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Cohomologies and deformations of Hom-Jacobi-Jordan algebras and relative Rota-Baxter operators

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Abstract: Representations of Hom-Jacobi-Jordan algebras are studied. In particular, adjoint representations and trivial representations are studied in detail. Derivations and central extensions of Hom-Jacobi-Jordan algebras are also discussed as an application. Morover, we introduce the cohomology theory on Hom-Jacobi-Jordan algebras as well as the one of relative Rota-Baxter operators on Hom-Jacobi-Jordan algebras. Finally, we use the cohomological approach to study deformations of Hom-Jacobi-Jordan algebras and those of relative Rota-Baxter operators.

Key words: Hom-Jacobi-Jordan algebra, relative Rota-Baxter operator, cohomology, deformation.

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1. Introduction

A commutative algebra satisfying the Jacobi identity called a Jacobi-Jordan algebra was introduced first in [21], where an important example of infinite-dimensional solvable but not nilpotent Jacobi-Jordan algebra was given. They are rather special objects in the jungle of non-associative algebras. Different names are used to study these algebras, indeed they are called mock-Lie algebras, Jordan algebras of nil index 3, Lie-Jordan algebras, pathological algebras or Jacobi-Jordan algebras in the literature [5, 6, 8, 9, 15, 18, 19, 22]. In contrary to associative and nonassociative algebras, results on the cohomology theory of Jacobi-Jordan algebras are relatively scarce for a long time. Recently, analogous to the existing theories for associative and Lie algebras, cohomology and deformation theories for Jacobi-Jordan algebras are developed [7] where it is observed that they have several properties not enjoyed by the Hochschild theory. Actually, this cohomology is called a zigzag coho-

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mology since its complex is defined by two sequences of operators. Moreover, a weighted Rota-Baxter Jacobi-Jordan algebra and their representations are studied in [2] where several new constructions of representations on these algebraic structures are given. Observe that the first appearance of Rota-Baxter operators on Hom-algebra is back to the paper [14]. Likewise, a generalized version of these operators called relative Rota-Baxter operators(or $\mathcal{O}$-operators), has subsequently been studied in [13, 17] (see also [3, 4]).

Due to the importance of Hom-algebras in several domains, the twisted generalization of Jacobi-Jordan algebras called Hom-Jacobi-Jordan algebras is initiated in [10] while its representation theory is introduced in [3]. Roughly, a Hom-type of a given algebra is obtained by a certain twisting of the defining identities by a twisting map, in such a way that when this twisting map is the identity map, then one recovers the original algebra. The first class of Hom-algebras is the one of Hom-Lie algebras by Hartwig, Larsson and Silverstory [11] to characterize the structures on deformations of the Witt and the Virasoro algebras. Hom-Jacobi-Jordan algebras are very close to Hom-Lie algebras having only the skew-symmetry condition replaced by the symmetry condition. Cohomology and deformation of associative and nonassociative algebras have been extended to Hom-associative and non-Hom-associative algebras in many papers these lastest years. Observe that in [16], a study of the low dimensional cohomology of metric Hom-Jacobi-Jordan algebras is given where n-coboundary operator is defined just for $n = 1, 2$. Moreover, it is established a one-to-one correspondence between the equivalence classes of abelian quadratic extensions of a Hom-Jacobi-Jordan algebra and its second cohomology group. The aim of this work is, among other things, to extend the cohomology theory of Hom-Jacobi-Jordan algebras started in [16]. More precisely, we define n-coboundary operator for any nonnegative integer n. Observe that, as Jacobi-Jordan algebras, we introduce cohomology theory of Hom-Jacobi-Jordan algebras and it is observed that this cohomology is also a zigzag cohomology since the complex is defined by two sequences of operators.

A description of the rest of this paper is as follows. In Section 2, some basic notions and concepts of Hom-Jacobi-Jordan algebras and their representations are given. In Section 3, derivations of Hom-Jacobi-Jordan algebras are studied and generalized to derivations of Hom-Jacobi-Jordan algebras with values in representations. For any nonnegative integer k, we define α^k-derivations of Hom-Jacobi-Jordan algebras. Several results are proved, in particular, a necessary and sufficient condition for the anti-commutator of two α-derivations (respectively α-antiderivations) to be an α-derivation (respectively α-antiderivation) is obtained. In Section 4, we introduce a cohomology for Hom-Jacobi-Jordan algebras. As Jacobi-Jordan algebras case, it is called a zigzag cohomology since it deals with two types of cochains and two sequences of operators. As applications, we explore the low degree cohomology spaces and explicit computations on examples are provided. Next, trivial and adjoint representations of Hom-Jacobi-Jordan algebras are studied and some results are obtained. Finally, in Section 5, as Jacobi-Jordan algebras case [4], we study cohomologies of relative Rota-Baxter operators of Hom-Jacobi-Jordan algebras with respect to a representation. Next, linear deformations and formal deformations of Hom-Jacobi-Jordan algebras as well as those of relative Rota-Baxter operators are explored.

Throughout this paper, all vector spaces and algebras are meant over a ground field $\mathbb{K}$ of characteristic 0.

2. Basic results on Hom-Jacobi-Jordan algebras

In this section, we introduce some notations and basics about Hom-Jacobi-Jordan algebras and their representations. Some results are proved and many examples of these Hom-algebras are given. For general results about Hom-Jacobi-Jordan algebras and their applications see for instance [3, 10, 16].

Definition 2.1. A Hom-algebra is a tuple (A,μ,α) in which (A,α) is a Hom-module, $\mu : A^{\otimes 2} \to A$ is a linear map. The Hom-algebra (A,μ,α) is said to be multiplicative if $\alpha\circ\mu = \mu\circ\alpha^{\otimes 2}$. A morphism $f : (A,\mu_A,\alpha_A) \to (B,\mu_B,\alpha_B)$ of Hom-algebras is a morphism of the underlying Hom-modules such that $f\circ\mu_A = \mu_B\circ f^{\otimes 2}$.

Definition 2.2. A Hom-associative algebra is a multiplicative Hom-algebra (A,μ,α) satisfying

$$\mu\big(\mu(x,y),\alpha(z)\big) = \mu\big(\alpha(x),\mu(y,z)\big), \qquad \forall\, x,y,z\in A.$$

Let us give the definition of the main Hom-algebra of this paper.

Definition 2.3. A Hom-Jacobi-Jordan algebra is a multiplicative Hom-algebra $(A,*,\alpha)$ such that

$$\begin{aligned} x*y &= y*x \text{ (commutativity)},\\ J_\alpha(x,y,z) &:= \circlearrowleft_{(x,y,z)} (x*y)*\alpha(z) = 0, \qquad \forall\, x,y,z\in A, \end{aligned} \tag{2.1}$$

where $\circlearrowleft_{(x,y,z)}$ is the sum over the cyclic permutation of x, y, z and J_α is called the Hom-Jacobian.

Remark 2.4. If $\alpha = Id$ (identity map) in a Hom-Jacobi-Jordan algebra $(A, *, \alpha)$, then it reduces to a usual Jacobi-Jordan algebra $(A, *)$. It follows that the category of Hom-Jacobi-Jordan algebras contains the one of Jacobi-Jordan algebras.

EXAMPLE 2.5. Let A be a 3-dimensional vector space generated by (e_1, e_2, e_3). Then $\mathcal{A}_1 := (A, *, \alpha)$ is a Hom-Jacobi-Jordan algebra where $\alpha(e_1) := 2e_3$, $\alpha(e_2) := 4e_2$, $\alpha(e_3) := 2e_1$ with non-zero products given by $e_1 * e_1 = e_3 * e_3 := 4e_2$.

It is easy to prove:

PROPOSITION 2.6. *Let $(A, *)$ be a Jacobi-Jordan algebra and $\alpha : A \to A$ be a morphism of the algebra $(A, *)$. Then $A_\alpha := (A, *_\alpha := \alpha \circ *, \alpha)$ is a Hom-Jacobi-Jordan algebra.*

EXAMPLE 2.7. Let $\mathcal{A}_2 = (A, *)$ be the Jacobi-Jordan algebra defined and denoted by $A_{1,4}$ in ([8], Proposition 3.4). It is defined with respect to a basis (e_1, e_2, e_3, e_4) by

$$e_1 * e_1 = e_2, \qquad e_1 * e_3 = e_3 * e_1 = e_4.$$

A linear map $\alpha : A \to A$ defined by

$$\alpha := \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

is a self-morphism of $\mathcal{A}_2$ if and only if

$$\alpha(e_i * e_j) = \alpha(e_i) * \alpha(e_j), \qquad \forall\, (i, j) \in \{1, 2, 3, 4\}^2 \text{ with } i \le j. \tag{2.2}$$

Observe that (2.2) holds for $\mathcal{A}_2$ if and only if

$$b_1 = b_3 = c_1 = d_1 = d_2 = d_3 = 0, \quad b_2 = a_1^2, \quad b_4 = 2a_1a_3, \quad d_4 = a_1c_3.$$

Hence,

$$\alpha := \begin{pmatrix} a_1 & 0 & 0 & 0 \\ a_2 & a_1^2 & c_2 & 0 \\ a_3 & 0 & c_3 & 0 \\ a_4 & 2a_1a_3 & c_4 & a_1c_3 \end{pmatrix}$$

and by Proposition 2.6, $\mathcal{A}_2$ is twisted into the Hom-Jacobi-Jordan algebra $(\mathcal{A}_2)_\alpha = (A, *_\alpha, \alpha)$ with non-zero products given by

$$e_1 *_\alpha e_1 = a_1^2 e_2 + 2a_1a_3e_4, \qquad e_1 *_\alpha e_3 = e_3 *_\alpha e_1 = a_1c_3e_4.$$

This Hom-Jacobi-Jordan algebra is non-zero if and only if $a_1 \neq 0$.

Let us give examples of 5-dimensional Hom-Jacobi-Jordan algebras.

Example 2.8. It is proved ([8], Proposition 4.1) that any non associative Jacobi-Jordan algebra of dimension 5 over an algebraically closed field of characteristic different from 2 and 3, is isomorphic to the following algebra, given by the products of the basis $(e_1, \dots, e_5)$ as follows:

$$\begin{aligned} &(A_1) : e_1 * e_1 = e_2, \qquad e_1 * e_4 = e_4 * e_1 = e_5, \\ &(A_2) : e_1 * e_5 = e_5 * e_1 = -\frac{1}{2}e_3, \qquad e_2 * e_4 = e_4 * e_2 = e_3, \end{aligned}$$

If $(A, *)$ is a 5-dimensional Jacobi-Jordan algebra with basis $(e_1, \dots, e_5)$, then a linear map $\alpha : A \to A$ defined by

$$\alpha := \begin{pmatrix} a_1 & b_1 & c_1 & d_1 & t_1 \\ a_2 & b_2 & c_2 & d_2 & t_2 \\ a_3 & b_3 & c_3 & d_3 & t_3 \\ a_4 & b_4 & c_4 & d_4 & t_4 \\ a_5 & b_5 & c_5 & d_5 & t_5 \end{pmatrix}$$

is a self-morphism of $(A, *)$ if and only if

$$\alpha(e_i * e_j) = \alpha(e_i) * \alpha(e_j), \qquad \forall\, (i, j) \in \{1, 2, 3, 4, 5\}^2 \text{ with } i \leq j. \tag{2.3}$$

Now, with the linear map α given as above, we shall discuss the conditions (2.3) for each of the types (A1) and (A2).

• Case of type (A_1):
For this type, the conditions (2.3) lead to the following simultaneous constraints on the coefficients a_i, b_j, c_k, d_l, t_s of α:

$$\begin{aligned}
&b_1 = b_3 = b_4 = c_1 = d_1 = t_1 = t_2 = t_3 = t_4 = 0,\\
&a_1^2 - b_2 = 2a_1a_2 - b_5 = a_1c_1 = a_1d_4 = a_1d_4 - t_5 = 0.
\end{aligned}$$

If $a_1 = 0$, from the equations above, we see that α is defined as

$$\alpha := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & c_2 & d_2 & 0 \\ a_3 & 0 & c_3 & d_3 & 0 \\ a_4 & 0 & c_4 & d_4 & 0 \\ a_5 & 0 & c_5 & d_5 & 0 \end{pmatrix}.$$

Then, applying Proposition 2.6, we get that (A1) is twisted into the zero Hom-Jacobi-Jordan. If $a_1 \neq 0$, the equations above give the morphism

$$\alpha := \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ \frac{b_5}{a_1} & a_1^2 & c_2 & d_2 & 0 \\ a_3 & 0 & c_3 & d_3 & 0 \\ a_4 & 0 & 0 & 0 & 0 \\ a_5 & b_5 & c_5 & d_5 & 0 \end{pmatrix}.$$

Hence, by Proposition 2.6, (A1) is twisted into the Hom-Jacobi-Jordan $(A, \alpha_\alpha, \alpha)$ with non-zero products given by:

$$e_1 *_\alpha e_1 := a_1^2 e_2 + b_5 e_5.$$

Observe that, this Hom-algebra is a Hom-associative algebra.

• Case of type (A_2):
For this type, the conditions (2.3) give the following simultaneous constraints on the coefficients a_i, b_j, c_k, d_l, t_s of α:

$$a_1 = b_1 = c_1 = c_2 = c_3 = c_4 = c_5 = d_1 = t_1 = 0.$$

Hence, the morphism α is defined by

$$\alpha := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & 0 & d_2 & t_2 \\ a_3 & b_3 & 0 & d_3 & t_3 \\ a_4 & b_4 & 0 & d_4 & t_4 \\ a_5 & b_5 & 0 & d_5 & t_5 \end{pmatrix}.$$

Then, applying Proposition 2.6, we get that (A_2) is twisted into the zero Hom-Jacobi-Jordan.

We have the following result.

PROPOSITION 2.9. *Let $(L,*,\alpha)$ be a Hom-Jacobi-Jordan algebra and $(A,\bullet,\beta)$ be a commutative Hom-associative algebra. Then $(L\otimes A,\circ,\theta:=\alpha\otimes\beta)$ is a Hom-Jacobi-Jordan algebra where*

$$(x\otimes a)\circ(y\otimes b)=(x*y)\otimes(a\bullet b),\qquad \forall\, x,y\in L \text{ and } a,b\in A,$$
$$\theta(x\otimes b)=\alpha(x)\otimes\beta(a),\qquad \forall\, x\in L \text{ and } a\in A.$$

Proof. First, the multiplicativity of θ with respect to $\circ$ follows from those of α and β with respect to $*$ and $\bullet$, respectively. Next, the commutativity of $\circ$ follows from the one of $*$ and $\bullet$. Finally, pick $x,y,z\in L$ and $a,b,c\in A$. Then,

$$\begin{aligned}
&\big((x\otimes a)\circ(y\otimes b)\big)\circ\theta(z\otimes c)+\big((y\otimes b)\circ(z\otimes c)\big)\circ\theta(x\otimes a)\\
&\qquad\qquad+\big((z\otimes c)\circ(x\otimes a)\big)\circ\theta(y\otimes b)\\
&=\big((x*y)\otimes(a\bullet b)\big)\circ\big(\alpha(z)\otimes\beta(c)\big)+\big((y*z)\otimes(b\bullet c)\big)\circ\big(\alpha(x)\otimes\beta(a)\big)\\
&\qquad\qquad+\big((z*x)\otimes(c\bullet a)\big)\circ\big(\alpha(y)\otimes\beta(b)\big)\\
&=\big((x*y)*\alpha(z)\big)\otimes\big((a\bullet b)\bullet\beta(c)\big)+\big((y*z)*\alpha(x)\big)\otimes\big((b\bullet c)\bullet\beta(a)\big)\\
&\qquad\qquad+\big((z*x)*\alpha(y)\big)\otimes\big((c\bullet a)\bullet\beta(b)\big)\\
&=\big((x*y)*\alpha(z)+(y*z)*\alpha(x)+(z*x)*\alpha(y)\big)\otimes\big((a\bullet b)\bullet\beta(c)\big)\\
&\qquad\text{(by the commutativity and the Hom-associativity of } (A,\bullet,\beta))\\
&=0\quad\text{(by (2.1))}.
\end{aligned}$$

Hence, $(L\otimes A,\circ,\theta:=\alpha\otimes\beta)$ is a Hom-Jacobi-Jordan algebra. ∎

Similarly, we can also prove the next result.

PROPOSITION 2.10. *Let $(L,*,\alpha)$ be a Hom-Jacobi-Jordan algebra and $(A,\bullet)$ be a commutative associative algebra. Then $(L\otimes A,\circ,\theta_0:=\alpha\otimes Id_A)$ is a Hom-Jacobi-Jordan algebra. Precisely:*

$$(x\otimes a)\circ(y\otimes b)=(x*y)\otimes(a\bullet b),\qquad \forall\, x,y\in L \text{ and } a,b\in A,$$
$$\theta_0(x\otimes b)=\alpha(x)\otimes a,\qquad \forall\, x\in L \text{ and } a\in A.$$

Definition 2.11. Let $(A, *, \alpha)$ be Hom-Jacobi-Jordan algebra and B be a vector subspace of A. Then, (B, α) is said to be a Hom-ideal of $(A, *, \alpha)$ if for all $(a, b) \in A \times B$, $\alpha(b) \in B$ and $a * b \in B$.

It is easy to prove the following.

Proposition 2.12. *Let $(A, *, \alpha)$ be Hom-Jacobi-Jordan. Then, the set*

$$\mathrm{HAnn}(A) := \{b \in A \,:\, \alpha(b) = b \text{ and } a * b = 0 \; \forall\, a \in A\}.$$

is a Hom-ideal of $(A, *, \alpha)$ with respect to α, called the Hom-annihilator of $(A, *, \alpha)$.

Now, we recall the definition of representations of a Hom-Jacobi-Jordan algebra.

Definition 2.13. [3] A representation of a Hom-Jacobi-Jordan algebra $(A, *, \alpha)$ is a tuple (V, ρ, ϕ) where V is a vector space, $\phi \in gl(V)$ and $\rho : A \to gl(V)$ is a linear map such that the following equalities hold for all $x, y \in A$,

$$\phi\rho(x) = \rho\big(\alpha(x)\big)\phi, \tag{2.4}$$

$$\rho(x * y)\phi = -\rho\big(\alpha(x)\big)\rho(y) - \rho\big(\alpha(y)\big)\rho(x), \qquad \forall\, x, y \in A. \tag{2.5}$$

A representation of $(A, *, \alpha)$ as above is denoted by (V, ρ, ϕ).

Remark 2.14. If $\alpha = Id_A,\ \phi = Id_V$, then (V, ρ, ϕ) reduces to a representation (V, ρ) of the Jacobi-Jordan algebra $(A, *)$ [1, 12].

Example 2.15. [3] Let $(A, *, \alpha)$ be a Hom-Jacobi-Jordan algebra and (B, α) be a Hom-ideal of $(A, *, \alpha)$. Set $\rho(a)b := a * b$ for all $(a, b) \in A \times B$, then (B, ρ, α) is a representation of $(A, *, \alpha)$.

It is easy to prove the following.

Proposition 2.16. *Let $(A, *, \alpha)$ be a Hom-Jacobi-Jordan algebra, (V_1, ρ_1, ϕ_1) and (V_2, ρ_2, ϕ_2) be two representations of $(A, *, \alpha)$. Then $(V_1 \oplus V_2, \rho_1 \oplus \rho_2, \phi_1 \oplus \phi_2)$ is a representation of $(A, *, \alpha)$ where*

$$(\rho_1 \oplus \rho_2)(x)(u + v) := \rho_1(x)u + \rho_2(x)v, \quad \forall\, x \in A \quad \text{and} \quad (u, v) \in V_1 \times V_2.$$

PROPOSITION 2.17. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra, (V,ρ,ϕ) be a representation of $(A,*,\alpha)$ and $D : A \to V$ be a linear map. Then $(V_D,\tilde{\rho},\tilde{\phi})$ is a representation of $(A,*,\alpha)$ where*

$$V_D := V + \mathbb{K}D, \quad \tilde{\phi} : V_D \to V_D \quad \text{and} \quad \tilde{\rho} : A \to gl(V_D)$$

with $\forall (r,x,u) \in \mathbb{K} \times A \times V$,

$$\tilde{\phi}(u + rD) := \phi(u) + r\phi D \quad \text{and} \quad \tilde{\rho}(x)(u + rD) = \rho(x)u + r\rho(x)D.$$

Proof. Pick $r \in \mathbb{K}$, $(x,y) \in A^{\times 2}$ and $u \in V$. Then by (2.4), we get

$$\begin{aligned}\tilde{\phi}\tilde{\rho}(x)(u + rD) &= \tilde{\phi}(\rho(x)u + r\rho(x)D) = \phi\rho(x)u + r\phi\rho(x)D\\ &= \rho(\alpha(x))\phi(u) + r\rho(\alpha(x))\phi D = \tilde{\rho}(\alpha(x))(\phi(u) = r\phi D)\\ &= \tilde{\rho}(\alpha(x))\tilde{\phi}(u + rD).\end{aligned}$$

Hence, $\tilde{\phi}\tilde{\rho} = \tilde{\rho}(\alpha(x))\tilde{\phi}$. Next, we obtain

$$\begin{aligned}\tilde{\rho}(x*y)\tilde{\phi}(u + rD) &= \tilde{\rho}(x * y)\big(\phi(u) + r\phi D\big) = \rho(x * y)\phi(u) + r\rho(x * y)\phi D\\ &= -\rho\big(\alpha(x)\big)\rho(y)u - \rho\big(\alpha(y)\big)\rho(x)u - r\rho\big(\alpha(x)\big)\rho(y)D\\ &\quad - r\rho\big(\alpha(y)\big)\rho(x)D \quad \text{(by (2.5))}\\ &= -\Big(\rho\big(\alpha(x)\big)\rho(y)u + r\rho\big(\alpha(x)\big)\rho(y)D\Big)\\ &\quad - \Big(\rho\big(\alpha(y)\big)\rho(x)u + r\rho\big(\alpha(y)\big)\rho(x)D\Big)\\ &= -\tilde{\rho}\big(\alpha(x)\big)\big(\rho(y)u + r\rho(y)D\big) - \tilde{\rho}\big(\alpha(y)\big)\big(\rho(x)u + r\rho(x)D\big)\\ &= -\tilde{\rho}\big(\alpha(x)\big)\tilde{\rho}(y)(u + rD) - \tilde{\rho}\big(\alpha(y)\big)\tilde{\rho}(x)(u + rD),\end{aligned}$$

i.e., $\tilde{\rho}(x * y)\tilde{\phi} = -\tilde{\rho}\big(\alpha(x)\big)\tilde{\rho}(y) - \tilde{\rho}\big(\alpha(y)\big)\tilde{\rho}(x)$. ∎

3. DERIVATIONS AND ANTIDERIVATIONS OF HOM-JACOBI-JORDAN ALGEBRAS

Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra. For any nonnegative integer k, denote by α^k the k-times composition of α, i.e.,

$$\alpha^k = \alpha \circ \cdots \circ \alpha \quad (k\text{-times}).$$

In particular, $\alpha^0 = Id$ and $\alpha^1 = \alpha$. If $(A,*,\alpha)$ is a regular Hom-Jacobi-Jordan algebra, we denote by α^{-k} the k-times composition of the inverse α^{-1}.

DEFINITION 3.1. Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra and k any nonnegative integer. A linear map $D : A \to A$ is called an

• α^k-derivation of $(A,*,\alpha)$ if

$$D \circ \alpha = \alpha \circ D, \tag{3.1}$$

$$D(u * v) = D(u) * \alpha^k(v) + \alpha^k(u) * D(v), \quad \forall\, u, v \in A. \tag{3.2}$$

• α^k-antiderivation of $(A,*,\alpha)$ if

$$D \circ \alpha = \alpha \circ D, \tag{3.3}$$

$$D(u * v) = -D(u) * \alpha^k(v) - \alpha^k(u) * D(v), \quad \forall\, u, v \in A. \tag{3.4}$$

For a regular Hom-Jacobi-Jordan algebra, α^{-k}-(anti)derivations can be defined similarly.

Denote by $\mathrm{Der}_{\alpha^k}(A)$ (resp. $\mathrm{ADer}_{\alpha^k}(A)$), the set of α^k-derivations (resp. α^k-antiderivations) of a Hom-Jacobi-Jordan algebra $(A,*,\alpha)$.

LEMMA 3.2. *For any nonnegative integer k and $u \in A$ satisfying $\alpha(u) = u$, define $D_k(u) : A \to A$ by*

$$D_k(u)(v) := u * \alpha^k(v), \qquad \forall\, v \in A.$$

Then $D_k(u)$ is an α^{k+1}-antiderivation called an inner α^{k+1}-antiderivation of $(A,*,\alpha)$.

Proof. Let $v, w \in A$. First, we have by the multiplicativity and the condition $\alpha(u) = u$,

$$\alpha\big(D_k(u)(v)\big) = \alpha(u) * \alpha^{k+1}(v) = u * \alpha^k\big(\alpha(v)\big) = D_k(u)\big(\alpha(v)\big).$$

Next, again by the multiplicativity and the condition $\alpha(u) = u$,

$$\begin{aligned} D_k(u)(v * w) &= \alpha(u) * \big(\alpha^k(v) * \alpha^k(w)\big) \\ &= -\alpha^{k+1}(v) * \big(\alpha^k(w) * u\big) - \alpha^{k+1}(w) * \big(u * \alpha^k(v)\big) \quad \text{(by (2.1))} \\ &= -\big(u * \alpha^k(v)\big) * \alpha^{k+1}(w) - \alpha^{k+1}(v) * \big(u * \alpha^k(w)\big). \end{aligned}$$

Therefore, $D_k(u)$ is an α^{k+1}-antiderivation of $(A,*,\alpha)$. ∎

Denote by $\mathrm{IADer}_{\alpha^k}(A)$, the set of inner α^k-antiderivations of $(A,*,\alpha)$.

$$\mathrm{IADer}_{\alpha^k}(A) := \{u * \alpha^k(-),\ u \in A \ \text{ and } \ \alpha(u) = u\}.$$

For any $D_1 \in \mathrm{Der}_{\alpha^k}(A) \cup \mathrm{ADer}_{\alpha^l}(A)$ and $D_2 \in \mathrm{Der}_{\alpha^r}(A) \cup \mathrm{ADer}_{\alpha^s}(A)$ define their commutator by

$$[D_1, D_2] := D_1 D_2 - D_2 D_1.$$

Then, it is easy to prove the following.

LEMMA 3.3. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra. Then,*

(i) $[\mathrm{Der}_{\alpha^k}(A), \mathrm{Der}_{\alpha^s}(A)] \subset \mathrm{Der}_{\alpha^{k+s}}(A)$,

(ii) $[\mathrm{ADer}_{\alpha^k}(A), \mathrm{ADer}_{\alpha^s}(A)] \subset \mathrm{Der}_{\alpha^{k+s}}(A)$,

(iii) $[\mathrm{ADer}_{\alpha^k}(A), \mathrm{Der}_{\alpha^s}(A)] \subset \mathrm{ADer}_{\alpha^{k+s}}(A)$.

Denoted by

$$\mathrm{Der}(A) := \oplus_{k\geq 0} \mathrm{Der}_{\alpha^k}(A).$$

Then, as in [20], we can prove that $\mathrm{Der}(A)$ is a Lie algebra.

For any linear map $D : A \to A$, consider the vector space $A + \mathbb{K}D$. Define a bilinear map $\diamond$ on $A + \mathbb{K}D$ by

$$\begin{aligned}(u + md) \diamond (v + nD) &:= u * v + mD(v) + nD(u),\\ u \diamond v = u * v, &\qquad D \diamond u = u \diamond D = D(u)\end{aligned}$$

and the linear map $\tilde{\alpha} : A + \mathbb{K}D \to A + \mathbb{K}D$ by

$$\tilde{\alpha}(u + nD) = \alpha(u) + nD.$$

Then, as in [20], we can prove:

PROPOSITION 3.4. *With the above notations, $(A + \mathbb{K}D, \diamond, \tilde{\alpha})$ is a Hom-Jacobi-Jordan algebra if and only if D is an α-antiderivation of the Hom-Jacobi-Jordan $(A,*,\alpha)$ with $D \circ D = 0$.*

Now, for any $D_1 \in \mathrm{Der}_{\alpha^k}(A) \cup \mathrm{ADer}_{\alpha^l}(A)$ and $D_2 \in \mathrm{Der}_{\alpha^r}(A) \cup \mathrm{ADer}_{\alpha^s}(A)$ define their anti-commutator by

$$\{D_1, D_2\} := D_1 D_2 + D_2 D_1.$$

In the two following propositions, we give a necessary and sufficient condition for the anti-commutator of two α-derivations(respectively α-antiderivations) to be an α-derivation (respectively α-antiderivation).

PROPOSITION 3.5. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra, $D_1 \in \mathrm{ADer}_{\alpha^k}(A)$ and $D_2 \in \mathrm{ADer}_{\alpha^s}(A)$. Then $\{D_1, D_2\} \in \mathrm{ADer}_{\alpha^{k+s}}(A)$ if and only if*

$$\begin{aligned}\{D_1, D_2\}(u*v) &= (\alpha^s \circ D_1)(u) * (\alpha^k \circ D_2)(v) \\ &\quad + (\alpha^k \circ D_2)(u) * (\alpha^s \circ D_1)(v), \quad \forall\, u, v \in A.\end{aligned} \tag{3.5}$$

Proof. Let $D_1 \in \mathrm{ADer}_{\alpha^k}(A)$, $D_2 \in \mathrm{ADer}_{\alpha^s}(A)$ and $u, v \in A$. Then by a straightforward computation, using (3.3) and (3.4) for D_1 and D_2 we get

$$\begin{aligned}\{D_1, D_2\}(u*v) &= D_1D_2(u*v) + D_2D_1(u*v) \\ &= D_1D_2(u) * \alpha^{k+s}(v) + (\alpha^k \circ D_2)(u) * (\alpha^s \circ D_1)(v) \\ &\quad + (\alpha^s \circ D_1)(u) * (\alpha^k \circ D_2)(v) + \alpha^{k+s}(u) * D_1D_2(v) \\ &\quad + D_2D_1(u) * \alpha^{k+s}(v) + (\alpha^s \circ D_1)(u) * (\alpha^k \circ D_2)(v) \\ &\quad + (\alpha^k \circ D_2)(u) * (\alpha^s \circ D_1)(v) + \alpha^{k+s}(u) * D_2D_1(v) \\ &= \{D_1, D_2\}(u) * \alpha^{k+s}(v) + \alpha^{k+s}(u) * \{D_1, D_2\}(v) \\ &\quad + 2\big((\alpha^s \circ D_1)(u) * (\alpha^k \circ D_2)(v) \\ &\quad + (\alpha^k \circ D_2)(u) * (\alpha^s \circ D_1)(v)\big).\end{aligned}$$

Hence, $\{D_1, D_2\} \in \mathrm{ADer}_{\alpha^{k+s}}(A)$ if and only if

$$\{D_1, D_2\}(u*v) = (\alpha^s \circ D_1)(u) * (\alpha^k \circ D_2)(v) + (\alpha^k \circ D_2)(u) * (\alpha^s \circ D_1)(v).$$

∎

In the case of two derivations of a Hom-Jacobi-Jordan algebra, we obtain.

PROPOSITION 3.6. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra, $D_1 \in \mathrm{Der}_{\alpha^k}(A)$ and $D_2 \in \mathrm{Der}_{\alpha^s}(A)$. Then $\{D_1, D_2\} \in \mathrm{Der}_{\alpha^{k+s}}(A)$ if and only if*

$$(\alpha^s \circ D_1)(u) * (\alpha^k \circ D_2)(v) + (\alpha^k \circ D_2)(u) * (\alpha^s \circ D_1)(v) = 0. \tag{3.6}$$

Proof. Let $D_1 \in \mathrm{Der}_{\alpha^k}(A)$, $D_2 \in \mathrm{Der}_{\alpha^s}(A)$ and $u, v \in A$. Then as in Proposition 3.5, using (3.1) and (3.2) for $D_1 \in \mathrm{Der}_{\alpha^k}(A)$ and $D_2 \in \mathrm{Der}_{\alpha^s}(A)$, we get

$$\begin{aligned}\{D_1, D_2\}(u * v) &= D_1D_2(u * v) + D_2D_1(u * v)\\ &= D_1D_2(u) * \alpha^{k+s}(v) + (\alpha^k \circ D_2)(u) * (\alpha^s \circ D_1)(v)\\ &\quad + (\alpha^s \circ D_1)(u) * (\alpha^k \circ D_2)(v) + \alpha^{k+s}(u) * D_1D_2(v)\\ &\quad + D_2D_1(u) * \alpha^{k+s}(v) + (\alpha^s \circ D_1)(u) * (\alpha^k \circ D_2)(v)\\ &\quad + (\alpha^k \circ D_2)(u) * (\alpha^s \circ D_1)(v) + \alpha^{k+s}(u) * D_2D_1(v)\\ &= \{D_1, D_2\}(u) * \alpha^{k+s}(v) + \alpha^{k+s}(u) * \{D_1, D_2\}(v)\\ &\quad + 2\big((\alpha^s \circ D_1)(u) * (\alpha^k \circ D_2)(v)\\ &\quad + (\alpha^k \circ D_2)(u) * (\alpha^s \circ D_1)(v)\big).\end{aligned}$$

Hence, $\{D_1, D_2\} \in \mathrm{Der}_{\alpha^{k+s}}(A)$ if and only if

$$(\alpha^s \circ D_1)(u) * (\alpha^k \circ D_2)(v) + (\alpha^k \circ D_2)(u) * (\alpha^s \circ D_1)(v).$$

∎

Similarly to the previous results, one can check the following.

Remarks 3.7. (i) For all $D_1 \in \mathrm{Der}_{\alpha^k}(A)$ and $D_2 \in \mathrm{Der}_{\alpha^s}(A)$, $\{D_1, D_2\} \in \mathrm{ADer}_{\alpha^{k+s}}$ if and only if (3.5) holds.

(ii) For all $D_1 \in \mathrm{ADer}_{\alpha^k}(A)$ and $D_2 \in \mathrm{ADer}_{\alpha^s}(A)$, $\{D_1, D_2\} \in \mathrm{Der}_{\alpha^{k+s}}$ if and only if (3.6) holds.

(iii) For all $D_1 \in \mathrm{ADer}_{\alpha^k}(A)$ and $D_2 \in \mathrm{Der}_{\alpha^s}(A)$, $\{D_1, D_2\} \in \mathrm{ADer}_{\alpha^{k+s}}$ if and only if (3.6) holds.

(iv) For all $D_1 \in \mathrm{ADer}_{\alpha^k}(A)$ and $D_2 \in \mathrm{Der}_{\alpha^s}(A)$, $\{D_1, D_2\} \in \mathrm{Der}_{\alpha^{k+s}}$ if and only if

$$\begin{aligned}\{D_1, D_2\}(u * v) = &-(\alpha^s \circ D_1)(u) * (\alpha^k \circ D_2)(v)\\ &- (\alpha^k \circ D_2)(u) * (\alpha^s \circ D_1)(v).\end{aligned} \tag{3.7}$$

Next, we introduce α^k-derivations (respectively α^k-antiderivations) of a Hom-Jacobi-Jordan algebra with values in a representation which generalize α^k-derivations (respectively α^k-antiderivations) of a Hom-Jacobi-Jordan algebra as follows.

Definition 3.8. Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra, (V,ρ,ϕ) be a representation of $(A,*,\alpha)$ and k be any nonnegative integer. A linear map $D : A \to V$ is called an

- α^k-derivation of/ $(A,*,\alpha)$ with values in (V,ρ,ϕ) if

$$D\circ\alpha = \phi\circ D, \tag{3.8}$$

$$D(u*v) = \rho\big(\alpha^k(u)\big)D(v) + \rho\big(\alpha^k(v)\big)D(u), \quad \forall\, u,v\in A. \tag{3.9}$$

- α^k-antiderivation of $(A,*,\alpha)$ with values in (V,ρ,ϕ) if

$$D\circ\alpha = \phi\circ D, \tag{3.10}$$

$$D(u*v) = -\rho\big(\alpha^k(u)\big)D(v) - \rho\big(\alpha^k(v)\big)D(u), \quad \forall\, u,v\in A. \tag{3.11}$$

For a regular Hom-Jacobi-Jordan algebra, α^{-k}-(anti)derivations of $(A,*,\alpha)$ with values in (V,ρ,ϕ) can be defined similarly.

Remark 3.9. Observe that for any integer k, α^k-derivations (respectively, α^k-antiderivations) of $(A,*,\alpha)$ with values in the adjoint representation (A,L,α) are α^k-derivations (respectively, α^k-antiderivations) of $(A,*,\alpha)$ (see Definition 3.1).

Similarly, denote by $\mathrm{Der}_{\alpha^k}(A,V)$ (resp. $\mathrm{ADer}_{\alpha^k}(A,V)$), the set of α^k-derivations (resp., α^k-antiderivations) of $(A,*,\alpha)$ with values in (V,ρ,ϕ). Then, we obtain:

Proposition 3.10. *For any nonnegative integer k and $u\in V$ satisfying $\phi(u)=u$, define $D_k(u) : A\to V$ by*

$$D_k(u)(v) := \rho\big(\alpha^k(v)\big)u, \qquad \forall\, v\in A.$$

Then $D_k(u)$ is an α^{k+1}-antiderivation of $(A,*,\alpha)$ with values in (V,ρ,ϕ) called an inner α^{k+1}-antiderivation of $(A,*,\alpha)$ with values in (V,ρ,ϕ).

Proof. Let $v,w\in A$. First by (2.4) and the condition $\phi(u)=u$, we have:

$$\begin{aligned}\phi\big(D_k(u)(v)\big) &= \phi\big(\rho(\alpha^k(v))u\big) = \rho\big(\alpha^{k+1}(v)\big)\phi(u)\\ &= \rho\big(\alpha^k(\alpha(v))\big)u = D_k(u)\big(\alpha(v)\big).\end{aligned}$$

Hence, $\phi\circ D_k(u)=D_k(u)\circ\alpha$. Next, by the multiplicativity and the condition $\phi(u)=u$, we compute

$$\begin{aligned} D_k(u)(v*w) &= \rho\big(\alpha^k(v)*\alpha^k(w)\big)(\phi(u)\\ &= -\rho\big(\alpha^{k+1}(v)\big)\rho\big(\alpha^k(w)\big)u-\rho\big(\alpha^{k+1}(w)\big)\rho\big(\alpha^k(v)\big)u \quad \text{(by (2.5))}\\ &= -\rho\big(\alpha^{k+1}(v)\big)D_k(u)(w)-\rho\big(\alpha^{k+1}(w)\big)D_k(u)(v). \end{aligned}$$

Therefore, $D_k(u)$ is an α^{k+1}-antiderivation of $(A,*,\alpha)$ with values in (V,ρ,ϕ). ∎

Denote by $\mathrm{IADer}_{\alpha^k}(A,V)$, the set of inner α^k-antiderivations of $(A,*,\alpha)$ with values in (V,ρ,ϕ).

$$\mathrm{IADer}_{\alpha^k}(A,V):=\big\{\rho\big(\alpha^k(-)\big)u\ :\ u\in V \text{ and } \ \phi(u)=u\big\}$$

and set

$$\mathrm{Der}(A,V):\oplus_{k\geq 0}\mathrm{Der}_{\alpha^k}(A,V).$$

Then $\mathrm{Der}(A,V)$ is a vector space called the vector space of α-antiderivations of $(A,*,\alpha)$ with values in (V,ρ,ϕ).

4. The zigzag cohomology of Hom-Jacobi-Jordan algebras

In this section basing on the cohomology theory of Jacobi-Jordan algebras [7], we develop a cohomology theory for Hom-Jacobi-Jordan algebras. First, we define the differential operators and then define the first and second cohomology groups. Finally, interpretations of these groups are given.

4.1. Definition of a zigzag cohomology.

For all nonnegative integer n, define

$$C^n(A,V):=\big\{f:A^{\otimes n}\to V\ :\ \ f \text{ is a } n\text{-linear map}\big\}$$

and

$$C^n_{\alpha,\phi}(A,V):=\big\{f\in C^n(A,V)\ :\ \phi\circ f=f\circ\alpha^{\otimes n}\big\}.$$

Denote by $S^n_{\alpha,\phi}(A,V)$ (resp. $A^n_{\alpha,\phi}(A,V)$) the set of symmetric (resp. α-skew-symmetric) maps of $C^n_{\alpha,\phi}(A,V)$ that is the set of any element $f\in C^n_{\alpha,\phi}(A,V)$ satisfying

$$f(x_{\sigma(1)},\ldots,x_{\sigma(n)})=f(x_1,\ldots,x_n)$$

for all σ in the symmetric group $\mathcal{S}_n$ $\big($resp. $\forall(i,j)\in\{1,\dots,n\}^2$

$$\begin{aligned} &f\big(x_1,\dots,x_i,\dots,x_{j-1},\alpha(x_j),x_{j+1},\dots,x_n\big)\\ &\qquad= -f\big(x_1,\dots,x_{i-1},x_j,x_{i+1},\dots,x_{j-1},\alpha(x_i),x_{j+1},\dots,x_n\big)\Big). \end{aligned}$$

Let us set

$$C^0_{\alpha,\phi}(A,V)(A,V)=A^0_{\alpha,\phi}(A,V)=S^0_{\alpha,\phi}(A,V)=\{v\in V\,:\,\phi(v)=v\}.$$

Furtheremore, for any nonnegative integer n, define the operators

$$d^n_\rho : C^n_{\alpha,\phi}(A,V)\longrightarrow C^{n+1}_{\alpha,\phi}(A,V)$$

and

$$\delta^n_\rho : A^n_{\alpha,\phi}(A,V)\longrightarrow C^{n+1}_{\alpha,\phi}(A,V)$$

by

$$d^0_\rho(v)(x)=\delta^0_\rho(v)(x):=\rho(x)(v),$$

$$\begin{aligned} d^n_\rho f(x_1,\dots,x_{n+1}) &= \sum_{i=1}^{n+1}\rho\big(\alpha^n(x_i)\big)f(x_1,\dots,\widehat{x_i},\dots,x_{n+1})\\ &+\sum_{1\le i<j\le n+1} f\big(x_i*x_j,\alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\widehat{\alpha(x_j)},\dots,\alpha(x_{n+1})\big) \end{aligned}$$

and

$$\begin{aligned} \delta^n_\rho f(x_1,\dots,x_{n+1}) &= \sum_{i=1}^{n+1}\rho\big(\alpha^n(x_i)\big)f(x_1,\dots,\widehat{x_i},\dots,x_{n+1})\\ &-\sum_{1\le i<j\le n+1} f\big(x_i*x_j,\alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\widehat{\alpha(x_j)},\dots,\alpha(x_{n+1})\big). \end{aligned}$$

Lemma 4.1. *With the above notations, for any* $(f,g)\in C^n_{\alpha,\phi}(A,V)\times A^n_{\alpha,\phi}(A,V)$, *we have*

$$d^n_\rho f\circ\alpha^{\otimes(n+1)}=\phi\circ d^n_\rho f\quad\text{and}\quad \delta^n_\rho g\circ\alpha^{\otimes(n+1)}=\phi\circ\delta^n_\rho g.$$

Proof. Let $(f,g) \in C^n_{\alpha,\phi}(A,V) \times A^n_{\alpha,\phi}(A,V)$ and $(x_1,\dots,x_{n+1}) \in A^{n+1}$, then

$$\begin{aligned}
d^n_\rho f\big(\alpha(x_1),\dots,\alpha(x_{n+1})\big) &= \sum_{i=1}^{n+1} \rho\big(\alpha^n(\alpha(x_i))\big) f\big(\alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\alpha(x_{n+1})\big) \\
&\quad + \sum_{1\le i<j\le n+1} f\big(\alpha(x_i)*\alpha(x_j),\alpha^2(x_1),\dots,\widehat{\alpha^2(x_i)},\dots,\widehat{\alpha^2(x_j)},\dots,\alpha^2(x_{n+1})\big) \\
&= \sum_{i=1}^{n+1} \phi\rho\big(\alpha^n(x_i)\big) f(x_1,\dots,\widehat{x_i},\dots,x_{n+1}) + \sum_{1\le i<j\le n+1} \phi f\big(x_i*x_j,\alpha(x_1),\dots, \\
&\qquad \widehat{\alpha(x_i)},\dots,\widehat{\alpha(x_j)},\dots,\alpha(x_{n+1})\big) \quad (\text{using } \phi\circ f = f\circ\alpha^{\otimes n} \text{ and } (2.4)) \\
&= \phi d^n_\rho f(x_1,\dots,x_{n+1}).
\end{aligned}$$

Hence we obtain $d^n_\rho f \circ \alpha^{\otimes(n+1)} = \phi \circ d^n_\rho f$. Similarly, we prove that $\delta^n_\rho g \circ \alpha^{\otimes(n+1)} = \phi \circ \delta^n_\rho g$. ∎

Theorem 4.2. *For $n \ge 1$, we have $d^n_\rho \circ \delta^{n-1}_\rho = 0$.*

Proof. Let f be an $(n-1)$-lineair map in $A^{n-1}_{\alpha,\phi}(A,V)$. We have

$$\begin{aligned}
d^n_\rho \circ \delta^{n-1}_\rho f(x_1,\dots,x_{n+1}) &= \sum_{i=1}^{n+1} \rho\big(\alpha^n(x_i)\big)\delta^{n-1}_\rho f(x_1,\dots,\widehat{x_i},\dots,x_{n+1}) \\
&\quad + \sum_{1\le i<j\le n+1} \delta^{n-1}_\rho f\big(x_i*x_j,\alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\widehat{\alpha(x_j)},\dots,\alpha(x_{n+1})\big).
\end{aligned}$$

First, we proceed for the first sum of the right hand side as follows:

$$\begin{aligned}
&\sum_{i=1}^{n+1} \rho\big(\alpha^n(x_i)\big)\delta^{n-1}_\rho f(x_1,\dots,\widehat{x_i},\dots,x_{n+1}) \\
&\quad = \sum_{1\le i<j\le n+1} \Big(\rho\big(\alpha^n(x_i)\big)\rho\big(\alpha^{n-1}(x_j)\big) + \rho\big(\alpha^n(x_j)\big)\rho\big(\alpha^{n-1}(x_i)\big)\Big) f(x_1,\dots,\widehat{x_i}, \\
&\qquad\qquad \dots,\widehat{x_j},\dots,x_{n+1}) \\
&\qquad - \sum_{\substack{1\le i,j,k\le n+1 \\ i\ne j,\, i\ne k,\, j<k}} \rho\big(\alpha^n(x_i)\big) f\big(x_i*x_j,\alpha(x_1),\dots,\widehat{\alpha(x_i)}, \\
&\qquad\qquad \dots,\widehat{\alpha(x_j)},\dots,\alpha(x_{n+1})\big).
\end{aligned}$$

Next, we proceed for the second sum of the right hand side as follows:

$$\begin{aligned}
&\sum_{1\le i<j\le n+1} \delta_\rho^{n-1} f\big(x_i * x_j, \alpha(x_1), \ldots, \widehat{\alpha(x_i)}, \ldots, \widehat{\alpha(x_j)}, \ldots, \alpha(x_{n+1})\big)\\
&\quad= \sum_{1\le i<j\le n+1} \rho\big(\alpha^{n-1}(x_i * x_j)\big) f\big(\alpha(x_1), \ldots, \widehat{\alpha(x_i)}, \ldots, \widehat{\alpha(x_j)}, \ldots, \alpha(x_{n+1})\big)\\
&\qquad+ \sum_{\substack{1\le i,j,k\le n+1\\ i\neq j,\, i\neq k,\, j<k}} \rho\big(\alpha^{n}(x_i)\big) f\big(x_i * x_j, \alpha(x_1), \ldots, \widehat{\alpha(x_i)}, \ldots, \widehat{\alpha(x_j)}, \ldots, \alpha(x_{n+1})\big)\\
&\qquad- \sum_{\substack{1\le i,j,k\le n+1\\ i\neq k,\, j\neq k,\, i<j}} f\big((x_i * x_j) * \alpha(x_k), \alpha(x_1), \ldots,\\
&\qquad\qquad\qquad \widehat{\alpha(x_i)}, \ldots, \widehat{\alpha(x_j)}, \ldots, \widehat{\alpha(x_k)}, \ldots, \alpha(x_{n+1})\big)\\
&\qquad- \sum_{\substack{1\le i,j,k,l\le n+1\\ \{i,j\}\cap\{k,l\}=\emptyset\\ j\neq k,\, i<j,\, k<l}} f\big(x_i * x_j, \alpha(x_k) * \alpha(x_l), \alpha(x_1), \ldots, \widehat{\alpha(x_i)}, \ldots, \widehat{\alpha(x_j)},\\
&\qquad\qquad\qquad \ldots, \widehat{\alpha(x_k)}, \ldots, \widehat{\alpha(x_l)}, \ldots, \alpha(x_{n+1})\big)\\
&\quad= \sum_{1\le i<j\le n+1} \rho\big(\alpha^{n-1}(x_i * x_j)\big)\phi f(x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{n+1})\\
&\qquad+ \sum_{\substack{1\le i,j,k\le n+1\\ i\neq j,\, i\neq k,\, j<k}} \rho\big(\alpha^{n}(x_i)\big) f\big(x_i * x_j, \alpha(x_1), \ldots, \widehat{\alpha(x_i)}, \ldots, \widehat{\alpha(x_j)}, \ldots, \alpha(x_{n+1})\big)
\end{aligned}$$

(by the Hom-Jacobi identity, the α-skew-symmetry of f and the condition $f \circ \alpha^{\otimes(n-1)} = \phi \circ f$).
Hence,

$$\begin{aligned}
&d_\rho^n \circ \delta_\rho^{n-1} f(x_1, \ldots, x_{n+1})\\
&\quad= \sum_{1\le i<j\le n+1} \Big(\rho\big(\alpha^n(x_i)\big)\rho\big(\alpha^{n-1}(x_j)\big) + \rho\big(\alpha^n(x_j)\big)\rho\big(\alpha^{n-1}(x_i)\big)\\
&\qquad\quad + \rho\big(\alpha^{n-1}(x_i * x_j)\big)\phi\Big) f(x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{n+1}) = 0 \quad \text{(by (2.5))}.
\end{aligned}$$

∎

Associated to the representation (V, ρ, ϕ) , we obtain the complex

$$\big(C^\bullet_{\alpha,\,\phi}(A, V),\ A^\bullet_{\alpha,\phi}(A, V),\ d^\bullet_\rho,\ \delta^\bullet_\rho\big).$$

Denote the set of closed k-Hom-cochains by

$$Z^k_{\alpha,\phi}(A, V) = \big\{f \in C^k_{\alpha,\phi}(A, V) \,:\, d^k_\rho f = 0\big\}$$

and the set of exact k-Hom-cochains by

$$B^k_{\alpha,\phi}(A,V) = \{\delta^{k-1}_\rho f \,:\, f \in A^{k-1}_{\alpha,\phi}(A,V)\}.$$

Then we define the k^{th} cohomology space of $(A, *, \alpha)$ with values/coefficients in (V, ρ, ϕ) as the quotient

$$H^k_{\alpha,\phi}(A,V) := Z^k_{\alpha,\phi}(A,V)/B^k_{\alpha,\phi}(A,V).$$

4.2. Low degree cohomology spaces. In this subsection, we give an interpretation of the low degree cohomology spaces in terms of algebraic properties of the Hom-Jacobi-Jordan algebra. In degree zero we have:

$$\begin{aligned} H^0_{\alpha,\phi}(A,V) &= \{m \in V \,:\, \phi(m) = m \ \text{ and } \ d^0_\rho m = 0\} \\ &= \{m \in V \,:\, \phi(m) = m \ \text{ and } \ \forall x \in A,\, \rho(x)m = 0\}. \end{aligned}$$

For example if $V = A$ is the adjoint representation of $(A, *, \alpha)$, we obtain

$$H^0_{\alpha,\alpha}(A,A) = \{m \in A \,:\, \alpha(m) = m \ \text{ and } \ \forall x \in A,\, x * m = 0\},$$

i.e.,

$$H^0_{\alpha,\alpha}(A,A) = \mathrm{HAnn}(A)$$

is the Hom-annihilator of $(A, *, \alpha)$.

Now, we shall investigate the first cohomology space. We have

$$H^1_{\alpha,\phi}(A,V) := Z^1_{\alpha,\phi}(A,V)/B^1_{\alpha,\phi}(A,V) = \mathrm{ADer}_{\alpha,\phi}(A,V)/\,\mathrm{IADer}_{\alpha,\phi}(A,V)$$

where

$$\begin{aligned} Z^1_{\alpha,\phi}(A,V) = \{f \in C^1_{\alpha,\phi}(A,V) \,:\, &\forall\,(x_1,x_2) \in A^2, \\ &f(x_1 * x_2) = -\rho\big(\alpha(x_1)\big)f(x_2) - \rho\big(\alpha(x_2)\big)f(x_1)\} \\ := \mathrm{ADer}_{\alpha,\phi}(A,V) & \end{aligned}$$

and

$$\begin{aligned} B^1_{\alpha,\phi}(A,V) &= \{\delta^0_\rho m \,:\, m \in C^0_{\alpha,\phi}(A,V)\} \\ &= \{f \in C^1_{\alpha,\phi}(A,V),\, \exists m \in V,\, \phi(m) = m,\, \forall\, x \in A, \\ &\qquad\qquad f(x) = \rho(x)m\} \\ &= \{\rho(-)m \,:\, m \in V,\ \phi(m) = m\}. \end{aligned}$$

A particular case is $V = A$ be the adjoint representation of $(A, *, \alpha)$. Then

$$\begin{aligned} Z^1_{\alpha,\alpha}(A, A) = \big\{ f \in C^1_{\alpha,\alpha}(A, A) \,:\, \forall\, (x_1, x_2) \in A^2, \\ f(x_1 * x_2) = -\alpha(x_1) * f(x_2) - \alpha(x_2) * f(x_1) \big\} \\ := \mathrm{ADer}_\alpha(A) \end{aligned}$$

and

$$B^1_{\alpha,\alpha}(A, A) = \big\{ m * (-) : m \in A,\ \alpha(m) = m \big\} := \mathrm{IADer}_\alpha(A).$$

Hence, the quotient space is then $H^1_{\alpha,\alpha}(A, A) = OADer_\alpha(A)$, the space of outer α-antiderivations of A.

Example 4.3. Consider the 2-dimensional Hom-Jacobi-Jordan algebra $(A, *, \alpha)$ defined by $\alpha(e_1) = e_1 + e_2,\ \alpha(e_2) = e_2$ with the only non-zero product $e_1 * e_1 = e_2$.

A linear map f of the Hom-Jacobi-Jordan algebra $(A, *, \alpha)$ is an α-antiderivation if and only if the matrix of f with respect to the basis (e_1, e_2) is given by

$$\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, \quad b \in \mathbb{K}.$$

Similarly we obtain $B^1_{\alpha,\alpha}(A, A) = \{0\}$. It follows that the dimension of $H^1_{\alpha,\alpha}(A, A)$ is 1 and elements of $H^1_{\alpha,\alpha}(A, A)$ can be written as the classes of the matrix

$$\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, \quad b \in \mathbb{K}.$$

Example 4.4. Let compute the first cohomology space $H^1_{\alpha,\phi}(\mathcal{A}_1, \mathcal{A}_1)$ for the Hom-Jacobi-Jordan algebra of Example 2.5. Let f be a linear map of A such that $f(e_1) = a_1e_1 + a_2e_2 + a_3e_3,\ f(e_2) = b_1e_1 + b_2e_2 + b_3e_3,\ f(e_3) = c_1e_1 + c_2e_2 + c_3e_3$. Then,

$$f \in Z^1_{\alpha,\alpha}(\mathcal{A}_1, \mathcal{A}_1) \iff f \text{ satisfies (3.3) and (3.4) with } k = 1.$$

Observe that f satisfies (3.3) if and only if

$$c_1 = a_3, \quad c_2 = a_2 = b_3 = b_1 = 0, \quad c_3 = a_1,$$

and f satisfies (3.4) if and only if

$$a_1 = 0, \quad b_2 = -4a_3.$$

Hence, $f \in Z^1_{\alpha,\alpha}(\mathcal{A}_1, \mathcal{A}_1)$ if and only if

$$f = \begin{pmatrix} 0 & 0 & a_3 \\ 0 & -4a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix}, \quad a_3 \in \mathbb{K}.$$

Next, we prove similarly that $B^1_{\alpha,\alpha}(\mathcal{A}_1, \mathcal{A}_1) = \{0\}$. Thus, the dimension of $H^1_{\alpha,\alpha}(\mathcal{A}_1, \mathcal{A}_1)$ is 1 and elements of $H^1_{\alpha,\alpha}(\mathcal{A}_1, \mathcal{A}_1)$ can be written as the classes of the following matrices:

$$\begin{pmatrix} 0 & 0 & a_3 \\ 0 & -4a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix}, \quad a_3 \in \mathbb{K}.$$

Let us consider another example of Hom-Jacobi-Jordan algebras whose set of 1-coboundaries is non-zero.

Example 4.5. Consider the Hom-Jacobi-Jordan algebra, denoted $\mathcal{A}_3 = (A, *, \alpha)$, obtained from the Hom-Jacobi-Jordan algebra of the Example 2.7, taking $a_1 = 1,\ a_2 = a_3 = a_4 = 0$ and $c_3 \in \mathbb{K} \setminus \{0, 1\}$. More precisely, we have $\mathcal{A}_3 = (A, *, \alpha)$ where

$$\alpha := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c_2 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & c_4 & c_3 \end{pmatrix}$$

and the non-zero products are:

$$e_1 *_\alpha e_1 = e_2, \quad e_1 *_\alpha e_3 = e_3 *_\alpha e_1 = c_3 e_4.$$

An endomorphism D on the Hom-Jacobi-Jordan algebra $\mathcal{A}_3$ is an antiderivation if and only if the matrix of D with respect to the basis $\{e_1, e_2, e_3, e_4\}$ is given by

$$\begin{pmatrix} 0 & 0 & -c_3 t_2 & 0 \\ x_2 & 0 & z_2 & t_2 \\ x_3 & 0 & -t_4 & 0 \\ x_4 & -2c_3 x_3 & z_4 & t_4 \end{pmatrix}, \quad x_2, x_3, x_4, z_2, z_4, t_2, t_4 \in \mathbb{K}.$$

Let $u = xe_1 + ye_2 + ze_3 + te_4 \in A$. then

$$\alpha(u) = u \iff u = xe_1 + ye_2$$

It follows that the matrix of an inner antiderivation $Lu : \mathcal{A}_3 \to \mathcal{A}_3$ defined by $L_u(v) := u * v$, $\forall v \in A$ has the following form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_3x & 0 \end{pmatrix}, \quad x \in \mathbb{K}.$$

Hence, $H^1(\mathcal{A}_3, \mathcal{A}_3)$ is 6-dimensional and elements of $H^1(\mathcal{A}_3, \mathcal{A}_3)$ can be written as the classes of the following matrices:

$$\begin{pmatrix} 0 & 0 & -c_3t_2 & 0 \\ x_2 & 0 & z_2 & t_2 \\ x_3 & 0 & -t_4 & 0 \\ x_4 & -2c_3x_3 & 0 & t_4 \end{pmatrix}, \quad x_2, x_3, x_4, z_2, t_2, t_4 \in \mathbb{K}.$$

4.3. The trivial representation of Hom-Jacobi-Jordan algebras. In this section, trivial representations of Hom-Jacobi-Jordan algebras are studied. As an application, it is shown that the central extension of a Hom-Jacobi-Jordan algebra $(A, *, \alpha)$ is controlled by the second cohomology group of A with coefficients in the trivial representation. Now let $V = \mathbb{R}$, then we have $gl(V) = \mathbb{R}$ and any $\phi \in gl(V)$ is exactly a real number which will be noted by r. Let $\rho : A \to gl(V) = \mathbb{R}$ be the zero map. It is clear that, ρ is a representation of the Hom-Jacobi-Jordan algebras $(A, *, \alpha)$ with respect to any $r \in \mathbb{R}$. Let assume in the sequel that $r = 1$. Then, this representation is called the trivial representation of the Hom-Jacobi-Jordan algebras $(A, *, \alpha)$. Associated to this representation, the set of k-cochains on A denoted by $C^k(A, \mathbb{R})$, is the set of k-linear maps from $A \times \cdots \times A$ to $\mathbb{R}$ and the one of k-Hom-cochains is then given by

$$C^k_{\alpha,\phi}(A, \mathbb{R}) := \left\{ f \in C^k(A, \mathbb{R}) \, : \, f \circ \alpha^{\otimes k} = f \right\}$$

with

$$C^0_{\alpha,\phi}(A, \mathbb{R}) = A^0_{\alpha}(A, \mathbb{R}) = \mathbb{R}.$$

The corresponding operators

$$d^n : C^n_{\alpha,\phi}(A, \mathbb{R}) \longrightarrow C^{n+1}_{\alpha,\phi}(A, \mathbb{R})$$

and

$$\delta^n : A^n_{\alpha,\phi}(A, \mathbb{R}) \longrightarrow C^{n+1}_{\alpha,\phi}(A, \mathbb{R})$$

are defined by

$$d^0(v)(x) = \delta^0(v)(x) := \rho(x)(v),$$

$$d^n f(x_1, \dots, x_{n+1}) = \sum_{1 \le i < j \le n+1} f\big(x_i * x_j, \alpha(x_1), \dots, \widehat{\alpha(x_i)}, \dots, \widehat{\alpha(x_j)}, \dots, \alpha(x_{n+1})\big)$$

and

$$\delta^n f(x_1, \dots, x_{n+1}) = -\sum_{1 \le i < j \le n+1} f\big(x_i * x_j, \alpha(x_1), \dots, \widehat{\alpha(x_i)}, \dots, \widehat{\alpha(x_j)}, \dots, \alpha(x_{n+1})\big).$$

Now, consider central extensions of the Hom-Jacobi-Jordan algebra $(A, *, \alpha)$. It is clear that, $(\mathbb{R}, 0, 1)$ is an abelian Hom-Jacobi-Jordan algebra with the trivial multiplication and the identity morphism. Let $\theta \in C^2_\alpha(A, \mathbb{R})$ such that $\theta(u, v) = \theta(v, u)$ (symmetry) and consider the direct sum $\mathcal{A} = A \oplus \mathbb{R}$ with the following multiplication and linear map

$$\mu_\theta(u + s, v + t) = u * v + \theta(u, v),$$
$$\tilde{\alpha}(u + s) = \alpha(u) + s.$$

Theorem 4.6. *The triple $(\mathcal{A}, \mu_\theta, \tilde{\alpha})$ is a Hom-Jacobi-Jordan algebras if and only if $\theta \in C^2(A, \mathbb{R})$ satisfies*

$$d^2\theta = 0.$$

This Hom-Jacobi-Jordan algebra $(\mathcal{A}, \mu_\theta, \tilde{\alpha})$ is called the central extension of $(A, *, \alpha)$ by the abelian Hom-Jacobi-Jordan algebras $(\mathbb{R}, 0, 1)$.

Proof. Similar to the one in [20]. ∎

Also, similarly to [20], we can prove:

Proposition 4.7. *For $\theta_1, \theta_2 \in C^2(A, \mathbb{R})$, if $\theta_1 - \theta_2$ is exact, the corresponding two central extensions $(\mathcal{A}, \mu_{\theta_1}, \tilde{\alpha})$ and $(\mathcal{A}, \mu_{\theta_2}, \tilde{\alpha})$ are isomorphic.*

4.4. Adjoint representations of Hom-Jacobi-Jordan algebras. Let $(A,*,\alpha)$ be a regular Hom-Jacobi-Jordan algebra and let consider the adjoint representation of $(A,*,\alpha)$. An interesting fact that we will see in the sequel, is that there exists many adjoint representations of a Hom-Jacobi-Jordan algebra.

Definition 4.8. For any integer s, the α^s-adjoint representation of the regular Hom-Jacobi-Jordan algebra $(A,*,\alpha)$, which we denote by ad_s, is defined by

$$ad_s(u)(v) := \alpha^s(u) * v, \qquad \forall\, u,v \in A.$$

Lemma 4.9. *With the above notations, it is obvious to prove:*

$$\begin{aligned} &ad_s\big(\alpha(u)\big)\circ\alpha = \alpha\circ ad_s,\\ &ad_s(u*v)\circ\alpha = -ad_s\big(\alpha(u)\big)\circ ad_s(v) - ad_s\big(\alpha(v)\big)\circ ad_s(u). \end{aligned}$$

Thus the definition of α^s-adjoint representation is well defined.

The set of k-Hom-cochains is given by

$$C^n_\alpha(A,A) := \big\{ f \in C^n(A,A) \,:\, f\circ\alpha^{\otimes n} = \alpha\circ f \big\}.$$

with

$$C^0_\alpha(A,A) = \big\{ v\in A \,:\, \alpha(v) = v \big\} = A^0_\alpha(A,A).$$

Associated to the α^s-adjoint representation the corresponding operators:

$$d^n_s : C^n_\alpha(A,A) \longrightarrow C^{n+1}_\alpha(A,A)$$

and

$$\delta^n_s : A^n_\alpha(A,A) \longrightarrow C^{n+1}_\alpha(A,A)$$

by

$$d^0_s(v)(x) = \delta^0_s(v)(x) := \rho(x)(v),$$

$$\begin{aligned} d^n_s f(x_1,\dots,x_{n+1}) &= \sum_{i=1}^{n+1} \alpha^{n+s}(x_i) * f(x_1,\dots,\widehat{x_i},\dots,x_{n+1})\\ &+ \sum_{1\le i<j\le n+1} f\big(x_i*x_j, \alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\widehat{\alpha(x_j)},\dots,\alpha(x_{n+1})\big) \end{aligned}$$

and

$$\delta_s^n f(x_1,\dots,x_{n+1}) = \sum_{i=1}^{n+1} \alpha^{n+s}(x_i) * f(x_1,\dots,\widehat{x_i},\dots,x_{n+1})$$
$$- \sum_{1\le i<j\le n+1} f\big(x_i * x_j, \alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\widehat{\alpha(x_j)},\dots,\alpha(x_{n+1})\big).$$

For the α^s-adjoint representation, ad_s we obtain the α^s-adjoint complex

$$\big(C_\alpha^\bullet(A,A),\ A_\alpha^\bullet(A,A),\ d_s^\bullet,\ \delta_s^\bullet\big).$$

PROPOSITION 4.10. *Associated to the α^s-adjoint representations ad_s of the regular Hom-Jacobi-Jordan algebra $(A,*,\alpha)$, $D \in C_\alpha(A,A)$ is a 1-cocycle if and only if D is an α^{s+1}-antiderivation, i.e., $D \in \mathrm{ADer}_{\alpha^{s+1}}(A)$.*

Proof. Similar to the one in [20]. ∎

5. DEFORMATIONS OF HOM-JACOBI-JORDAN ALGEBRAS AND RELATIVE ROTA-BAXTER OPERATORS

One-parameter formal deformations introduced for associative and for Lie algebras, have been extended in the case of Hom-algebras. In this section we introduce a formal deformations theory for Hom-Jacobi-Jordan algebras and those of their relative Rota-Baxter operators. We prove that the zigzag cohomology defined above is suitable to establish their connections to cohomology groups.

5.1. COHOMOLOGIES OF A RELATIVE ROTA-BAXTER OPERATOR ON A HOM-JACOBI-JORDAN ALGEBRA. In this part, we define the cohomology theory of a relative Rota-Baxter operator on a Hom-Jacobi-Jordan algebra in terms of the cohomology of the Hom-Jacobi-Jordan algebra on the representation space.

DEFINITION 5.1. ([3]) Let (V,ρ,ϕ) be a representation of a Hom-Jacobi-Jordan algebra $(A,*,\alpha)$. A linear map $T: V \to A$ is called a relative Rota-Baxter operator with respect to (V,ρ,ϕ) if it satisfies

$$T\phi = \alpha T, \tag{5.1}$$

$$T(u) * T(v) = T\Big(\rho\big(T(u)\big)v + \rho\big(T(v)\big)u\Big), \quad \forall\, u,v \in V. \tag{5.2}$$

Observe that Rota-Baxter operators on Hom-Jacobi-Jordan algebras are relative Rota-Baxter operators with respect to the regular representation.

PROPOSITION 5.2. ([3]) *Let $(A, *, \alpha)$ be a Hom-Jacobi-Jordan algebra and $T : V \to A$ a relative Rota-Baxter operator with respect to a representation (V, ρ, ϕ). Then $(V, *_T, \phi)$ is a Hom-Jacobi-Jordan algebra, where*

$$u *_T v := \rho\big(T(u)\big)v + \rho\big(T(v)\big)u, \qquad \forall\, u, v \in V. \tag{5.3}$$

PROPOSITION 5.3. ([3]) *Let $T : V \to A$ be a relative Rota-Baxter operator on the Hom-Jacobi-Jordan algebra $(A, *, \alpha)$ with respect to the representation (V, ρ, ϕ). Let define a map $\rho_T : V \to gl(A)$ by*

$$\rho_T(u)x := T(u) * x - T\big(\rho(x)u\big), \qquad \forall\, (u, x) \in V \times A.$$

Then, the tuple (A, ρ_T, α) is a representation of the Hom-Jacobi-Jordan algebra $(V, *_T, \phi)$.

Let

$$d^n_{\rho_T} : C^n_{\phi,\alpha}(V, A) \longrightarrow C^{n+1}_{\phi,\alpha}(V, A)$$

and

$$\delta^n_{\rho_T} : A^n_{\phi,\alpha}(V, A) \longrightarrow C^{n+1}_{\phi,\alpha}(V, A)$$

be the corresponding coboundary operators of the Hom-Jacobi-Jordan algebra $(V, *_T, \phi)$ with coefficients in the representation (A, ρ_T, α). More precisely,

$$d^0_{\rho_T}(x)(v) = \delta^0_{\rho_T}(x)(v) := \rho_T(v)(x) = T(v) * x - T\big(\rho(x)v\big).$$

Furthermore, for any $n \in \mathbb{N} \setminus \{0\}$,

$$\begin{aligned} d^n_{\rho_T} f(u_1, \ldots, u_{n+1}) &= \sum_{i=1}^{n+1} \rho_T\big(\phi^n(u_i)\big) f(u_1, \ldots, \widehat{u_i}, \ldots, u_{n+1}) \\ &+ \sum_{1 \le i < j \le n+1} f\big(u_i *_T u_j, \phi(u_1), \ldots, \widehat{\phi(u_i)}, \ldots, \widehat{\phi(u_j)}, \ldots, \phi(u_{n+1})\big) \end{aligned}$$

and

$$\begin{aligned} \delta^n_{\rho_T} f(u_1, \ldots, u_{n+1}) &= \sum_{i=1}^{n+1} \rho_T\big(\phi^n(u_i)\big) f(u_1, \ldots, \widehat{u_i}, \ldots, u_{n+1}) \\ &- \sum_{1 \le i < j \le n+1} f\big(u_i *_T u_j, \phi(u_1), \ldots, \widehat{\phi(u_i)}, \ldots, \widehat{\phi(u_j)}, \ldots, \phi(u_{n+1})\big). \end{aligned}$$

Definition 5.4. Let T be a relative Rota-Baxter operator of the Hom-Jacobi-Jordan algebra $(A, *, \alpha)$ with respect to a representation (V, ρ, ϕ). The complex cochain

$$\Big(C^\bullet_{\phi,\alpha}(V,A) = \bigoplus_{k=0}^{+\infty} C^k_{\phi,\alpha}(V,A),\ A^\bullet_{\phi,\alpha}(V,A) = \bigoplus_{k=0}^{+\infty} A^k_{\phi,\alpha}(V,A),\ d^\bullet_{\rho_T},\ \delta^\bullet_{\rho_T}\Big)$$

is called the cohomology complex for the relative Rota-Baxter operator T.

The p^{th} cohomology space of V with values/coefficients in A is given by the quotient

$$H^k_{\phi,\alpha}(V,A) := Z^k_{\phi,\alpha}(V,A)/B^k_{\phi,\alpha}(V,A).$$

It is obvious that $x \in A$ is closed if and only if

$$\alpha(x) = x \quad \text{and} \quad L_x \circ T - T \circ p(x) = 0,$$

and $f \in C^1_{\phi,\alpha}(V,A)$ is closed if and only if

$$\begin{aligned}\alpha\big(T(u)\big) * f(v) + \alpha\big(T(v)\big) * f(u) - T\Big(\rho\big(f(u)\big)(\phi(v)) + \rho\big(f(v)\big)(\phi(u))\Big)\\ - f\Big(\rho\big(T(u)\big)(v) + \rho\big(T(v)\big)(u)\Big) = 0.\end{aligned}$$

5.2. Linear deformations of Hom-Jacobi-Jordan algebras and relative Rota-Baxter operators. This subsection is devoted to linear deformations studies of Hom-Jacobi-Jordan algebras and those of their relative Rota-Baxter operators using the cohomology theory given for Hom-Jacobi-Jordan algebras.

Definition 5.5. Let $(A, *, \alpha)$ be a Hom Jacobi-Jordan algebra and $T : V \to A$ be a relative Rota-Baxter operator with respect to a representation (V, ρ, ϕ). A linear map $\mathcal{Z} : V \to A$ is said to generate a linear deformation of T if $\forall t \in \mathbb{K}, T_t = T + t\mathcal{Z}$ is a relative Rota-Baxter operateur of $(A, *, \alpha)$ with respect to (V, ρ, ϕ).

Observe that a linear map $\mathcal{Z} : V \to A$ generates a linear deformation of T

if and only if:

$$\mathcal{Z}\phi = \alpha\mathcal{Z}, \tag{5.4}$$

$$\mathcal{Z}u * \mathcal{Z}v = \mathcal{Z}\big(\rho(\mathcal{Z}u)v + \rho(\mathcal{Z}v)u\big), \tag{5.5}$$

$$\begin{aligned} Tu * \mathcal{Z}v + Tv * \mathcal{Z}u - T\Big(\rho\big(\mathcal{Z}(u)\big)v + \rho\big(\mathcal{Z}(v)\big)u\Big) \\ - \mathcal{Z}\Big(\rho\big(T(u)\big)v + \rho\big(T(v)\big)u\Big) = 0. \end{aligned} \tag{5.6}$$

The identities (5.4) and(5.5) mean that $\mathcal{Z}$ is a relative Rota-Baxter operateur of $(A, *, \alpha)$ with respect to (V, ρ, ϕ). Observe that (5.6) is equivalent to

$$\mathcal{Z}(u *_T v) = \rho_T(u)\mathcal{Z}v + \rho_T(v)\mathcal{Z}u.$$

Therefore, (5.4) and (5.6) mean that $\mathcal{Z} \in \mathrm{Der}_{\phi^0}(V, A)$, i.e., $\mathcal{Z}$ is a ϕ^0-derivation of the Hom-Jacobi-Jordan algebra $(V, *_T, \phi)$ with values in the representation (A, ρ_T, α).

Definition 5.6. Let $(A, *, \alpha)$ be a regular Hom-Jacobi-Jordan algebra. A bilinear map $\psi \in C^2_\alpha(A, A)$ is said to generate a linear deformation of the multiplication $*$ of the regular Hom-Jacobi-Jordan algebra $(A, *, \alpha)$ if the t-parametrized family of bilinear operations

$$\mu_t(u, v) = u * v + t\psi(u, v) \quad \text{for } t \in \mathbb{K}$$

gives to (A, α) the structure of Hom-Jacobi-Jordan algebras, i.e., (A, μ_t, α) is a Hom-Jacobi-Jordan algebra for all $t \in \mathbb{K}$.

By computing the Hom-Jordan-Jacobi identity of μ_t and using Hom-Jacobi identity for $*$ we obtain:

$$\begin{aligned} t\Big(\alpha(x) * \psi(y, z) + \alpha(y) * \psi(z, x) + \alpha(z) * \psi(x, y) + \psi\big(\alpha(x), y * z\big) \\ + \psi\big(\alpha(y), z * x\big) + \psi\big(\alpha(z), x * y\big)\Big) \\ + t^2\Big(\psi\big(\psi(x, y), \alpha(z)\big) + \psi\big(\psi(y, z), \alpha(x)\big) + \psi\big(\psi(z, x), \alpha(y)\big)\Big). \end{aligned}$$

Hence, (A, μ_t, α) is a Hom-Jacobi-Jordan algebra for all $t \in \mathbb{K}$ if and only if:

$$\alpha\psi = \psi\alpha^{\otimes 2}, \tag{5.7}$$

$$\phi(x,y) = \psi(y,x), \tag{5.8}$$

$$\psi\big(\psi(x,y),\alpha(z)\big) + \psi\big(\psi(y,z),\alpha(x)\big) + \psi\big(\psi(z,x),\alpha(y)\big) = 0, \tag{5.9}$$

$$\begin{aligned}&\alpha(x) * \psi(y,z) + \alpha(y) * \psi(z,x) + \alpha(z) * \psi(x,y)\\ &\quad + \psi\big(\alpha(x), y * z\big) + \psi\big(\alpha(y), z * x\big) + \psi\big(\alpha(z), x * y\big) = 0.\end{aligned} \tag{5.10}$$

Obviously, (5.7), (5.8) and (5.9) means that ψ must itself defines a Hom-Jacobi algebra structure on (A, α). Furthermore, (5.7) and (5.10) mean that ψ is closed with respect to the α^{-1}-adjoint representation ad_{-1}, i.e., $d^2_{-1}\psi = 0$.

Definition 5.7. Let $(A, *, \alpha)$ be a regular Hom-Jacobi-Jordan algebra. Two linear deformations $\mu^1_t = * + t\psi_1$ and $\mu^2_t = * + t\psi_2$ are said to be equivalent if there exists a linear operator $N \in gl(A)$ such that $T_t = Id + tN$ is a Hom-Jacobi-Jordan algebras morphism from (A, μ^2_t, α) to (A, μ^1_t, α). In particular, a linear deformation $\mu_t = * + t\psi$ of a regular Hom-Jacobi-Jordan algebra $(A, *, \alpha)$ is said to be trivial if there exists a linear operator $N \in gl(A)$ such that for all $t \in \mathbb{K}$, $T_t = Id + tN$ is a Hom-Jacobi-Jordan algebras morphism from (A, μ_t, α) to $(A, *, \alpha)$.

For all $t \in \mathbb{K}$, $T_t = Id + tN$ being a morphism of Hom-algebras is equivalent to

$$N\alpha = \alpha N, \tag{5.11}$$

$$\psi_2(x,y) - \psi_1(x,y) = x * N(y) + N(x) * y - N(x * y), \tag{5.12}$$

$$\begin{aligned}&\psi_1\big(x, N(y)\big) + \psi_1\big(N(x), y\big) = N\big(\psi_2(x,y)\big) - N(x) * N(y),\\ &\psi_1\big(N(x), N(y)\big) = 0.\end{aligned} \tag{5.13}$$

Observe that (5.12) means that $\psi_2 - \psi_1 = \delta_{-1}N \in B^2(A, A)$. Hence, it follows:

Theorem 5.8. *Let $(A, *, \alpha)$ be a regular Hom-Jacobi-Jordan algebra. If two linear deformations $\mu^1_t = * + t\psi_1$ and $\mu^2_t = * + t\psi_2$ are equivalent, then ψ_1 and ψ_2 are in the same cohomology class of $H^2_\alpha(A, A)$.*

Definition 5.9. Let $(A, *, \alpha)$ be a regular Hom-Jacobi-Jordan algebra.A linear operator $N \in C^1_\alpha(A, A)$ is called a Nijenhuis operator if

$$N(u) * N(v) = N(u *_N v)$$

where

$$u *_N v := N(u) * v + u * N(v) - N(u * v).$$

THEOREM 5.10. *Let $(A, *, \alpha)$ be a regular Hom-Jacobi-Jordan algebra and $N \in C^1_\alpha(A, A)$ be a Nijenhuis operator. Then, a deformation μ_t of $(A, *, \alpha)$ can be obtained by putting*

$$\psi(u, v) := \delta_{-1}N(u, v) = u *_N v.$$

Furthermore, this deformation is trivial.

Proof. Similar to the one in [20]. ∎

Now, let give the link between these two deformations.

PROPOSITION 5.11. *Let $(A, *, \alpha)$ be a Hom Jacobi-Jordan algebra and $T : A \to A$ be a relative Rota-Baxter operator with respect to a representation (V, ρ, ϕ). If a linear map $\mathcal{Z} : V \to A$ generates a linear deformation of T, then the bilinear map $\psi_{\mathcal{Z}} : V^2 \to V$ defined by*

$$\psi_{\mathcal{Z}}(u, v) = \rho(\mathcal{Z}u)v + \rho(\mathcal{Z}v)u, \qquad \forall\, u, v \in V,$$

generates a linear deformation of the associated Hom-Jacobi-Jordan algebra $(V, *_T, \phi)$.

Proof. Let denote by $*_{T_t}$ the corresponding Hom-Jacobi-Jordan algebra structure associated to the relative Rota-Baxter operator $T_t := T + tT$. Then we obtain

$$\begin{aligned} u *_{T_t} v &= \rho(T_t u)v + \rho(T_t v)u \\ &= \rho(Tu)v + t\rho(\mathcal{Z}u)v + \rho(Tv)u + t\rho(\mathcal{Z}v)u \\ &= u *_T v + t\psi_{\mathcal{Z}}(u, v), \qquad \forall\, u, v \in V. \end{aligned}$$

Hence, $\psi_{\mathcal{Z}}$ generates a linear deformation of $(V, *_T, \phi)$. ∎

DEFINITION 5.12. Let T and T' be two relative Rota-Baxter operators of $(A, *, \alpha)$ with respect to a representation (V, ρ, ϕ). A morphism from T' to T is a couple (ϕ_A, ϕ_V) where $\phi_A : A \to A$ is a morphism of Hom-Jacobi-Jordan algebras and $\phi_V : V \to V$ is a linear map such as:

$$\phi_V \phi = \phi \phi_V, \tag{5.14}$$

$$T\phi_V = \phi_A T', \tag{5.15}$$

$$\phi_V \rho(x)u = \rho\big(\phi_A(x)\big)\phi_V(u), \qquad \forall\, u \in V, \forall x \in A. \tag{5.16}$$

Remark 5.13. If ϕ_A et ϕ_V are invertible, we say that (ϕ_A, ϕ_V) is an isomorphism from T to T'.

PROPOSITION 5.14. *Let T and T' be relative Rota-Baxter operators on a Hom-Jacobi-Jordan algebra $(A, *, \alpha)$ with respect to a representation (V, ρ, ϕ). If (ϕ_A, ϕ_V) is an isomorphism from T to T', then $(\phi_A^{-1}, \phi_V^{-1})$ is an isomorphism from T' to T.*

Proof. It is clear that $\phi_A^{-1} : A \to A$ is a morphism of Hom-algebras and $\phi_V^{-1} : V \to V$ is a linear map. Next, we have $\phi_V^{-1}\phi = \phi\phi_V^{-1}$ and $\phi_A^{-1}T = T'\phi_V^{-1}$ by (5.14) and (5.15) respectively. Finally, let $x \in A$ and $u \in V$. Then, there exists $(y, v) \in A \times V$ such that $x = \phi_A(y)$ and $u = \phi_V(v)$ i.e. $y = \phi_A^{-1}(x)$ and $v = \phi_V^{-1}(u)$. Hence,

$$\begin{aligned}
(5.16) \iff\ & \phi_V \rho(y)v = \rho\big(\phi_A(y)\big)\phi_V(v) \\
\iff\ & \phi_V \rho\big(\phi_A^{-1}(x)\big)\phi_V^{-1}(u) = \rho(x)u \\
\iff\ & \phi_V^{-1} \rho(x)u = \rho\big(\phi_A^{-1}(x)\big)\phi_V^{-1}(u).
\end{aligned}$$

∎

PROPOSITION 5.15. *Let T and T' be relative Rota-Baxter operators on a Hom-Jacobi-Jordan algebra $(A, *, \alpha)$ with respect to a representation (V, ρ, ϕ). If (ϕ_A, ϕ_V) is a morphism (resp., isomorphism) from T to T', then ϕ_V is a morphism (resp., isomorphism) of Hom-Jacobi-Jordan algebras from $(V, *_{T'}, \phi)$ to $(V, *_T, \phi)$.*

Proof. Let $u, v \in V$. It is clear that $\phi\phi_V = \phi_V\phi$. Next, using (2.4) and (5.1), we compute:

$$\begin{aligned}
\phi_V(u *_{T'} v) &= \phi_V\big(\rho(T'u)v + \rho(T'v)u\big) \\
&= \phi_V\big(\rho(T'u)v\big) + \phi_V\big(\rho(T'v)u\big) \\
&= \rho\big(\phi_A(T'u)\big)\phi_V(v) + \rho\big(\phi_A(T'v)\big)\phi_V(u) \\
&= \rho\big(T\phi_V(u)\big)\phi_V(v) + \rho\big(T\phi_V(v)\big)\phi_V(u) \\
&= \phi_V(u) *_T \phi_V(v).
\end{aligned}$$

Hence, ϕ_V is a morphism from $(V, *_{T'}, \phi)$ to $(V, *_T, \phi)$. ∎

PROPOSITION 5.16. *Let $(A, *, \alpha)$ be a Hom-Jacobi-Jordan algebra, T be a relative Rota-Baxter operator with respect to a representation (V, ρ, ϕ),*

$\phi_A : A \to A$ be an isomorphism of Hom-algebras and $\phi_A : V \to V$ an isomorphism of vector spaces such that:

$$\phi_V \phi = \phi \phi_V, \tag{5.17}$$

$$\phi_V \rho(x) u = \rho\big(\phi_A(x)\big)\phi_{\phi_V}(u), \qquad \forall\, x \in A,\ \forall u \in V. \tag{5.18}$$

Then, the map $T' := \phi_A^{-1} T \phi_V : V \to A$ is a relative Rota-Baxter operator with respect to the representation (V, ρ, ϕ).

Proof. First, thanks to conditions (5.17), (5.1) and $\alpha\phi_A = \phi_A\alpha$, we get

$$T'\phi = \phi_A^{-1} T \phi_V \phi = \phi_A^{-1} T \phi \phi_V = \phi_A^{-1} \alpha T \phi_V = \alpha \phi_A^{-1} T \phi_V = \alpha T'.$$

Next, pick $u, v \in V$. Then, using ϕ_A is a morphism of Hom-algebras, we compute:

$$\begin{aligned}
T'u * T'v &= \big(\phi_A^{-1} T \phi_V\big)(u) * \big(\phi_A^{-1} T \phi_V\big)(v)\\
&= \phi_A^{-1}\big(T\phi_V(u)\big) * \phi_A^{-1}\big(T\phi_V(v)\big)\\
&= \phi_A^{-1}\big(T\phi_V(u) * T\phi_V(v)\big)\\
&= \phi_A^{-1}\Big(T\Big(\rho\big(T\phi_V(u)\big)\phi_V(v) + \rho\big(T\phi_V(v)\big)\phi_V(u)\Big)\Big) \quad \text{(by (5.2))}\\
&= \phi_A^{-1} T\Big(\rho\big(T\phi_V(u)\big)\phi_v(v) + \rho\big(T\phi_V(v)\big)\phi_V(u)\Big)\\
&= \phi_A^{-1} T\big(\rho(\phi_A T'u)\phi_V(v) + \rho(\phi_A T'v)\phi_V(u)\big) \quad (\text{by } T\phi_V = \phi_A T')\\
&= \phi_A^{-1} T\big(\phi_V \rho(T'u)v + \phi_V \rho(T'v)u\big) \quad \text{(by (5.18))}\\
&= \phi_A^{-1} T \phi_V\big(\rho(T'u)v + \rho(T'v)u\big)\\
&= T'\big(\rho(T'u)v + \rho(T'v)u\big).
\end{aligned}$$

∎

5.3. Formal deformations of Hom-Jacobi-Jordan algebras. Given a $\mathbb{K}$-vector space A, $\mathbb{K}[[t]]$ designates the power series ring in one variable t whose coefficients are elements of $\mathbb{K}$ and $A[[t]]$ be the set of formal series with coefficients in the vector space A, ($A[[t]]$ is gotten by extending the coefficients domain of A from $\mathbb{K}$ to $\mathbb{K}[[t]]$). Any $\mathbb{K}$-bilinear map $f : A \times A \to A$ has naturally an extension to $\mathbb{K}[[t]]$-bilinear map $f : A[[t]] \times A[[t]] \to A[[t]]$. More precisely if $x = \sum_{i=0}^{+\infty} x_i t^i$ and $y = \sum_{j=0}^{+\infty} y_i t^j$ then $f(x, y) = \sum_{i\geq 0, j\geq 0} t^{i+j} f(x_i, y_j)$. For linear maps, the same holds.

DEFINITION 5.17. Let (A, μ, α) be a Hom-Jacobi-Jordan algebra. A one-paramater formal deformation of (A, μ, α) is given by a $\mathbb{K}[[t]]$-bilinear map $\mu_t : A[[t]] \times A[[t]] \to A[[t]]$ of the form

$$\mu_t = \sum_{i=0}^{+\infty} \mu_i t^i$$

where each μ_i is a $\mathbb{K}$-bilinear map $\mu_i : A \times A \to A$ (extended to $\mathbb{K}[[t]]$-bilinear), $\mu_0 = \mu$ such that the following conditions hold for all $x, y, z \in A$:

$$\begin{aligned} \alpha \circ \mu_t &= \mu_t \circ \alpha^{\otimes 2} \quad \text{(multiplicativity)}, \\ \mu_t(x, y) &= \mu_t(y, x) \quad \text{(symmetry)}, \\ \circlearrowleft_{(x,y,z)} \mu_t\big(\mu_t(x, y), \alpha(z)\big) &= 0 \quad \text{(Hom-Jacobi-Jordan identity)}. \end{aligned} \tag{5.19}$$

The deformation is said to be of order $k \in \mathbb{N}$ if $\mu_t = \sum_{i=0}^{k} \mu_i t^i$, and infinitesimal if $k = 1$.

Remark 5.18. The symmetry of μ_t is equivalent to the symmetry of all μ_i, for $i \geq 0$. In addition, the multiplicativity of μ_t with respect to α is equivalent to the one of μ_i with respect to α for all i.

EXAMPLE 5.19. Let (A, μ, α) be a Hom-Jacobi-Jordan algebra.

1. The bilinear map $\mu_t = \sum_{i=0}^{+\infty} \mu_i t^i$ defined by $\mu_0 = \mu$ and $\mu_i = 0$, for all $i \in \mathbb{N}^*$, is a formal deformation of (A, μ, α) of order 0.
2. The bilinear map $\mu_t = \sum_{i=0}^{+\infty} \mu_i t^i$ defined by $\mu_i = \mu$, $\forall i \geq 0$ is a formal deformation of (A, μ, α).

The identity (5.19) is called the deformation equation of the Hom-Jacobi-algebra. It is equivalent to

$$\circlearrowleft_{(x,y,z)} \sum_{i \geq 0, j \geq 0} t^{i+j} \mu_i\big(\mu_j(x, y), \alpha(z)\big) = 0,$$

i.e.

$$\circlearrowleft_{(x,y,z)} \sum_{i \geq 0, s \geq 0} t^{s} \mu_i\big(\mu_{s-i}(x, y), \alpha(z)\big) = 0,$$

or

$$\sum_{s \geq 0} t^s \circlearrowleft_{(x,y,z)} \sum_{i \geq 0} \mu_i\big(\mu_{s-i}(x, y), \alpha(z)\big) = 0,$$

which is equivalent to the following infinite system

$$\circlearrowleft_{(x,y,z)} \sum_{i\geq 0} \mu_i\big(\mu_{s-i}(x,y),\alpha(z)\big) = 0 \qquad \text{for } s = 0,1,2,\dots . \tag{5.20}$$

In particular for $s = 0$, we have

$$\circlearrowleft_{(x,y,z)} \mu_0\big(\mu_0(x,y),\alpha(z)\big) = 0,$$

which is the Hom-Jacobi-Jordan identity of (A,μ,α). The previous equation for $s = 1$, leads to the following equation

$$\begin{aligned}\mu\big(\mu_1(x,y),\alpha(z)\big) + \mu\big(\mu_1(y,z),\alpha(x)\big) + \mu\big(\mu_1(z,x),\alpha(y)\big)\\ + \mu_1\big(\mu(x,y),\alpha(z)\big) + \mu_1\big(\mu(y,z),\alpha(x)\big) + \mu_1\big(\mu(z,x),\alpha(y)\big) = 0,\end{aligned}$$

which means that μ_1 is closed with respect to the α^{-1}-adjoint representation ad_{-1}, i.e., $d^2_{-1}\mu_1 = 0$. Hence, μ_1 is a 2-Hom-cocycle, i.e, $\mu_1 \in \mathcal{Z}^2_\alpha(A,A)$.

Definition 5.20. Let (A,μ,α) be a Hom-Jacobi-Jordan algebra. Two one-paramater deformations $A_t = (A,\mu_t,\alpha)$ and $A'_t = (A,\mu'_t,\alpha)$ of (A,μ,α) where $\mu_t = \sum_{i=0}^{+\infty}\mu_i t^i$ and $\mu'_t = \sum_{i=0}^{+\infty}\mu'_i t^i$ with $\mu_0 = \mu'_0 = \mu$, are said to be equivalent if there exists a formal automorphism $(\phi_t)_{t\geq 0} : A[[t]] \to A[[t]]$ that may be written in the form

$$\phi_t = \sum_{k=0}^{+\infty} \phi_k t^k$$

where each $\phi_i \in gl(A)$ (extended to $\mathbb{K}[[t]]$-linear), $\phi_0 = Id_A$ such that:

$$\phi_t\big(\alpha(x)\big) = \alpha\big(\phi_t(x)\big), \tag{5.21}$$

$$\phi_t\big(\mu_t(x,y)\big) = \mu'_t\big(\phi_t(x),\phi_t(y)\big) \qquad \forall\, x,y \in A[[t]]. \tag{5.22}$$

A one-paramater formal deformation $A_t = (A,\mu_t,\alpha)$ of (A,μ,α) is said to be trivial if A_t is equivalent to (A,μ,α) (viewed as an algebra over $A[[t]]$).

The identity (5.22) may be written for all $x,y \in A$ as:

$$\sum_{i\geq 0,\, j\geq 0} t^{i+j}\Big(\phi_i\big(\mu_j(x,y)\big)\Big) - \sum_{i\geq 0,\, j\geq 0,\, k\geq 0} t^{i+j+k}\Big(\mu'_j\big(\phi_i(x),\phi_k(y)\big)\Big) = 0,$$

i.e.,

$$\sum_{i\geq 0,\, s\geq 0} t^{s}\Big(\phi_i\big(\mu_{s-i}(x,y)\big)\Big) - \sum_{i\geq 0,\, j\geq 0,\, s\geq 0} t^{s}\Big(\mu'_j\big(\phi_i(x),\phi_{s-i-j}(y)\big)\Big) = 0.$$

Therefore,

$$\sum_{i\geq 0}\Big(\phi_i\big(\mu_{s-i}(x,y)\big)-\sum_{j\geq 0}\mu'_j\big(\phi_i(x),\phi_{s-i-j}(y)\big)\Big)=0 \qquad \text{for } s=0,1,2,\dots .$$

In particular, for $s=0$ we have $\mu_0=\mu'_0$, and for $s=1$ we have

$$\begin{aligned}\phi_0\big(\mu_1(x,y)\big)+\phi_1\big(\mu_0(x,y)\big)=\mu'_0\big(\phi_0(x),\phi_1(y)\big)+\mu'_0\big(\phi_1(x),\phi_0(y)\big)\\ +\mu'_1\big(\phi_0(x),\phi_0(y)\big).\end{aligned}$$

Since $\phi_0=Id_A$, $\mu_0=\mu'_0=\mu$, in the case of a regular Hom-Jacobi-Jordan algebra (A,μ,α) we have:

$$\mu'_1(x,y)=\mu_1(x,y)+\phi_1\big(\mu(x,y)\big)-\mu\big(x,\phi_1(y)\big)-\mu\big(\phi_1(x),y\big), \tag{5.23}$$

i.e.

$$\mu'_1-\mu_1=\delta_{-1}\phi_1.$$

Therefore the two 2-Hom-cocycles corresponding to the two equivalent deformations are cohomologous.

Definition 5.21. Let (A,μ,α) be a regular Hom-Jacobi-Jordan algebra and $\mu_1\in\mathcal{Z}^2_\alpha(A,A)$. Then, the 2-Hom-cocycle μ_1 is said to be integrable if there exists a family $(\mu_t)_{t\geq 0}$ such that $\mu_t=\sum_{i=0}^{+\infty}\mu_i t^i$ defines a formal deformation of (A,μ,α).

Thank to identity (5.23), the integrability of μ_1 depends only on its cohomology class. Hence, it is easy to prove the following:

Theorem 5.22. *Let (A,μ,α) a regular Hom-Jacobi-Jordan algebra and $(\mu_t)_{t\geq 0}$ be a one-paramater formal deformation of (A,μ,α). Then there exists a one-paramater formal deformation $(\mu'_t)_{t\geq 0}$ of (A,μ,α) which is equivalent to $(\mu_t)_{t\geq 0}$ such that $\mu'_1\in\mathcal{Z}^2_\alpha(A,A)$ and $\mu'_1\notin\mathcal{B}^2_\alpha(A,A)$. In addition, if $\mathcal{H}^2_\alpha(A,A)=\{0\}$, then any one-paramater formal deformation of (A,μ,α) is equivalent to the trivial one.*

Remark 5.23. The Hom-Jacobi-Jordan algebra whose all formal deformations are trivial are said analytically rigid. The nullity of the second cohomology group gives a sufficient criterion for rigidity.

5.4. Formal deformations of relative Rota-Baxter operators on Hom-Jacobi-Jordan algebras. In this subsection, we study formal deformations of relative Rota-Baxter operators using the cohomology theory given for Hom-Jacobi-Jordan algebras.

Observe first that if (A,μ,α) is a Hom-Jacobi-Jordan algebra over $\mathbb{K}$, then there is a $\mathbb{K}[[t]]$-Hom-Jacobi-Jordan algebra structure on $A[[t]]$ given by

$$\alpha\Big(\sum_{i=0}^{+\infty} x_it^i\Big) = \sum_{i=0}^{+\infty}\alpha(x_i)t^i, \tag{5.24}$$

$$\mu\Big(\sum_{i=0}^{+\infty} x_it^i, \sum_{i=0}^{+\infty} y_jt^j\Big) = \sum_{k=0}^{+\infty}\sum_{i+j=k}\mu(x_i,y_j)t^k, \qquad \forall\, x_i,y_j\in A. \tag{5.25}$$

Let (V,ρ,ϕ) be a representation of a Hom-Jacobi-Jordan algebra (A,μ,α). There is a representation $(V[[t]],\rho,\phi)$ of the $\mathbb{K}[[t]]$-Hom-Jacobi-Jordan algebra $A[[t]]$ given by

$$\phi\Big(\sum_{i=0}^{+\infty} v_it^i\Big) = \sum_{i=0}^{+\infty}\phi(v_i)t^i, \tag{5.26}$$

$$\rho\Big(\sum_{i=0}^{+\infty} x_it^i\Big)\Big(\sum_{j=0}^{+\infty} v_jt^j\Big) = \sum_{k=0}^{+\infty}\sum_{i+j=k}\rho(x_i)(v_j)t^k, \qquad \forall\, x_i\in A,\, v_j\in V. \tag{5.27}$$

Let T be a relative Rota-Baxter operator on a Hom-Jacobi-Jordan algebra (A,μ,α) with respect to a representation (V,ρ,ϕ). We consider a power series

$$T_t = \sum_{i=0}^{+\infty} T_it^i, \quad T_i\in \mathrm{Hom}_{\mathbb{K}}(V,A), \tag{5.28}$$

i.e., $T_t\in \mathrm{Hom}_{\mathbb{K}}(V,A)[[t]]$. Extend it to be a $\mathbb{K}[[t]]$-module map from $V[[t]]$ to $A[[t]]$ which is still denoted by T_t.

Definition 5.24. Let (A,μ,α) be a Hom-Jacobi-Jordan algebra, (V,ρ,ϕ) be a representation of (A,μ,α) and $T: V\to A$ be a relative Rota-Baxter operator of (A,μ,α) with respect to (V,ρ,ϕ). If $T_t=\sum_{i=0}^{+\infty}T_it^i$, with $T_0=T$, satisfies

$$T_t\phi = \alpha T_t, \qquad \forall\, t\in\mathbb{K}, \tag{5.29}$$

$$\mu(T_tu,T_tv) = T_t\big(\rho(T_tu)v+\rho(T_tv)u\big), \qquad \forall\, u,v\in V, \tag{5.30}$$

we say that T_t is a formal deformation of the relative Rota-Baxter operator T.

Remark 5.25. (a) The order of the deformation $(T_t)_{t\in\mathbb{K}}$ is k, if $T_t = \sum_{i=0}^{k} T_i t^i$, where k is a nonnegative integer.

(b) If $k = 1$, the deformation is said to be infinitesimal.

The identity (5.29) is equivalent to

$$\sum_{i\geq 0} \big(T_i\phi - \alpha T_i\big)t^i = 0,$$

i.e.,

$$T_i\phi = \alpha T_i \qquad \text{for } i = 0, 1, 2, \dots . \tag{5.31}$$

The identity (5.30) is equivalent to

$$\sum_{i\geq 0, j\geq 0} \Big(\mu(T_iu, T_jv) - T_i\big(\rho(T_ju)v + \rho(T_jv)u\Big)t^{i+j} = 0,$$

i.e.,

$$\sum\sum_{i\geq 0} \Big(\mu(T_iu, T_{s-i}v) - T_i\big(\rho(T_{s-i}u)v + \rho(T_{s-i}v)u\Big) = 0 \qquad \text{for } s = 0, 1, 2, \dots .$$

In particular, for $s = 0$ we have

$$\mu(T_0u, T_0v) = T_0\big(\rho(T_0u)v + \rho(T_0v)u\big)$$

and therefore T_0 is a relative Rota-Baxter operator of (A, μ, α) with respect to the representation (V, ρ, ϕ).

For $s = 1$ we get

$$\mu(T_0u, T_1v) - T_0\big(\rho(T_1u)v + \rho(T_1v)u\big) + \mu(T_1u, T_0v) - T_1\big(\rho(T_0u)v + \rho(T_0v)u\big),$$

i.e., if set $\mu_{T_0} = *_{T_0}$, we obtain

$$T(u *_{T_0} v) = \rho_{T_0}(u)T_1v + \rho_{T_0}(v)T_1u. \tag{5.32}$$

Hence, by (5.31) and (5.32), it follows that $T_1 \in \mathrm{Der}_{\phi^0}(V, A)$, i.e., T_1 is a ϕ^0-derivation of the Hom-Jacobi-Jordan algebra $(V, *_{T_0}, \phi)$ with values in the representation (A, ρ_{T_0}, α). Hence, we get, Proposition 5.26. *Let $T_t = \sum_{i=0}^{+\infty} T_i t^i$ be a formal deformation of a relative Rota-Baxter operator T on a Hom-Jacobi-Jordan algebra (A, μ, α) with respect to a representation (V, ρ, ϕ). Then T_1 is a ϕ^0-derivation of the Hom-Jacobi-Jordan algebra $(V, *_T, \phi)$ with values in the representation (A, ρ_T, α).*

Building on the relationship between relative Rota-Baxter operators and Hom-Jacobi-Jordan algebras, we have

Proposition 5.27. *If $(T_t)_{t\in\mathbb{K}}$ is a formal deformation of T, then the bilinear map μ_{T_t} défined by*

$$\mu_{T_t}(u, v) = \sum_{i=0}^{+\infty} \big(\rho(T_i u)v + \rho(T_i v)u\big)t^i \tag{5.33}$$

is a formal deformation of (V, μ_T, ϕ) where

$$\mu_T(u, v) = \rho(Tu)v + \rho(Tv)u, \qquad \forall\, u, v \in V.$$

Proof. Applying Eqs. (5.25), (5.27) and (5.28) to expand Eq. (5.33) and collecting coefficients of t^k, we see that Eq. (5.33) is equivalent to the system of equations

$$\sum_{\substack{i+j=k,\\ i,j\geq 0}}^{+\infty} \Big(\mu\big(T_i(u), T_j(v)\big) - T_i\big(\rho(T_j(u))v + \rho\big(T_j(v)\big)u\Big) = 0, \quad \forall\, k \geq 0,\ u, v \in V.$$

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6. Summary and Outlook

In this work, we have established representation and cohomology theories of Hom-Jacobi-Jordan algebras as well as those of relative Rota-Baxter operators of these Hom-algebras. Deformations of Hom-Jacobi-Jordan algebras and their relative Rota-Baxter operators are also studied using the cohomological approach. Future work may explore the representation and cohomology theory of (Hom-)Jacobi-Jordan bialgebras.

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