréchet space), it is called a Banach-Lie groupoid (resp. Fréchet-Lie groupoid).

4.3. EXAMPLES OF CONVENIENT LIE GROUPOIDS. We will adapt to the convenient context several classical examples of finite-dimensional Lie groupoids (cf. [5]).

4.3.1. CONVENIENT LIE GROUPS. Any convenient Lie group G is a convenient Lie groupoid: the set of arrows $\mathcal{G}$ is the set G and the set of objects M consists of a single point $\{\mathbf{1}\}$, where $\mathbf{1} \in G$ denotes the unit element.

4.3.2. CONVENIENT LIE PAIR GROUPOID. Given a convenient manifold M , set $\mathcal{G} := M \times M$ and let $\mathbf{s}$ and $\mathbf{t}$ be the projections of $M \times M$ onto the first and the second factor, respectively. The multiplication map $\mathbf{m}$ and the inverse $\mathbf{i}$ are respectively defined by

$$\mathbf{m}((x, y), (y, z)) = (x, z) \quad \text{and} \quad \mathbf{i}(x, y) = (y, x)$$

and are smooth. Finally, the unit map $\mathbf{1} : x \mapsto (x, x)$ is smooth as well. We thus obtain a convenient-Lie groupoid $M \times M \rightrightarrows M$.

4.3.3. GENERAL LINEAR BANACH-LIE GROUPOIDS. Let $\pi : \mathcal{A} \rightarrow M$ be a Banach vector bundle. The general linear Banach groupoid $\mathcal{GL}(\mathcal{A}) \rightrightarrows M$ is the Banach groupoid such that $\mathcal{GL}(\mathcal{A})$ is the set of linear isomorphisms $g : \mathcal{A}_x \rightarrow \mathcal{A}_y$ between each pair of fibres $(\mathcal{A}_x, \mathcal{A}_y)$. The source map and the target map are obvious and the multiplication is the composition of linear isomorphisms and $\mathbf{1}_x = \text{Id}_{\mathcal{A}_x}$.

4.3.4. DISJOINT UNION OF n.n.H. CONVENIENT LIE GROUPOIDS. Let $\{\mathcal{G}_\lambda \rightrightarrows M_\lambda\}_{\lambda \in \mathbf{L}}$ be a family of n.n.H. convenient-Lie groupoids. We denote $\mathcal{G} := \bigsqcup_{\lambda \in \mathbf{L}} \mathcal{G}_\lambda$ and $M := \bigsqcup_{\lambda \in \mathbf{L}} M_\lambda$. Since we consider here disjoint unions, the structure of the n.n.H. convenient-Lie groupoid $\mathcal{G} \rightrightarrows M$ is clearly defined by the collection of structure maps for each particular $\mathcal{G}_\lambda \rightrightarrows M_\lambda$ for $\lambda \in \mathbf{L}$.

For example, given a finite sequence of convenient bundles $\mathcal{A}_i \rightarrow M_i$, for $i \in \{1, \dots, n\}$, then we have the natural structure of a n.n.H. convenient-Lie groupoid $\bigsqcup_{i=1}^n \mathcal{GL}(\mathcal{A}_i) \rightrightarrows \bigsqcup_{i=1}^n M_i$.

More generally, given any convenient-Lie n.n.H. groupoid $\mathcal{G} \rightrightarrows M$, if $M = \bigsqcup_{\lambda \in \mathbf{L}} M_\lambda$ is a partition of M into open $\mathcal{G}$ -invariant submanifolds, then the corresponding groupoids $\mathcal{G}_\lambda \rightrightarrows M_\lambda$ obtained by restriction, are n.n.H. Banach-Lie groupoids and $\mathcal{G} \rightrightarrows M$ is the disjoint union of the family $\{\mathcal{G}_\lambda \rightrightarrows M_\lambda\}_{\lambda \in \mathbf{L}}$.

4.3.5. ACTION OF A CONVENIENT LIE GROUP. To each smooth action

$$\begin{aligned} A : G \times M &\longrightarrow M \\ (g, x) &\longmapsto g \cdot x \end{aligned}$$

of a convenient Lie group on a convenient manifold, is associated a convenient Lie groupoid $\mathcal{G} \rightrightarrows M$ defined in the following way:

- $\mathcal{G} := M \times G$;
- $\mathbf{s}(x, g) := x$ and $\mathbf{t}(x, g) := A(g, x) = g \cdot x$;
- if $y = g \cdot x$ then $\mathbf{m}((y, h), (x, g)) := (x, hg)$;
- $\mathbf{i}(x, g) := (g \cdot x, g^{-1})$;
- $\mathbf{1}_x = (x, \mathbf{1})$.

The smoothness of the action immediately implies the smoothness of all the structure maps defined above.

It is straightforward to verify that for any $x_0, y_0 \in M$ one has:

$$\mathcal{G}(x_0, y_0) = \{x_0\} \times \{g \in G : g \cdot x_0 = y_0\},$$

and, in particular, the isotropy group at x_0 is given by:

$$\mathcal{G}(x_0) = \{x_0\} \times G(x_0)$$

where $G(x_0) := \{g \in G : g \cdot x_0 = x_0\}$.

It is worth pointing out that the above groupoid needs not split. To obtain a concrete example, let $\mathcal{X}$ be any real Banach space with a closed linear subspace $\mathcal{X}_0$ which has no supplement in $\mathcal{X}$, e.g., the space of all bounded sequences of real numbers $\mathcal{X} = \ell^\infty(\mathbb{R})$ with its subspace $\mathcal{X}_0 = c_0$ consisting of the sequences which converge to 0 (cf. [8, Example 1.2]). Then the abelian Banach-Lie group $G = (\mathcal{X}, +)$ acts smoothly transitively on the Banach manifold $M = \mathcal{X} \setminus \mathcal{X}_0$ via $A(g, x + \mathcal{X}_0) := g + x + \mathcal{X}_0$ for all $g, x \in \mathcal{X}$. If we define the groupoid $\mathcal{G} := M \times G \rightrightarrows M$ as above, then the isotropy group at the point $\mathcal{X}_0 \in M$, that is,

$$\mathcal{G}(\mathcal{X}_0) = \{\mathcal{X}_0\} \times \mathcal{X}_0 \subseteq \mathcal{G}$$

is not a submanifold of $\mathcal{G} = M \times \mathcal{X}$ since $\mathcal{X}_0$ is not a supplemented subspace of $\mathcal{X}$.

4.3.6. THE GAUGE GROUPOID OF A BANACH PRINCIPAL BUNDLE. A Banach principal bundle is a locally trivial fibration $\pi : P \rightarrow M$ over a connected Banach manifold whose typical fibre is a Banach-Lie group G . We then have a right action of G on P whose corresponding quotient P/G is canonically diffeomorphic to M . We get a right diagonal action of G on $P \times P$ in an evident way. The *gauge groupoid* is the set of orbits of this action, that is, the quotient $\mathcal{G} := (P \times P)/G$ provided with the quotient topology. The source map (resp. the target map) of the equivalence class of (v, u) is $\pi(u)$ (resp. $\pi(v)$) (cf. [30]).

4.3.7. FUNDAMENTAL GROUPOID OF A BANACH MANIFOLD. Recall that to any connected topological space M one associates its corresponding *fundamental topological groupoid* $\Pi(M) \rightrightarrows M$ where $\Pi(M)$ is the set of homotopy classes of paths with fixed end points. The source map (resp. the target map) assigns to a class the initial (resp. terminal) point of a representative path, the multiplication is given by concatenation of paths whenever defined, and the inverse is given by reversing the direction of a path.

When M is a connected Banach manifold, each source fibre is a universal covering of M and, in particular, carries a natural Banach manifold structure. Moreover, it is a principal bundle over M with structural group $\pi_1(M)$. The associated gauge groupoid of this principal bundle can then be identified with $\Pi(M)$, yielding a structure of n.n.H. Banach-Lie groupoid on $\Pi(M)$.

4.4. ADDITIONAL PROPERTIES OF BANACH-LIE GROUPOIDS. We recall below some properties of a Banach-Lie groupoid (cf. [5, Theorem 3.3]).

THEOREM 4.4. *Let $\mathcal{G} \rightrightarrows M$ be a n.n.H. (pure) Banach-Lie groupoid and define $\mathbf{t}_x$ as the restriction of $\mathbf{t}$ to $\mathcal{G}(x, -)$.*

For any $x \in M$, the following assertions hold.

- (1) *For all $x \in M$ and any $y \in \mathcal{G} \cdot x$ the set $\mathcal{G}(x, y)$ is a closed submanifold of $\mathcal{G}$. In particular, the isotropy group $\mathcal{G}(x)$ is a Banach-Lie group and its Lie algebra is isomorphic to $\ker T_{1_x} \mathbf{s} \cap \ker T_{1_x} \mathbf{t}$.*
- (2) *For any $x \in M$ such that $\mathcal{G}(x, y)$ is a Banach submanifold of $\mathcal{G}$, for all $y \in \mathcal{G} \cdot x = \mathbf{t}_x(\mathcal{G}(x, -))$, then the orbit $\mathcal{G} \cdot x$ is a pure Banach manifold and its inclusion map $\mathcal{G} \cdot x \hookrightarrow M$ is smooth.*
- (3) *Under the assumption of (2), $\mathbf{t}_x : \mathcal{G}(x, -) \rightarrow \mathcal{G} \cdot x$ is a Banach principal $\mathcal{G}(x)$ -bundle.*

Remark 4.5. Note that the assumptions in (2) are satisfied for all $(x, y) \in M^2$ whenever $\mathcal{G} \rightrightarrows M$ is split.

Proof. (Sketch of the proof) To prove (1), we consider the distribution

$$\Delta_g = \ker T_g \mathbf{s} \cap \ker T_g \mathbf{t} \quad \text{for all } g \in \mathcal{G}(x, -),$$

and we get a Banach subbundle of $\Delta \subseteq T\mathcal{G}(x, -)$. This distribution is integrable. Thus, from¹⁹ [8, Theorem 8.1], there exists a partition of $\mathcal{G}(x, -)$ into

¹⁹This theorem was proved in the context of (Hausdorff) Banach manifolds but all the arguments used in the proof of this result are based on local charts; hence the result also holds for n.n.H. Banach manifolds.

closed immersed n.n.H. Banach manifolds, each of which is modelled on the Banach space $\Delta_{\mathbf{1}_x}$.

Moreover, since $\Delta_g = \ker T_g \mathbf{t}_x$, the maximal leaf through $g \in \mathcal{G}(x, -)$ is a connected component of $\mathbf{t}^{-1}(\mathbf{t}(g))$ and hence is a closed subset of the Banach manifold $\mathcal{G}(x, -)$. In particular, the isotropy group $\mathcal{G}(\mathbf{t}(g))$ is a union of such leaves and hence is a closed immersed n.n.H. Banach manifold. Since each point in $\mathcal{G}(\mathbf{t}(g))$ is closed, thus it follows from [7] that $\mathcal{G}(\mathbf{t}(g))$ is in fact Hausdorff. Consequently, $\mathcal{G}(\mathbf{t}(g))$ carries the structure of a Banach-Lie group.

For the proof of (2), we consider the map

$$\begin{aligned} \Psi: \mathcal{G}(x, -)/\mathcal{G}(x) &\longrightarrow \mathcal{G} \cdot x \\ g \cdot \mathcal{G}(x) &\longmapsto \mathbf{t}(g). \end{aligned}$$

It is easy to prove that Ψ is bijective. We can show that, under the assumption that $\mathcal{G}(x, y)$ is a Banach submanifold of $\mathcal{G}$, for all $y \in \mathcal{G} \cdot x$, the quotient set $\mathcal{G}(x, -)/\mathcal{G}(x)$ has the structure of a smooth manifold for which the map $\Psi: \mathcal{G}(x, -)/\mathcal{G}(x) \rightarrow M$ is smooth (cf. proof of [5, Theorem 3.3]).

The last point 3. is a consequence of [5, Lemma 2.5]. ■

Unfortunately, the properties of a Banach-Lie groupoid described in Theorem 4.4 do not hold in general for a convenient Lie groupoid, even when it is split. The essential obstruction is the absence, in the convenient setting, of a general existence theorem for local flows of vector fields. We will show that, under some strong assumptions on projective or direct limits of Banach-Lie groupoids, these properties can nevertheless be recovered.

4.5. LIE ALGEBROID ASSOCIATED WITH A CONVENIENT LIE GROUPOID.

Analogously to the Banach-Lie groupoid setting, any convenient Lie groupoid $\mathcal{G} \rightrightarrows M$ admits a natural convenient Lie algebroid structure over M . We now briefly recall its construction.

Denote by $T^s \mathcal{G}$ the vertical bundle associated with the submersion $\mathbf{s}: \mathcal{G} \rightarrow M$ and by $\Gamma_{\text{inv}}^r(\mathcal{G})$ the vector space of right invariant vector fields. The space $\Gamma_{\text{inv}}^r(\mathcal{G})$ is endowed with the Lie bracket of vector fields, thereby forming a Lie subalgebra of the Lie algebra of vertical vector fields. The tangent map $T\mathbf{t}$ induces a morphism of Lie algebras from $\Gamma_{\text{inv}}^r(\mathcal{G})$ to the Lie algebra $\Gamma(M)$ of vector fields on M . We consider the pull-back $\mathcal{AG} \rightarrow M$ of the convenient bundle $T^s \mathcal{G} \rightarrow \mathbf{1}(M)$ along the unit map $\mathbf{1}: M \rightarrow \mathcal{G}$. $\mathcal{AG}$ may be viewed as the restriction of $T^s \mathcal{G}$ to $\mathbf{1}(M)$ identified with $M \subset \mathcal{G}$.

Note that since M is Hausdorff, it follows that $\mathcal{AG}$ is a Hausdorff convenient bundle.

On the one hand, there is a canonical convenient vector bundle morphism $\mathbf{1}_* : \mathcal{AG} \rightarrow T^s\mathcal{G}$ from the bundle $\mathcal{AG} \rightarrow M$ to the bundle $T^s\mathcal{G} \rightarrow \mathbf{1}(M)$ which is an isomorphism in restriction to each fibre. On the other hand, if $\Gamma_M(T^s\mathcal{G})$ is the vector space of smooth sections of $T^s\mathcal{G} \rightarrow \mathbf{1}(M)$, then $T\mathbf{1}\mathfrak{t}$ gives rise to an isomorphism from $\Gamma_M(T^s\mathcal{G})$ onto $\Gamma_{\text{inv}}^r(\mathcal{G})$ which is also a morphism of Lie algebras. Now, the morphism $\mathbf{1}_*$ gives rise to a Lie bracket on the vector space $\Gamma(\mathcal{AG})$ of smooth sections of $\mathcal{AG} \rightarrow M$, which will be denoted by $[\cdot, \cdot]$. Finally, by composition, we obtain an isomorphism of Lie algebras between $\Gamma_M(T^s\mathcal{G})$ and $\Gamma_{\text{inv}}^r(\mathcal{G})$. Now, $\mathcal{AG}$ is provided with the anchor $\rho = T\mathfrak{t}|_M$. From the property of Lie bracket of vector fields on $\mathcal{G}$, it follows that the Lie bracket $[\cdot, \cdot]$ on $\mathcal{AG}$ is localizable by using the same arguments as in the proof of [5, Theorem 4.19]. Moreover, since this Lie bracket of kinematic vector fields only depends on the 1-jets of such vector fields, it follows that $(\mathcal{AG}, M, \rho, [\cdot, \cdot])$ is a convenient Lie algebroid (cf. [8, Definition 3.78]).

In a dual way, we can consider the vertical bundle $T^t\mathcal{G}$ associated with the submersion $\mathfrak{t}$ and the Lie algebra $\Gamma_{\text{inv}}^l(\mathcal{G})$ of left invariant vector fields. Then the tangent map $T\mathfrak{s}$ induces a morphism of Lie algebras from $\Gamma_{\text{inv}}^l(\mathcal{G})$ to the Lie algebra $\mathfrak{X}(M)$ of vector fields on M . As previously, we consider the restriction $\overline{\mathcal{AG}}$ of $T^s\mathcal{G}$ to $\mathbf{1}(M)$ identified with M . If $\rho' := T\mathfrak{s}|_M$ is an anchor for $\overline{\mathcal{AG}}$, the isomorphism between $\Gamma_1(\mathcal{G}^t)$ and $\Gamma_{\text{inv}}^l(\mathcal{G})$ provides the set of sections of $\overline{\mathcal{AG}}$ with a Lie bracket denoted $[\cdot, \cdot]$; thus we get a convenient Lie algebroid $(\overline{\mathcal{AG}}, M, \overline{\rho}, [\cdot, \cdot])$. Since $T\mathfrak{i}$ transforms a right invariant vector field into a left invariant vector field, this map induces an isomorphism of Lie algebroid from $(\mathcal{AG}, M, \rho, [\cdot, \cdot])$ to $(\overline{\mathcal{AG}}, M, \overline{\rho}, [\cdot, \cdot])$.

4.6. CONVENIENT PARTIAL POISSON GROUPOIDS. Consider a Banach-Lie groupoid $\mathcal{G} \rightrightarrows M$. The multiplication $\mathbf{m}$ is a map from $\mathcal{G}^{(2)}$ to $\mathcal{G}$. Since $\mathcal{G}^{(2)}$ is a closed submanifold of $\mathcal{G} \times \mathcal{G}$, the graph $G_{\mathbf{m}}$ of $\mathbf{m}$ is a closed submanifold of $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$.

DEFINITION 4.6. 1. A groupoid $\mathcal{G} \rightrightarrows M$ is called a *partial Banach Poisson-Lie groupoid* if it carries a partial Poisson structure associated with a bundle morphism $P : T^b\mathcal{G} \rightarrow T\mathcal{G}$ such that the graph $G_{\mathbf{m}}$ of the multiplication map is a coisotropic submanifold of the partial Poisson manifold $\mathcal{G} \times \mathcal{G} \times \mathcal{G}^-$.

2. A groupoid $\mathcal{G} \rightrightarrows M$ is called a *partial symplectic* (resp. *strong symplectic*) *Banach-Lie groupoid* if there exists an isomorphism P from $T^b\mathcal{G}$ to $T\mathcal{G}$ (resp. $T'\mathcal{G}$ to $T\mathcal{G}$) such that for the associated Poisson structure

$\mathcal{G}^+ \times \mathcal{G}^+ \times \mathcal{G}^-$ on $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$, the submanifold $G_{\mathbf{m}}$ is Lagrangian.

For a convenient partial Poisson-Lie groupoid, the following properties hold.

In fact, a partial symplectic Banach-Lie groupoid enjoys stronger properties than a general partial Poisson convenient Lie groupoid. For this reason, we shall state the corresponding results separately in a subsequent theorem.

THEOREM 4.7. *Let $\mathcal{G} \rightrightarrows M$ be a partial Poisson Banach-Lie groupoid. Then following assertions hold:*

1. *The submanifold M is a coisotropic submanifold of $\mathcal{G}$.*
2. *The inversion map on $\mathcal{G}$ is an anti-Poisson morphism.*
3. *There exists a unique Lie Poisson algebras sheaf on M for which the source map $\mathbf{s}$ (resp. the target map $\mathbf{t}$) is a Poisson morphism (resp. anti-Poisson morphism). Moreover if $T^b\mathcal{G} = T'\mathcal{G}$ or if $(T^b\mathcal{G}, T'\mathcal{G}, P)$ is partial symplectic, then there exists a unique partial Poisson structure on M such that $\mathbf{s}$ (resp. $\mathbf{t}$) is a Poisson morphism (resp. anti-Poisson morphism).*

Proof. The proof of this theorem is an adaptation to our setting of the proof of [41, Theorem 4.2.3].

1. By axiom **(TGr3)** of Definition 4.1, M is precisely the set:

$$\{x \in \mathcal{G} : \exists y \in \mathcal{G}, yx = y\}.$$

Let C be the diagonal $\Delta_{\mathcal{G}}$ of $\mathcal{G}^2$ and let $R : \mathcal{G}^2 \rightarrow \mathcal{G}$ be the relation whose graph is $\{(yx, y, x) : (y, x) \in \mathcal{G} \times \mathcal{G}\}$.

Then one has:

$$R(C) = \{x \in \mathcal{G} : \exists y \in \mathcal{G}, (y, y) \in \Delta_{\mathcal{G}} \text{ and } (y, y, x) \in R\}$$

which coincides with M by the preceding characterization.

On the one hand, the identity $\text{Id}_{\mathcal{G}}$ is a Poisson morphism. Therefore its graph, namely the diagonal $\Delta_{\mathcal{G}}$, is a coisotropic submanifold of $\mathcal{G}^- \times \mathcal{G}^+$.

On the other hand, since $G_{\mathbf{m}}$ is coisotropic in $\mathcal{G}^+ \times \mathcal{G}^+ \times \mathcal{G}^-$, it follows that R , whose graph is $G_{\mathbf{m}}^{-1}$, is also coisotropic in $\mathcal{G}^- \times \mathcal{G}^+ \times \mathcal{G}^+$. Hence, by Corollary 2.44, $R(C) = M$ is coisotropic in $\mathcal{G}^+$ provided that (R, C) is a clean pair.

According to Definition 2.39 and Definition 2.43, if $\Delta_{\mathcal{G}^2}$ denotes the diagonal of $\mathcal{G}^2$, we must show first that $(\Delta_{\mathcal{G}} \times R) \cap (\Delta_{\mathcal{G}^2} \times \mathcal{G})$ is a closed submanifold of $(\mathcal{G}^2) \times \mathcal{G}^2 \times \mathcal{G}$ whose tangent bundle is given by $(T\Delta_{\mathcal{G}} \times TR) \cap (T\Delta_{\mathcal{G}^2} \times T\mathcal{G})$.

But we have:

$$(\Delta_{\mathcal{G}} \times R) \cap (\Delta_{\mathcal{G}^2} \times \mathcal{G}) = \{(y, y, y, y, x) \in C \times R : yx = y\}$$

This set is the image of the embedding of $\mathcal{G}$ in $\mathcal{G}^2 \times \mathcal{G}^2 \times \mathcal{G}$ given by:

$$\mathbf{j} : y \longmapsto (y, y, y, y, \mathbf{s}(y)).$$

Now the tangent space of this submanifold at a point (y, y, y, y, x) is the range $T_y \mathbf{j}$. To show that this space is exactly

$$(T_{(y,y,y,y,x)} \Delta_{\mathcal{G}} \times TR) \cap (T_{(y,y,y,y,x)} \Delta_{\mathcal{G}^2} \times T\mathcal{G})$$

it suffices to prove that $T_{(y,y,x)} R \cap (T(y, y) \Delta_{\mathcal{G}} \times T_x \mathcal{G})$ is of the form:

$$\{(u, u, v) : u \in T_y \mathcal{G}, v = T_y \mathbf{s}(u)\}.$$

First, any vector in $T_{(y,y,x)} R \cap (T(y, y) \Delta_{\mathcal{G}} \times T_x \mathcal{G})$ is of type (u, u, v) for $u \in T_y \mathcal{G}$ and $v \in T_x \mathcal{G}$. Let $x : t \mapsto x(t)$ and $y : t \mapsto y(t)$ be two smooth curves in $\mathcal{G}$, defined on $] - \varepsilon, \varepsilon[$ such that $x(0) = x, y(0) = y, \dot{x}(0) = v, \dot{y}(0) = u$ and such that the derivative of the product satisfies:

$$\frac{d}{dt} (y(t)x(t))|_{t=0} = u.$$

Since $y(0)x(0) = y(0)$, this implies that $x(0)$ belongs to M . We can write

$$x(t) = \mathbf{t}(\mathbf{L}_{y(t)^{-1}}(yx(t))).$$

Hence we obtain $v = \dot{x}(0) = T_y \mathbf{t}(u)$.

Since $\frac{d}{dt}(y(t))_{t=0} = \frac{d}{dt}(y(t)x(t))_{t=0}$, the last condition (2) (ii) of Definition 2.39 is clear. This completes the proof of 1.

2. Once again, the graph of the inversion is of the form $R(C)$ where $R : \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$ is the relation whose graph is $\{(gh, g, h), : g, h \in \mathcal{G}\}$ and $C = M$, since

$$R(C) = \{(g, h) \in \mathcal{G} \times \mathcal{G} : \exists x \in M, (x, g, h) \in R\}.$$

Now, M is coisotropic in $\mathcal{G}^+$ and R is coisotropic in $\mathcal{G}^+ \times \mathcal{G}^- \times \mathcal{G}^-$. Thus, if the pair (R, C) is clean, then $R(C)$ is coisotropic in $\mathcal{G}^- \times \mathcal{G}^-$. This is equivalent to

saying that $R(C)$, i.e. the graph of $\mathbf{i}$, is coisotropic in $\mathcal{G}^+ \times \mathcal{G}^+$. Consequently, $\mathbf{i}$ is an anti-Poisson morphism.

The intersection $(M \times R) \cap (\Delta_{\mathcal{G}} \times \mathcal{G}^2)$ is the set

$$\{(x, x, g, h) \in M \times R\}.$$

This is precisely the image of the embedding

$$\begin{aligned} \mathbf{j} : M \times \mathcal{G}^2 &\longrightarrow \mathcal{G}^2 \times \mathcal{G}^2 \\ (x, g, h) &\longmapsto (x, x, g, h). \end{aligned}$$

The same type of argument used in the proof of 1. permits to show that

$$T(M \times R) \cap (\Delta_{\mathcal{G}} \times \mathcal{G}^2) = T(M \times R) \cap T(\Delta_{\mathcal{G}} \times \mathcal{G}^2)$$

and the verification left to the reader. Finally, condition (2) (ii) of Definition 2.39 is satisfied, and thus the proof 2. is complete.

3. We have $\mathbf{s}(g) = \mathbf{s}(g')$ if and only if there exists $h \in \mathcal{G}$ such that $g' = hg$; for instance, one may take $h = g'g^{-1}$.

We interpret the graph of $\mathbf{s} \circ \mathbf{s}^{-1}$ as $R(C)$, where $R : \mathcal{G} \times \mathcal{G} \rightrightarrows \mathcal{G}$ is the relation whose graph is

$$\{(h, g, hg) : h, g \in \mathcal{G}\},$$

and $C = \mathcal{G}$. Indeed, we have

$$\mathbf{s} \circ \mathbf{s}^{-1} = \{(g, g') \in \mathcal{G}^2 : \exists h \in \mathcal{G}, g' = gh\}.$$

Since M is coisotropic in $\mathcal{G}^+$ and R is coisotropic in $\mathcal{G}^+ \times \mathcal{G}^+ \times \mathcal{G}^-$, if the pair (R, C) is clean, then $R(C)$ is coisotropic in $\mathcal{G}^+ \times \mathcal{G}^-$. Hence $\mathbf{s} \circ \mathbf{s}^{-1}$ is a Poisson relation. The result follows from Theorem 2.55. The proof of the cleanliness of the pair (R, C) is elementary and is left to the reader.

Since $\mathbf{t} = \mathbf{s} \circ \mathbf{i}$ and $\mathbf{i}$ is an anti-Poisson morphism, it follows that $\mathbf{t}$ is an anti-Poisson morphism. ■

Remark 4.8. In the context of Theorem 4.7, assume that $T^b\mathcal{G} = T'\mathcal{G}$ or that $(T^b\mathcal{G}, T\mathcal{G}, P)$ is partial symplectic. Then there exists a unique partial Poisson structure on M such $\mathbf{s}$ (resp. $\mathbf{t}$) is a Poisson morphism (resp. anti-Poisson morphism) is such that the Lie-Poisson algebras sheaf on M built in 1. is exactly the Poisson Lie algebras sheaf associated with this partial Poisson structure on M_2 . However, even if the partial Poisson structure on $\mathcal{G}$ is symplectic the unique induced Poisson structure on M is not, in general symplectic, even in finite dimension (cf. [1]).

For a partial symplectic Lie convenient groupoid, the following properties hold:

THEOREM 4.9. *Let $\mathcal{G} \rightrightarrows M$ be a partial symplectic convenient Lie groupoid. Then we have:*

1. *The submanifold M is Lagrangian in $\mathcal{G}$.*
2. *The inversion map in $\mathcal{G}$ is an anti-symplectic map.*
3. *There exists a unique strong Poisson structure on M such that $\mathbf{s}$ (resp. $\mathbf{t}$) is a Poisson morphism (resp. anti-Poisson morphism).*
4. *There exists a weak symplectic 2-form ω such that $\omega^\flat : T\mathcal{G} \rightarrow T'\mathcal{G}$ has image $T^\flat M$ and satisfies $\omega^\flat|_{T^\flat M} = P$.*
5. *For each $g \in \mathcal{G}$, the vertical spaces $T_g\mathcal{G}(\mathbf{s}(g), -)$ and $T_g\mathcal{G}(-, \mathbf{t}(g))$ are symplectic orthogonal subspaces of the symplectic vector space $(T_g\mathcal{G}, \omega)$.*
6. *Moreover, if $\mathcal{G} \rightrightarrows M$ is a split Banach-Lie groupoid, ω induces a weak symplectic form on each orbit of $\mathcal{G}$ in M .*

Following [5, 6.2], a convenient Lie groupoid $\mathcal{G} \rightrightarrows M$ is called *transitive* if the map $(\mathbf{s}, \mathbf{t}) : \mathcal{G} \rightarrow M \times M$ is a surjective submersion. In this case, the action of $\mathcal{G}$ on M has a single orbit, namely M itself.

Thus, by application of Theorem 4.9, we obtain:

COROLLARY 4.10. *Let $\mathcal{G} \rightrightarrows M$ be a transitive partial symplectic Banach-Lie groupoid. Then, via the identification of M with the unique orbit of $\mathcal{G}$, the weak symplectic 2-form ω on $\mathcal{G}$ induces a weak symplectic structure on M .*

Remark 4.11. Consider a partial symplectic Banach-Lie groupoid $\mathcal{G} \rightrightarrows M$. The 2-form induced by ω on $T_g\mathcal{G}(\mathbf{s}(g), -)$ (resp. $T_g\mathcal{G}(-, \mathbf{t}(g))$) is not symplectic since the intersection of these two tangent spaces is isomorphic to the Lie algebra of the isotropic group of $\mathbf{s}$ under the assumptions of Theorem 4.4. In this context, this is true if and only if the isotropy group of $\mathbf{s}(g)$ is reduced to the identity. Moreover, the partial Poisson structure on M is not, in general, weakly symplectic (see item 6.).

Combining items 1. and 3. of Theorem 4.9, Proposition 2.59 and Remark 2.60, we obtain:

COROLLARY 4.12. *Let $\mathcal{G} \rightrightarrows M$ be a partial symplectic Banach-Lie groupoid. For any open set U in M , let $(\mathcal{E}_M(U), \{ , \}_U)$ be the Poisson Lie algebra on U associated with the partial Poisson structure on M and $\{ , \}_P$ the Poisson bracket on $\mathcal{G}$ associated with the partial symplectic structure on $\mathcal{G}$. Then any pair (f_1, f_2) in $\mathcal{E}_M(U)$ is in involution, i.e. $\{f_1, f_2\}_U = 0$.*

Proof of Theorem 4.9. The proof of 1. is analogous to that of Theorem 4.7. 1., under the additional assumption that $G_{\mathbf{m}}$ is Lagrangian. Since, in this case, $\Delta_{\mathcal{G}}$ is also Lagrangian in $\mathcal{G}^- \times \mathcal{G}^+$, we invoke the second part of Corollary 2.44. In fact, in this case, the Lie-Poisson algebras sheaf on M_2 is exactly the sheaf associated with the partial Poisson structure on M (cf. Remark 4.8).

2. and 3. are exactly the result of 2. and 3. of Theorem 4.7.

4. is a consequence of Theorem 2.23 applied to the characteristic foliation whose leaves are the connected components of $\mathcal{G}$.

For 5., we will adapt the proof of [29, Theorem 3.6]. According to 4., we have a weak symplectic form ω on $\mathcal{G}$ such that: $\omega(w, w') = \langle P^{-1}(w), w' \rangle$ for any w, w' in $T_g \mathcal{G}$ and any $g \in \mathcal{G}$. In this way, on $\mathcal{G}^+ \times \mathcal{G}^+ \times \mathcal{G}^-$, the partial symplectic structure is an isomorphism P^3 from $(T^b \mathcal{G})^3$ to $(T\mathcal{G})^3$ given by:

$$P^3(w_1, w_2, w_3) = (P(w_1), P(w_2), -P(w_3)).$$

Hence, we obtain a weak symplectic form ω^3 on $(\mathcal{G})^3$ associated with P^3 given by:

$$\omega^3((w_1, w_2, w_3), (w'_1, w'_2, w'_3)) = \omega(w_1, w'_1) + \omega(w_2, w'_2) - \omega(w_3, w'_3). \quad (4.1)$$

For each $g \in \mathcal{G}$, we set $A_g = \mathcal{G}(\mathbf{s}(g), -)$ and $B_g = \mathcal{G}(-, \mathbf{t}(g))$ the fibres of $\mathbf{s}$ and $\mathbf{t}$ respectively.

Fix $k \in \mathcal{G}$. For any $(g, h) \in \mathcal{G}^{(2)}$, we set $k = gh$. Then we have $\mathbf{R}_h(A_h) = A_k$ and $\mathbf{L}_g(B_g) = B_k$. For $u \in T_g A_g$ and $v \in T_h A_h$, if w_1 and w_2 belong to $T_k \mathcal{G}$, the vectors $(0, u, w_1)$ and $(0, v, w_2)$ belongs to $T_{(g,h,k)} \mathcal{G}^3$ and are tangent to the graph $G_{\mathbf{m}}$ of $\mathbf{m}$ if and only if

$$w_1 = T\mathbf{R}_h(u) \quad \text{and} \quad w_2 = T\mathbf{L}_g(v). \quad (4.2)$$

By assumption, $G_{\mathbf{m}}$ is Lagrangian in $\mathcal{G}^+ \times \mathcal{G}^+ \times \mathcal{G}^-$. Thus, according to Proposition 2.6, the definition of a Lagrangian manifold, the relations (4.1) and (4.2), we must have:

$$\omega(u, 0) + \omega(0, v) - \omega(T\mathbf{R}_h(u), T\mathbf{L}_g(v)) = 0,$$

for all $u \in T_g A_g$ and $v \in T_h B_h$. Since $T\mathbf{R}_h(T_h A_h) = T_k A_k$ and $T\mathbf{L}_g(T_g B_g) = T_k B_k$, it follows that

$$T_k B_k = (T_k A_k)^{\perp_P} \quad \text{and} \quad T_k A_k = (T_k B_k)^{\perp_P}$$

according to Proposition 2.6 which completes the proof of 5.

6. Moreover, if $\mathcal{G} \rightrightarrows M$ is split, for each $x \in M$, from Theorem 4.4, $\mathcal{G}(x, -)$ is a closed split submanifold of $\mathcal{G}$. For any $g \in \mathcal{G}(x, -)$, let H_g be a supplement of $T_g \mathcal{G}(x, -) \cap T_g \mathcal{G}(-, x)$ in $T_g \mathcal{G}(-, x)$. Then from 5. of Theorem 4.9 and point (ii) of Proposition 2.6, ω induces on H_g a symplectic form. Since we have a canonical isomorphism from $T\mathcal{G}(x, -)/T\mathcal{G}(x)$ to $T\mathcal{G}.x$ (cf. Proof of 3. of [5, Theorem 3.3]), by using the same arguments as in the proof of [35, Proposition 2.4.5], we obtain a symplectic form $\tilde{\omega}$ on $T\mathcal{G}.x$. This means that the set of connected components of the orbit of $\mathcal{G}$ is exactly the characteristic foliation of the partial Poisson structure on M . Since this partial Poisson structure is strong, it follows that the almost symplectic form on each orbit is closed (cf. Theorem 2.23). ■

5. PARTIAL CONORMAL BUNDLES AND CONVENIENT LIE ALGEBROIDS

In this section, we study partial conormal bundles in the Banach and convenient Poisson settings. We first establish their basic properties for coisotropic submanifolds and derive the associated partial dual Lie algebroids. We then prove that if the partial conormal bundle of the unit manifold is a split Banach subbundle, the Lie algebroid of a Banach Lie-Poisson groupoid and its partial dual form a partial bialgebroid.

5.1. THE PARTIAL CONORMAL BUNDLE OF A COISOTROPIC MANIFOLD.

Let $(T^{\flat}M, TM, P, \{ , \}_P)$ be a partial Poisson manifold. For any closed submanifold C of M , we denote by $T'_C M$ (resp. $T^{\flat}_C M$) the pullback of $T'M$ (resp. $T^{\flat}M$) along the inclusion $C \hookrightarrow M$.

DEFINITION 5.1. (1) The *conormal bundle* $\mathcal{N}^a(C)$ is the subbundle of $T'_C M$ defined by:

$$\mathcal{N}^a(C) = \{(x, \eta) \in T'_C M : \forall u \in T_x C, \langle \eta, u \rangle = 0\}.$$

(2) The *partial conormal bundle* $\mathcal{N}^0(C)$ of $T^{\flat}_C M$ is defined by:

$$\mathcal{N}^0(C) = \{(x, \eta) \in T^{\flat}_C M : \forall u \in T_x C, \langle \eta, u \rangle = 0\}.$$

Remark 5.2. It is clear that $\mathcal{N}^a(C)$ is a subbundle of $T'_C M$, whereas, in general, $\mathcal{N}^0(C)$ is not a vector bundle over C .

THEOREM 5.3. *Let C be a coisotropic submanifold of M .*

- (1) *The partial conormal $\mathcal{N}^0(C)$ is exactly $(TC)^\perp$. If $T^\flat M = T' M$ or if P is partial symplectic and C is Lagrangian, then $\mathcal{N}^0(C)$ is a subbundle of $T^\flat M$.*
- (2) *If $\mathcal{N}^0(C)$ is a closed sub-bundle of $T^\flat_C M$, it can be provided with a Lie algebroid structure whose anchor is the restriction of P to $\mathcal{N}^0(C)$.*

Proof. (1) From the definition 2.4, we have $(T_x M)^\perp_{P_x} = P_x((T_x M)^0)$ and hence $\mathcal{N}^0(M) = (TM)^\perp$. First, assume that $T^\flat M = T' M$. Then $\mathcal{N}^a(C) = \mathcal{N}^0(C)$ and hence it is a subbundle of $T' M$. Assume now that P is partial symplectic. Since C is Lagrangian, for each $x \in C$, we have $P_x(\mathcal{N}^0_x(C)) = T_x C$. Thus $\mathcal{N}^0(C) = P^{-1}(TC)$ and since P is an isomorphism, it follows that $\mathcal{N}^0(C)$ is a subbundle of $T^\flat_C M$.

(2) Let P_C be the restriction of P to $\mathcal{N}^0(C)$. Since C is coisotropic, we have already seen that $P(\mathcal{N}^0(C)) \subset TC$ and hence P_C takes values in TC . We first show that the bracket defined in (2.9), when restricted to sections of $\mathcal{N}^0(C)$, takes values in $\mathcal{N}^0(C)$. Fix some open set U in M . Using the identity $L_X \omega = d\omega(X, \cdot) + d(\omega(X))$ and the definition of $[\cdot, \cdot]_P$, for any α and β in $\Gamma(T^\flat M_U)$, we obtain:

$$[\alpha, \beta] = d\beta(P\alpha, \cdot) - d\alpha(P\beta, \cdot) - d\beta(P\alpha, \cdot) - d\langle \alpha, P\beta \rangle.$$

It follows that, for $X \in T(C \cap U)$ and for any section α and β of $\mathcal{N}^0(C)$ over $C \cap U$, we have $\langle \alpha, P\beta \rangle \equiv 0$ on $C \cap U$. Hence, we obtain $[\alpha, \beta](X) = d\beta(P\alpha, X) - d\alpha(P\beta, X)$. But since $P\alpha$ and X are tangent to C , their Lie bracket $[P\alpha, X]$ is also tangent to C . It follows that $d\beta(P\alpha, X) = 0$. By the same argument, we also obtain $d\alpha(P\beta, X) = 0$. Thus $[\alpha, \beta]$ belongs to $\mathcal{N}^0(C)$. Since the Jacobiator of $[\cdot, \cdot]_P$ vanishes on sections of $T^\flat M$ and since the restriction of $[\cdot, \cdot]_P$ to sections of $\mathcal{N}^0(C)$ takes values in $\mathcal{N}^0(C)$, it follows that the Jacobiator also vanishes on sections of $\mathcal{N}^0(C)$. Finally the Leibniz rule on $\Gamma(T^\flat M_U)$ implies the Leibniz rule upon restriction to $\Gamma(\mathcal{N}^0(C \cap U))$. ■

5.2. PARTIAL DUAL LIE ALGEBROIDS. Let $(\mathcal{AG}, M, \rho, [\cdot, \cdot, \cdot])$ be the Lie algebroid associated with a groupoid $\mathcal{G} \rightrightarrows M$. We denote by $\mathcal{AG}' \rightarrow M$ the dual bundle of $\mathcal{AG} \rightarrow M$. We have the following properties:

THEOREM 5.4. *Assume that $\mathcal{G} \rightrightarrows M$ is a convenient Lie-Poisson groupoid such that the partial conormal bundle $\mathcal{N}^0(M)$ is a split convenient subbundle of $T_M^\flat \mathcal{G}$. Then*

- (1) *If $T^\flat \mathcal{G} = T' \mathcal{G}$, the above assumption is always satisfied and $\mathcal{N}^0(M) = \mathcal{N}^a(M)$ is canonically isomorphic to the dual $\mathcal{AG}' \rightarrow M$ of the Lie algebroid $\mathcal{AG} \rightarrow M$.*
- (2) *The inclusion of $\mathcal{N}^0(M)$ into the conormal bundle $\mathcal{N}^a(M)$ is a morphism of vector bundles.*
- (3) *$\mathcal{N}^0(M)$ admits a Lie algebroid structure whose anchor is the restriction of P to $\mathcal{N}^0(M)$.*

COROLLARY 5.5. *If a Banach Lie groupoid $\mathcal{G} \rightrightarrows M$ satisfies one of the following assumptions:*

- (i) *$\mathcal{G} \rightrightarrows M$ is a Lie-Poisson groupoid such that $T^\flat \mathcal{G} = T' \mathcal{G}$;*
- (ii) *The kernel of P restricted to M is a split subbundle of $T_M^\flat \mathcal{G}$ and its range is $T_M \mathcal{G}$;*
- (iii) *$\mathcal{G} \rightrightarrows M$ is a partial symplectic Lie groupoid;*

then $\mathcal{N}^0(M)$ is a split subbundle of $T^\flat \mathcal{G}$.

If we identify $\mathcal{AG}'$ with $\mathcal{N}^a(M)$, then $\mathcal{N}^0(M)$ is a weak subbundle of $\mathcal{AG}^\flat$ and carries a structure of convenient Lie algebroid.

Remark 5.6. According to the decomposition (5.1), if $\mathcal{G}$ is a Hilbert groupoid, or if TM is finite dimensional or finite codimensional, then the assumptions of Theorem 5.4 are also satisfied.

Under the assumptions of Theorem 5.4, if we identify $\mathcal{AG}'$ with $\mathcal{N}^a(M)$, then $\mathcal{N}^0(M)$ is a weak subbundle of $\mathcal{AG}^\flat$ and carries a structure of Banach Lie algebroid.

We therefore introduce in the Banach framework:

DEFINITION 5.7. Assume that $\mathcal{G} \rightrightarrows M$ is a Banach Lie-Poisson groupoid such that the partial conormal bundle $\mathcal{N}^0(M)$ is a split Banach subbundle of $T_M^\flat \mathcal{G}$. Then $\mathcal{N}^0(M)$ is called *the partial dual* of $\mathcal{AG}$ and is denoted $\mathcal{AG}^\flat$.

Proof of Theorem 5.4. (1) Assume that $T^\flat \mathcal{G} = T' \mathcal{G}$. Then $\mathcal{N}^0(M) = \mathcal{N}^a(M)$. First, since $\mathbf{s} : \mathcal{G} \rightarrow M$ is a submersion, as M is viewed as a submanifold of $\mathcal{G}$, it follows that

$$T_M \mathcal{G} = T_M^\mathbf{s} \mathcal{G} \oplus TM = \mathcal{AG} \oplus TM. \quad (5.1)$$

If $p_M^s : T_M\mathcal{G} \rightarrow T\mathcal{G}^s$ is the associated projection, we obtain an isomorphism $\widehat{p_M^s} : T\mathcal{G}^s \rightarrow T_M\mathcal{G}/TM$. It follows that $T'_M\mathcal{G}$ is isomorphic to $T'_M\mathcal{G}^s \oplus T'M$. Thus $\mathcal{N}^a(M)$ can be identified with $T'_M\mathcal{G}^s$. It is a split subbundle of $T^b\mathcal{G}$. Since $\mathcal{AG} \equiv T'_M\mathcal{G}^s$, $\mathcal{N}^0(M)$ can be identified with $\mathcal{AG}'$.

(2) For simplicity, we assume that M is connected and work in the setting of the previous constructions.

Let $\mathbb{G}^s$ the typical fibre of $T_M\mathcal{G}^s$ and let $\mathbb{M}$ be the typical model of M . Then, for each $x \in M$, there exists a chart (U, ψ) of x in $\mathcal{G}$ such that $\psi(U) = \bar{U}_1 \times \bar{U}_2 \subset \mathbb{M} \times \mathbb{G}^s$. In this way, we obtain a trivialization:

$$T'\psi^{-1} : T'\mathcal{G}_U \longrightarrow (\bar{U}_1 \times \bar{U}_2) \times (\mathbb{M}' \oplus (\mathbb{G}^s)').$$

In this chart, we have $(T'\psi)^{-1}(\mathcal{N}^a(M)|_U) = \bar{U}_1 \times \bar{U}_2 \times (\mathbb{G}^s)'$. Without loss of generality, we may assume that $\bar{U}_1$ and $\bar{U}_2$ are simply connected. Now, since $\mathcal{N}^0(M)$ is a closed subbundle of $T^b\mathcal{G}$, there exists a local trivialization Ψ of $T^b\mathcal{G}_U$ such that $\Psi(T^b\mathcal{G}_U) = \bar{U}_1 \times \bar{U}_2 \times \mathbb{G}^b$ where $\mathbb{G}^b$ is the typical fibre of $T^b\mathcal{G}_U$. On the other hand, since $\mathcal{N}^0(M)$ is a closed subbundle of $T_M^b\mathcal{G}$, if we identify $\mathbb{G}^b$ with $T_x^b\mathcal{G}$, we have a decomposition $\mathbb{G}^b = \mathbb{N}^b \oplus \mathbb{H}$ where $\mathbb{N}^b$ is the typical fibre of $\mathcal{N}^0(M)$. Thus, we may assume that $\Psi(T^b\mathcal{G}_U) = (\bar{U}_1 \times \bar{U}_2) \times (\mathbb{N}^b \oplus \mathbb{H})$. Since we have $\mathcal{N}^0(M) = \mathcal{N}^a(M) \cap T'_M\mathcal{G}$, it follows that $\mathbb{N}^b = (\mathbb{G}^s)' \cap \mathbb{G}^b$. Thus, in these local coordinates associated with the chosen trivializations, the inclusion of $\mathcal{N}^0(M)|_U$ into $\mathcal{N}^a(M)|_U$ is given by:

$$(y, \eta, 0) \mapsto (y, \eta, 0).$$

Now consider another chart (U', ψ') around x and a corresponding trivialization $\Psi'(T^b\mathcal{G}_U) = \bar{U}'_1 \times \bar{U}'_2 \times \mathbb{G}^b$ with the same properties and decompositions. Since $\mathcal{N}^0(M)$ (resp. $\mathcal{N}^a(M)$) is a split subbundle of $T^b\mathcal{G}$ (resp. $T'\mathcal{G}$), the transition maps $\Psi' \circ \Psi^{-1}$ (resp. $(T'\psi')^{-1} \circ (T'\psi)$) preserve the vector space $(\mathbb{G}^s)' \cap \mathbb{G}^b$ (resp. $(\mathbb{G}^s)'$). This completes the proof of (2).

(3) Is a direct application of Theorem 5.3 (2). ■

Proof of Corollary 5.5. This corollary is a direct consequence of Theorem 4.9 (1), Theorem 5.3 (1) and Theorem 5.4 provided that under assumption (i), (ii) or (iii), $\mathcal{N}^0(M)$ is a split subbundle of $T_M^b\mathcal{G}$.

(i) Note that according to the proof of Theorem 5.4 (1), under assumption (i), $\mathcal{N}^0(M) = \mathcal{N}^a(M)$ and hence it is a closed split subbundle of $T_M^b\mathcal{G} = T'_M\mathcal{G}$.

(ii) Under assumption (ii), we denote by K^bM the kernel of P restricted to $T_M^b\mathcal{G}_M$ and H^bM is supplementary of K^bM in $T_M^b\mathcal{G}$. We denote by P_H the restriction of P to H^bM . Since P is surjective, it follows that $P_H : H^bM \rightarrow$

$T_M\mathcal{G}$ is an isomorphism. Taking into account the decomposition (5.1), we set $H^0M = P_H^{-1}(TM)$ and $K^0M = P_H^{-1}(\mathcal{AG})$. By construction, for any $\alpha \in K^0M$ (resp. $\alpha \in H^0M$), we have $\langle \alpha, u \rangle = 0$, for all $u \in TM$ (resp. $u \in \mathcal{AG}$). Note that $\mathcal{N}^0(M) = K^bM \oplus H^0M$ and so $\mathcal{N}^0(M)$ is a split subbundle of $T^b\mathcal{G}$.

(iii) Under assumption (iii), M is a particular case of the context of assumption (ii). ■

We are now in a position to prove the following result which generalizes a result in finite dimension for Poisson Lie groupoids (cf. [27]):

THEOREM 5.8. *Let $\mathcal{G} \rightrightarrows M$ be a Banach Lie-Poisson groupoid such that partial conormal $\mathcal{N}^0(M)$ is a split Banach subbundle of $T'_M\mathcal{G}$. Then $(\mathcal{AG}, \mathcal{AG}^b)$ is a partial bialgebroid.*

Proof. Since $T^b\mathcal{G}$ carries a partial Poisson structure, Proposition 2.30 implies that, for any open set $U \subset \mathcal{G}$, we have:

$$d_P[X, Y] = L_X d_P Y - L_Y d_P X \quad (5.2)$$

for all vector fields X and Y on U viewed as 1-forms on $T^b\mathcal{G}$.

On the other hand, we have seen in the proof of Theorem 5.3, 2., for each open set U in $\mathcal{G}$ that the restriction of $[\cdot, \cdot]_P$ to $\mathcal{AG}_{U \cap M}^b$ takes values in $\mathcal{AG}_{U \cap M}^b$. Moreover, by (2.16) and Proposition 2.30 (1), if X is a section of $\mathcal{AG}$ restricted to $\mathcal{AG}_{U \cap M}^b$ is a 2-form on $\mathcal{AG}_{U \cap M}^b$. Finally, by (2.18) and Proposition 2.30 (2), it follows that the restriction of each member of (5.2) to $\mathcal{AG}_{U \cap M}^b$ induces a 2-form on $\mathcal{AG}_{U \cap M}^b$ which satisfies relation (5.2). This completes the proof according to the definition of a partial bialgebroid. ■

Remark 5.9. All the previous results are not true in general in the convenient setting. The essential reason is that even if $\mathbb{F}$ is a split convenient subspace of a convenient space $\mathbb{E}$ and $\mathbb{E} = \mathbb{F} \oplus \mathbb{G}$, then the canonical projection of $\mathbb{G}$ onto $\mathbb{E}/\mathbb{F}$ is not, in general, a convenient isomorphism.

In particular, $\mathbb{F}^*$ is not isomorphic to the annihilator of $\mathbb{G}$ and therefore $\mathcal{N}^a(M)$ cannot be identified with $\mathcal{AG}'$.

Consequently the whole proof of Theorem 5.4 fails in this setting. However, in the convenient setting, if $T^b\mathcal{G} = T'\mathcal{G}$ and $P = T^b\mathcal{G} \rightarrow T\mathcal{G}$ is a convenient bundle isomorphism, i.e. it is a strong symplectic convenient groupoid, the typical fibre $\mathbb{G}$ of $\mathcal{G}$ is reflexive. In particular, $\mathcal{AG}$ is isomorphic to $\mathcal{AG}'$. In fact, we also can show that $(\mathcal{AG}, \mathcal{AG}')$ is a partial bialgebroid, with a proof formally identical to that of Theorem 5.8. Note that we will see that Theorem 5.4 remains valid for direct limit of finite dimensional Poisson groupoids.

6. CONVENIENT SUB-POISSON GROUPOIDS

This section is devoted to the study of the cotangent bundle of convenient Lie groupoids and Lie groups. We prove that it carries a natural groupoid structure and a canonical partial symplectic groupoid structure, yielding an associated partial Lie bialgebroid. We also establish a characterization of sub-Poisson groupoids in terms of partial Poisson structures. Finally, we analyze in detail the cotangent bundle of a regular convenient Lie group and the induced Poisson geometry on the dual of its Lie algebra.

6.1. PARTIAL SYMPLECTIC POISSON STRUCTURE ON THE COTANGENT BUNDLE. Let $\mathcal{G} \rightrightarrows M$ be a convenient Lie groupoid. As in finite dimension, the conormal bundle $\mathcal{N}^a(M)$ is the subbundle of $T'_M \mathcal{G}$ defined by:

$$\mathcal{N}^a(M) = \{(x, \eta) \in T'_M \mathcal{G} : \forall u \in T_x M, \langle \eta, u \rangle = 0\}.$$

It is clearly a closed subbundle of $T'M$. From the decomposition (5.1), we have:

$$T'_M \mathcal{G} = \mathcal{N}^a(M) \oplus (\mathcal{A}\mathcal{G})^a \quad (6.1)$$

where $(\mathcal{A}\mathcal{G})^a$ denotes the annihilator of $\mathcal{A}\mathcal{G}$. It follows that $\mathcal{N}^a(M)$ is canonically isomorphic to the dual bundle of $\mathcal{A}\mathcal{G}$. Therefore, $\mathcal{N}^a(M)$ will be regarded as the dual bundle of $\mathcal{A}\mathcal{G}$ and will be denoted by $\mathcal{A}\mathcal{G}'$. Note that the typical fibre of $\mathcal{A}\mathcal{G}'$ is isomorphic to $\mathbb{M}^a$ and, therefore, to $\mathbb{G}'$. Likewise, the typical fibre of $\mathcal{A}\mathcal{G}^a$ is isomorphic to $\mathbb{G}^a$ and hence to $\mathbb{M}'$. As in finite dimension, we have:

THEOREM 6.1. *Let $\mathcal{G} \rightrightarrows M$ be a convenient Lie groupoid. Then we have*

- (1) $T'\mathcal{G} \rightrightarrows \mathcal{A}\mathcal{G}'$ admits a groupoid structure.
- (2) $T'\mathcal{G} \rightrightarrows \mathcal{A}\mathcal{G}'$ carries a canonical partial symplectic groupoid structure. Consequently, the pair $(\mathcal{A}\mathcal{G}, \mathcal{A}\mathcal{G}')$ forms a partial Lie bialgebroid.

Proof. (1) For the proof, we shall use local charts in this context. For simplicity, we assume that M is connected. Let $\mathbb{G}$ be the typical fibre of $T_M^s \mathcal{G}$ and $\mathbb{M}$ the typical model of M . Note that since $\mathbf{s}_{\mathcal{B}}$ and $\mathbf{t}_{\mathcal{B}}$ are diffeomorphisms on U_1 and V_1 respectively on their images, it follows that $F_{\mathcal{B}} := \mathbf{t}_{\mathcal{B}} \circ \mathbf{s}_{\mathcal{B}}^{-1}$ is a diffeomorphism from U_1 onto V_1 . Therefore $\mathbb{G}$ is also the typical fibre of $T_M^t \mathcal{G}$. Fix some $y_0 \in M$. There exists a chart (V, ψ) of $\mathcal{G}$ at y_0 with simply connected domain, such that $\psi(V) = \bar{V}_1 \times \bar{V}_2 \subset \mathbb{M} \times \mathbb{G}$ and $\psi(y_0) = (0, 0)$. In

this way, ψ can be written as a pair (ψ_1, ψ_2) . For $i \in \{1, 2\}$, we set $\psi_i(y) = \bar{y}_i$; thus, $\bar{y}_i : V \rightarrow \bar{V}_i$ is a smooth map.

If $T^*\psi$ denotes the adjoint of $T\psi$, we obtain a trivialization:

$$(T'\psi)^{-1} : T'\mathcal{G}|_V \longrightarrow \bar{V}_1 \times \bar{V}_2 \times (\mathbb{M} \times \mathbb{G})'.$$

However, we have:

$$(\mathbb{M} \times \mathbb{G})' = \mathbb{M}^a \times \mathbb{G}^a \equiv \mathbb{G}' \times \mathbb{M}'$$

where $\mathbb{E}^a$ denotes the annihilator of a closed subspace $\mathbb{E}$ of $\mathbb{M} \times \mathbb{G}$.

If we set $V_1 = \psi^{-1}(\bar{V}_1 \times \{0\})$, then $(V_1, (\psi_1)|_{V_1})$ is a chart around $z \in M$. In this chart, we have:

$$(T'\psi)^{-1}(\mathcal{AG}'_{|V_1}) = \bar{V}_1 \times \mathbb{G}' \quad \text{and} \quad (T'\psi)^{-1}(\mathcal{AG}'_{|V_1}) = \bar{V}_1 \times \mathbb{M}'.$$

On the one hand, for any $\xi \in T'M|_V$, $(T'\psi)^{-1}(\xi)$ can be written as a pair (ξ_1, ξ_2) with $\xi_1 : V \rightarrow \mathbb{G}'$ and $\xi_2 : V \rightarrow \mathbb{M}'$ are smooth maps.

Since $\mathbf{t} : \mathcal{G} \rightarrow M$ is a submersion, around any $g_0 \in \mathcal{G}$, we have a chart $(\mathcal{V}, \Psi)$ with simply connected domain and such that:

$$\Psi(\mathcal{V}) = \bar{\mathcal{V}}_1 \times \bar{\mathcal{V}}_2 \subset \mathbb{M} \times \mathbb{G}$$

with $\Psi(g_0) = 0$. Again, write $\Psi = (\Psi_1, \Psi_2)$. Define $\mathcal{V}_1 = \Psi^{-1}(\bar{\mathcal{V}}_1 \times 0)$ and $\mathcal{V}_2 = \Psi^{-1}(0 \times \bar{\mathcal{V}}_2)$. With respect to the local coordinates induced by Ψ , the target map $\mathbf{t}|_{\mathcal{V}}$ takes the form $(\bar{y}, \bar{g}) \mapsto \bar{y}$. Assume that $\mathbf{t}(g_0) = y_0$. After possibly shrinking the domains $\mathcal{V}$ and V , and composing with a local diffeomorphism in $\mathbb{M}$ fixing $0 \in \mathbb{M}$ and a diffeomorphism of $\mathbb{G}$ fixing $0 \in \mathbb{G}$, we may assume that the charts (V, ψ) and $(\mathcal{V}, \Psi)$ are chosen so that $\bar{\mathcal{V}}_1 = \bar{V}_1$. In the corresponding local coordinates, let $\bar{y} = \psi_1(\mathbf{t}(g))$ and $\bar{g}_2 = \Psi_2(g)$. Then target map $\mathbf{t}$ is still represented by $(\bar{y}, \bar{g}_2) \mapsto \bar{y}$. Here $\bar{y} : \mathcal{V} \rightarrow \mathbb{M}$ and $\bar{g}_2 : \mathcal{V} \rightarrow \mathbb{G}$ are smooth maps.

At the level of the tangent bundle of $\mathcal{G}$, and with the above notation, the chart Ψ induces a local trivialization

$$T\Psi : T\mathcal{V} \longrightarrow (\bar{V}_1 \times \mathbb{M}) \times (\bar{\mathcal{V}}_2 \times \mathbb{G})$$

and the following commutative diagram:

$$\begin{array}{ccc} T\mathcal{G}|_{\mathcal{V}} & \xrightarrow{T\Psi} & (\bar{V}_1 \times \mathbb{M}) \times (\bar{\mathcal{V}}_2 \times \mathbb{G}) \\ \downarrow T\mathbf{t} & & \downarrow pr_{V_1} \\ TM|_V & \xrightarrow{T\psi|_{TV}} & \bar{V}_1 \times \mathbb{M} \end{array} \quad (6.2)$$

with $T\Psi((g, Y)) = ((\bar{y}, \bar{Y}_1), (\bar{g}_2, \bar{Y}_2))$ and $T\Psi \circ T\mathbf{t}((g, Y)) = (\bar{y}, \bar{Y}_1)$ where $\bar{y} : \mathcal{V} \rightarrow \mathcal{V}_1$ and $\bar{Y}_1 : \mathcal{V} \rightarrow \mathbb{M}$ are smooth maps.

Since $\mathbf{s}$ is also a submersion, an analogous construction yields a chart (U, ϕ) of $x_0 = \mathbf{s}(g_0)$ such that $\Phi(U) = \bar{U}_1 \times \bar{U}_2 \subset \mathbb{M} \times \mathbb{G}$ together with a chart $(\mathcal{U}, \Phi)$ around $g \in \mathcal{G}$ such that $\Phi(\mathcal{U}) = \bar{U}_1 \times \bar{U}_2$. Moreover, the pair of charts (U, ϕ) and $(\mathcal{U}, \Phi)$ enjoys the same properties as the pair (V, ψ) and $(\mathcal{V}, \Psi)$ constructed above. As we have seen (up to the choice of a bisection $\mathcal{B}$), there exists a diffeomorphism $F : U_1 \rightarrow V_1$ such that $F(y_0) = x_0$.

On the one hand, for any ξ , $(T'\psi)^{-1}(\xi)$ can be written as a pair $(\bar{\xi}_1, \bar{\xi}_2)$ with $\bar{\xi}_1 : V_1 \rightarrow \mathbb{G}'$ and $\bar{\xi}_2 : V_1 \rightarrow \mathbb{M}'$.

On the other hand, let $1_M : M \rightarrow \mathcal{G}$ denote the unit map of the groupoid $\mathcal{G} \rightrightarrows M$, and identify $1_M(M)$ with a submanifold of $\mathcal{G}$. Thus every $Z \in T_x \mathcal{G}$ can be written uniquely as $Z = (a, X)$ where $X \in T_x M$ and $a \in \mathcal{AG}_x$. Then the unit map $\tilde{1} : \mathcal{AG}' \rightarrow T' \mathcal{G}$ is defined by

$$\langle \tilde{1}(\xi), (a, X) \rangle = \langle \xi, a \rangle.$$

According to the previous notations, it follows that, in local coordinates, $\tilde{1}$ is the identity of $\bar{V}_1 \times \mathbb{M}'$. It follows that $1_M \rightarrow T' \mathcal{G}$ is an embedding and its image $\tilde{1}(\mathcal{AG}')$ is a subbundle of $T_M \mathcal{G}$, namely the annihilator of TM .

Without loss of generality, in the above setting, we may assume that $\mathcal{V}_1 = \bar{V}_1 \subset \mathbb{M}$ and $\mathcal{V}_2 = \bar{V}_2 \subset \mathbb{G}$. Moreover, we set $T_V \mathcal{G} := T\mathcal{G}|_{\mathcal{V}} = (V_1 \times \mathcal{V}_2 \times \mathbb{M} \times \mathbb{G})$ and in these coordinates the differential of the target map satisfies $T\mathbf{t}|_{T_V \mathcal{G}} = \text{pr}_{V_1}$. In particular, each $Z \in T_V \mathcal{G}$ can be written as a pair (Z_1, Z_2) where $Z_1 : V \rightarrow \mathbb{M}$ and $Z_2 : V \rightarrow \mathbb{G}$ are smooth maps. Moreover, by diagram (6.2), we obtain $T\mathbf{t} = p_{V_1}$.

Dually, we have $T'_V \mathcal{G} = V_1 \times \mathcal{V}_2 \times \mathbb{G}' \times \mathbb{M}'$. In particular,

$$(\mathcal{AG}'^a_{|V_1}) = \bar{V}_1 \times \mathbb{M}' \quad \text{and} \quad \mathcal{AG}'_{|V_1} = \bar{V}_1 \times \mathbb{G}'. \quad (6.3)$$

Therefore, for any $g = (y, g_2) \in \mathcal{V}$ any $\xi \in T'_g \mathcal{V}$ can be written as a pair $(\xi_1, \xi_2) \in \{(y, g_2)\} \times \mathbb{G}' \times \{(y, g_2)\} \times \mathbb{M}'$ where ξ_1 is a smooth map from $\mathcal{V}$ to $\mathbb{G}'$ and ξ_2 is a smooth map from $\mathcal{V}$ to $\mathbb{M}'$.

Also, for any $\xi \in T'_{U_1} \mathcal{G}$, its components along $\mathcal{AG}|_{U_1}$ is defined by its evaluation on any section a of $\mathcal{AG}|_{U_1}$.

On the other hand, let b be a section of $\mathcal{AG}|_{V_1}$, i.e. a smooth map $b \rightarrow \mathbb{G}$. Then $TR_g(b)$ belongs to $T\mathbf{s}^{-1}(\mathbf{s}(g))$. Similarly, let a be a section of $\mathcal{AG}|_{U_1}$. Then $a - T\mathbf{t}(a)$ is a section of $\ker T\mathbf{t}|_{U_1}$, and hence $TL_g(a - \rho(a))$ belongs to $T\mathbf{t}^{-1}(\mathbf{t}(g))$. Moreover, the maps $(b, g) \mapsto TR_g(b)$ and $(a, g) \mapsto TL_g(a - \rho(a))$ are locally smooth in the convenient setting (using smooth curves).

According to previous arguments, as in finite dimension, the source and the target maps $\tilde{s}, \tilde{t} : T'\mathcal{G} \rightrightarrows \mathcal{AG}'$ are well defined as follows:

$$\begin{aligned} \langle \tilde{s}(\xi), a \rangle (g) &= \langle \xi, TL_g(a - \rho(a)) \rangle, \\ \langle \tilde{t}(\xi), b \rangle &= \langle \xi, TR_g(b) \rangle \end{aligned} \quad (6.4)$$

for all $\xi \in T'_g\mathcal{G}$, $a \in \mathcal{AG}_{s(g)}$ and $b \in \mathcal{AG}_{t(g)}$.

We show that the maps $\tilde{s}$ and $\tilde{t}$ are submersions starting with $\tilde{t}$.

In fact, in the previous context, if ξ belongs to $T'_g\mathcal{G}$, then ξ can be identified with a pair (ξ_1, ξ_2) where ξ_1 is a smooth map from $\mathcal{V}$ to $\mathbb{G}'$ and ξ_2 is a smooth map from $\mathcal{V}$ to $\mathbb{M}'$. Moreover, we have $\langle \tilde{t}(\xi), b \rangle = \langle \xi, TR_g(b) \rangle = \langle \xi_1, TR_g(b) \rangle$ and, since $TR_g(b)$ belongs to $\mathbb{G}$ and ξ_2 belongs to $\mathbb{M}' = \mathbb{G}^a$, the pairing $\langle \xi_2, TR_g(b) \rangle$ vanishes. It follows that $\tilde{t}$ takes values in $\mathbb{G}' \equiv \mathcal{AG}'_g$ which is smooth. Indeed, for any smooth curve $\gamma : \mathbb{R} \rightarrow V_1$, the map $t \mapsto R_{g(\gamma(t))}^*(\xi(\gamma(t)))$ is a smooth section of $\gamma^*(T'\tilde{t}^{-1}(M)) \equiv \mathbb{R} \times \mathbb{M}'$. Note that, for any $\eta \in \mathcal{AG}'_t(g)$, if we set $\xi = (R_g^{-1})^*\eta$, then $\tilde{t}(\xi) = \eta$. Thus $\tilde{t}$ is surjective and hence a submersion.

Let $\mathbf{i} : \mathcal{G} \rightarrow \mathcal{G}$ be the inversion in $\mathcal{G}$. Then $T\mathbf{i}$ is the inversion for the induced Lie groupoid structure $T\mathcal{G} \rightrightarrows TM$. The adjoint $\tilde{\mathbf{i}} := \mathbf{i}^* := T'\mathbf{i}$ is a diffeomorphism of $T'\mathcal{G}$. Since $\mathbf{i} \circ L_g = R_g$ it follows that $R_g^* = L_g^* \circ \tilde{\mathbf{i}}$. Moreover, from the definition of $\tilde{s}$, it is easy to see that $\tilde{s} = \tilde{t} \circ \tilde{\mathbf{i}}$ and we leave the proof as an exercise. By Proposition 4.2, it follows that $\tilde{s}$ is a submersion.

As in finite dimension, the multiplication $\tilde{\mathbf{m}}$ on $T'\mathcal{G}$ is given by

$$\langle \xi \bullet \eta, X \bullet Y \rangle := \langle \xi, X \rangle + \langle \eta, Y \rangle$$

if $\tilde{t}(\xi) = \tilde{s}(\eta)$. By Proposition 4.2, we know that the set of such pairs (ξ, η) is a submanifold $(T'\mathcal{G})^{(2)}$ of $T'\mathcal{G} \times T'\mathcal{G}$. If $s(g) = t(h)$, on the one hand, the bilinear map $T_g\mathcal{G} \times T_h\mathcal{G} \rightarrow T_{\mathbf{m}(g,h)}\mathcal{G} : (X, Y) \mapsto X + Y$ is a bounded surjective map. On the other hand, its graph $T_g\mathcal{G} \times T_h\mathcal{G} \rightarrow T_{\mathbf{m}(g,h)}\mathcal{G} : (X, Y) \mapsto X + Y$ can be viewed as a relation from $T_g\mathcal{G} \times T_h\mathcal{G}$ to $T_{\mathbf{m}(g,h)}\mathcal{G}$. From the surjectivity of the multiplication $T\mathbf{m}$ in $T'\mathcal{G}$ and the definition of the multiplication $\tilde{\mathbf{m}}$ in $T'\mathcal{G}$, it follows that the graph of $\tilde{\mathbf{m}}$ in $T'\mathcal{G}$ is equal to the annihilator $\text{Ann}(G_{T\mathbf{m}})$. The associativity of $T\mathbf{m}$ implies the associativity of $\tilde{\mathbf{m}}$.

Finally, under the previous local context (even in the convenient setting), from its definition, it follows that the map $(\xi\eta) \mapsto \xi \bullet \eta$ is a smooth map from $(T'\mathcal{G})^{(2)}$ to $T'\mathcal{G}$, which completes the proof of (1).

Remark 6.2. $T'\mathcal{G} \rightrightarrows \mathcal{AG}'$ is a $\mathcal{VB}$ -groupoid dual of $\mathcal{VB}$ -groupoid $T\mathcal{G} \rightrightarrows TM$.

(2) Let ω be the canonical 2-form on $T'\mathcal{G}$. In a local trivialization of $T'\mathcal{G}$, any vector $X \in T_{(\xi,g)}T'\mathcal{G}$ can be written as a pair (v, v') , where $v \in T_g\mathcal{G}$ and $v' \in T'_g\mathcal{G}$. We then have:

$$\omega_{(\xi,g)}(X, Y) = \langle w', v \rangle - \langle v', w \rangle, \quad (6.5)$$

if $X = (v, v')$ and $Y = (w, w')$. Let $T^\flat(T'\mathcal{G})$ be the image of the map $\omega^\flat : T(T'\mathcal{G}) \rightarrow T'(T'\mathcal{G})$. Thus, $P := (\omega^\flat)^{-1} : T^\flat(T'\mathcal{G}) \rightarrow T(T'\mathcal{G})$ is an isomorphism.

It follows that P is a Poisson anchor and hence $T'\mathcal{G} \rightrightarrows \mathcal{N}^a(M)$ is a partial symplectic groupoid.

Finally, Proposition 2.30 implies that $(\mathcal{AG}, \mathcal{AG}')$ is a partial Lie bialgebroid. ■

6.2. SUB-POISSON STRUCTURE ON A SUB-GROUPOID OF THE COTANGENT BUNDLE. In the Banach framework used in [32], the authors introduce a notion of sub-Poisson structure (p. 29). The following definition is an adaptation of this notion to the convenient setting:

DEFINITION 6.3. A convenient Lie groupoid $\mathcal{G} \rightrightarrows M$ has a *sub-Poisson groupoid structure* if there exists a Poisson anchor $P : T^\flat\mathcal{G} \rightarrow T\mathcal{G}$ such that $T^\flat\mathcal{G}$ admits a convenient subgroupoid structure $T^\flat\mathcal{G} \rightrightarrows \mathcal{AG}^\flat$ of $T'\mathcal{G} \rightrightarrows \mathcal{AG}'$ where $\mathcal{AG}^\flat$ is a weak subbundle of $\mathcal{AG}'$. Moreover, there exists a bundle morphism $P^\flat : \mathcal{AG}^\flat \rightarrow TM$ such that the following diagram is commutative:

$$\begin{array}{ccc} T^\flat\mathcal{G} & \xrightarrow{P} & T\mathcal{G} \\ \Downarrow & & \Downarrow \\ \mathcal{AG}^\flat & \xrightarrow{P^\flat} & TM \end{array} \quad (6.6)$$

and P is a groupoid morphism over $P^\flat$.

We have the following characterization:

PROPOSITION 6.4. Consider a convenient Lie groupoid $\mathcal{G} \rightrightarrows M$ endowed with a Poisson anchor $P : T^\flat\mathcal{G} \rightarrow T\mathcal{G}$. Assume that the partial conormal set $\mathcal{N}^0(M)$ is a closed sub-bundle of $T_M^\flat\mathcal{G}$, and let denote $P^\flat$ the restriction of P

to $\mathcal{N}^0(M)$. Suppose the following diagram is commutative:

$$\begin{array}{ccc} T^b\mathcal{G} & \xrightarrow{P} & T\mathcal{G} \\ \tilde{s}_{T^b\mathcal{G}} \downarrow \tilde{t}_{|T^b\mathcal{G}} & & Ts \downarrow Tt \\ \mathcal{N}^0(M) & \xrightarrow{P^b} & TM \end{array} \quad (6.7)$$

Then the convenient Lie groupoid $\mathcal{G} \rightrightarrows M$ is a sub-Poisson groupoid if and only if $\mathcal{G} \rightrightarrows M$ admits a partial Poisson structure.

COROLLARY 6.5. *Under any of the assumptions of Corollary 5.5, any Lie groupoid is a sub-Poisson groupoid if and only if it is a partial Poisson groupoid.*

Remark 6.6. We can regard Proposition 6.4 as a generalization of a classical characterization of Poisson groupoid (cf. [27, Proposition 8.1]). From Proposition 6.4, a sub-Poisson groupoid admits a partial Poisson structure. In particular, all examples of partial Poisson -Lie groupoids given in § 4.4 are sub-Poisson Lie groupoids. But, *a priori*, the converse is not true in general without the assumption that $\mathcal{N}^0(M)$ is a closed sub-bundle of $T_M^b\mathcal{G}$. Unfortunately, we have no example of a partial Poisson-Lie groupoid for which $\mathcal{N}^0(M)$ is not a closed sub-bundle of $T_M^b\mathcal{G}$.

Proof of Proposition 6.4. Assume that $T^b\mathcal{G} \rightrightarrows \mathcal{AG}'$ is a sub-Poisson Lie groupoid. Since P is a groupoid morphism, it satisfies the condition

$$P \circ \tilde{\mathbf{m}}(\xi, \eta) = T\mathbf{m}(P(\xi), P(\eta)), \quad (6.8)$$

for all $(\xi, \eta) \in (T^b\mathcal{G})^{(2)}$. Now, since we have

$$\langle (\xi, \eta, -\tilde{\mathbf{m}}(\xi, \eta)), (X, Y, T\mathbf{m}(X, Y)) \rangle = 0$$

for all $(\xi, \eta) \in T_{(g,h)}^b\mathcal{G}^{(2)}$ and $(X, Y) \in T_{(g,h)}\mathcal{G}^{(2)}$, the same holds for all (ξ, η) and (ξ', η') in $T_{(g,h)}^b\mathcal{G}^{(2)}$, since $T^b\mathcal{G} \rightrightarrows \mathcal{AG}^b$ is a sub-groupoid of $T'\mathcal{G} \rightrightarrows \mathcal{AG}'$. By applying this relation to the left hand-side of (6.8), it follows that $T'\mathcal{G} \rightrightarrows \mathcal{AG}'$ is a partial Poisson groupoid. On the other hand, $\mathcal{AG}^b$ is a subgroupoid of $\mathcal{AG}'$ (in particular $\mathcal{N}^0(M) = \mathcal{AG}^b$ is a closed subbundle of $T_M^b\mathcal{G}$) and therefore P^b must be the restriction of P to $\mathcal{AG}^b$. The Diagram (8.1) is then identical to the diagram (6.7).

Conversely, assume that $\mathcal{G} \rightrightarrows M$ has a partial Poisson structure. On the one hand, since the graph of $\mathbf{m}$ is coisotropic, this implies that the restriction $\tilde{\mathbf{m}}^b$ of $\tilde{\mathbf{m}}$ to $T_{g,h}^b \mathcal{G}^{(2)}$ takes values in $T^b \mathcal{G}$, and its graph is a closed submanifold of $\{T^b \mathcal{G}\}^3$. It follows that the projection of the graph of $\tilde{\mathbf{m}}^b$ onto $T^b \mathcal{G}$ is smooth which implies that $\tilde{\mathbf{m}}^b$ is smooth. Note that the coisotropy of the graph means that:

$$\langle (\xi, P(\eta), -P(\xi \bullet \eta)), (\xi', \eta', -\xi' \bullet \eta') \rangle = 0$$

for all (ξ, η) and (ξ', η') in $(T^b \mathcal{G})^{(2)}$. But, for any $g \in \mathcal{G}$, the annihilator of $T^b \mathcal{G}$ is zero. Since $\tilde{\mathbf{m}}$ is surjective, it follows that $P(\xi \bullet \eta) = (P(\xi) + P(\eta))$. This implies that P satisfies the relation (6.8).

On the other hand, since M is coisotropic and $\mathcal{N}^0(M)$ is a closed subbundle of $T_M^b \mathcal{G}$, it follows from Theorem 5.3 that $\mathcal{N}^0(M)$ carries a Lie algebroid structure whose anchor is precisely P^b . In fact, it is a subalgebroid of $\mathcal{AG}'$ (cf. Proof of Theorem 5.3).

Now, from Diagram (6.7), we have $P^b(\tilde{\mathbf{t}}|_{T^b \mathcal{G}})(\xi) = T\mathbf{t} \circ P(\xi)$ for all $\xi \in T^b \mathcal{G}$. Recall that, for any $\xi \in T'_g \mathcal{G}$, we have $\tilde{\mathbf{t}}(\xi)(R_g^*(\xi)) = \tilde{\mathbf{t}}(\xi)$ which belongs to $(\mathcal{AG}')'_{\mathbf{t}(g)}$. Therefore, $\tilde{\mathbf{t}}(\circ R_g^*)^{-1} = Id_{\mathcal{AG}'_{\mathbf{t}(g)}}$ for all $g \in \mathcal{G}$. This implies that

$$\tilde{\mathbf{t}}|_{T^b \mathcal{G}} \circ (R_g^*)^{-1} = Id_{\mathcal{AG}^b_{\mathbf{t}(g)}}$$

and it follows easily that $\tilde{\mathbf{t}}|_{T^b \mathcal{G}}$ is a submersion. The same arguments apply to $\tilde{\mathbf{s}}|_{T^b \mathcal{G}}$ as well. Finally, $T^b \mathcal{G} \rightrightarrows \mathcal{AG}^b$ is a subgroupoid of $T' \mathcal{G} \rightrightarrows \mathcal{AG}'$. Since P satisfies the relation (6.8) as shown in the first part of the proof, it follows that P is a groupoid morphism. This completes the proof. ■

6.3. THE COTANGENT GROUPOID OF A LIE GROUP

6.3.1. SOME PRELIMINARIES. If G is a Lie group, then it acts naturally on its kinematic cotangent bundle $T'G$ by right and left translations. The right action is given by $(g, (h, \xi)) \mapsto (gh, T_e^* L_g(\xi))$ while the left action is given by $(g, (h, \xi)) \mapsto (gh, T_e^* R_h(\xi))$. These actions induce the corresponding coadjoint action of G on $T'G$.

The right action (resp. the left action) gives rise to an isomorphism $\tilde{R}$ (resp. $\tilde{L}$) from $\mathfrak{g}' \times G$ (resp. $G \times \mathfrak{g}'$) to $T'G$, defined by $\tilde{R}(g, \xi) = (\tilde{\mathbf{s}}_g(\xi), g)$ (resp. $\tilde{L}(g, \xi) = (g, \tilde{\mathbf{t}}_g(\xi))$), which respects the submersion $\tilde{\mathbf{s}}$ (resp. $\tilde{\mathbf{t}}$). We then

have the following commutative diagrams:

$$\begin{array}{ccc} \mathfrak{g}' \times G & \xrightarrow{\tilde{R}} & T'G \\ \downarrow & & \downarrow \tilde{s} \\ \mathfrak{g}' & \xrightarrow{Id_{\mathfrak{g}'}} & \mathfrak{g}' \end{array} \quad , \quad \begin{array}{ccc} \mathfrak{g}' \times G & \xrightarrow{\tilde{L}} & T'G \\ \downarrow & & \downarrow \tilde{t} \\ \mathfrak{g}' & \xrightarrow{Id_{\mathfrak{g}'}} & \mathfrak{g}' \end{array} . \quad (6.9)$$

Note that the arguments given in the finite-dimensional framework in [2, 2.3], remain valid in the convenient setting for regular convenient Lie groups (see also [21] for such a context). Thus, throughout the sequel, we assume that G is provided with an exponential map, for example by requiring that it be a convenient regular Lie group. We have the coadjoint action of G on $\mathfrak{g}'$, given by $g \mapsto \text{Ad}_g^*(\xi)$. Since $\tilde{L}$ is an isomorphism from $T'G$ onto $G \times \mathfrak{g}'$, we can provide $T'G$ with a semi-direct product given by:

$$(g, \xi) \cdot (h, \eta) := (gh, \text{Ad}_{h^{-1}}^*(\eta) + \xi).$$

The coadjoint action of G on $\mathfrak{g}'$ induces a smooth action of the Lie algebra $\mathfrak{g}$ on $\mathfrak{g}'$ in the following way: any $X \in \mathfrak{g}$ gives rise to a vector field $X^\sharp$ on $\mathfrak{g}'$ whose value at $\xi \in \mathfrak{g}'$ is characterized by

$$\langle X_\xi^\sharp, Y \rangle := \langle \xi, [X, Y] \rangle = \text{ad}_X^*(Y) \quad (6.10)$$

for any $Y \in \mathfrak{g}$.

Thus, for any $\xi \in \mathfrak{g}'$, we define a bilinear alternating form ω_ξ on $\mathfrak{g}$ by:

$$\sigma_\xi(X, Y) = \langle \xi, [X, Y] \rangle = \text{ad}_X^*(Y). \quad (6.11)$$

The kernel of σ_ξ consists of all $X \in \mathfrak{g}$ such that $\langle X_\xi^\sharp, Y \rangle = 0$, for all $Y \in \mathfrak{g}$. Thus, this kernel is the Lie algebra $\mathfrak{g}_\xi$ of the stabilizer G_ξ under the coadjoint action at ξ . Let $\mathcal{O}_\xi = \{\text{Ad}_g(\xi) : g \in G\}$ be the coadjoint orbit through ξ . Unfortunately, in general, $\mathcal{O}_\xi$ does not carry a convenient manifold structure. Even when G is a Banach Lie group, $\mathcal{O}_\xi$ need not admit a Banach manifold structure. This holds if $\mathfrak{g}_\xi$ is a supplemented Banach subalgebra of $\mathfrak{g}$. Of course, this is always the case in the Hilbert setting for a finite dimensional Lie group G . In this case, σ is a symplectic form in restriction to each orbit $\mathcal{O}_\xi$ and defines a Poisson structure on $\mathfrak{g}'$ known as the *Kostant-Kirolov-Souriau Poisson structure* on $\mathfrak{g}'$. We will see in the next section a generalization of this result in the Banach setting.

As in [2], we consider the map $\kappa^r : T\mathbf{G} \rightarrow \mathfrak{g}$ (resp. $\kappa^l : T\mathbf{G} \rightarrow \mathfrak{g}$) defined by $\kappa_g^r = T_e R_{g^{-1}}$ (resp. $\kappa_g^l = T_e L_{g^{-1}}$). Note that these maps are 1-forms with values in $\mathfrak{g}$ and that they satisfy the Maurer-Cartan equations:

$$d\kappa^r + \frac{1}{2}\kappa^r \wedge \kappa^r = 0 \quad (\text{resp. } d\kappa^l - \frac{1}{2}\kappa^l \wedge \kappa^l = 0).$$

Remark that κ^r (resp. κ^l) is right invariant (resp. left invariant). In fact, we have $\kappa^l = \mathbf{s}$ and $\kappa^r = \mathbf{t}$.

We consider the following diagram:

$$\begin{array}{ccc} T(T'\mathbf{G}) & \xrightarrow{T\pi_{\mathbf{G}}} & T\mathbf{G} \\ \pi_{T'\mathbf{G}} \downarrow & & \downarrow \pi_{\mathbf{G}} \\ T'\mathbf{G} & \xrightarrow{\pi} & \mathbf{G} \end{array}$$

Under these notations, we have:

$$\tilde{R} = \mathbf{s} \times \pi, \quad \tilde{L} = \pi \times \mathbf{t} \quad \text{and} \quad \tilde{\mathbf{s}}_g = ((\kappa_g^l)^{-1})^*, \quad \tilde{\mathbf{t}}_g = ((\kappa_g^r)^{-1})^*.$$

Let λ be the Liouville form on $T'\mathbf{G}$. Then the canonical symplectic form on $T'\mathbf{G}$ is given by $\omega = -d\lambda$.

To show that $T'\mathbf{G}$ carries a structure of partial symplectic groupoid, let pr_1 (resp. pr_2) denote the projection of $T'\mathbf{G} \times T'\mathbf{G}$ on the first factor (resp. second factor). Then by Remark 3.18, it suffices to prove that:

$$pr_1^* \omega + pr_2^* \omega = \mathbf{m}^* \omega. \quad (6.12)$$

Since $\omega = -d\lambda$, it suffices to verify relation (6.12) for λ .

Given $\xi \in T'_g \mathbf{G}$ (resp. $\eta \in T'_h \mathbf{G}$) and $\Xi \in T'_\xi(T'\mathbf{G})$ (resp. $\Theta \in T'_\eta(T'\mathbf{G})$), we have:

$$T'\mathbf{G}(\Xi) \in T'_g \mathbf{G} \quad (\text{resp. } T'\mathbf{G}(\Theta) \in T'_h \mathbf{G}).$$

From the expression of the multiplication on the tangent space of a groupoid, we obtain $T_{(g,h)} \tilde{\mathbf{m}}(\Xi, \Theta) = \Xi_1 + \Xi_2$ which yields:

$$\tilde{\mathbf{m}}_{(g,h)}^* \lambda(\Xi_1, \Theta) = \lambda_g(\Xi_1) + \lambda_h(\Theta).$$

This shows that $T'\mathbf{G}$ is endowed with a partial symplectic structure defined by the Poisson anchor:

$$P = (\omega^\flat)^{-1} : T^\flat(T'\mathbf{G}) := \omega^\flat(T(T'\mathbf{G})) \longrightarrow T(T'\mathbf{G}).$$

6.3.2. THE COTANGENT BUNDLE OF A LIE GROUP AS A PARTIAL SYMPLECTIC GROUPOID. We consider the structure of a partial symplectic groupoid on $T'\mathbf{G} \rightrightarrows \mathfrak{g}'$ as described in the previous section.

Essentially, by application of Theorem 4.9, we have:

THEOREM 6.7. *Let $\mathbf{G}$ be a regular convenient Lie group. Its cotangent bundle $T'\mathbf{G}$ carries a structure of partial symplectic Lie groupoid over its dual Lie algebra $\mathfrak{g}'$. We then have:*

- (1) $\mathfrak{g}'$, when considered as the zero section of $T'\mathbf{G}$, is Lagrangian in $T'\mathbf{G}$.
- (2) The inversion in $T'\mathbf{G}$ is an anti-symplectic map.
- (3) Let ω be the canonical symplectic 2-form on $T'\mathbf{G}$ and consider the associated partial Poisson structure $P : T^b(T'\mathbf{G}) := \omega^b(\mathbf{G}) \rightarrow T\mathbf{G}$. In fact, $T'\mathbf{G} \rightrightarrows \mathfrak{g}'$ has a sub-Poisson groupoid structure. In particular, $T^b(T'\mathbf{G}) \rightrightarrows \mathfrak{g}' \times \mathfrak{g}^b$ is a Lie groupoid²⁰.
- (4) For each $g \in \mathcal{G}$, each vector space $T_g(\tilde{\mathbf{s}}^{-1}(g))$ and $T_g(\tilde{\mathbf{t}}^{-1}(g))$ are orthogonal symplectic subspaces of the linear symplectic space $(T_g(T'\mathbf{G}), \omega)$.
- (5) There exists a unique Poisson anchor $P_{\mathfrak{g}'} = T\mathbf{s} \circ P \circ T\tilde{\mathbf{s}} = T\mathbf{t} \circ P \circ T\tilde{\mathbf{t}}$ on $\mathfrak{g}'$ such that $\mathbf{s}$ (resp. $\mathbf{t}$) is a Poisson morphism (resp. anti-Poisson morphism). More precisely, if σ is the induced 2 form on $\mathfrak{g}'$ by ω , we have:
 - (a) σ is the Poisson bivector associated with $P_{\mathfrak{g}'}$.
 - (b) For each coadjoint orbit $\mathcal{O}_\xi$ in $\mathfrak{g}'$, for any $\xi \in \mathcal{O}$, then σ_ξ is exactly the form defined by relation (6.11) and its kernel $\mathfrak{g}_\xi$ is a closed Lie subalgebra of $\mathfrak{g}$.
- (6) If $\mathbf{G}$ is a Banach Lie group such that $\ker \sigma_\xi$ is split in $\mathfrak{g}$, for all $\xi \in \mathfrak{g}'$, we have:
 - (c) For any $\xi \in \mathfrak{g}'$, the isotropy group $\mathbf{G}_\xi$ of ξ , under the coadjoint action, is a Lie subgroup of $\mathbf{G}$ whose Lie algebra is $\mathfrak{g}_\xi$.
 - (d) Each orbit $\mathcal{O}_\xi$ is a split submanifold of $\mathfrak{g}'$ which is diffeomorphic to the homogeneous space $\mathbf{G}/\mathbf{G}_\xi$.
 - (e) ω induces a weak symplectic form on each orbit $\mathcal{O}_\xi$.
 - (f) The pair $(\mathbf{s}, -\mathbf{t})$ is a dual pair.

²⁰cf. left vertical part of Diagram (6.7).

Remarks 6.8. (a) Theorem 6.3.2 can be considered as a generalization to the Banach setting of a finite-dimensional classical result concerning the groupoid structure of the cotangent bundle of a finite dimensional Lie group G and its connection to the Kostant-Kirilov-Souriau Poisson structure on the dual of the Lie algebra of G (see for instance [29, 1]).

(b) In general, in the convenient setting, the last assertion (6) of Theorem 6.3.2 is not true without these complementary assumptions. However, we will exhibit a context in which this result holds without these additional assumptions.

Proof. Assertions (1) and (2) are direct applications of Theorem 4.9.

Assertion (3) will follow from Proposition 6.4 if we show that Diagram (6.7) commutes in this context.

Since $\tilde{R}$ is an isomorphism, $\bar{\lambda} = \tilde{R}^*\lambda$ is the Liouville 1-form on $\mathfrak{g}' \times G$ and thus $\bar{\lambda}(X, \xi) = \langle \xi, X \rangle$. Therefore, $\bar{\omega} = \tilde{R}^*\omega$ is the canonical symplectic form on $\mathfrak{g}' \times G$. Thus, if $\mathfrak{X}_1 = (\eta_1, X_1)$ and $\mathfrak{X}_2 = (\eta_2, X_2)$ belong to $T_\xi \mathfrak{g}' \times T_g G$, we have

$$\bar{\omega}(\mathfrak{X}_1, \mathfrak{X}_2) = \langle \eta_1, (X_2)_e \rangle - \langle \eta_2, (X_1)_e \rangle + \langle \xi, [X_1, X_2] \rangle$$

where $(X_i)_e \in \mathfrak{g}$ is such that $T_e R_g((X_i)_e) = X_i$ for $i \in \{1, 2\}$.

Therefore, for $\mathfrak{X} = (\eta, X) \in T_\xi \mathfrak{g}' \times T_g G$, we have:

$$(\bar{\omega})^\flat(\mathfrak{X}) = (\mathfrak{i}(\eta + \xi \circ \text{ad}_{X_e}), -X) \in T'(\mathfrak{g}' \times G) = \mathfrak{g}' \times \mathfrak{g}'' \times T'G, \quad (6.13)$$

where $\mathfrak{i}$ is the canonical inclusion of $\mathfrak{g}$ into $\mathfrak{g}''$ and we set $\mathfrak{g}^\flat = \mathfrak{i}(\mathfrak{g})$. This implies that the range of $(\bar{\omega})^\flat$ is $(\mathfrak{g}' \times \mathfrak{g}^\flat) \times T'G$. Thus $T^\flat(T'G) := \omega^\flat(T(T'G))$ is equal to $T^*\tilde{R}((\mathfrak{g}' \times \mathfrak{g}^\flat) \times T'G)$ and hence is a closed subbundle of $T'(T'G)$. Let $\bar{P} := ((\bar{\omega})^\flat)^{-1}$ and $P = ((\omega)^\flat)^{-1}$. Then the following diagram commutes:

$$\begin{array}{ccccc}
 T^\flat(T'G) & \xrightarrow{T^*\tilde{R}} & \mathfrak{g}' \times \mathfrak{g}^\flat \times T'G & & \\
 \downarrow T\tilde{\mathfrak{s}}|_{T^\flat(T'G)} & \searrow P & \downarrow & \searrow \bar{P} & \\
 & & T(T'G) & \xleftarrow{T\tilde{R}} & \mathfrak{g}' \times \mathfrak{g}' \times TG \\
 & & \downarrow Ts & \downarrow p_{\mathfrak{g}' \times \mathfrak{g}^\flat} & \downarrow p_{\mathfrak{g}' \times \mathfrak{s}} \\
 \mathfrak{g}' \times \mathfrak{g}^\flat & \xrightarrow{P_{\mathfrak{g}'}} & \mathfrak{g}' \times \mathfrak{g}' & \xrightarrow{Id_{\mathfrak{g}' \times \mathfrak{g}^\flat}} & \mathfrak{g}' \times \mathfrak{g}^\flat \\
 & \searrow & \downarrow Id_{\mathfrak{g}' \times \mathfrak{g}'} & \searrow \bar{P}_{\mathfrak{g}'} & \\
 & & \mathfrak{g}' \times \mathfrak{g}' & \xleftarrow{Id_{\mathfrak{g}' \times \mathfrak{g}'}} & \mathfrak{g}' \times \mathfrak{g}'
 \end{array} \quad (6.14)$$

where $s = T_e L : TG \rightarrow \mathfrak{g}$. By the same arguments, we also have the diagram:

$$\begin{array}{ccccc}
 T^b(T'G) & \xrightarrow{T^* \tilde{L}} & \mathfrak{g}' \times \mathfrak{g}^b \times T'G & & \\
 \downarrow T\tilde{\mathbf{t}}|_{T^b(T'G)} & \searrow P & \downarrow & \searrow \bar{P} & \\
 & T(T'G) & \xleftarrow{T\tilde{L}} & \mathfrak{g}' \times \mathfrak{g}' \times TG & \\
 & \downarrow T\mathbf{t} & \downarrow p_{\mathfrak{g}' \times \mathfrak{g}^b} & \downarrow p_{\mathfrak{g}' \times \mathfrak{t}} & \\
 \mathfrak{g}' \times \mathfrak{g}^b & \xrightarrow{P_{\mathfrak{g}'}} & \mathfrak{g}' \times \mathfrak{g}^b & \xrightarrow{\bar{P}_{\mathfrak{g}'}} & \mathfrak{g}' \times \mathfrak{g}' \\
 & \downarrow & \downarrow Id_{\mathfrak{g}' \times \mathfrak{g}^b} & \downarrow Id_{\mathfrak{g}' \times \mathfrak{g}'} & \\
 & \mathfrak{g}' \times \mathfrak{g}' & \xleftarrow{Id_{\mathfrak{g}' \times \mathfrak{g}'}} & \mathfrak{g}' \times \mathfrak{g}' &
 \end{array} \tag{6.15}$$

Now, note that it is clear that $\mathcal{N}^0(G)$ is nothing but $\mathfrak{g}' \times \mathfrak{g}^b$. By superposing the diagrams (6.14) and (6.15), the vertical right face of the resulting diagram implies that its left vertical face gives rise to Diagram 8.1 in this context. Therefore, the assumptions of Proposition 6.4 are satisfied, thus ending the proof of assertion (3).

Assertion (4) is a direct application of Theorem 4.9 (4).

Assertion (5) is a direct consequence of Theorem 4.9 (3), which guarantees the existence, uniqueness, and explicit expression of $P_{\mathfrak{g}'}$.

From this point until the end of the proof of assertion (5), we identify $\mathfrak{g}^b$ with $\mathfrak{g}$.

We now work in the setting of Theorem 2.21 with $\mathcal{A} = T'G$, endowed with the Lie bracket of vector fields and the anchor being the identity. Therefore, by relation [8, (7.19)], for every $\mathfrak{X} = (\eta, X) \in T_{(\xi, g)}^b(T'G)$, we have:

$$P(\eta, X) = (\eta + \xi \circ \text{ad}_X, -X) = (\eta + \text{ad}_X^* \xi, -X) \tag{6.16}$$

Since $\mathfrak{g}' \times \mathfrak{g}'$ is identified with $\mathfrak{g}' \times \mathfrak{g}' \times \{0\}$, the map $P_{\mathfrak{g}'}$ may be viewed as a map from $\mathfrak{g}' \times \mathfrak{g}$ to $\mathfrak{g}' \times \mathfrak{g}'$, given by:

$$(P_{\mathfrak{g}'})_{\xi}(X) = \text{ad}_X^* \xi \tag{6.17}$$

On the one hand, from (6.13), the 2-form $\omega_{\mathfrak{g}'}$ induced by ω on $\mathfrak{g}'$ is given by:

$$(\omega_{\mathfrak{g}'})_{\xi}(X, Y) = \langle \text{ad}_X^* \xi, Y \rangle = \langle \xi, [X, Y] \rangle = \sigma_{\xi}(X, Y).$$

On the other hand, we have:

$$\omega'_{\mathfrak{g}}(X, Y) = \langle P_{\mathfrak{g}'}(X), Y \rangle.$$

Since $(P_{\mathfrak{g}'}(\{\xi\} \times \mathfrak{g}))^0 = \{X \in \mathfrak{g} : \langle \text{ad}_X^* \xi, Y \rangle = 0, \forall Y \in \mathfrak{g}\}$, we obtain $(P_{\mathfrak{g}'}(\{\xi\} \times \mathfrak{g}))^0 = \{X \in \mathfrak{g} : \text{ad}_X^* \xi = 0\} = \ker \sigma_\xi$. Since the pullback of a closed 2-form is also closed, it follows that σ is closed and thus, its kernel $\mathfrak{g}_\xi$ is a Lie algebra. This completes the proof of assertion (5).

For the proof of assertion (6), we assume that G is a Banach-Lie group and that, for every $\xi \in \mathfrak{g}'$, the Lie algebra $\mathfrak{g}_\xi$ is split. Then assertion (6) follows from Theorem 4.9 (5). Nevertheless, we provide a complete proof in this setting in order to make explicit the connection between the Lie groupoid approach and the coadjoint action.

Under the previous assumptions, it follows that the Lie algebra of the isotropy group is G_ξ of ξ under the coadjoint action is $\mathfrak{g}_\xi$ and hence G_ξ is a Banach Lie subgroup of G . This implies that the orbit $\mathcal{O}_\xi$ is a Banach submanifold of G , since its tangent space at ξ (and hence at every point of $\mathcal{O}_\xi$) is isomorphic to $\mathfrak{g}/\mathfrak{g}_\xi$ and is therefore supplemented. Hence, $\mathcal{O}_\xi$ is a submanifold of $\mathfrak{g}'$ and is diffeomorphic to G/G_ξ (cf. [5, Example 3.5] or [4]).

Finally according to Definition 2.58, the pair $(\mathbf{s}, -\mathbf{t})$ is a dual pair. ■

Remark 6.9. $\mathfrak{g}' \times \mathfrak{g}^b$ is the Lie algebroid of the Lie groupoid $T^b(T'G)$ and the anchor is $P_{\mathfrak{g}'}$. Thus, when G is a Banach Lie group, the assumption " $\mathfrak{g}_\xi$ is split " is equivalent to the condition " $\ker P_{\mathfrak{g}'}$ is split at ξ ". Therefore, under the assumptions of assertion (6), if G is connected, coadjoint orbits coincide with the orbits of the Lie algebroid $\mathfrak{g}' \times \mathfrak{g}^b$ acting on $\mathfrak{g}'$.

7. DIRECT SEQUENCES OF PARTIAL POISSON-LIE GROUPOIDS

In this section, we study the direct limits of ascending sequences of finite-dimensional Poisson-Lie groupoids and endow them with convenient structures.

DEFINITION 7.1. An *ascending sequence of finite-dimensional Poisson-Lie groupoids* $(\mathcal{G}_i \rightrightarrows M_i, M_i, P_i)_{i \in \mathbb{N}}$ consists of the following data:

$(\mathcal{G}_i \rightrightarrows M_i)_{i \in \mathbb{N}}$ is a sequence of finite-dimensional Lie groupoids, together with close embedding structures maps:

$$\varphi_i^j : \mathcal{G}_i \longrightarrow \mathcal{G}_j \quad \text{and} \quad \varepsilon_i^j : M_i \longrightarrow M_j, \quad i \leq j,$$

fulfilling the following properties:

(ASPLG1) $(M_i)_{i \in \mathbb{N}}$ is an ascending sequence of finite-dimensional manifolds

with bonding maps $(\varepsilon_i^j)_{i \leq j}$, and $(\mathcal{G}_i \rightrightarrows M_i)_{i \in \mathbb{N}}$ is an ascending sequence of finite-dimensional Lie groupoids.

(ASPLG2) For every $i \leq j$, the pair $(\varphi_i^j, \varepsilon_i^j) : \mathcal{G}_i \rightrightarrows M_i \longrightarrow \mathcal{G}_j \rightrightarrows M_j$ is a Lie groupoid morphism²¹ and the maps φ_i^j and ε_i^j are compatible with units, inverses and multiplication.

(ASPLG3) Each $\mathcal{G}_i$ is endowed with a Poisson structure π_i such that $(\mathcal{G}_i \rightrightarrows M_i, \pi_i)$ is a Poisson-Lie groupoid, and for every $i \leq j$, the map φ_i^j is a Poisson map.

(ASPLG4) The structure maps are compatible with the inclusions, namely

$$s_j|_{\varphi_i^j(\mathcal{G}_i)} = \varepsilon_i^j \circ s_i \circ (\varphi_i^j)^{-1} \quad \text{and} \quad t_j|_{\varphi_i^j(\mathcal{G}_i)} = \varepsilon_i^j \circ t_i \circ (\varphi_i^j)^{-1},$$

and similarly for the unit, inversion and multiplication maps.

PROPOSITION 7.2. *Let $(\mathcal{G}_i \rightrightarrows M_i)_{i \in \mathbb{N}}$ be a direct sequence of finite dimensional Lie groupoids. Then:*

- (1) *The direct limits $\mathcal{G} = \varinjlim \mathcal{G}_i$ and $M = \varinjlim M_i$ carry natural structures of Hausdorff convenient manifolds.*
- (2) *The direct limit $\mathbf{1} = \varinjlim \mathbf{1}_i$ is a smooth closed immersion $M = \varinjlim M_i \rightarrow \mathcal{G}$;*
- (3) *the direct limits $\mathbf{s} = \varinjlim s_i$ and $\mathbf{t} = \varinjlim t_i$ are smooth submersions from $\mathcal{G}$ onto M ;*
- (4) *$\mathbf{m} = \varinjlim \mathbf{m}_i$ is a smooth map from the convenient manifold $\mathcal{G}^{(2)} = \varinjlim \mathcal{G}_i^{(2)}$ to $\mathcal{G}$;*
- (5) *$\mathbf{i} = \varinjlim \mathbf{i}_i$ is a smooth diffeomorphism of $\mathcal{G}$.*

Consequently, $\mathcal{G} \rightrightarrows M$ inherits a natural structure of Hausdorff convenient Lie groupoid, called the direct limit Lie groupoid of the direct sequence $(\mathcal{G}_i \rightrightarrows M_i)_{i \in \mathbb{N}}$.

Moreover, its Lie algebroid is naturally isomorphic to the direct limit of the corresponding sequence of finite-dimensional Lie algebroids $(\mathcal{A}\mathcal{G}_i, M_i, \rho_i, [\cdot, \cdot]_i)_{i \in \mathbb{N}}$.

²¹Equivalently, $s_j \circ \varphi_i^j = \varepsilon_i^j \circ s_i$ and $t_j \circ \varphi_i^j = \varepsilon_i^j \circ t_i$.

Remark 7.3. Since $\mathcal{G}$ (resp. M) is obtained as the direct limit of an ascending sequence of finite-dimensional manifolds, its typical modelling space is the strict inductive limit $\mathbb{R}^\infty := \varinjlim \mathbb{R}^n$.

By [24], $\mathbb{R}^\infty$ is a convenient reflexive locally convex vector space. Therefore, $\mathcal{G}$ and M inherit natural structures of convenient manifolds modelled on $\mathbb{R}^\infty$.

THEOREM 7.4. *Let $(\mathcal{G}_i \rightrightarrows M_i)_{i \in \mathbb{N}}$ be an ascending sequence of finite dimensional Lie groupoids and $\mathcal{G} \rightrightarrows M$ its direct limit Lie groupoid. Then, for each $x = x = \varinjlim x_i \in M$, the following properties hold:*

- (1) *The isotropy group $\mathcal{G}(x)$ is a convenient Lie group which is the direct limit of the sequence of isotropy groups $(\mathcal{G}_i(x_i))_{i \in \mathbb{N}}$.*
- (2) *The orbit $\mathcal{G}.x$ is an immersed convenient submanifold of M , and $\mathcal{G}.x = \varinjlim (\mathcal{G}_i.x_i)$.*
- (3) *The restriction $\mathbf{t}_x := \mathbf{t}_{|\mathcal{G}(x, -)} : \mathcal{G}(x, -) \rightarrow M$ is a $\mathcal{G}(x)$ principal bundle which is the direct limit of the sequence of principal bundles $(\mathbf{t}_i)_{x_i} : \mathcal{G}_i(x_i, -) \rightarrow M_i$.*
- (4) *The associated convenient Lie algebroid $(\mathcal{AG}, M, \rho, [\ , \])$ is split and, for any $x \in M$, its orbit $\mathcal{G}.x$ is a closed immersed submanifold of M whose connected components are orbits of the Banach-Lie algebroid $\mathcal{AG}$.*

EXAMPLE 7.5. Direct limit of finite-dimensional cotangent groupoids.

Let $(\mathcal{G}_i \rightrightarrows M_i)_{i \in \mathbb{N}}$ be an ascending sequence of finite-dimensional Poisson-Lie groupoids, where each structure map $\varphi_i^j : \mathcal{G}_i \rightarrow \mathcal{G}_j$ is an injective immersion and a morphism of Lie groupoids, identifying $\mathcal{G}_i$ with a Lie subgroupoid of $\mathcal{G}_j$.

Assume that, for every $i \leq j$, there exists a smooth vector bundle splitting along $\mathcal{G}_i$:

$$T\mathcal{G}_j|_{\mathcal{G}_i} = T\mathcal{G}_i \oplus N_i^j,$$

where N_i^j is a smooth choice of a supplement, and that the sequence (N_i^j) is compatible in the sense that for all $i \leq j \leq k$,

$$T\mathcal{G}_k|_{\mathcal{G}_i} = T\mathcal{G}_i \oplus N_i^k \quad \text{and} \quad N_i^k|_{\mathcal{G}_j} = N_i^j \oplus N_j^k.$$

Define, for each $i \leq j$, a bundle morphism $\iota_i^j : T_i^{\mathcal{G}} \rightarrow T_j^{\mathcal{G}}$ as follows.

For $g \in \mathcal{G}_i$ and $\alpha_g \in T_g^*\mathcal{G}_i$, the covector $\iota_i^j(\alpha_g) \in T_{\varphi_i^j(g)}^*\mathcal{G}_j$ is the unique element satisfying:

$$(T_g\varphi_i^j)^* \left(\iota_i^j(\alpha_g) \right) = \alpha_g \quad \text{and} \quad \iota_i^j(\alpha_g)|_{N_i^j(g)} = 0.$$

Then each ι_i^j is a smooth injective vector bundle morphism, and the compatibility of the splittings implies:

$$\iota_i^i = \text{Id}_{T^*\mathcal{G}_i} \quad \text{and} \quad \iota_j^k \circ \iota_i^j = \iota_i^k \quad (i \leq j \leq k).$$

Each cotangent bundle $T^*\mathcal{G}_i \rightrightarrows \mathcal{AG}_i^*$ carries its canonical symplectic Lie groupoid structure. The maps ι_i^j are Lie groupoid morphisms compatible with these structures, hence

$$\left(T^*\mathcal{G}_i \rightrightarrows \mathcal{AG}_i^*, \iota_i^j \right)_{i \in \mathbb{N}}$$

forms an ascending sequence of finite-dimensional symplectic Lie groupoids.

Moreover, $\mathcal{G} = \varinjlim \mathcal{G}_i$ exists in the category of convenient manifolds, and the direct limits

$$T'\mathcal{G} := \varinjlim T^*\mathcal{G}_i, \quad \mathcal{AG}' := \varinjlim \mathcal{AG}_i^*$$

exist in this category, and the induced groupoid

$$T'\mathcal{G} \rightrightarrows \mathcal{AG}'$$

inherits a natural structure of strong convenient symplectic Lie groupoid.

8. VON NEUMANN ALGEBRAS AND LIE-POISSON STRUCTURES

Von Neumann algebras were originally introduced by John von Neumann in the context of operator theory and quantum mechanics. These structures since become fundamental objects in functional analysis, non commutative geometry, and mathematical physics (see [12]). Their geometric aspects, including the Banach-Lie groupoids naturally associated with W^* -algebras, have been extensively studied.

In this section, we apply the Poisson-geometric framework developed in the previous sections to groupoids arising from von Neumann algebras. We first recall the construction of the groupoid associated with a von Neumann algebra and then consider the orbit groupoid of projections. Next, we introduce Lie-Poisson structures on the corresponding weak cotangent bundle groupoid. Finally, we illustrate these constructions in the important example $\mathfrak{M} = \mathcal{L}_\infty(\mathcal{H})$, where the associated Fredholm groupoid provides a concrete realization of the theory.

8.1. VON NEUMANN ALGEBRAS. Let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and associated norm $\|\cdot\|$. We denote by $\mathcal{L}_\infty(\mathcal{H})$ the Banach algebra of all bounded linear operators on $\mathcal{H}$, equipped with the operator norm $\mathcal{L}_\infty(\mathcal{H})$ defined by $\|A\|_\infty = \sup_{\|x\| \leq 1} \|Ax\|$.

It is an involutive Banach algebra with involution given by the adjoint $A \mapsto A^*$, and a Lie algebra structure induced by the commutator

$$[A, B] := AB - BA,$$

where AB denotes composition $A \circ B$.

We recall several standard operator topologies on $\mathcal{L}_\infty(\mathcal{H})$:

- (1) The *strong operator topology*.

A basis of neighbourhoods of $A \in \mathcal{L}_\infty(\mathcal{H})$ for this topology is

$$V(A, x_1, \dots, x_n, \varepsilon) := \{B : \forall i \in \{1, \dots, n\}, \|(B - A)x_i\|_\infty < \varepsilon\}.$$

- (2) The *weak operator topology*.

A basis of neighbourhood of $A \in \mathcal{L}_\infty(\mathcal{H})$ for this topology is

$$V(A, x_1, \dots, x_n, \eta_1, \dots, \eta_n, \varepsilon) := \{B : \forall i \in \{1, \dots, n\}, |\langle (B - A)x_i | \eta_i \rangle| < \varepsilon\}.$$

- (3) The *weak topology*.

It is the topology of pointwise convergence on $\mathcal{H}$ in the "weak topology" on $\mathcal{H}$ according to the inner product.

We have the following order between these topologies

$$\{\text{weak operator topology}\} < \{\text{strong operator topology}\} < \{\text{norm topology}\}.$$

A unital²² $*$ -subalgebra of $\mathcal{L}_\infty(\mathcal{H})$ is a unital subalgebra $\mathfrak{b} \subset \mathcal{L}_\infty(\mathcal{H})$ that is closed under the adjoint operation:

$$A \in \mathfrak{b} \implies A^* \in \mathfrak{b}.$$

DEFINITION 8.1. A *von Neumann algebra* is a unital $*$ -subalgebra $\mathfrak{M} \subset \mathcal{L}_\infty(\mathcal{H})$ which is closed in the weak operator topology.

²²That is the subalgebra contains the identity operator $\text{Id}_\mathcal{H}$.

Remark 8.2. For any Banach space $\mathbb{E}$, a predual of $\mathbb{E}$ is a Banach space $\mathbb{E}_*$ such that $(\mathbb{E}_*)'$ is isomorphic to $\mathbb{E}$.

In general, a Banach space may admit several non-isomorphic preduals. A remarkable result due to S. Sakai (cf. [37]) states that a C^* -algebra $\mathfrak{M}$ is a von Neumann algebra if and only if it admits a unique predual $\mathfrak{M}_*$. Thus, von Neumann algebras may be characterized intrinsically as those C^* -algebras that are dual Banach spaces.

A fundamental example is provided by the trace-class operators $L_1(\mathcal{H})$, whose dual space is canonically identified with $L_\infty(\mathcal{H})$:

$$(L_1(\mathcal{H}))' \cong L_\infty(\mathcal{H}).$$

Hence $L_1(\mathcal{H})$ is the predual of the von Neumann algebra $L_\infty(\mathcal{H})$. Moreover, the dual of the space $L_K(\mathcal{H})$ of compact operators is canonically identified with $L_1(\mathcal{H})$.

EXAMPLE 8.3. Any finite dimensional unital $*$ -subalgebra of $L_\infty(\mathcal{H})$ is a von Neumann algebra.

EXAMPLE 8.4. Let $([0, 1], \mathcal{B}([0, 1]), dx)$ be the unit interval equipped with the Lebesgue measure, and consider the Hilbert space $\mathcal{H} = L^2([0, 1], dx)$. The algebra $L^\infty([0, 1], dx)$ can be identified with the algebra of bounded multiplication operators on $\mathcal{H}$. More precisely, for each $f \in L^\infty([0, 1], dx)$, the operator $M_f : L^2([0, 1], dx) \rightarrow L^2([0, 1], dx)$ is defined, for every $g \in L^2([0, 1], dx)$ by:

$$(M_f(g))(x) = f(x)g(x), \quad g \in L^2([0, 1], dx).$$

The collection $\{M_f : f \in L^\infty([0, 1], dx)\}$ forms a commutative von Neumann algebra.

8.2. GROUPOID ASSOCIATED WITH A VON NEUMANN ALGEBRA. For this paragraph, we refer to [34].

Let $\mathfrak{M}$ be a von Neumann algebra. We denote by $\mathcal{L}(\mathfrak{M})$ the lattice²³ of projections of $\mathfrak{M}$ and by $\mathcal{U}(\mathfrak{M})$ the unitary group of $\mathfrak{M}$, namely:

$$\mathcal{L}(\mathfrak{M}) := \{p \in \mathfrak{M} : p = p^2 = p^*\} \quad \text{and} \quad \mathcal{U}(\mathfrak{M}) := \{u \in \mathfrak{M} : u^*u = uu^* = 1\}.$$

²³The set $\mathcal{L}(\mathfrak{M})$ is partially ordered by $p \leq q \Leftrightarrow pq = p$. In fact, $\mathcal{L}(\mathfrak{M})$ is a lattice: a partially ordered set $(L, \leq)$ is called a *lattice* if every two-element subset $\{a, b\} \subset L$ admits a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$.

For $x \in \mathfrak{M}$, let $x = u|x|$ be its polar decomposition, where $|x| = (x^*x)^{1/2} \in \mathfrak{M}^+$ and $\mathfrak{M}^+$ denotes the cone of positive elements of $\mathfrak{M}$.

The operator $u \in \mathfrak{M}$ is a partial isometry²⁴, uniquely determined by the conditions

$$u^*u = s(x), \quad uu^* = t(x),$$

where $s(x), t(x) \in \mathcal{L}(\mathfrak{M})$. In particular, these are projections in the lattice $\mathcal{L}(\mathfrak{M})$.

For each projection $p \in \mathcal{L}(\mathfrak{M})$, let $G(p\mathfrak{M}p)$ denote the group of invertible elements of the von Neumann algebra $p\mathfrak{M}p$, and let $U(p\mathfrak{M}p)$ denote its unitary group.

In particular, when $p = 1$, one has $p\mathfrak{M}p = \mathfrak{M}$, and we simply write $G(\mathfrak{M})$ and $U(\mathfrak{M})$.

For any $x \in \mathfrak{M}$, let $x = u|x|$ be its polar decomposition, where $|x| = (x^*x)^{1/2} \in \mathfrak{M}^+$ and u is a partial isometry. If $p = s(x) := u^*u$, then $|x| \in p\mathfrak{M}p$, since $p|x| = |x| = |x|p$.

We define

$$\mathcal{G}(\mathfrak{M}) := \{u \in \mathfrak{M} : u^*u \in \mathcal{L}(\mathfrak{M}) \text{ and } uu^* \in \mathcal{L}(\mathfrak{M})\}.$$

The following theorem summarizes some essential results proved in [32], 2:

THEOREM 8.5. 1. $\mathcal{G}(\mathfrak{M})$ has a canonical structure of groupoid over $\mathcal{L}(\mathfrak{M})$, where:

- (i) the identity section $\mathbf{1} : \mathcal{L}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ is the natural inclusion;
- (ii) the source and target maps $\mathbf{s}, \mathbf{t} : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ are given, for $x = u|x|$, by

$$\mathbf{s}(x) = u^*u \quad \text{and} \quad \mathbf{t}(x) = uu^*$$

- (iii) the multiplication is the product in $\mathfrak{M}$, defined for $x, y \in \mathcal{G}(\mathfrak{M})$ whenever $\mathbf{s}(x) = \mathbf{t}(y)$, and in this case $xy \in \mathcal{G}(\mathfrak{M})$;
- (iv) the inverse $\mathbf{i} : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ is given by $\mathbf{i}(x) = x^*$.

2. $\mathcal{G}(\mathfrak{M})$ is a wide subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

²⁴An element $u \in \mathfrak{M}$ is called a *partial isometry* if u^*u is a projection (equivalently, uu^* is a projection).

8.3. THE ORBIT GROUPOID OF PROJECTIONS IN A VON NEUMANN ALGEBRA. For this subsection, we again refer to [32].

Given a Von Neumann algebra $\mathfrak{M}$, recall that two projections $p, p' \in \mathfrak{M}$ are called *Murray-von Neumann equivalent* and denoted $p \sim p'$ if there exists $u \in \mathfrak{M}$ such that $p = uu^*$ and $p' = u^*u$. Fix some $p_0 \in \mathcal{L}(\mathfrak{M})$ and set

$$\begin{aligned}\mathcal{L}_{p_0} &:= \{p \in \mathcal{L}(\mathfrak{M}) : p \sim p_0\}, \\ \mathcal{G}_{p_0}(\mathfrak{M}) &:= \mathbf{t}^{-1}(\mathcal{L}_{p_0}(\mathfrak{M})) \cap \mathbf{s}^{-1}(\mathcal{L}_{p_0}(\mathfrak{M})).\end{aligned}$$

Then $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ is a subgroupoid of $\mathcal{G}(\mathfrak{M} \rightrightarrows \mathcal{L}(\mathfrak{M}))$. If $p \sim p_0$, then $\mathcal{G}_p(\mathfrak{M}) \rightrightarrows \mathcal{L}_p(\mathfrak{M})$ and $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ coincide. From Theorem 4.4 and [32] on the other hand, we have the following illustration of Theorem 4.4 in this context:

- THEOREM 8.6. 1. Each $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ is an open and closed subgroupoid of the Banach-Lie groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.
2. The Banach-Lie groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a disjoint union of the Banach-Lie groupoids $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ for $p_0 \in \mathcal{L}(\mathfrak{M})$. In particular, each set $\mathcal{L}_{p_0}(\mathfrak{M})$ is an orbit of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.
3. Let $P_0 := \mathbf{s}^{-1}(p_0) \cap \mathcal{G}_{p_0}(\mathfrak{M})$. Then $P_0 \rightarrow \mathcal{L}_{p_0}(\mathfrak{M})$, given by the restriction of the target map, is a principal bundle with structure group $G_0 := \{g \in \mathcal{G}_{p_0}(\mathfrak{M}) : \mathbf{s}(g) = p_0, \mathbf{t}(g) = p_0\}$, i.e. the isotropy group at p_0 .

To clarify the connection between the proofs in [32] and direct applications of Theorem 4.4, we conclude this subsection by interpreting these results in the setting of von Neumann algebras, following the results of Section 1 in [32].

For a fixed $p_0 \in \mathcal{L}(\mathfrak{M})$, the set P_0 is an open subset of $\mathfrak{M}p_0$. Moreover, G_0 denotes the group $G(p_0\mathfrak{M}p_0)$ of invertible elements of the von Neumann subalgebra $p_0\mathfrak{M}p_0$.

There is a natural free right action of G_0 on P_0 , which induces a G_0 -principal bundle

$$P_0 \longrightarrow P_0/G_0 \quad \text{and} \quad (P_0 \times P_0) \longrightarrow (P_0 \times P_0)/G_0,$$

both of which are isomorphic to the principal bundle $P_0 \rightarrow \mathcal{L}_{p_0}(\mathfrak{M})$. In particular, $\mathcal{L}_{p_0}(\mathfrak{M})$ (respectively $\mathcal{G}_{p_0}(\mathfrak{M})$) is isomorphic to P_0/G_0 (respectively $(P_0 \times P_0)/G_0$). In fact, we have the following diagram of groupoid isomor-

phisms (see [32, Proposition 1.4]):

$$\begin{array}{ccc} (P_0 \times P_0)/G_0 & \longrightarrow & \mathcal{G}_{p_0}(\mathfrak{M}) \\ \Downarrow & & \Downarrow \\ P_0/G_0 & \longrightarrow & \mathcal{L}_{p_0}(\mathfrak{M}) \end{array}$$

8.4. LIE-POISSON STRUCTURES ON THE WEAK COTANGENT BUNDLE GROUPOID. For fixed $p_0 \in \mathcal{L}(\mathfrak{M})$, we set $P_0 = \mathbf{s}^{-1}(p_0)$. Then P_0 is an open subset of the manifold $\mathfrak{M}p_0$. Thus the tangent bundle TP_0 is isomorphic to $P_0 \times \mathfrak{M}p_0$, and its cotangent bundle is isomorphic to $P_0 \times p_0\mathfrak{M}^*$.

Consider the predual $(\mathfrak{M}p_0)_* = p_0\mathfrak{M}_*$ of $(\mathfrak{M}p_0)^* = p_0\mathfrak{M}^*$. Hence the bundle $P_0 \times p_0\mathfrak{M}_*$ is a weak Banach subbundle of $P_0 \times p_0\mathfrak{M}^*$, which defines a weak Banach subbundle T_*P_0 of T^*P_0 . We denote by $\iota : T_*P_0 \hookrightarrow T^*P_0$ the inclusion morphism.

Now, according to [32, Proposition 5.2 and Remark 5.1], if ω denotes the canonical symplectic form on T^*P_0 , then $\omega_0 = \iota^*\omega$ is a weak symplectic form on T_*P_0 . More precisely, identifying T_*P_0 with $P_0 \times p_0\mathfrak{M}_*$ and T^*P_0 with $P_0 \times p_0\mathfrak{M}^*$, we obtain:

$$T(T_*P_0) \simeq (P_0 \times p_0\mathfrak{M}_*) \times (\mathfrak{M}p_0 \times p_0\mathfrak{M}_*),$$

and the image of $\omega_0^\flat$ is

$$\begin{aligned} T^\flat(T_*P_0) &\simeq (P_0 \times p_0\mathfrak{M}_*) \times (\mathfrak{M}p_0 \times p_0\mathfrak{M}_*) \\ &\subset (P_0 \times p_0\mathfrak{M}_*) \times (\mathfrak{M}p_0 \times (\mathfrak{M}p_0)^*) \simeq T^*(T_*P_0). \end{aligned}$$

Finally, the weak symplectic form ω_0 defines a Poisson-Lie structure on the algebra $\mathfrak{A}(T_*P_0)$ associated with $T^\flat(T_*P_0)$. In particular, $\mathbf{P} := (\omega_0^\flat)^{-1}$ is a Poisson anchor.

On the other hand, we consider the pair groupoid $T_*P_0 \times T_*P_0 \rightrightarrows T_*P_0$ and the Poisson anchor

$$\mathbf{P} \times \mathbf{P} : T^\flat(T_*P_0 \times T_*P_0) \longrightarrow T(T_*P_0 \times T_*P_0).$$

In [32, P. 40] it is proved:

PROPOSITION 8.7. *The groupoid $T_*P_0 \times T_*P_0 \rightrightarrows T_*P_0$ admits a sub-Poisson groupoid structure associated with the following commutative diagram of groupoid morphisms:*

$$\begin{array}{ccc} T^b(T_*P_0 \times T_*P_0) & \xrightarrow{\mathbf{P} \times \mathbf{P}} & T(T_*P_0 \times T_*P_0) \\ \Downarrow & & \Downarrow \\ T^b(T_*P_0) & \xrightarrow{\mathbf{P}} & T(T_*P_0) \end{array} \quad (8.1)$$

8.5. FREDHOLM GROUPOID ASSOCIATED WITH $\mathfrak{M} = \mathcal{L}_\infty(\mathcal{H})$. In this section, we consider the von Neumann algebra $\mathfrak{M} = \mathcal{L}_\infty(\mathcal{H})$ and recall results from [32, Section 6.3].

Recall that the Banach predual $(\mathcal{L}_\infty(\mathcal{H}))_*$ is $\mathcal{L}_1(\mathcal{H})$, and that the dual pairing between $\mathcal{L}_1(\mathcal{H})$ and $\mathcal{L}_\infty(\mathcal{H})$ is given by

$$\langle A, B \rangle := \text{Tr}(AB).$$

The set $\mathcal{L}(\mathcal{L}_\infty(\mathcal{H}))$ of orthogonal projections is naturally identified with $\mathcal{L}(\mathcal{H})$. The associated groupoid $\mathcal{G}(\mathcal{L}_\infty(\mathcal{H})) \rightrightarrows \mathcal{L}(\mathcal{L}_\infty(\mathcal{H}))$ can therefore be identified with the groupoid $\mathcal{G}(\mathcal{H}) \rightrightarrows \mathcal{L}(\mathcal{H})$, where the source and target maps are given by:

$$\mathbf{s}(A) = (\ker A)^\perp \quad \text{and} \quad \mathbf{t}(A) = \overline{\text{Im } A}.$$

For $N \in \mathbb{N}$, let $\mathcal{L}_N(\mathcal{H})$ be the set of projections of rank N , and define

$$\mathcal{L}_{\text{fin}}(\mathcal{H}) := \bigcup_{N=1}^{\infty} \mathcal{L}_N(\mathcal{H}).$$

We then set

$$\mathcal{G}_{\text{fin}}(\mathcal{H}) := \mathbf{s}^{-1}(\mathcal{L}_{\text{fin}}(\mathcal{H})) \cap \mathbf{t}^{-1}(\mathcal{L}_{\text{fin}}(\mathcal{H})).$$

Thus $\mathcal{G}_{\text{fin}}(\mathcal{H}) \rightrightarrows \mathcal{L}_{\text{fin}}(\mathcal{H})$ is the groupoid of finite-rank operators.

On the other hand, consider the groupoid $\mathcal{G}_\infty(\mathcal{H}) \rightrightarrows \mathcal{L}_\infty(\mathcal{H})$ of partially invertible operators with infinite-dimensional range.

We define the involution $\perp: \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ by $\perp(p) := p^\perp := 1 - p$, for every $p \in \mathcal{L}(\mathcal{H})$. This map is an involutive automorphism of $\mathcal{L}(\mathcal{H})$, i.e. $\perp \circ \perp = \text{Id}_{\mathcal{L}(\mathcal{H})}$.

We define the subgroupoid $\mathcal{G}_{\text{Fred}}(\mathcal{H}) \rightrightarrows \mathcal{L}_{\text{Fred}}(\mathcal{H})$ of $\mathcal{G}_\infty(\mathcal{H}) \rightrightarrows \mathcal{L}_\infty(\mathcal{H})$ by setting

$$\mathcal{L}_{\text{Fred}}(\mathcal{H}) := \perp(\mathcal{L}_{\text{fin}}(\mathcal{H})),$$

and

$$\mathcal{G}_{\text{Fred}}(\mathcal{H}) := \mathbf{s}^{-1}(\mathcal{L}_{\text{Fred}}(\mathcal{H})) \cap \mathbf{t}^{-1}(\mathcal{L}_{\text{Fred}}(\mathcal{H})).$$

Now, as before, for any fixed $p_0 \in \mathcal{L}(\mathcal{H})$, we consider

$$P_0^{\text{Fred}} := \mathbf{s}^{-1}(p_0) \cap \mathcal{G}_{\text{Fred}}(\mathcal{H}).$$

To P_0^{Fred} is associated a weak Banach subbundle $T_*P_0^{\text{Fred}}$ of $T^*P_0^{\text{Fred}}$, whose fibres are the preduals of the fibres of TP_0^{Fred} .

Then, by the same arguments as in the proof of Proposition 8.7, we obtain:

PROPOSITION 8.8. *The groupoid $T_*P_0^{\text{Fred}} \times T_*P_0^{\text{Fred}} \rightrightarrows T_*P_0^{\text{Fred}}$ admits a sub-Poisson groupoid structure associated with the following commutative diagram of groupoid morphisms:*

$$\begin{array}{ccc} T^b(T_*P_0^{\text{Fred}} \times T_*P_0^{\text{Fred}}) & \xrightarrow{\mathbf{P} \times \mathbf{P}} & T(T_*P_0^{\text{Fred}} \times T_*P_0^{\text{Fred}}) \\ \Downarrow & & \Downarrow \\ T^b(T_*P_0^{\text{Fred}}) & \xrightarrow{\mathbf{P}} & T(T_*P_0^{\text{Fred}}) \end{array} \quad (8.2)$$

A. BOUNDED OPERATORS ON A COMPLEX HILBERT SPACE

A.1. BASIC PRELIMINARIES. The following results on the space of bounded operators on a complex separable Hilbert space are classical. They can be found in the concise and complete notes [18] (see also [12]).

A.1.1. BASIC PROPERTIES OF BOUNDED OPERATORS ON A SEPARABLE COMPLEX HILBERT SPACE. Let $\mathcal{H}$ be a separable complex Hilbert space. We denote by $\langle \cdot, \cdot \rangle$ its inner product and $\|\cdot\|$ its associated norm. Let $\mathbf{L}_\infty(\mathcal{H})$ be the *Lie algebra Banach space of bounded linear operators* on $\mathcal{H}$ endowed with the classical operator norm $\|A\|_\infty = \sup_{\|x\| \leq 1} \|Ax\|$ and the Lie bracket of

commutator $[A, B] = AB - BA$ where AB means the composition $A \circ B$.

We now recall some classical properties of bounded operators.

The adjoint of $A \in \mathbf{L}_\infty(\mathcal{H})$ is denoted by A^* .

An operator $A \in \mathbf{L}_\infty(\mathcal{H})$ is called *self-adjoint* if $A = A^*$.

An operator $p \in \mathbf{L}_\infty(\mathcal{H})$ is called a *projection* if $p^2 = p = p^*$.

An operator $A \in \mathbf{L}_\infty(\mathcal{H})$ is called *positive* if $\langle Ax|x \rangle \geq 0$ for all $x \in \mathcal{H}$; in this case, A is self-adjoint.

For $A, B \in \mathbf{L}_\infty(\mathcal{H})$, we write $A \geq B$ if $A - B$ is positive.

An operator $u \in \mathcal{L}(\mathcal{H})$ is an *isometry* if $uu^* = \text{Id}$.

An operator $u \in \mathcal{L}(\mathcal{H})$ is *unitary* if $uu^* = u^*u = \text{Id}$.

An operator $u \in \mathcal{L}(\mathcal{H})$ is a *partial isometry* if u^*u is a projection.

LEMMA A.1. *If u is a partial isometry then so is u^* .*

The subspace $u^\mathcal{H}$ is closed and is called the initial domain of u , while the subspace $u\mathcal{H}$ is also closed and is called the final domain of u .*

If A is positive, then $A = A^*$ and it admits a unique positive square root $A^{1/2}$, i.e. an operator B such that $B^2 = A$.

For any $A \in \mathcal{L}(\mathcal{H})$, we set $|A| = (A^*A)^{1/2}$.

LEMMA A.2. *For any $A \in \mathcal{L}(\mathcal{H})$, there exists a partial isometry u such that $A = u|A|$, and u is unique under the condition that its initial space is $(\ker A)^\perp$. The final space of u is $A\mathcal{H} = (\ker A^*)^\perp$. Such a decomposition is called the polar decomposition of A .*

A.1.2. SOME CLASSICAL BANACH SETS OF BOUNDED OPERATORS. The trace of a positive operator is defined by:

$$\text{Tr } A := \sum_{n=0}^{+\infty} \langle e_n | A e_n \rangle$$

where $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$. This value does not depend on the choice of the orthonormal basis.

The set $\mathcal{L}_2(\mathcal{H})$ of *Hilbert-Schmidt operators* is the subspace of $\mathcal{L}_\infty(\mathcal{H})$ consisting of operators A such that

$$\|A\|_2 := (\text{Tr}(A^*A))^{1/2} < \infty.$$

It is a Banach space for the norm $\|\cdot\|_2$, stable under the commutator bracket, and it is a two-sided ideal of $\mathcal{L}_\infty(\mathcal{H})$.

For any $A \in \mathcal{L}_\infty(\mathcal{H})$, the operator A^*A is positive and admits a square root, so the set $\mathcal{L}_1(\mathcal{H})$ of *trace-class operators* is defined as the space of operators $A \in \mathcal{L}_\infty(\mathcal{H})$ such that

$$\|A\|_1 := \text{Tr} \left((A^*A)^{1/2} \right) < \infty.$$

It is a Banach space for the norm $\|\cdot\|_1$, stable under the commutator bracket. Moreover, we have

$$|\text{Tr}(A)| \leq \|A\|_1,$$

and thus $\mathcal{L}_1(\mathcal{H}) \hookrightarrow \mathcal{L}_2(\mathcal{H})$ continuously.

For any $1 < p < \infty$, let $\mathcal{L}_p(\mathcal{H})$ denote the space of operators $A \in \mathcal{L}_\infty(\mathcal{H})$ such that:

$$\|A\|_p := \left(\text{Tr}((A^*A)^{p/2}) \right)^{1/p} < \infty.$$

It is a Banach space for the norm $\|\cdot\|_p$, stable under the commutator bracket.

For $1 < p < 2 < q < \infty$, we have the continuous inclusions:

$$\mathcal{L}_1(\mathcal{H}) \hookrightarrow \mathcal{L}_p(\mathcal{H}) \hookrightarrow \mathcal{L}_2(\mathcal{H}) \hookrightarrow \mathcal{L}_q(\mathcal{H}) \hookrightarrow \mathcal{L}_\infty(\mathcal{H}). \quad (\text{A.1})$$

Note that $\mathcal{L}_p(\mathcal{H})$ is dense in $\mathcal{L}_2(\mathcal{H})$, and it is continuously embedded in $\mathcal{L}_\infty(\mathcal{H})$, but it is not supplemented in general.

The bilinear map $(A, B) \mapsto \text{Tr}(AB)$ defines:

- a strong duality pairing between $\mathcal{L}_2(\mathcal{H})$ and itself;
- a weak duality pairing between $\mathcal{L}_1(\mathcal{H})$ and $\mathcal{L}_\infty(\mathcal{H})$, and, in particular, $\mathcal{L}_\infty(\mathcal{H})$ can be identified with the dual of $\mathcal{L}_1(\mathcal{H})$;
- a strong duality pairing between $\mathcal{L}_p(\mathcal{H})$ and $\mathcal{L}_q(\mathcal{H})$ when $\frac{1}{p} + \frac{1}{q} = 1$, so that $(\mathcal{L}_p(\mathcal{H}))^* \simeq \mathcal{L}_q(\mathcal{H})$ and conversely.

A.2. FINITE RANK AND COMPACT OPERATORS. Throughout, we fix an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of $\mathcal{H}$.

Let $\mathcal{L}_{\text{fin}}(\mathcal{H})$ denote the subspace of operators $A \in \mathcal{L}_\infty(\mathcal{H})$ whose range is finite-dimensional.

An operator $A \in \mathcal{L}_\infty(\mathcal{H})$ is called *compact* if the image of the closed unit ball in $\mathcal{H}$ under A is relatively compact. We write $\mathcal{L}_K(\mathcal{H})$ the space of compact operators on $\mathcal{H}$.

The following properties of $\mathcal{L}_K(\mathcal{H})$ are classical.

- The space $\mathcal{L}_K(\mathcal{H})$ is the closure of $\mathcal{L}_{\text{fin}}(\mathcal{H})$ in $\mathcal{L}_\infty(\mathcal{H})$ relative to the norm $\|\cdot\|_\infty$. In particular, $\mathcal{L}_K(\mathcal{H})$ is closed in $\mathcal{L}_\infty(\mathcal{H})$.
- $\mathcal{L}_K(\mathcal{H})$ is an ideal of the algebra $\mathcal{L}_\infty(\mathcal{H})$. In particular, $\mathcal{L}_K(\mathcal{H})$ is a Lie subalgebra of $\mathcal{L}_\infty(\mathcal{H})$.
- One has $\mathcal{L}_2(\mathcal{H}) \subset \mathcal{L}_K(\mathcal{H})$.
- The continuous dual of $\mathcal{L}_K(\mathcal{H})$ is canonically identified with $\mathcal{L}_1(\mathcal{H})$:

$$\mathcal{L}_K(\mathcal{H})' = \mathcal{L}_1(\mathcal{H}),$$

and the continuous dual of $\mathcal{L}_1(\mathcal{H})$ is

$$\mathcal{L}_1(\mathcal{H})' = \mathcal{L}_\infty(\mathcal{H}).$$

Remark A.3. Consider the map $\Phi : \ell^\infty(\mathbb{N}) \rightarrow \mathcal{L}_\infty(\mathcal{H})$ given by:

$$\Phi(\lambda_n)(x) = \sum_{n=0}^{+\infty} \lambda_n \langle x | e_n \rangle e_n.$$

Then we have:

- $\Phi^{-1}(\mathcal{L}_{\text{fin}}(\mathcal{H})) = c_{00}(\mathbb{N})$, where $c_{00}(\mathbb{N})$ denotes the space of finitely supported complex sequences;
- $\Phi^{-1}(\mathcal{L}_K(\mathcal{H})) = c_0(\mathbb{N})$, where $c_0(\mathbb{N})$ denotes the space of complex sequences converging to 0.

Moreover, in each previous situation, Φ is an isometry.

It is well known that, for $1 < p < 2 < q < \infty$, we have the continuous inclusions:

$$\ell^1(\mathbb{N}) \hookrightarrow \ell^p(\mathbb{N}) \hookrightarrow \ell^2(\mathbb{N}) \hookrightarrow \ell^q(\mathbb{N}) \hookrightarrow c_0(\mathbb{N}) \hookrightarrow \ell^\infty(\mathbb{N})$$

where each embedding is continuous. On the other hand, $c_{00}(\mathbb{N}) \subset \ell^1(\mathbb{N})$ is a convenient space which can be viewed as algebraic direct limit of the increasing sequence of finite-dimensional spaces

$$\ell(n) := \{x = (x_k)_{k \in \mathbb{N}} : x_k = 0 \text{ for } k > n\}$$

(cf. proof of Proposition A.4). Moreover, $c_{00}(\mathbb{N})$ is dense in $\ell^1(\mathbb{N})$ and the inclusion $c_{00}(\mathbb{N}) \hookrightarrow \ell^1(\mathbb{N})$ is continuous with respect to its convenient topology.

As in Remark A.3, an analogous situation holds for the corresponding spaces of bounded operators:

PROPOSITION A.4. *The space $\mathcal{L}_{\text{fin}}(\mathcal{H})$ carries a natural convenient vector space structure, and is dense in $\mathcal{L}_1(\mathcal{H})$. Moreover, the inclusion*

$$\mathcal{L}_{\text{fin}}(\mathcal{H}) \hookrightarrow \mathcal{L}_1(\mathcal{H})$$

is continuous when $\mathcal{L}_{\text{fin}}(\mathcal{H})$ is equipped with its convenient topology and $\mathcal{L}_1(\mathcal{H})$ with its Banach space norm topology.

In particular, for $1 < p < 2 < q < \infty$, we obtain the chain of continuous inclusions:

$$\mathcal{L}_{\text{fin}}(\mathcal{H}) \hookrightarrow \mathcal{L}_1(\mathcal{H}) \hookrightarrow \mathcal{L}_p(\mathcal{H}) \hookrightarrow \mathcal{L}_2(\mathcal{H}) \hookrightarrow \mathcal{L}_q(\mathcal{H}) \hookrightarrow \mathcal{L}_K(\mathcal{H}) \hookrightarrow \mathcal{L}_\infty(\mathcal{H}), \quad (\text{A.2})$$

Moreover, for every $1 \leq k \leq \infty$, the diagonal identification Φ satisfies:

$$\Phi^{-1}(\mathcal{L}_k(\mathcal{H})) = \ell^k(\mathbb{N}).$$

Proof. Let $\mathcal{H}_n$ be the finite-dimensional vector subspace of $\mathcal{H}$ generated by $\{e_0, \dots, e_n\}$. We consider the canonical inclusion $\iota_n : \mathcal{H}_n \hookrightarrow \mathcal{H}_{n+1}$. If $\mathbf{L}(\mathcal{H}_n)$ denotes the Banach space of continuous endomorphisms of $\mathcal{H}_n$, then we obtain an induced inclusion $\iota_n^L : \mathbf{L}(\mathcal{H}_n) \hookrightarrow \mathbf{L}(\mathcal{H}_{n+1})$. Thus, the sequence $\{\mathbf{L}(\mathcal{H}_n), \iota_n^L\}_{n \in \mathbb{N}}$ forms an ascending sequence of Banach spaces and its direct limit $\varinjlim \mathbf{L}(\mathcal{H}_n) = \bigcup_{n \in \mathbb{N}} \mathbf{L}(\mathcal{H}_n)$ is contained in a convenient subspace of $\mathbf{L}_1(\mathcal{H})$.

Since $\mathbf{L}(\mathcal{H}_n)$ is a Banach subspace, each inclusion ι_n^L of $\mathbf{L}(\mathcal{H}_n)$ into $\mathbf{L}_1(\mathcal{H})$ is continuous and so is the inclusion $\varinjlim \iota_n^L$ of $\varinjlim \mathbf{L}(\mathcal{H}_n)$ in $\mathbf{L}_1(\mathcal{H})$ (cf. [8, 5.4]).

Note that $\varinjlim \mathbf{L}(\mathcal{H}_n) = \bigcup_{n \in \mathbb{N}} \mathbf{L}(\mathcal{H}_n)$ is also contained in $\mathbf{L}_{\text{fin}}(\mathcal{H})$. Consider $A \in \mathbf{L}_{\text{fin}}(\mathcal{H})$. Then the Kernel K of A is finite codimensional and its range $R = A(E)$ is finite dimensional. Therefore, there exists $n \in \mathbb{N}$ such that $K + \mathcal{H}_n = \mathcal{H}$ and $R \subset \mathcal{H}_n$. Thus, the restriction A_n of A to $\mathcal{H}_n$ belongs to $\mathbf{L}(\mathcal{H}_n)$. For any $m \geq n$, the restriction A_m to $\mathcal{H}_m$ also belongs to $\mathbf{L}(\mathcal{H}_n)$ and the restriction of A_m to $\mathcal{H}_n$ is A_n . It follows that $A = \varinjlim A_n$ and, finally, we have $\varinjlim \mathbf{L}(\mathcal{H}_n) = \mathbf{L}_{\text{fin}}(\mathcal{H})$. This completes the proof of the first part.

Relation (A.2) is then a direct consequence of the previous result and (A.1). Finally, the last part is a consequence of Remark A.3 and the definition of each Banach space $\mathbf{L}_k(\mathcal{H})$ for $1 \leq k \leq \infty$. ■

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