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Exact categories of coalgebras

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Abstract: Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor and, write $\mathcal{C}_F$ the category of F-coalgebras. The endofunctor F is called a covarietor when the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ has a right adjoint. Assuming $\mathcal{C}$ is Barr-exact (i.e., regular and internal equivalence relations are effective), we prove that $\mathcal{C}_F$ is Barr-exact under the condition F is a covarietor and preserves pullbacks.

Key words: Barr-exact category, coalgebra, effective equivalence relation, factorization system, pullback.

MSC (2020): 18E10, 68Q85, 46M18, 18A32, 18A30.

1. Introduction

A category $\mathcal{C}$ admitting finite limits, (regular epi)-mono factorizations, and coequalizers of equivalence relations is called effective if every equivalence relation in $\mathcal{C}$ is effective (i.e., the kernel pair of its coequalizer). Given an equivalence relation R on an object A of $\mathcal{C}$, the diagram

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A \xrightarrow{\;e\;} B$$

is called an exact sequence whenever R is the kernel pair of e and e is the coequalizer of R. It's also said that an effective category has exact sequences.

A category will be called regular if every finite diagram has a limit, if the kernel pair of any morphism admits a coequalizer and if regular epimorphisms are stable under pullbacks (see [9]). In that way, regular categories recapture many properties of abelian categories like the existence of images, without requiring additivity. At the same time, regular categories provide a foundation for the study of a fragment of first-order logic, known as regular logic (see [6]). An important aspect of regular categories is that in these categories

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the calculus of relations provides a powerful method to prove new results, possibly satisfying some additional exactness conditions (see [7], [8] and [19]). An illustration of this method is given by the Mal'tsev axiom: in a regular category this axiom is equivalent to the permutability of the composition of relations, in the sense that any pair R and S of equivalence relations on a given object are such that $R \circ S = S \circ R$. Indeed, regular categories are important not just for Mal'tsev categories but also for Goursat categories, arithmetical categories, the categorical algebra of congruence distributivity, and others (see [10], [11], [12], [13] and [21]).

A category which is both regular and effective is called effective regular or Barr-exact. Every elementary topos is a Barr-exact category (see [15]). However, not every Barr-exact category is an elementary topos. A counter-example is the category $KHaus$ of compact Hausdorff spaces (see [20]). Categories monadic over sets are Barr-exact (see [5]). In particular, varieties of algebras are Barr-exact, so that categories such as *Grp* of groups and *Rng* of rings are Barr-exact. These are also not elementary toposes. Abelian categories are Barr-exact.

Coalgebras have emerged in the last years as a significant concept in computer science, particularly in the areas of system modeling, semantics, and logic. Indeed, they provide a powerful framework to study systems that are more concerned with behavior and interaction than with static structure, making them fundamental to many modern applications in computing.

Given an endofunctor $F : \mathcal{C} \to \mathcal{C}$. An F-coalgebra (A, a) consists in an object A of $\mathcal{C}$ together with a $\mathcal{C}$-morphism $a : A \to FA$. A homomorphism between F-coalgebras (A, a) and (B, b) is a $\mathcal{C}$-morphism $f : A \to B$ such that $F(f) \circ a = b \circ f$. We write $\mathcal{C}_F$ the category of F-coalgebras. The endofunctor F is called a covarietor when the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ has a right adjoint. Assuming $\mathcal{C}$ is Barr-exact, we prove that $\mathcal{C}_F$ is Barr-exact under the condition F is a covarietor and preserves pullbacks.

2. Factorization systems

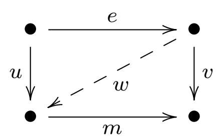
Roughly speaking, a factorization system on a category consists of two classes of morphisms, E and M, such that every morphism factors into an E-morphism followed by an M-morphism, and the E-morphisms and M-morphisms satisfy some lifting or diagonal fill-in property.

Formally, a *factorization system* for a category $\mathcal{C}$ consists of a pair (E,M) of classes of morphisms in $\mathcal{C}$ such that:

FS1. E and M contain all isomorphisms of $\mathcal{C}$ and are closed under composition.

FS2. $\mathcal{C}$ has E-M factorizations (of morphisms); i.e., every morphism f of $\mathcal{C}$ can be factored as $f = m \circ e$ for some morphisms $e \in E$ and $m \in M$.

FS3. $\mathcal{C}$ has the unique E-M diagonalization property; i.e., for each commutative square



with $e \in E$ and $m \in M$, there is a unique arrow w making both triangles commute.

Definition 2.1. (See [3]) A category admitting a factorization system (E,M) is said to be *(E,M)-structured*.

For instance, the categories *Set* of sets, *Grp* of groups and *Top* of topological spaces are (regular epi, mono)-structured. Every category has the unique (regular epi)-mono diagonalization property.

Definition 2.2. Let $G : \mathcal{A} \to \mathcal{B}$ be a functor, where the category $\mathcal{B}$ has (regular epi)-mono factorizations. We say that G *creates (regular epi)-mono factorizations* when, given any morphism $f : A \to A'$ and a (regular epi)-mono factorization $G(f) : G(A) \overset{e}{\to} B \overset{m}{\rightarrowtail} G(A')$, there exists a (regular epi)-mono factorization $A \overset{e'}{\to} \bar{A} \overset{m'}{\rightarrowtail} A'$ and an isomorphism $\varphi : G(\bar{A}) \overset{\backsimeq}{\longrightarrow} B$ such that $\varphi \circ G(e') = e$ and $m \circ \varphi = G(m')$.

3. Subobjects

The notion of subobject is a generalization of concepts such subsets from set theory, subgroup from group theory subspaces from topology.

In detail, let A be an object of some category. Given two monomorphisms $u : S \to A$ and $v : T \to A$ with codomain A, we define an equivalence relation by $u \equiv v$, if there exists an isomorphism $\phi : S \to T$ with $u = v \circ \phi$.

Equivalently, we write $u \leq v$ if u factors through v; that is, if there exists $\phi : S \to T$ such that $u = v \circ \phi$. The binary relation $\equiv$ defined by

$$u \equiv v \quad \Longleftrightarrow \quad u \leq v \text{ and } v \leq u$$

is an equivalence relation on the monomorphisms with codomain A.

Definition 3.1. An equivalence class of monomorphisms with codomain A is called a *subobject* of A.

The dual concept to a subobject is a quotient object. It may also refer to a co-subobject; that is, a subobject in the opposite category.

Definition 3.2. In a category with binary products, a *binary relation* between A and B is a subobject of $A \times B$. This is represented by a monomorphism $m : R \rightarrowtail A \times B$ or equivalently, by a pair of arrows

$$\begin{array}{ccc} & \overset{r_1}{\nearrow} & A \\ R & & \\ & \underset{r_2}{\searrow} & B \end{array}$$

with the property that the induced arrow $\langle r_1, r_2 \rangle : R \to A \times B$ is a monomorphism. So we have $r_1 = p_1 \circ \langle r_1, r_2 \rangle$ and $r_2 = p_2 \circ \langle r_1, r_2 \rangle$; p_1 and p_2 being the projections of the product of A and B. A binary relation between A and A is called a binary relation on A.

The *pullback* of two morphisms $f : A \to C$ and $g : B \to C$ with common codomain (or the pullback of g along f) is a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{p_1} & B \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

with the following property: if $u : D \to A$ and $v : D \to B$ are morphisms with $f \circ u = g \circ v$, then there is exactly one morphism $w : D \to P$ with $u = p_1 \circ w$ and $v = p_2 \circ w$. In particular, the pullback of f and f is called the *kernel pair* of f. We say that a category $\mathcal{C}$ has pullbacks if it has a pullback for any two morphisms.

Definition 3.3. A span $(A \xleftarrow{f} X \xrightarrow{g} B)$ is called a *mono source*, if for any two arrows $h_1, h_2 : Y \to X$ we have:

$$((f \circ h_1 = f \circ h_2) \text{ and } (g \circ h_1 = g \circ h_2)) \quad \Rightarrow \quad h_1 = h_2.$$

Binary products and pullbacks are mono sources. In a category with binary products, binary relations are mono sources and vice-versa. Given A

in a category admitting binary products, pullbacks and (regular epi)-mono factorizations. Consider a binary relation R on A. Form the pullback of its projections r_1 and r_2.

$$\begin{CD} R\times_A R @>{t_1}>> R @>{r_1}>> A \\ @V{t_2}VV @VV{r_2}V \\ R @>>{r_1}> A \\ @V{r_2}VV \\ A \end{CD}$$

Then R is called an *equivalence relation* if it is *reflexive* (i.e., the diagonal map $\langle 1_A, 1_A\rangle : A \to A \times A$ factors through R), *symmetric* (i.e., there is an arrow $\tau : R \to R$ such that $r_1 \circ \tau = r_2$ and $r_2 \circ \tau = r_1$) and *transitive* (i.e., $\langle r_1 \circ t_1, r_2 \circ t_2\rangle : R \times_A R \to A \times A$ factors through R).

4. Coalgebras and their homomorphisms

Consider an endofunctor $F : \mathcal{C} \to \mathcal{C}$. An *$F$-coalgebra* consists in an object A of $\mathcal{C}$ together with a morphism $a : A \to FA$ in $\mathcal{C}$. The object A is called the *underlying object* of (A, a) and the morphism a is called its *coalgebra structure*.

A *homomorphism* between (A, a) and (B, b) is a morphism $f : A \to B$ that respects the coalgebra structures; that is, the following diagram commutes

$$\begin{CD} A @>{f}>> B \\ @V{a}VV @VV{b}V \\ FA @>{F(f)}>> FB \end{CD}$$

The category of F-coalgebras and their homomorphisms is denoted $\mathcal{C}_F$. The forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ maps every F-coalgebra (A, a) to its underlying object A.

Example 4.1. Consider the *Set*-endofunctor

$$\begin{array}{rcl} F : Set & \longrightarrow & Set \\ A & \longmapsto & Z \times A \end{array}$$

for a fixed nonempty set Z. A coalgebra for this functor consists of a set A together with a mapping

$$a : A \longrightarrow Z \times A$$

Equivalently, an F-coalgebra is given by a set A and two mappings

$$hd_A : A \longrightarrow Z \qquad \text{and} \qquad tl_A : A \longrightarrow A.$$

Given any such coalgebra, each element $x \in A$ gives rise to a sequence over Z, namely the sequence

$$\big(hd_A(x), hd_A \circ tl_A(x), hd_A \circ tl_A^2(x), \dots\big).$$

An example of such an F-coalgebra is given by $A = Z^\omega$, the set of all infinite sequences over Z. A homomorphism between two F-coalgebras $(A, \langle hd_A, tl_A\rangle)$ and $(B, \langle hd_B, tl_B\rangle)$ is a mapping $f : A \to B$ satisfying

$$hd_A(x) = hd_B(f(x)) \qquad \text{and} \qquad f(tl_A(x)) = tl_B(f(x))$$

with $x \in A$.

EXAMPLE 4.2. (RUTTEN [22]) A Moore machine with input alphabet Σ can be viewed as a coalgebra for the functor

$$\begin{array}{rcl} F : Set & \longrightarrow & Set \\ A & \longmapsto & B \times A^\Sigma \end{array}$$

where B is any set. For the case of $B = 2 = \{0, 1\}$, an F-coalgebra is given by a set A together with two mappings

$$o : A \to 2 \qquad \text{and} \qquad t : A \to A^\Sigma.$$

Any such coalgebra is known as a *deterministic automata*. The output function o indicates whether a state s in A is accepting (also called final): $o(s) = 1$, or not: $o(s) = 0$. The transition function t assigns to a state s a function $t(s) : \Sigma \to A$ which specifies the state $t(s)(e)$ that is reached after an input symbol e has been consumed. A homomorphism between two F-coalgebras $(A, \langle o, t\rangle)$ and $(B, \langle o', t'\rangle)$ is a mapping $f : A \to B$ satisfying the condition

$$F(f) \circ \langle o, t\rangle = \langle o', t'\rangle \circ f$$

which is equivalent to

$$o(s) = o'(f(s)) \qquad \text{and} \qquad f(t(s)(e)) = t'(f(s))(e)$$

for all $s \in A$ and $e \in \Sigma$.

Example 4.3. (Jacobs [14]) A context-free grammar is a basic tool in computer science to describe the syntax of programming languages via so-called production rules. These rules are of the form $v \to \sigma$, where v is a non-terminal symbol and σ is finite list of terminal and non-terminal symbols. Let $*$ denote the Kleene star operator. If we write V for the set of non-terminal symbols, and A for the terminals, then a context-free grammar $g : V \longrightarrow \mathcal{P}((V + A)^*)$ is a coalgebra for the functor

$$\begin{array}{rcl} F : Set & \longrightarrow & Set \\ X & \longmapsto & \mathcal{P}((X + A)^*). \end{array}$$

It sends each non-terminal v to a set $g(v) \subseteq (V + A)^*$ of right-hand-sides $\sigma \in g(v)$ in productions $v \to \sigma$. A word $\tau \in A^*$ with terminals only can be generated by a context-free grammar g if there is a non-terminal $v \in V$ from which τ arises by applying rules repeatedly. The collection of such strings is the language generated by the grammar. A simple example is the grammar with $V = \{v\}$, $A = \{a, b\}$ and $g(v) = \{\langle\rangle, avb\}$. It thus involves two productions $v \to \langle\rangle$ and $v \to a.v.b$. This grammar generates the language of words $a^n b^n$ consisting of a number of a's followed by an equal number of b's.

4.1. (Regular epi)-mono factorizations Recall that every category admitting (regular epi)-mono factorizations is (regular epi, mono)-structured.

Definition 4.4. (See [18]) Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor and let $\mathcal{D}$ be a small category. We say that

- *F creates $\mathcal{D}$-(co)limits* if, for every diagram $J : \mathcal{D} \to \mathcal{C}$, whenever (L', φ') is a (co)limiting (co)cone to $F \circ J$, then there is a unique (co)limiting (co)cone (L, φ) to J such that $FL = L'$ and $F\varphi = \varphi'$.
- *F creates (co)limits* when it creates $\mathcal{D}$-(co)limits, for all small categories $\mathcal{D}$.

A functor is said to preserve pullbacks if it transforms every pullback into a pullback. For instance, every polynomial functor preserves pullbacks (see [22, Theorem 10.1]). A morphism $f : A \to B$ in any category is a monomorphism if and only if

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

is the kernel pair of f. Thus every functor that preserves pullbacks also preserves monomorphisms.

PROPOSITION 4.5. *The following assertions hold:*

(i) *The forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ creates colimits.*

(ii) *When $\mathcal{C}$ has pullbacks and F preserves them, U creates pullbacks. In particular, it creates kernel pairs of homomorphisms.*

Proof. Assertion (i) is Proposition 1.1 in [4]. For assertion (ii), see Remark 4.4-(a) in [1]. As a consequence, the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ creates kernel pairs of homomorphisms. ∎

PROPOSITION 4.6. (SEE [2, 17]) *Suppose that $\mathcal{C}$ has (regular epi)-mono factorizations and $F : \mathcal{C} \to \mathcal{C}$ preserves monomorphisms (e.g. if F preserves pullbacks). Then $\mathcal{C}_F$ has (regular epi)-mono factorizations and the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ creates them.*

4.2. LIMITS We are interested in the existence of limits in a category of coalgebras.

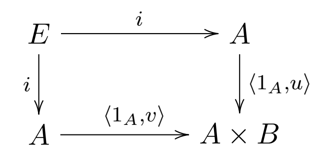
DEFINITION 4.7. A functor $F : \mathcal{C} \to \mathcal{D}$ *preserves relations* if it transforms every binary relation R between A and B in $\mathcal{C}$ into a binary relation FR between FA and FB in $\mathcal{D}$.

LEMMA 4.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be two finitely complete categories. Then every functor $F : \mathcal{C} \to \mathcal{D}$ that preserves pullbacks also preserves relations.*

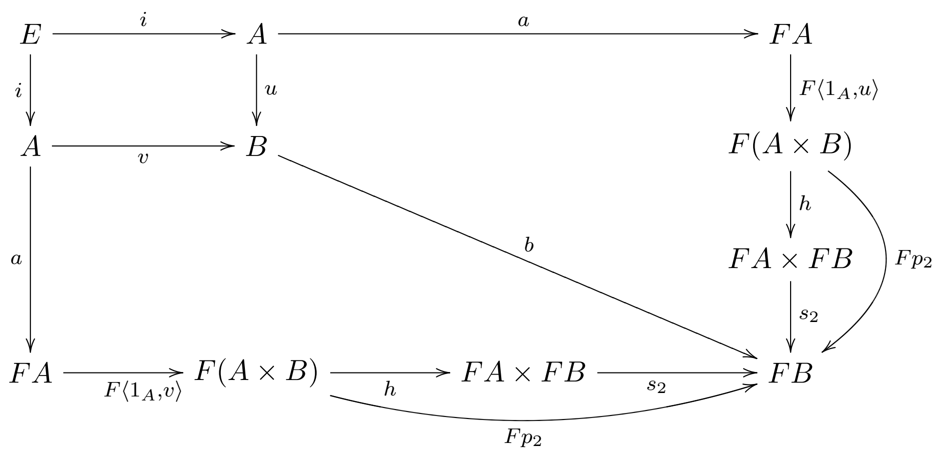
Proof. Given a binary relation R between A and B in $\mathcal{C}$; that is, a subobject of $A \times B$. Also, $A \times B$ is a pullback because $\mathcal{C}$ is finitely complete (see [3] and [23]). Since F preserves pullbacks, it transforms $A \times B$ into a pullback, and due to $\mathcal{D}$ has binary products, $F(A \times B)$ is a subobject of $FA \times FB$. Furthermore, FR is a subobject of $F(A \times B)$ as F preserves monomorphisms. It follows that FR is a subobject of $FA \times FB$, and hence a binary relation between FA and FB. ∎

PROPOSITION 4.9. *Let $\mathcal{C}$ be a finitely complete category and $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor that preserves pullbacks. Then $\mathcal{C}_F$ has equalizers, and the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ creates them.*

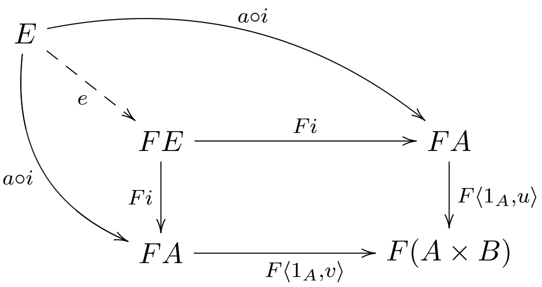
Proof. Consider a pair $u, v : (A, a) \to (B, b)$ of homomorphisms. Let $i : E \to A$ be their equalizer in the category $\mathcal{C}$. Then E is a regular subobject of A. Denote by $(A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B)$ the product of A and B. From universality of the product, there is a unique arrow $\langle 1_A, u\rangle : A \to A \times B$ such that $p_1 \circ \langle 1_A, u\rangle = 1_A$ and $p_2 \circ \langle 1_A, u\rangle = u$. Similarly, there is a unique arrow $\langle 1_A, v\rangle : A \to A \times B$ such that $p_1 \circ \langle 1_{A_1}, v\rangle = 1_A$ and $p_2 \circ \langle 1_A, v\rangle = v$. Since $\mathcal{C}$ is finitely complete, the following diagram is a pullback.



Conversely, the pullback of $\langle 1_A, u\rangle$ and $\langle 1_A, v\rangle$ is the equalizer of u and v. By the fact that F preserves pullbacks, FE together with Fi as projections is the pullback of $F\langle 1_A, u\rangle$ and $F\langle 1_A, v\rangle$. Furthermore, there is a unique arrow $\langle 1_{FA}, Fu\rangle : FA \to FA \times FB$ such that $s_1 \circ \langle 1_{FA}, Fu\rangle = 1_{FA}$ and $s_2 \circ \langle 1_{FA}, Fu\rangle = Fu$ with s_1 and s_2 the projections of the product of FA and FB. Also, there is a unique arrow $\langle 1_{FA}, Fv\rangle : FA \to FA \times FA$ such that $s_1 \circ \langle 1_{FA}, Fv\rangle = 1_{FA}$ and $s_2 \circ \langle 1_{FA}, Fv\rangle = Fv$. Let $h : F(A \times B) \to FA \times FB$ be the unique arrow such that $s_1 \circ h = Fp_1$ and $s_2 \circ h = Fp_2$. We have that $s_1 \circ (h \circ F\langle 1_A, u\rangle \circ a \circ i) = s_1 \circ (h \circ F\langle 1_A, v\rangle \circ a \circ i)$ due to $Fp_1 \circ F\langle 1_A, u\rangle = 1_{FA} = Fp_1 \circ F\langle 1_A, v\rangle$ and, $s_2 \circ (h \circ F\langle 1_A, u\rangle \circ a \circ i) = s_2 \circ (h \circ F\langle 1_A, v\rangle \circ a \circ i)$ by commutativity of the following diagram.



Since s_1 and s_2 form a mono source, we have that $h \circ F\langle 1_A, u\rangle \circ a \circ i = h \circ F\langle 1_A, v\rangle \circ a \circ i$. But h is a monomorphism as F preserves relations; this follows from Lemma 4.8. So $F\langle 1_A, u\rangle \circ a \circ i = F\langle 1_A, v\rangle \circ a \circ i$.



By the universal property of pullbacks, there is a unique arrow $e : E \to FE$ such that $F(i) \circ e = a \circ i$. Hence (E, e) is the object of the equalizer of the homomorphisms u and v. Consequently, $\mathcal{C}_F$ has equalizers. ∎

Definition 4.10. An endofunctor $F : \mathcal{C} \to \mathcal{C}$ is called a *covarietor* if the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ has a right adjoint.

Corollary 4.11. *Let $\mathcal{C}$ be a (finitely) complete category and $F : \mathcal{C} \to \mathcal{C}$ be a covarietor that preserves pullbacks. Then $\mathcal{C}_F$ is (finitely) complete.*

Proof. By Proposition 4.9, the category $\mathcal{C}_F$ has equalizers. Moreover, it has (finite) products since F is a covarietor (see [1, Remark 4.4-(c)]), and hence $\mathcal{C}_F$ is (finitely) complete (see [3] and [23]). ∎

5. Exact categories

We discuss the conditions under which a category of coalgebras can be made Barr-exact.

Definition 5.1. A finitely complete category $\mathcal{C}$ is called effective when it has (regular epi)-mono factorizations and coequalizers of equivalence relations, and every equivalence relation in $\mathcal{C}$ is effective (i.e., the kernel pair of its coequalizer). When in addition regular epimorphisms are stable under pullbacks, $\mathcal{C}$ is called *Barr-exact.*

Given an equivalence relation R on an object A, the diagram

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A \xrightarrow{\ e\ } B$$

is called an exact sequence whenever R is the kernel pair of e and e is the coequalizer of R. It's also said that an effective category has exact sequences.

Every elementary topos is a Barr-exact category (see [15]). However, not every Barr-exact category is an elementary topos. A counter-example is the category $KHaus$ of compact Hausdorff spaces (see [20]). Categories monadic over sets are Barr-exact (see [5]). In particular, varieties of algebras are Barr-exact, so that categories such as Grp of groups and Rng of rings are Barr-exact. These are also not elementary toposes. Abelian categories are Barr-exact.

PROPOSITION 5.2. (SEE [16, 24]) *If $\mathcal{C}$ is an elementary topos and $F : \mathcal{C} \to \mathcal{C}$ is a covarietor that preserves pullbacks, then $\mathcal{C}_F$ is an elementary topos.*

DEFINITION 5.3. A *bisimulation* between F-coalgebras (A, a) and (B, b) is a binary relation R between A and B such that there is a morphism $r : R \to FR$ making the following diagram commute.

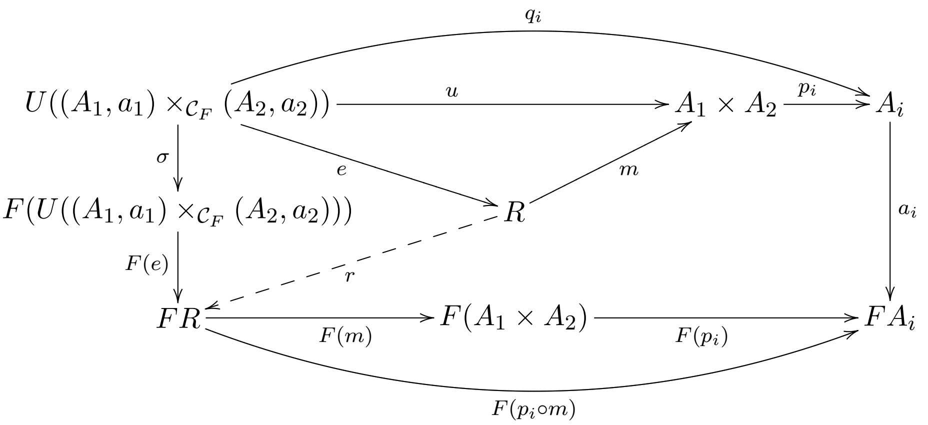
$$\begin{array}{ccccc} A & \xleftarrow{r_1} & R & \xrightarrow{r_2} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle b} \\ FA & \xleftarrow{Fr_1} & FR & \xrightarrow{Fr_2} & FB \end{array}$$

A bisimulation which is an equivalence relation is called a *bisimulation equivalence*.

LEMMA 5.4. *Let $F : \mathcal{C} \to \mathcal{C}$ be a covarietor that preserves pullbacks. Suppose that $\mathcal{C}$ is a finitely complete category (and hence so is $\mathcal{C}_F$ by Corollary 4.11). If moreover $\mathcal{C}$ has (regular epi)-mono factorizations, the canonical morphism $U((A_1, a_1) \times_{\mathcal{C}_F} (A_2, a_2)) \to U(A_1, a_1) \times U(A_2, a_2) = A_1 \times A_2$ is a monomorphism in $\mathcal{C}$, where $(A_1, a_1) \times_{\mathcal{C}_F} (A_2, a_2)$ denotes the product of (A_1, a_1) and (A_2, a_2), for all F-coalgebras (A_1, a_1) and (A_2, a_2).*

Proof. Given two F-coalgebras (A_1, a_1) and (A_2, a_2), their product $(A_1, a_1) \times_{\mathcal{C}_F} (A_2, a_2)$ exists due to Corollary 4.11. We write $q_1 : (A_1, a_1) \times_{\mathcal{C}_F} (A_2, a_2) \to (A_1, a_1)$ and $q_2 : (A_1, a_1) \times_{\mathcal{C}_F} (A_2, a_2) \to (A_2, a_2)$ its projections. From universality of the product, there exists a unique arrow $u : U((A_1, a_1) \times_{\mathcal{C}_F} (A_2, a_2)) \to A_1 \times A_2$ such that $p_1 \circ u = q_1$ and $p_2 \circ u = q_2$, where p_1 and p_2 are the projections of the product of A_1 and A_2. Factorize u

into a regular epimorphism e followed by a monomorphism $m : R \rightarrowtail A_1 \times A_2$. For each $i = 1, 2$, the following diagram commutes



where $\sigma : U((A_1, a_1) \times_{\mathcal{C}_F} (A_2, a_2)) \to F(U((A_1, a_1) \times_{\mathcal{C}_F} (A_2, a_2)))$ is the coalgebra structure on $U((A_1, a_1) \times_{\mathcal{C}_F} (A_2, a_2))$. By definition, e is the coequalizer of a pair of parallel morphisms v and w in $\mathcal{C}$. So for each $i = 1, 2$, we have that $(F(p_i) \circ F(m)) \circ ((F(e) \circ \sigma) \circ v) = (F(p_i) \circ F(m)) \circ ((F(e) \circ \sigma) \circ w)$ or equivalently $F(p_i \circ m) \circ ((F(e) \circ \sigma) \circ v) = F(p_i \circ m) \circ ((F(e) \circ \sigma) \circ w)$. Also, $F(p_1 \circ m)$ and $F(p_2 \circ m)$ form a mono source; this is because F preserves relations as it preserves pullbacks. Whence $(F(e) \circ \sigma) \circ v = (F(e) \circ \sigma) \circ w$. By the universal property of coequalizers, there is a unique arrow $r : R \to FR$ such that $F(e) \circ \sigma = r \circ e$. It follows that for each $i = 1, 2$, $(F(p_i \circ m) \circ r) \circ e = (a_i \circ (p_i \circ m)) \circ e$, and hence $F(p_i \circ m) \circ r = a_i \circ (p_i \circ m)$ because e is an epimorphism. Thus R is equipped with an F-coalgebra structure $r : R \to FR$ turning its projections $r_1 = p_1 \circ m$ and $r_2 = p_2 \circ m$ into homomorphisms; that is, R is a bisimulation between (A_1, a_1) and (A_2, a_2). As a consequence, there exists a unique homomorphism $s : (R, r) \to (A_1, a_1) \times_{\mathcal{C}_F} (A_2, a_2)$ such that $q_1 \circ s = r_1$ and $q_2 \circ s = r_2$. Since q_1 and q_2 form a mono source in $\mathcal{C}_F$, the composite $s \circ e : (A_1, a_1) \times_{\mathcal{C}_F} (A_2, a_2) \to (A_1, a_1) \times_{\mathcal{C}_F} (A_2, a_2)$ is then the unique homomorphism such that $q_1 \circ (s \circ e) = q_1$ and $q_2 \circ (s \circ e) = q_2$; that is, $s \circ e = 1_{U((A_1,a_1) \times_{\mathcal{C}_F} (A_2,a_2))}$. Consequently, e is an isomorphism as a regular epimorphism and a coretraction. This shows that u is a monomorphism. ∎

Corollary 5.5. *Let $F : \mathcal{C} \to \mathcal{C}$ be a covarietor that preserves pullbacks. Suppose that $\mathcal{C}$ is a finitely complete category with (regular epi)-mono factorizations and coequalizers of equivalence relations. Then $\mathcal{C}_F$ is effective provided that $\mathcal{C}$ so is.*

Proof. By Corollary 4.11, $\mathcal{C}_F$ is finitely complete. Given an equivalence relation (R, r) on an F-coalgebra (A, a). Under condition that F preserves pullbacks, its underlying object $R = U(R, r)$ is a subobject of $U((A, a) \times_{\mathcal{C}_F} (A, a))$ due to Proposition 4.6, and hence R is a binary relation on A as a consequence of Lemma 5.4. Also, the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ preserves equivalence relations as F preserves pullbacks. Thus R is an equivalence relation on A. Let φ be the coequalizer of R in $\mathcal{C}$. It exists as $\mathcal{C}$ is effective. From Proposition 4.5-(i), φ is also the coequalizer of (R, r) in $\mathcal{C}_F$. Since R is the kernel pair of φ in $\mathcal{C}$, it follows that (R, r) is the kernel pair of φ in $\mathcal{C}_F$. Hence $\mathcal{C}_F$ is an effective category. ∎

PROPOSITION 5.6. *Let $F : \mathcal{C} \to \mathcal{C}$ be a covarietor that preserves pullbacks, where $\mathcal{C}$ is a Barr-exact category. Then $\mathcal{C}_F$ is also Barr-exact.*

Proof. First, $\mathcal{C}_F$ is effective due to Corollary 5.5. So it only remains to prove that regular epimorphisms in $\mathcal{C}_F$ are stable under pullbacks. But this is a direct consequence of the fact that the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ creates (regular epi)-mono factorizations and pullbacks (see Proposition 4.5 (ii) and Proposition 4.6). ∎

References

[1] J. Adámek, Introduction to coalgebra, *Theory Appl. Categ.* **14** (2005), 157–199.

[2] J. Adámek, F. Bonchi, M. Hülsbusch, B. König, S. Milius, A. Silva, A coalgebraic perspective on minimization and determinization, in "Foundations of software science and computational structures", Lecture Notes in Comput. Sci., 7213, Springer, Heidelberg, 2012, 58–73. doi:10.1007/978-3-642-28729-9_4

[3] J. Adámek, H. Herrlich, G.E. Strecker, "Abstract and concrete categories", John Wiley & Sons, Inc., New York, 1990.

[4] M. Barr, Terminal coalgebras in well-founded set theory, *Theoret. Comput. Sci.* **114** (2) (1993), 299–315.

[5] M. Barr, P.A. Grillet, D.H. Osdol, "Exact Categories and Categories of Sheaves", Lecture Notes in Math., 236, Springer-Verlag, Berlin, 1971. doi:10.1007/BFb0058579

[6] C. Butz, "Regular categories and regular logic", BRICS Lecture Series LS-98-2, Aarhus, Denmark 1998.

[7] A. Carboni, J. Lambeck, M.C. Pedicchio, Diagram chasing in Mal'cev categories, *J. Pure Appl. Algebra* **69** (1990), 271–284.

[8] A. Carboni, M.C. Pedicchio, N. Pirovano, Internal graphs and internal groupoids in Mal'cev categories, in "Category theory 1991" (Proceedings of the International Summer Meeting held in Montreal, Quebec, June 23-30, 1991), CMS Conf. Proc., 13, Ottawa, ON, 1992, 97 – 109.

[9] M. Gran, Notes on regular, exact and additive categories, Summer School on Category Theory and Algebraic Topology, Ecole Polytechnique Fédérale de Lausanne, 11-13 September 2014, Université Catholique de Louvain.

[10] M. Gran, Z. Janelidze, D. Rodelo, A. Ursini, Symmetry of regular diamonds, the Goursat property, and subtractivity, *Theory Appl. Categ.* **27** (6) (2012), 80 – 96.

[11] M. Gran, D. Rodelo, A new characterisation of Goursat categories, *Appl. Categ. Structures* **20** (2012), 229 – 238.

[12] M. Hoefnagel, Characterizations of majority categories, *Appl. Categ. Structures* **28** (2020), 113 – 134.

[13] M. Hoefnagel, Majority categories, *Theory Appl. Categ.* **34** (10) (2019), 249 – 268.

[14] B. Jacobs, "Introduction to coalgebra. Towards mathematics of states and observation", Cambridge Tracts Theoret. Comput. Sci., 59, Cambridge University Press, Cambridge, 2017.

[15] P.T. Johnstone, "Topos Theory", London Math. Soc. Monogr., Vol. 10, Academic Press, London-New York, 1977.

[16] P. Johnstone, J. Power, T. Tsujishita, H. Watanabe, J. Worrell, On the structure of categories of coalgebras, *Theoret. Comput. Sci.* **260** (1-2) (2001), 87 – 117. Coalgebraic methods in computer science (Lisbon, 1998). doi:10.1016/S0304-3975(00)00124-9

[17] A. Kurz, Logics for Coalgebras and Applications to Computer Science, PhD thesis, Ludwig-Maximilians-Universität München, 2000.

[18] S. Mac Lane, "Categories for the working mathematician", Grad. Texts in Math., 5, Springer-Verlag, New York-Berlin, 1971.

[19] A.I. Mal'cev, On the general theory of algebraic systems, *Mat. Sb. (N.S.)* **35** (77) (1954), 3 – 20.

[20] V. Marra, L. Reggio, A characterisation of the category of compact Hausdorff spaces, *Theory Appl. Categ.* **35** (2020), Paper No. 51, 1871 – 1906.

[21] M.C. Pedicchio, Arithmetical categories and commutator theory, *Appl. Categ. Structures* **4** (2-3) (1996), 297 – 305.

[22] J.J.M.M. Rutten, Universal coalgebra: a theory of systems, *Theoret. Comput. Sci.* **249** (2000), 3 – 80.

[23] H. Schubert, "Categories", Springer-Verlag, New York-Heidelberg, 1972.

[24] J. Worrell, Toposes of coalgebras and hidden algebras, in "First Workshop on Coalgebraic Methods in Computer Science (CMCS'98)" (Lisbon, March 28-29, 1998), Electron. Notes Theor. Comput. Sci., 11, Elsevier Science B.V., Amsterdam, 1998, 1 – 19.