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Ternary Nambu F-algebras: generalizing F-manifolds and Nambu-Poisson structures

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Abstract: In this paper, we introduce the notion of ternary Nambu F-algebras, which extend both F-algebras (F-manifold algebras) and Nambu-Poisson algebras. Their representation theory is developed, with particular attention to the definition of dual representations, requiring additional conditions analogous to those in the binary setting. We also establish the concept of coherent ternary Nambu F-algebras and investigate their construction from underlying F-algebras. Moreover, we define and study relative Rota-Baxter operators associated with representations, and show how they naturally give rise to ternary pre-Nambu F-algebras.

Key words: Ternary Nambu F-algebra, representation, relative Rota-Baxter operator, ternary pre-Nambu F-algebra.

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1. Introduction

Ternary Lie algebras and more generally n-ary Lie algebras, are natural extensions of classical Lie algebras. They were first introduced and studied by Filippov [18]. Such algebras also arise in the algebraic formulation of mechanics [30], where they generalize Hamiltonian mechanics by involving two Hamiltonians (see [19, 33]). Furthermore, 3-Lie algebras play an important role in string theory and M-theory: Basu and Harvey proposed replacing the Lie algebra in the Nahm equation with a 3-Lie algebra in the lifted Nahm equations [8].

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In recent years, n-Lie algebras have been investigated in various algebraic contexts. In [6, 7], the authors developed constructions, realizations, and classifications of both 3-Lie and n-Lie algebras. Representation theory for n-Lie algebras was initiated by Kasymov [25], while the cohomology theory of Filippov algebras was explored in [3]. The adjoint representation is defined by fixing two elements in the ternary bracket.

The notion of Rota-Baxter operators on associative algebras was introduced in 1960 by G. Baxter [9] in the context of fluctuation theory in probability. Since then, Rota-Baxter operators have found numerous applications, notably in Connes-Kreimer's algebraic approach to renormalization in perturbative quantum field theory [12]. Within Lie theory, a Rota-Baxter operator of weight zero appeared independently in the 1980s as the operator form of the classical Yang-Baxter equation, which itself plays a central role in many areas of mathematics and mathematical physics, including quantum groups and integrable systems [11, 32]. Rota-Baxter operators on superalgebras were studied in [1], where connections were established between associative superalgebras, Lie superalgebras, L-dendriform superalgebras, and pre-Lie superalgebras. The notion of relative Rota-Baxter operators on 3-Lie algebras, defined with respect to a representation and connected to solutions of the 3-Lie classical Yang-Baxter equation, was investigated in [5]. In particular, Rota-Baxter operators on 3-Lie algebras introduced in [4] are precisely relative Rota-Baxter operators with respect to the adjoint representation.

In this context, to provide a geometric interpretation of the WDVV equations, Dubrovin introduced the concept of Frobenius manifolds [16]. Later, Hertling and Manin defined F-manifolds as a relaxation of the Frobenius manifold axioms [22]. These structures appear in a wide range of mathematical areas, including singularity theory [21], quantum K-theory [26], integrable systems [13, 14, 24], and operads [28]. Inspired by the study of algebraic structures underlying F-manifolds, Dotsenko introduced the concept of F-algebras [15], defined as a tuple $(A, \cdot, [\cdot,\cdot])$ where $(A, \cdot)$ is a commutative associative algebra and $(A, [\cdot,\cdot])$ is a Lie algebra, subject to the Hertling-Manin relation:

$$\mathcal{L}(x \cdot y, z, w) = x \cdot \mathcal{L}(y, z, w) + y \cdot \mathcal{L}(x, z, w), \qquad \forall\, x, y, z, w \in A, \tag{1.1}$$

where the Leibnizator $\mathcal{L}$ is given by

$$\mathcal{L}(x, y, z) = [x, y \cdot z] - [x, y] \cdot z - y \cdot [x, z]. \tag{1.2}$$

More recently, extensions such as F-manifold color algebras and Hom-F-manifold algebras, along with various constructions, have been introduced in [10, 29].

This naturally raises the question of how to introduce a ternary version of Nambu F-algebras. The aim of this article is to address this problem. Specifically, we propose a notion of ternary Nambu F-algebras by (i) defining a structure that can be derived from an F-algebra analogously to 3-Lie algebras, and (ii) ensuring that the structure generalizes ternary Nambu-Poisson algebras.

The paper is organized as follows. In Section 2, we recall some basic notions of 3-Lie algebras, 3-pre-Lie algebras, ternary Nambu-Poisson algebras, and F-algebras. Section 3 introduces ternary Nambu F-algebras and provides a construction result using an F-algebra equipped with a trace function. In Section 4, we study representations and dual representations of ternary Nambu F-algebras. Section 5 is devoted to the introduction of ternary pre-Nambu F-algebras and relative Rota-Baxter operators on ternary Nambu F-algebras. We show that a relative Rota-Baxter operator on a ternary Nambu F-algebra naturally induces a ternary pre-Nambu F-algebra, while such algebras can also be constructed via symplectic structures.

Throughout this paper, all vector spaces are considered over a field $\mathbb{K}$ of characteristic 0.

2. Preliminaries

In this section, we recall some basics on associative, Zinbiel, and 3-(pre-) Lie algebras and their representations. An associative algebra is a pair $(A, \cdot)$, where A is a vector space, and $\cdot : A \otimes A \to A$ is a linear map satisfying that for all $x, y, z \in A$, the associator $\mathfrak{as}(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z) = 0$, i.e.

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

Furthermore, if $x \cdot y = y \cdot x$ for all $x, y \in A$, then $(A, \cdot)$ is called a commutative associative algebra.

A representation of commutative associative algebra $(A, \cdot)$ on a vector space V is a linear map $\mu : A \to \mathfrak{gl}(V)$, such that for any $x, y \in A$, the following equalities are satisfied:

$$\mu(x \cdot y) = \mu(x)\mu(y), \qquad \forall\, x, y \in A. \tag{2.1}$$

If there exists an element $1_A \in A$ satisfying $x \cdot 1_A = 1_A \cdot x = x$, then A is called unital with unit 1_A.

Lemma 2.1. *Let $(V;\mu)$ be a representation of a commutative associative algebra $(A,\cdot)$. Define $\mu^*: A \to \mathfrak{gl}(V^*)$ by*

$$\langle \mu^*(x)\xi, v\rangle = -\langle \xi, \mu(x)v\rangle, \qquad \forall\, x \in A, \xi \in V^*, v \in V.$$

Then $(V^*, -\mu^*)$ is a representation of $(A,\cdot)$.

The above representation is called the dual representation of $(V;\mu)$.

A Zinbiel algebra is a pair $(A,\diamond)$, where A is a vector space, $\diamond : A\otimes A \to A$ is a binary multiplication such that for all $x, y, z \in A$,

$$x \diamond (y \diamond z) = (y \diamond x) \diamond z + (x \diamond y) \diamond z.$$

Lemma 2.2. *Let $(A,\diamond)$ be a Zinbiel algebra. Then $(A,\cdot)$ is a commutative associative algebra, where $x \cdot y = x \diamond y + y \diamond x$. Moreover, for $x \in A$, define $L_\diamond(x) : A \to \mathfrak{gl}(A)$ by*

$$L_\diamond(x)(y) = x \diamond y, \qquad \forall\, y \in A. \tag{2.2}$$

Then $(A; L_\diamond)$ is a representation of the commutative associative algebra $(A,\cdot)$.

Definition 2.3. A 3-Lie algebra consists of a vector space A equipped with a skew-symmetric linear map called 3-Lie bracket $[\cdot,\cdot,\cdot] : \otimes^3 A \to A$ such that the following Fundamental Identity holds (for $x_i \in A, 1 \le i \le 5$)

$$[x_1, x_2, [x_3, x_4, x_5]] = [[x_1, x_2, x_3], x_4, x_5] + [x_3, [x_1, x_2, x_4], x_5] + [x_3, x_4, [x_1, x_2, x_5]]$$

In other words, for $x_1, x_2 \in A$, the operator

$$\mathrm{ad}_{x_1,x_2} : A \to A, \quad \mathrm{ad}_{x_1,x_2} x := [x_1, x_2, x], \qquad \forall\, x \in A, \tag{2.3}$$

is a derivation in the sense that, for all $x_3, x_4, x_5 \in A$,

$$\mathrm{ad}_{x_1,x_2}[x_3, x_4, x_5] = [\mathrm{ad}_{x_1,x_2} x_3, x_4, x_5] + [x_3, \mathrm{ad}_{x_1,x_2} x_4, x_5] + [x_3, x_4, \mathrm{ad}_{x_1,x_2} x_5].$$

A morphism between 3-Lie algebras is a linear map that preserves the 3-Lie brackets.

Example 2.4. [18] Consider 4-dimensional 3-Lie algebra A generated by (e_1, e_2, e_3, e_4) with the following multiplication

$$[e_1, e_2, e_3] = e_4, \quad [e_1, e_2, e_4] = e_3, \quad [e_1, e_3, e_4] = e_2, \quad [e_2, e_3, e_4] = e_1.$$

The notion of a representation of an n-Lie algebra was introduced in [25]. See also [17].

Definition 2.5. A representation of a 3-Lie algebra A on a vector space V is a skew-symmetric linear map $\rho : \otimes^2 A \to gl(V)$ satisfying

$$\rho(x_1, x_2)\rho(x_3, x_4) - \rho(x_3, x_4)\rho(x_1, x_2) = \rho([x_1, x_2, x_3], x_4) - \rho([x_1, x_2, x_4], x_3),$$

$$\rho([x_1, x_2, x_3], x_4) = \rho(x_1, x_2)\rho(x_3, x_4) + \rho(x_2, x_3)\rho(x_1, x_4) + \rho(x_3, x_1)\rho(x_2, x_4),$$

for $x_i \in A, 1 \leq i \leq 4$.

Proposition 2.6. *Let (V, ρ) be a representation of a 3-Lie algebra A. Define $\rho^* : \otimes^2 A \to \mathfrak{gl}(V^*)$ by*

$$\langle \rho^*(x_1, x_2)\xi, v\rangle = -\langle \xi, \rho(x_1, x_2)v\rangle, \qquad \forall\, \xi \in V^*,\ x_1, x_2 \in A,\ v \in V.$$

Then (V^*, ρ^*) is a representation of A, called the dual representation.

Example 2.7. Let A be a 3-Lie algebra. The linear map $\mathrm{ad} : \otimes^2 A \to \mathfrak{gl}(A)$ with $x_1, x_2 \to \mathrm{ad}_{x_1,x_2}$ for any $x_1, x_2 \in A$ defines a representation (A, ad) which is called the adjoint representation of A, where ad_{x_1,x_2} is given by (2.3). The dual representation (A^*, ad^*) of the adjoint representation (A, ad) of a 3-Lie algebra A is called the coadjoint representation.

Definition 2.8. A 3-pre-Lie algebra is a pair $(A, \{\cdot, \cdot, \cdot\})$ consisting of a a vector space A and a linear map $\{\cdot, \cdot, \cdot\} : A \otimes A \otimes A \to A$ such that the following identities hold:

$$\{x, y, z\} = -\{y, x, z\}, \tag{2.4}$$

$$\begin{aligned}\{x_1, x_2, \{x_3, x_4, x_5\}\} = &\{[x_1, x_2, x_3]^C, x_4, x_5\} + \{x_3, [x_1, x_2, x_4]^C, x_5\} \\ &+ \{x_3, x_4, \{x_1, x_2, x_5\}\},\end{aligned} \tag{2.5}$$

$$\begin{aligned}\{[x_1, x_2, x_3]^C, x_4, x_5\} = &\{x_1, x_2, \{x_3, x_4, x_5\}\} + \{x_2, x_3, \{x_1, x_4, x_5\}\} \\ &+ \{x_3, x_1, \{x_2, x_4, x_5\}\},\end{aligned} \tag{2.6}$$

where $x, y, z, x_i \in A, 1 \leq i \leq 5$ and $[\cdot, \cdot, \cdot]^C$ is defined by

$$[x, y, z]^C = \{x, y, z\} + \{y, z, x\} + \{z, x, y\}, \qquad \forall\, x, y, z \in A. \tag{2.7}$$

Proposition 2.9. *Let $(A, \{\cdot, \cdot, \cdot\})$ be a 3-pre-Lie algebra. Then the induced 3-commutator given by (2.7) defines a 3-Lie algebra.*

Let $(A, \{\cdot, \cdot, \cdot\})$ be a 3-pre-Lie algebra. Define a skew-symmetric linear map $\mathbb{L} : \otimes^2 A \to \mathfrak{gl}(A)$ by

$$\mathbb{L}(x, y)z = \{x, y, z\}, \quad \forall\, x, y, z \in A. \tag{2.8}$$

By the definitions of a 3-pre-Lie algebra and a representation of a 3-Lie algebra, we have:

LEMMA 2.10. *The pair $(A, \mathbb{L})$ is a representation of the 3-Lie algebra $(A, [\cdot, \cdot, \cdot]^C)$.*

DEFINITION 2.11. [27] A ternary Nambu-Poisson algebra is a triple $(A, \cdot, [\cdot, \cdot, \cdot])$ consisting of a $\mathbb{K}$-vector space A, two linear maps $\cdot : A \otimes A \to A$ and $[\cdot, \cdot, \cdot] : A \otimes A \otimes A \to A$ such that

1. $(A, \cdot)$ is a commutative associative algebra,
2. $(A, [\cdot, \cdot, \cdot])$ is a 3-Lie algebra,
3. the following Leibniz rule

$$[x_1, x_2, x_3 \cdot x_4] = x_3 \cdot [x_1, x_2, x_4] + [x_1, x_2, x_3] \cdot x_4$$

holds for all $x_1, x_2, x_3 \in A$.

Now, we recall the notion of F-algebra given in [15].

Let $(A, \cdot, [\cdot, \cdot])$ be an F-algebra, $(V; \rho)$ be a representation of the Lie algebra $(A, [\cdot, \cdot])$ and $(V; \mu)$ be a representation of the commutative associative algebra $(A, \cdot)$. Define the two linear maps $\mathcal{L}_1, \mathcal{L}_2 : A \otimes A \otimes V \to V$ given by

$$\mathcal{L}_1(x, y, u) = \rho(x)\mu(y)(u) - \mu(y)\rho(x)(u) - \mu([x, y])(u), \tag{2.9}$$

$$\mathcal{L}_2(x, y, u) = \mu(x)\rho(y)(u) + \mu(y)\rho(x)(u) - \rho(x \cdot y)(u), \tag{2.10}$$

for all $x, y \in A$ and $u \in V$.

DEFINITION 2.12. With the above notations, the tuple $(V; \rho, \mu)$ is a representation of A if the following conditions hold:

$$\mathcal{L}_1(x \cdot y, z, u) = \mu(x)\mathcal{L}_1(y, z, u) + \mu(y)\mathcal{L}_1(x, z, u), \tag{2.11}$$

$$\mu(\mathcal{L}(x, y, z))(u) = \mathcal{L}_2(y, z, \mu(x)(u)) - \mu(x)\mathcal{L}_2(y, z, u), \tag{2.12}$$

for all $x, y, z \in A$ and $u \in V$.

3. Ternary Nambu F-algebras

In this section, we introduce the notion of ternary Nambu F-algebras as a generalization of Nambu F-algebras given in [15]. They are the generalization of F-algebra and ternary Nambu-Poisson algebra.

Definition 3.1. A tuple $(A, \cdot, [\cdot,\cdot,\cdot])$ is called a ternary Nambu F-algebra if $(A, \cdot)$ is a commutative associative algebra and $(A, [\cdot,\cdot,\cdot])$ is a 3-Lie algebra, such that for all $x_1, x_2, x_3, x_4, x_5 \in A$, holds:

$$\mathcal{L}(x_1 \cdot x_2, x_3, x_4, x_5) = x_1 \cdot \mathcal{L}(x_2, x_3, x_4, x_5) + x_2 \cdot \mathcal{L}(x_1, x_3, x_4, x_5), \qquad (3.1)$$

where $\mathcal{L}(x_1, x_2, x_3, x_4)$ is the 3-Leibnizator define by

$$\mathcal{L}(x_1, x_2, x_3, x_4) = [x_1, x_2, x_3 \cdot x_4] - x_3 \cdot [x_1, x_2, x_4] - [x_1, x_2, x_3] \cdot x_4.$$

Example 3.2. Every ternary Nambu-Poisson algebra is a ternary F-manifold.

Definition 3.3. Let $(A, \cdot_A, [\cdot,\cdot,\cdot]_A)$ and $(B, \cdot_B, [\cdot,\cdot,\cdot]_B)$ be two ternary Nambu F-algebras. A homomorphism between A and B is a linear map $f : A \to B$ such that

$$\begin{aligned} f(x_1 \cdot_A x_2) &= f(x_1) \cdot_B f(x_2), \\ f[x_1, x_2, x_3]_A &= [f(x_1), f(x_2), f(x_3)]_B, \qquad \forall\, x_1, x_2, x_3 \in A. \end{aligned}$$

Proposition 3.4. *Let $(A, \cdot_A, [\cdot,\cdot,\cdot]_A)$ and $(B, \cdot_B, [\cdot,\cdot,\cdot]_B)$ be two ternary Nambu F-algebras. Then $(A \oplus B, \cdot_{A\oplus B}, [\cdot,\cdot,\cdot]_{A\oplus B})$ is a ternary Nambu F-algebra, where the product $\cdot_{A\oplus B}$ and bracket $[\cdot,\cdot,\cdot]_{A\oplus B}$ are given by*

$$\begin{aligned} (x_1 + y_1) \cdot_{A\oplus B} (x_2 + y_2) &= x_1 \cdot_A x_2 + y_1 \cdot_B y_2, \\ [(x_1 + y_1), (x_2 + y_2), (x_3 + y_3)]_{A\oplus B} &= [x_1, x_2, x_3]_A + [y_1, y_2, y_3]_B \end{aligned}$$

for all $x_i \in A, y_i \in B, \forall i = 1, 2, 3$.

Proof. It easy to check that $(A \oplus B, \cdot_{A\oplus B})$ is a commutative associative algebra and $(A \oplus B, [\cdot,\cdot,\cdot]_{A\oplus B})$ is a 3-Lie algebra and for all $x_i \in A$, $y_i \in B$, $i = 1, \dots, 4$

$$\mathcal{L}_{A\oplus B}(x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) = \mathcal{L}_A(x_1, x_2, x_3, x_4) + \mathcal{L}_B(y_1, y_2, y_3, x_4).$$

Using the above identity, we have

$$\begin{aligned}
\mathcal{L}_{A\oplus B}&((x_1+y_1)\cdot_{A\oplus B}(x_2+y_2),x_3+y_3,x_4+y_4,x_5+y_5)\\
&=\mathcal{L}_{A\oplus B}(x_1\cdot_A x_2+y_1\cdot_B y_2,x_3+y_3,x_4+y_4,x_5+y_5)\\
&=\mathcal{L}_A(x_1\cdot_A x_2,x_3,x_4,x_5)+\mathcal{L}_B(y_1\cdot_B y_2,y_3,y_4,y_5)\\
&=x_1\cdot_A \mathcal{L}_A(x_2,x_3,x_4,x_5)+x_2\cdot_A \mathcal{L}_A(x_1,x_3,x_4,x_5)\\
&\quad+y_1\cdot_B \mathcal{L}_B(y_2,y_3,y_4,y_5)+y_2\cdot_B \mathcal{L}_B(y_1,y_3,y_4,y_5)\\
&=(x_1+y_1)\cdot_{A\oplus B}\mathcal{L}_{A\oplus B}(x_2+y_2,x_3+y_3,x_4+y_4,x_5+y_5)\\
&\quad+(x_2+y_2)\cdot_{A\oplus B}\mathcal{L}_{A\oplus B}(x_1+y_1,x_3+y_3,x_4+y_4,x_5+y_5).
\end{aligned}$$

Then, (3.1) holds. ∎

PROPOSITION 3.5. *Let $(A,\cdot_A,[\cdot,\cdot,\cdot]_A)$ be a ternary Nambu F-algebra and $(B,\cdot_B)$ be commutative associative algebra. Then $(A\otimes B,\cdot_{A\otimes B},[\cdot,\cdot,\cdot]_{A\otimes B})$ is a ternary Nambu F-algebra, where the product $\cdot_{A\otimes B}$ is defined by*

$$(x_1\otimes y_1)\cdot_{A\otimes B}(x_2\otimes y_2)=(x_1\cdot_A x_2)\otimes(y_1\cdot_B y_2),$$
$$[x_1\otimes y_1,x_2\otimes y_2,x_3\otimes y_3]_{A\otimes B}=[x_1,x_2,x_3]_A\otimes(y_1\cdot_B y_2\cdot_B y_3),$$

for all $x_i\in A, y_i\in B, \forall i=1,2,3$.

Proof. It is obvious to check that $(A\otimes B,\cdot_{A\otimes B})$ is a commutative associative algebra and $(A\otimes B,[\cdot,\cdot,\cdot]_{A\otimes B})$ is a 3-Lie algebra. For any $x_i,y_i\in A$, $i=1,\dots,4$ we have

$$\begin{aligned}
\mathcal{L}_{A\otimes B}(x_1\otimes y_1,&x_2\otimes y_2,x_3\otimes y_3,x_4\otimes y_4)\\
&=[x_1\otimes y_1,x_2\otimes y_2,(x_3\otimes y_3)\cdot_{A\otimes B}(x_4\otimes y_4)]_{A\otimes B}\\
&\quad-(x_3\otimes y_3)\cdot_{A\otimes B}[x_1\otimes y_1,x_2\otimes y_2,x_4\otimes y_4]\\
&\quad-[x_1\otimes y_1,x_2\otimes y_2,x_3\otimes y_3]\cdot_{A\otimes B}(x_4\otimes y_4)\\
&=[x_1\otimes y_1,x_2\otimes y_2,(x_3\cdot_A x_4)\otimes(y_3\cdot_B y_4)]_{A\otimes B}\\
&\quad-(x_3\otimes y_3)\cdot_{A\otimes B}([x_1,x_2,x_4]_A\otimes(y_1\cdot_B y_2\cdot_B y_4))\\
&\quad-([x_1,x_2,x_3]_A\otimes(y_1\cdot_B y_2\cdot_B y_3))\cdot_{A\otimes B}(x_4\otimes y_4)\\
&=[x_1,x_2,(x_3\cdot_A x_4)]\otimes(y_1\cdot_B y_2\cdot_B y_3\cdot_B y_4))\\
&\quad-(x_3\cdot_A[x_1,x_2,x_4]_A)\otimes(y_1\cdot_B y_2\cdot_B y_3\cdot y_4)\\
&\quad-([x_1,x_2,x_3]_A\cdot_A x_4\otimes(y_1\cdot_B y_2\cdot_B y_3\cdot_B y_4)\\
&=\mathcal{L}_A(x_1,x_2,x_3,x_4)\otimes(y_1\cdot_B y_2\cdot_B y_3\cdot_B y_4).
\end{aligned}$$

Using the above equality, we obtain,

$$
\begin{aligned}
&\mathcal{L}_{A\otimes B}((x_1\otimes y_1)\cdot_{A\otimes B}(x_2\otimes y_2),(x_3\otimes y_3),(x_4\otimes y_4),(x_5\otimes y_5))\\
&\quad=\mathcal{L}_{A\otimes B}((x_1\cdot_A x_2)\otimes(y_1\cdot_B y_2),(x_3\otimes y_3),(x_4\otimes y_4),(x_5\otimes y_5))\\
&\quad=\mathcal{L}_A(x_1\cdot_A x_2,x_3,x_4,x_5)\otimes(y_1\cdot_B y_2\cdot_B y_3\cdot_B y_4\cdot_B y_5)\\
&\quad=(x_1\cdot_A\mathcal{L}_A(x_2,x_3,x_4,x_5)\otimes(y_1\cdot_B y_2\cdot_B y_3\cdot_B y_4\cdot_B y_5)\\
&\qquad+x_2\cdot_A\mathcal{L}_A(x_1,x_3,x_4,x_5)\otimes(y_1\cdot_B y_2\cdot_B y_3\cdot_B y_4\cdot_B y_5)\\
&\quad=(x_1\otimes y_1)\cdot_{A\otimes B}\mathcal{L}_A(x_2,x_3,x_4,x_5)\otimes(y_2\cdot_B y_3\cdot_B y_4\cdot_B y_5)\\
&\qquad+(x_2\cdot_B y_2)\cdot_{A\otimes B}\mathcal{L}_A(x_1,x_3,x_4,x_5)\otimes(y_1\cdot_B y_3\cdot_B y_4\cdot_B y_5)\\
&\quad=(x_1\otimes y_1)\cdot_{A\otimes B}\mathcal{L}_{A\otimes B}(x_2\otimes y_2,x_3\otimes y_2,x_4\otimes y_4,x_5\otimes y_5)\\
&\qquad+(x_2\cdot_B y_2)\cdot_{A\otimes B}\mathcal{L}_{A\otimes B}(x_1\otimes y_1,x_3\otimes y_3,x_4\otimes y_4,x_5\otimes y_5).
\end{aligned}
$$

Then, (3.1) holds and $(A\otimes B,\cdot_{A\otimes B},[\cdot,\cdot,\cdot]_{A\otimes B})$ is a ternary Nambu F-algebra. ∎

PROPOSITION 3.6. *Let $(A,\cdot,[\cdot,\cdot,\cdot])$ be a ternary Nambu F-algebra. Fixed $a\in A$ and define a bracket $[\cdot,\cdot]_a : A\otimes A\to A$ by $[x,y]_a=[x,a,y]$ for all $x,y\in A$. Then $(A,\cdot,[\cdot,\cdot]_a)$ is a Nambu F-algebra.*

Proof. Thanks to [31], we have $(A,[\cdot,\cdot]_a)$ is a Lie algebra. Now, we will show (1.1) for any $x,y,z,w\in A$, observe that

$$
\begin{aligned}
\mathcal{L}_a(x,y,z)&=[x,y\cdot z]_a-[x,y]_a\cdot z-y\cdot[x,z]_a\\
&=[x,a,y\cdot z]-[x,a,y]\cdot z-y\cdot[x,a,z]\\
&=\mathcal{L}(x,a,y,z).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\mathcal{L}_a(x\cdot y,z,w)-x\cdot\mathcal{L}_a(y,z,w)+y\cdot\mathcal{L}_a(x,z,w)\\
&\qquad=\mathcal{L}(x\cdot y,a,z,w)-x\cdot\mathcal{L}(y,a,z,w)+y\cdot\mathcal{L}(x,a,z,w)=0.
\end{aligned}
$$

Therefore, $(A,\cdot,[\cdot,\cdot]_a)$ is an F-algebra. ∎

Now, we generalize the result of [27] to the framework of F-algebras.

THEOREM 3.7. ([2]) *Let $(A,[\cdot,\cdot])$ be a Lie algebra and let $\tau : A\to\mathbb{K}$ be a linear map satisfying $\tau([x,y])=0$ for all $x,y\in A$ which is called a trace. Define the ternary bracket, for any $x_1,x_2,x_3\in A$, by*

$$[x_1,x_2,x_3]_\tau=\tau(x_1)[x_2,x_3]-\tau(x_2)[x_1,x_3]+\tau(x_3)[x_1,x_2].$$

Then $(A,[\cdot,\cdot,\cdot]_\tau)$ *is a 3-Lie algebra.*

THEOREM 3.8. *Let* $(A,\cdot,[\cdot,\cdot])$ *be an* F*-algebra and let* τ *be a trace. Then* $(A,\cdot,[\cdot,\cdot,\cdot]_\tau)$ *is a ternary Nambu* F*-algebra if and only if*

$$\tau(x_1\cdot x_2)\mathcal{L}(x_3,x_4,x_5)=\tau(x_2)x_1\cdot\mathcal{L}(x_3,x_4,x_5)+\tau(x_1)x_2\cdot\mathcal{L}(x_3,x_4,x_5). \tag{3.2}$$

Proof. We have $(A,\cdot)$ commutative associative algebra and $(A,[\cdot,\cdot,\cdot]_\tau)$ is a 3-Lie algebra. For any $x_i, i=1,\dots,5$ we have

$$\begin{aligned}\mathcal{L}_\tau(x_1,x_2,x_3,x_4)&=[x_1,x_2,x_3\cdot x_4]_\tau-x_3\cdot[x_1,x_2,x_4]_\tau-[x_1,x_2,x_3]_\tau\cdot x_4\\&=\tau(x_1)\cdot[x_2,x_3\cdot x_4]-\tau(x_2)\cdot[x_1,x_3\cdot x_4]+\tau(x_3\cdot x_4)\cdot[x_1,x_2]\\&\quad-x_3\cdot(\tau(x_1)\cdot[x_2,x_4]-\tau(x_2)\cdot[x_1,x_4]+\tau(x_4)\cdot[x_1,x_2])\\&\quad-(\tau(x_1)\cdot[x_2,x_3]-\tau(x_2)[x_1,x_3]+\tau(x_3)\cdot[x_1,x_2])\cdot x_4\\&=\tau(x_1)([x_2,x_3\cdot x_4]-x_3\cdot[x_2,x_4]-[x_2,x_3]\cdot x_4)\\&\quad-\tau(x_2)([x_1,x_3\cdot x_4]-x_3\cdot[x_1,x_4]-[x_1,x_3]\cdot x_4)\\&\quad+\tau(x_3\cdot x_4)[x_1,x_2]-\tau(x_4)x_3\cdot[x_1,x_2]-\tau(x_3)[x_1,x_2]\cdot x_4\\&=\tau(x_1)\mathcal{L}(x_2,x_3,x_4)-\tau(x_2)\mathcal{L}(x_1,x_3,x_4)\\&\quad+\tau(x_3\cdot x_4)[x_1,x_2]-\tau(x_4)x_3\cdot[x_1,x_2]-\tau(x_3)[x_1,x_2]\cdot x_4.\end{aligned}$$

Using the above relation, we obtain

$$\begin{aligned}&\mathcal{L}_\tau(x_1\cdot x_2,x_3,x_4,x_5)-x_1\cdot\mathcal{L}_\tau(x_2,x_3,x_4,x_5)-x_2\cdot\mathcal{L}_\tau(x_1,x_3,x_4,x_5)\\&=\tau(x_1\cdot x_2)\mathcal{L}(x_3,x_4,x_5)-\tau(x_3)\mathcal{L}(x_1\cdot x_2,x_4,x_5)-\tau(x_2)x_1\cdot\mathcal{L}(x_3,x_4,x_5)\\&\quad-\tau(x_3)x_1\cdot\mathcal{L}(x_2,x_4,x_5))-\tau(x_1)x_2\cdot\mathcal{L}(x_3,x_4,x_5)-\tau(x_3)x_2\cdot\mathcal{L}(x_1,x_4,x_5)\\&\quad+\tau(x_4\cdot x_5)[x_1\cdot x_2,x_3]-\tau(x_5)x_4\cdot[x_1\cdot x_2,x_3]-\tau(x_4)[x_1\cdot x_2,x_3]\cdot x_5\\&\quad-\tau(x_4\cdot x_5)x_1\cdot[x_2,x_3]+\tau(x_5)x_1\cdot x_4\cdot[x_2,x_3]+\tau(x_4)x_1\cdot[x_2,x_3]\cdot x_5\\&\quad-\tau(x_4\cdot x_5)x_2\cdot[x_1,x_3]+\tau(x_5)x_2\cdot x_4\cdot[x_1,x_3]+\tau(x_4)x_2\cdot[x_1,x_3]\cdot x_5\\&=\tau(x_1\cdot x_2)\mathcal{L}(x_3,x_4,x_5)-\tau(x_3)\mathcal{L}(x_1\cdot x_2,x_4,x_5)-\tau(x_2)x_1\cdot\mathcal{L}(x_3,x_4,x_5)\\&\quad-\tau(x_3)x_1\cdot\mathcal{L}(x_2,x_4,x_5))-\tau(x_1)x_2\cdot\mathcal{L}(x_3,x_4,x_5)-\tau(x_3)x_2\cdot\mathcal{L}(x_1,x_4,x_5))\\&\quad+\tau(x_4\cdot x_5)\mathcal{L}(x_1,x_2,x_3)-\tau(x_5)x_4\cdot\mathcal{L}(x_1,x_2,x_3)-\tau(x_4)\mathcal{L}(x_1,x_2,x_3)\cdot x_5\\&=\tau(x_1\cdot x_2)\mathcal{L}(x_3,x_4,x_5)-\tau(x_1)x_2\cdot\mathcal{L}(x_3,x_4,x_5)-\tau(x_2)x_1\cdot\mathcal{L}(x_3,x_4,x_5)\\&\quad+\tau(x_4\cdot x_5)\mathcal{L}(x_1,x_2,x_3)-\tau(x_4)\mathcal{L}(x_1,x_2,x_3)\cdot x_5-\tau(x_5)x_4\cdot\mathcal{L}(x_1,x_2,x_3).\end{aligned}$$

Then $(A,\cdot,[\cdot,\cdot,\cdot]_\tau)$ is a ternary F-algebra if and only if (3.2) holds. ∎

COROLLARY 3.9. *Let $(A, \cdot, [\cdot, \cdot])$ be a F-algebra and τ a trace. If $(A, \cdot)$ is unital with unit 1_A and*

$$\tau(x \cdot y)1_A = \tau(x)y + \tau(y)x.$$

Then $(A, \cdot, [\cdot, \cdot, \cdot]_\tau)$ is a ternary Nambu F-algebra.

4. Representations of ternary Nambu F-algebras

In this section, we develop the representation theory of ternary Nambu F-algebras, focusing in particular on the definition of dual representations, which necessitates supplementary conditions similar to those in the binary case. We further introduce the notion of coherent ternary Nambu F-algebras and explore their construction from underlying F-algebras.

Let $(A, \cdot, [\cdot, \cdot, \cdot])$ be a ternary Nambu F-algebra, $(V; \rho)$ be a representation of the 3-Lie algebra $(A, [\cdot, \cdot, \cdot])$ and $(V; \mu)$ be a representation of the commutative associative algebra $(A, \cdot)$.

Define the three linear maps $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 : A \otimes A \otimes A \otimes V \to V$ given by

$$\mathcal{L}_1(x, y, z, u) = \rho(x, y)\mu(z)(u) - \mu(z)\rho(x, y)(u) - \mu([x, y, z])u, \tag{4.1}$$

$$\mathcal{L}_2(x, y, z, u) = \mu(z)\rho(x, y)(u) + \mu(y)\rho(x, z)(u) - \rho(x, y \cdot z)u, \tag{4.2}$$

$$\mathcal{L}_3(x, y, z, u) = \rho(x, y)\mu(z)(u) + \rho(x, z)\mu(y)(u) - \rho(x, y \cdot z)u \tag{4.3}$$

for all $x, y \in A$ and $u \in V$. Note that, if we consider $V = A$, then $\mathcal{L}_1 = \mathcal{L}$ and $\mathcal{L}_2 = \mathcal{L}\sigma$, where $\sigma(x, y, z) = (z, x, y)$.

DEFINITION 4.1. With the above notations, the tuple $(V; \rho, \mu)$ is a representation of A if the following conditions hold:

$$\mathcal{L}_1(x_1 \cdot x_2, x_3, x_4, u) = \mu(x_1)\mathcal{L}_1(x_2, x_3, x_4, u) + \mu(x_2)\mathcal{L}_1(x_1, x_3, x_4, u), \tag{4.4}$$

$$\mathcal{L}_2(x_1 \cdot x_2, x_3, x_4, u) = \mu(x_1)\mathcal{L}_2(x_2, x_3, x_4, u) + \mu(x_2)\mathcal{L}_2(x_1, x_3, x_4, u), \tag{4.5}$$

$$\mu(\mathcal{L}(x_1, x_2, x_3, x_4))u = \mathcal{L}_2(x_2, x_3, x_4, \mu(x_1)(u)) - \mu(x_1)\mathcal{L}_2(x_2, x_3, x_4, u), \tag{4.6}$$

for all $x_1, x_2, x_3, x_4 \in A$ and $u \in V$.

EXAMPLE 4.2. Let $(A, \cdot, [\cdot, \cdot, \cdot])$ be a ternary Nambu F-algebra. Then (A, ad, L) is a representation of A called the adjoint representation.

EXAMPLE 4.3. Let $(V; \rho, \mu)$ be a representation of a ternary Nambu--Poisson algebra $(P, \cdot, [\cdot, \cdot, \cdot])$, i.e., $(V; \rho, \phi)$ is a representation of the 3-Lie algebra $(P, [\cdot,\cdot,\cdot])$ and $(V;\mu)$ is a representation of the commutative associative algebra $(P,\cdot)$ satisfying

$$\mathcal{L}_1(x_1,x_2,x_3,u) = \mathcal{L}_2(x_1,x_2,x_3,u) = 0, \qquad \forall\ x_1,x_2,x_3 \in P, u \in V.$$

Then $(V;\rho,\mu)$ is also a representation of the ternary Nambu F-algebra given by this ternary Nambu-Poisson algebra P.

Example 4.4. Let $(A, \cdot, [\cdot,\cdot,\cdot])$ be a ternary Nambu F-algebra and $(V;\rho,\mu)$ be a representation of A. Fixed $a \in A$, we define a linear map $\rho_a : A \to \mathrm{End}(V)$ as $\rho_a(x)v = \rho(x,a)v$. Then, $(V;\rho_a,\mu)$ is a representation of the F-algebra defined in Proposition 3.6.

Proposition 4.5. *Let $(A,\cdot,[\cdot,\cdot,\cdot])$ be a ternary Nambu F-algebra. Then $(V;\rho,\mu)$ is a representation of A if and only if $(A\oplus V, \cdot_\mu, [\cdot,\cdot,\cdot]_\rho)$ is a ternary Nambu F-algebra, where $(A\oplus V,\cdot_\mu)$ is the semi-direct product commutative associative algebra $A \ltimes_\mu V$, i.e.,*

$$(x_1+v_1)\cdot_\mu (x_2+v_2) = x_1\cdot x_2 + \mu(x_1)v_2 + \mu(x_2)v_1,$$

and $(A\oplus V, [\cdot,\cdot,\cdot]_\rho)$ is the semi-direct product 3-Lie algebra $A\ltimes_\rho V$, i.e.,

$$[x_1+v_1, x_2+v_2, x_3+v_3]_\rho = [x_1,x_2,x_3] + \rho(x_1,x_2)v_3 - \rho(x_1,x_3)v_2 + \rho(x_2,x_3)v_1,$$

for all $x_1,x_2,x_3 \in A,\ v_1,v_2,v_3 \in V$.

Proof. It is obvious to check that $(A\oplus V,\cdot_\mu)$ is a commutative associative algebra and $(A\oplus V,[\cdot,\cdot,\cdot]_\rho)$ is a 3-Lie algebra. Observe that for any $x_i \in A$ and $u_i \in V$, $i = 1,\cdots,4$

$$\begin{aligned}&\mathcal{L}_{A\oplus V}(x_1+u_1,x_2+u_2,x_3+u_3,x_4+u_4)\\ &\quad= \mathcal{L}(x_1,x_2,x_3,x_4) + \mathcal{L}_1(x_1,x_2,x_3,u_4) + \mathcal{L}_1(x_1,x_2,x_4,u_3)\\ &\qquad + \mathcal{L}_2(x_1,x_3,x_4,u_2) - \mathcal{L}_2(x_2,x_3,x_4,u_1).\end{aligned}$$

So, we have

$$\begin{aligned}
&\mathcal{L}_{A\oplus V}((x_1+u_1)\cdot_{A\oplus V}(x_2+u_2),x_3+u_3,x_4+u_4,x_5+u_5)\\
&\quad-(x_1+u_1)\cdot_{A\oplus V}\mathcal{L}_{A\oplus V}(x_2+u_2,x_3+u_3,x_4+u_4,x_5+u_5)\\
&\quad-(x_2+u_2)\cdot_{A\oplus V}\mathcal{L}_{A\oplus V}(x_1+u_1,x_3+u_3,x_4+u_4,x_5+u_5)\\
&=\mathcal{L}_{A\oplus V}(x_1\cdot x_2+\mu(x_1)u_2+\mu(x_2)u_1,x_3+u_3,x_4+u_4,x_5+u_5)\\
&\quad-(x_1+u_1)\cdot_{A\oplus V}\big(\mathcal{L}(x_2,x_3,x_4,x_5)+\mathcal{L}_1(x_2,x_3,x_4,u_5)\\
&\quad+\mathcal{L}_1(x_2,x_3,x_5,u_4)+\mathcal{L}_2(x_2,x_4,x_5,u_3)-\mathcal{L}_2(x_3,x_4,x_5,u_2)\big)\\
&\quad-(x_2+u_2)\cdot_{A\oplus V}\big(\mathcal{L}(x_1,x_3,x_4,x_5)+\mathcal{L}_1(x_1,x_3,x_4,u_5)\\
&\quad+\mathcal{L}_1(x_1,x_3,x_5,u_4)+\mathcal{L}_2(x_1,x_4,x_5,u_3)-\mathcal{L}_2(x_3,x_4,x_5,u_1)\big)\\
&=\mathcal{L}(x_1\cdot x_2,x_3,x_4,x_5)+\mathcal{L}_1(x_1\cdot x_2,x_3,x_4,u_5)+\mathcal{L}_1(x_1\cdot x_2,x_3,x_5,u_4)\\
&\quad+\mathcal{L}_2(x_1\cdot x_2,x_4,x_5,u_3)-\mathcal{L}_2(x_3,x_4,x_5,\mu(x_1)u_2)-\mathcal{L}_2(x_3,x_4,x_5,\mu(x_2)u_1)\\
&\quad-x_1\cdot\mathcal{L}(x_2,x_3,x_4,x_5)-\mu(\mathcal{L}(x_2,x_3,x_4,x_5))u_1-\mu(x_1)\mathcal{L}_1(x_2,x_3,x_4,u_5)\\
&\quad-\mu(x_1)\mathcal{L}_1(x_2,x_3,x_5,u_4)-\mu(x_1)\mathcal{L}_2(x_2,x_4,x_5,u_3)+\mu(x_1)\mathcal{L}_2(x_3,x_4,x_5,u_2)\\
&\quad-x_2\cdot\mathcal{L}(x_1,x_3,x_4,x_5)-\mu(\mathcal{L}(x_1,x_3,x_4,x_5))u_2-\mu(x_2)\mathcal{L}_1(x_1,x_3,x_4,u_5)\\
&\quad-\mu(x_2)\mathcal{L}_1(x_1,x_3,x_5,u_4)-\mu(x_2)\mathcal{L}_2(x_1,x_4,x_5,u_3)+\mu(x_2)\mathcal{L}_2(x_3,x_4,x_5,u_1).
\end{aligned}$$

Then, $(V;\rho,\mu)$ is a representation of A if and only if $(A\oplus V,\cdot_\mu,[\cdot,\cdot,\cdot]_\rho)$ is a ternary Nambu F-algebra. ∎

Let $(V;\rho,\mu)$ be a representation of a ternary Nambu-Poisson algebra $(P,\cdot_P,\{\,\cdot\,,\,\cdot\,,\,\cdot\,\}_P)$. In this case, the dual space $(V^*;\,\rho^*,\,\mu^*)$ also carries a representation of P (see [20]). However, this property does not extend to ternary Nambu F-algebras. Indeed, we have:

Lemma 4.6. *Let $(A,\cdot,[\cdot,\cdot,\cdot])$ be a ternary Nambu F-algebra and $(V;\rho,\mu)$ be a representation of A. Define two linear maps $\mathcal{L}_1^*,\mathcal{L}_2^*:A\otimes A\otimes A\otimes V^*\to V^*$ given by*

$$\begin{aligned}
\mathcal{L}_1^*(x,y,z,\xi)&=-\rho^*(x,y)\mu^*(z)(\xi)+\mu^*(z)\rho^*(x,y)(\xi)+\mu^*([x,y,z])(\xi),\\
\mathcal{L}_2^*(x,y,z,\xi)&=-\mu^*(x)\rho^*(y,z)(\xi)-\mu^*(y)\rho^*(x,z)(\xi)+\rho^*(x,y\cdot z)(\xi)
\end{aligned}$$

for all $x,y,z\in A$ and $\xi\in V^*$. Then we have

$$\begin{aligned}
\langle\mathcal{L}_1^*(x,y,z,\xi),u\rangle&=\langle\xi,\mathcal{L}_1(x,y,z,u)\rangle,\\
\langle\mathcal{L}_2^*(x,y,z,\xi),u\rangle&=-\langle\xi,\mathcal{L}_3(x,y,z,u)\rangle.
\end{aligned}$$

for all $x, y, z \in A$ and $\xi \in V^*$.

Proof. Let $x, y, z \in A$ and $\xi \in V^*$, we have

$$\begin{aligned}
\langle \mathcal{L}_1^*(x,y,z,\xi), u\rangle &= \langle -\rho^*(x,y)\mu^*(z)(\xi) + \mu^*(z)\rho^*(x,y)(\xi) + \mu^*([x,y,z])(\xi), u\rangle \\
&= -\langle \rho^*(x,y)\mu^*(z)(\xi), u\rangle + \langle \mu^*(z)\rho^*(x,y)(\xi), u\rangle + \langle \mu^*([x,y,z])(\xi), u\rangle \\
&= -\langle \xi, \mu(z)\rho(x,y)u\rangle + \langle \xi, \rho(x,y)\mu(z)(u)\rangle - \langle \xi, \mu([x,y,z])(u)\rangle \\
&= \langle \xi, \mathcal{L}_1(x,y,z,u)\rangle.
\end{aligned}$$

Similarly, we obtain

$$\langle \mathcal{L}_2^*(x,y,z,\xi), u\rangle = -\langle \xi, \mathcal{L}_3(x,y,z,u)\rangle.$$

∎

Proposition 4.7. *Let $(A, \cdot, [\cdot,\cdot,\cdot])$ be a ternary Nambu F-algebra. If the tuple $(V;\rho,\mu)$ is a representation of A satisfying the following axioms*

$$\mathcal{L}_1(x\cdot y, z, t, u) = \mathcal{L}_1(y, z, t, \mu(x)u) + \mathcal{L}_1(x, z, t, \mu(y)u), \tag{4.7}$$

$$\mathcal{L}_3(x\cdot y, z, t, u) = -\mathcal{L}_3(y, z, t, \mu(x)u) - \mathcal{L}_3(x, z, t, \mu(y)u), \tag{4.8}$$

$$\mu(\mathcal{L}(x,y,z,t))u = \mathcal{L}_3(y, z, t, \mu(x)u) - \mu(x)\mathcal{L}_3(y,z,t,u) \tag{4.9}$$

for any $x, y, z, t \in A$ and $u \in V$. Then $(V^*;\rho^*, -\mu^*)$ is a representation of A called the dual representation.

Proof. For any $x, y, z, t \in A, v \in u, \xi \in V^*$, we have

$$\begin{aligned}
&\langle \mathcal{L}_1^*(x\cdot y, z, t, \xi) + \mu^*(x)\mathcal{L}_1^*(y,z,t,\xi) + \mu^*(y)\mathcal{L}_1^*(x,z,t,\xi), u\rangle \\
&\qquad = \langle \xi, \mathcal{L}_1(x\cdot y, z, t, u) - \mathcal{L}_1(y,z,t,\mu(x)u) - \mathcal{L}_1(x,z,t,\mu(y)u)\rangle, \\
&\langle \mathcal{L}_2^*(x\cdot y, z, t, \xi) + \mu^*(x)\mathcal{L}_2^*(y,z,t,\xi) + \mu^*(y)\mathcal{L}_2^*(x,z,t,\xi), u\rangle \\
&\qquad = \langle \xi, -\mathcal{L}_3(x\cdot y, z, t, u) + \mathcal{L}_3(y,z,t,\mu(x)u) + \mathcal{L}_1(x,z,t,\mu(y)u)\rangle, \\
&\langle -\mu^*(\mathcal{L}(x,y,z,t)\xi + \mathcal{L}_2^*(y,z,t,\mu^*(x)\xi - \mu^*(x)\mathcal{L}_2^*(y,z,t,\xi), u\rangle \\
&\qquad = \langle \xi, \mu(\mathcal{L}(x,y,z,t))u + \mu(x)\mathcal{L}_3(y,z,t,u) - \mathcal{L}_3(y,z,t,\mu(x)u)\rangle.
\end{aligned}$$

According to (4.7)–(4.9) and Definition 4.1, the conclusion follows immediately. ∎

COROLLARY 4.8. *Let* $(A, \cdot, [\cdot,\cdot,\cdot])$ *be a ternary Nambu* F*-algebra such that the following relations hold:*

$$\mathcal{L}(x \cdot y, z, t, u) = \mathcal{L}(y, z, t, x \cdot u) + \mathcal{L}(x, z, t, y \cdot u), \tag{4.10}$$

$$\mathcal{K}(x \cdot y, z, t \cdot u) = -\mathcal{K}(x, z, t, y \cdot u) - \mathcal{K}(y, z, t, x \cdot u), \tag{4.11}$$

$$\mathcal{L}(x, y, z, t) \cdot u = \mathcal{K}(y, z, t, x \cdot u) - x \cdot \mathcal{K}(y, z, t, u) \tag{4.12}$$

for all $x, y, z, t, u \in A$*, where* $\mathcal{K} : \otimes^4 A \to A$ *is defined by*

$$\mathcal{K}(x, y, z, u) = \circlearrowleft_{y,z,u} [x, y, z \cdot u].$$

Then $(A^*; \mathrm{ad}^*, -L^*)$ *is a representation of* A *called the coadjoint representation.*

We now introduce the notion of coherence for ternary Nambu F-algebras, which is characterized by the validity of a specific set of structural conditions

DEFINITION 4.9. A coherent ternary Nambu F-algebra is a ternary Nambu F-algebra such that (4.10)–(4.12) hold.

Remark 4.10. The coherence conditions ((4.10)–(4.12)) are not arbitrary. They are precisely designed to guarantee that the dual of a representation of a ternary Nambu F-algebra is again a representation, namely the dual representation. Thus, coherence provides the structural framework needed to extend representation theory in a consistent way to the dual level.

5. Ternary pre-Nambu F-algebras

In this section, we introduce ternary pre-Nambu F-algebras and investigate relative Rota-Baxter operators on ternary Nambu F-algebras. We establish that a relative Rota-Baxter operator on a ternary Nambu F-algebra naturally gives rise to a ternary pre-Nambu F-algebra, and we further show that such algebras can also be constructed from symplectic structures.

5.1. Definition and elementary results

DEFINITION 5.1. A ternary pre-Nambu F-algebra is a triple $(A, \diamond, \{\cdot,\cdot,\cdot\})$, where $(A, \diamond)$ is a Zinbiel algebra and $(A, \{\cdot,\cdot,\cdot\})$ is a 3-pre-Lie algebra, such that:

$$F_1(x_1\cdot x_2,x_3,x_4,x_5)=x_1\diamond F_1(x_2,x_3,x_4,x_5)+x_2\diamond F_1(x_1,x_3,x_4,x_5), \quad (5.1)$$

$$F_2(x_1\cdot x_2,x_3,x_4,x_5)=x_1\diamond F_1(x_2,x_3,x_4,x_5)+x_2\diamond F_1(x_1,x_3,x_4,x_5), \quad (5.2)$$

$$\mathcal{L}(x_1,x_2,x_3,x_4)\diamond x_5=F_2(x_2,x_3,x_4,x_1\diamond x_5)-x_1\diamond F_2(x_2,x_3,x_4,x_5) \quad (5.3)$$

where $F_1,F_2,\mathcal{L}:\otimes^4 A\to A$ are defined by

$$F_1(x_1,x_2,x_3,x_4)=\{x_1,x_2,x_3\diamond x_4\}-x_3\diamond\{x_1,x_2,x_4\}-[x_1,x_2,x_3]\diamond x_4, \quad (5.4)$$

$$F_2(x_1,x_2,x_3,x_4)=x_3\diamond\{x_1,x_2,x_4\}+x_2\diamond\{x_1,x_3,x_4\}-\{x_1,x_2\cdot x_3,x_4\}, \quad (5.5)$$

$$\mathcal{L}(x_1,x_2,x_3,x_4)=[x_1,x_2,x_3\cdot x_4]-x_3\cdot[x_1,x_2,x_4]-[x_1,x_2,x_3]\cdot x_4 \quad (5.6)$$

and the operation $\cdot$ and the bracket $[\cdot,\cdot,\cdot]$ are defined by

$$x\cdot y=x\diamond y+y\diamond x,\qquad [x,y,z]=\circlearrowleft_{x,y,z}\{x,y,z\} \quad (5.7)$$

for all $x,y,z,\in A$.

If $F_1=F_2=0$ in the above definition, then we obtain a particular case of ternary pre-Nambu F-algebras. It will be called ternary pre- Nambu-Poisson algebra since it can be obtained by splitting of a ternary Nambu-Poisson algebra introduced in [27].

Definition 5.2. A ternary pre-Nambu-Poisson algebra is a tuple $(A,\diamond,\{\cdot,\cdot,\cdot\})$, where $(A,\diamond)$ is a Zinbiel algebra and $(A,\{\cdot,\cdot,\cdot\})$ is a 3-pre-Lie algebra, such that:

$$\{x_1,x_2,x_3\diamond x_4\}=x_3\diamond\{x_1,x_2,x_4\}+[x_1,x_2,x_3]\diamond x_4,$$
$$\{x_1,x_2\cdot x_3,x_4\}=x_3\diamond\{x_1,x_2,x_4\}+x_2\diamond\{x_1,x_3,x_4\}$$

for all $x_1,x_2,x_3,x_4\in A$.

We now show how every ternary pre-Nambu F-algebra gives rise to a canonical ternary Nambu F-algebra structure, called its sub-adjacent algebra.

Theorem 5.3. *Let $(A,\diamond,\{\cdot,\cdot,\cdot\})$ be a ternary pre-Nambu F-algebra. Then $(A,\cdot,[\cdot,\cdot,\cdot])$ is a ternary Nambu F-algebra, where the operation $\cdot$ and bracket $[\cdot,\cdot,\cdot]$ are given by* (5.7)*, which is called the sub-adjacent ternary Nambu F-algebra of $(A,\diamond,\{\cdot,\cdot,\cdot\})$ and denoted by A^c.*

Proof. Thanks to Lemma 2.2 and Proposition 2.9, we have that $(A, \cdot)$ is a commutative associative algebra and $(A, [\cdot,\cdot,\cdot])$ is a 3-Lie algebra. Using (5.4)–(5.6), we have

$$\begin{aligned}
&\mathcal{L}(x_1, x_2, x_3, x_4)\\
&= [x_1, x_2, x_3 \cdot x_4] - x_3 \cdot [x_1, x_2, x_4] - [x_1, x_2, x_3] \cdot x_4\\
&= \{x_1, x_2, x_3 \cdot x_4\} + \{x_2, x_3 \cdot x_4, x_1\} + \{x_3 \cdot x_4, x_1, x_2, \}\\
&\quad - x_3 \diamond [x_1, x_2, x_4] - [x_1, x_2, x_4] \diamond x_3 - [x_1, x_2, x_3] \diamond x_4 - x_4 \diamond [x_1, x_2, x_3]\\
&\quad - x_3 \diamond [x_1, x_2, x_4] - [x_1, x_2, x_4] \diamond x_3 - [x_1, x_2, x_3] \diamond x_4 - x_4 \diamond [x_1, x_2, x_3]\\
&= \{x_1, x_2, x_3 \diamond x_4\} + \{x_1, x_2, x_4 \diamond x_3\} + \{x_2, x_3 \cdot x_4, x_1\} - \{x_1, x_3 \cdot x_4, x_2\}\\
&\quad - x_3 \diamond \{x_1, x_2, x_4\} - x_3 \diamond \{x_2, x_4, x_1\} - x_3 \diamond \{x_4, x_1, x_2\}\\
&\quad - x_4 \diamond \{x_1, x_2, x_3\} - x_4 \diamond \{x_2, x_3, x_1\} - x_4 \diamond \{x_3, x_1, x_2\}\\
&\quad - [x_1, x_2, x_4] \diamond x_3 - [x_1, x_2, x_3] \diamond x_4\\
&= F_1(x_1, x_2, x_3, x_4) + F_1(x_1, x_2, x_4, x_3)\\
&\quad + F_2(x_1, x_3, x_4, x_2) - F_2(x_2, x_3, x_4, x_1).
\end{aligned}$$

According to (5.1), (5.2) and (5.3), we get

$$\begin{aligned}
&\mathcal{L}(x_1 \cdot x_2, x_3, x_4, x_5) - x_1 \cdot \mathcal{L}(x_2, x_3, x_4, x_5) - x_2 \cdot \mathcal{L}(x_1, x_3, x_4, x_5)\\
&\quad = F_1(x_1 \cdot x_2, x_3, x_4, x_5) + F_1(x_1 \cdot x_2, x_3, x_5, x_4)\\
&\quad\quad + F_2(x_1 \cdot x_2, x_4, x_5, x_3) - F_2(x_3, x_4, x_5, x_1 \cdot x_2)\\
&\quad\quad - x_1 \diamond \mathcal{L}(x_2, x_3, x_4, x_5) - \mathcal{L}(x_2, x_3, x_4, x_5) \diamond x_1\\
&\quad\quad - x_2 \diamond \mathcal{L}(x_1, x_3, x_4, x_5) - \mathcal{L}(x_1, x_3, x_4, x_5) \diamond x_2\\
&\quad = F_1(x_1 \cdot x_2, x_3, x_4, x_5) + F_1(x_1 \cdot x_2, x_3, x_5, x_4) + F_2(x_1 \cdot x_2, x_4, x_5, x_3)\\
&\quad\quad - F_2(x_3, x_4, x_5, x_1 \diamond x_2) - F_2(x_3, x_4, x_5, x_2 \diamond x_1)\\
&\quad\quad - x_1 \diamond F_1(x_2, x_3, x_4, x_5) - x_1 \diamond F_1(x_2, x_3, x_5, x_4)\\
&\quad\quad - x_1 \diamond F_2(x_2, x_4, x_5, x_3) + x_1 \diamond F_2(x_3, x_4, x_5, x_2)\\
&\quad\quad - \mathcal{L}(x_2, x_3, x_4, x_5) \diamond x_1 - x_2 \diamond F_1(x_1, x_3, x_4, x_5)\\
&\quad\quad - x_2 \diamond F_1(x_1, x_3, x_5, x_4) - x_2 \diamond F_2(x_1, x_4, x_5, x_3)\\
&\quad\quad + x_2 \diamond F_2(x_3, x_4, x_5, x_1) - \mathcal{L}(x_1, x_3, x_4, x_5) \diamond x_2 \;=\; 0.
\end{aligned}$$

Therefore, $(A, \cdot, [\cdot,\cdot,\cdot])$ is a ternary Nambu F-algebra. ∎

The following observation is a direct consequence of the above Theorem.

COROLLARY 5.4. *Let* $(A, \diamond, \{\cdot,\cdot,\cdot\})$ *be a ternary pre-Nambu-Poisson algebra. Then* $(A, \cdot, [\cdot,\cdot,\cdot])$ *is a ternary Nambu-Poisson algebra, where the operation* $\cdot$ *and bracket* $[\cdot,\cdot,\cdot]$ *are given by* (5.7).

PROPOSITION 5.5. *The tuple* $(A; \mathbb{L}, L_\diamond)$ *is a representation of the sub-adjacent ternary Nambu* F*-algebras* A^c*, where* $\mathbb{L}$ *and* $L_\diamond$ *are given by* (2.8) *and* (2.2)*, respectively.*

Proof. Thanks to Lemma 2.2 and Lemma 2.10, $(A; L_\diamond)$ is a representation of the commutative associative algebra $(A, \cdot)$ as well as $(A; \mathbb{L})$ is a representation of the sub-adjacent 3-Lie algebra A^c.

According to (5.1)–(5.3), we can easily check (4.4)–(4.6). Therefore, $(A; \mathbb{L}, L_\diamond)$ is a representation of the sub-adjacent ternary Nambu F-algebra A^c. ∎

5.2. RELATIVE ROTA-BAXTER OPERATORS ON TERNARY NAMBU F-ALGEBRAS. The notion of relative Rota-Baxter operators (called also $\mathcal{O}$-operators) was first given for Lie algebras by Kupershmidt in [23] as a natural generalization of the classical Yang-Baxter equation and then defined by analogy in other various (associative, alternative, Jordan…). Later, this notion is extended to 3-Lie algebras.

A linear map $T : V \to A$ is called a relative Rota-Baxter operator on a commutative associative algebra $(A, \cdot)$ with respect to a representation $(V; \mu)$ if T satisfies

$$Tu \cdot Tv = T(\mu(Tu)v + \mu(Tv)u), \qquad \forall\ u, v \in V.$$

In particular, a relative Rota-Baxter operator on a commutative associative algebra $(A, \cdot)$ with respect to the adjoint representation is called a Rota-Baxter operator of weight zero or briefly a Rota-Baxter operator on A.

It is obvious to obtain the following result.

LEMMA 5.6. *Let* $(A, \cdot)$ *be a commutative associative algebra and* $(V; \mu)$ *a representation. Let* $T : V \to A$ *be a relative Rota-Baxter operator on* $(A, \cdot)$ *with respect to* $(V; \mu)$*. Then there exists a Zinbiel algebra structure on* V *given by*

$$u \diamond v = \mu(Tu)v, \qquad \forall\ u, v \in V.$$

A linear map $T : V \to A$ is called a relative Rota-Baxter operator on a 3-Lie algebra $(A, [\cdot,\cdot,\cdot])$ with respect to a representation $(V;\rho)$ if T satisfies

$$[Tu,Tv,Tw] = T\big(\rho(Tu,Tv)w + \rho(Tv,Tw)u + \rho(Tw,Tu)v\big), \quad \forall\, u,v,w \in V.$$

In particular, a relative Rota-Baxter operator on a 3-Lie algebra $(A, [\cdot,\cdot,\cdot])$ with respect to the adjoint representation ad is called a Rota-Baxter operator of weight zero or briefly a Rota-Baxter operator on A.

LEMMA 5.7. ([5]) *Let $T : V \to A$ be a relative Rota-Baxter operator on a 3-Lie algebra $(A, [\cdot,\cdot,\cdot])$ with respect to a representation $(V;\rho)$. Define the bracket $\{\cdot,\cdot,\cdot\}$ on V by*

$$\{u,v,w\} = \rho(Tu,Tv)w, \qquad \forall\ u,v,w \in V.$$

Then $(V, \{\cdot,\cdot,\cdot\})$ is a 3-pre-Lie algebra.

Let $(V;\rho,\mu)$ be a representation of a ternary Nambu F-algebra $(A,\cdot,[\cdot,\cdot,\cdot])$.

DEFINITION 5.8. A linear operator $T : V \to A$ is called a relative Rota--Baxter operator on $(A,\cdot,[\cdot,\cdot,\cdot])$ if T is both a relative Rota-Baxter operator on the commutative associative algebra $(A,\cdot)$ and a relative Rota-Baxter operator on the 3-Lie algebra $(A,[\cdot,\cdot,\cdot])$.

In particular, a linear operator $\mathcal{R} : A \to A$ is called a Rota-Baxter operator of weight zero or briefly a Rota-Baxter operator on A, if $\mathcal{R}$ is both a Rota-Baxter operator on the commutative associative algebra $(A,\cdot)$ and a Rota-Baxter operator on the 3-Lie algebra $(A,[\cdot,\cdot,\cdot])$.

In the next result, we construct ternary pre-Nambu F-algebras by means of relative Rota-Baxter operators on ternary Nambu F-algebras. Equivalently, this provides a dendrification of ternary Nambu F-algebras through the use of relative Rota-Baxter operators.

THEOREM 5.9. *Let $(A,\cdot,[\cdot,\cdot,\cdot])$ be a ternary Nambu F-algebra and $T : V \to A$ be a relative Rota-Baxter operator on A with respect to the representation $(V;\rho,\mu)$. Define two operations $\diamond$ and $\{\cdot,\cdot,\cdot\}$ on V as follow:*

$$u \diamond v = \mu(Tu)v, \quad \{u,v,w\} = \rho(Tu,Tv)w, \qquad \forall\ u,v,w \in V.$$

Then $(V,\diamond,\{\cdot,\cdot,\cdot\})$ is a ternary pre-Nambu F-algebra and T is a homomorphism from V^c to $(A,\cdot,[\cdot,\cdot,\cdot])$.

Proof. Since T is a relative Rota-Baxter operator on the commutative a ssociative algebra $(A,\cdot)$ as well as a relative Rota-Baxter operator on the 3-Lie algebra $(A,[\cdot,\cdot,\cdot])$ with respect to the representations $(V;\mu)$ and $(V;\rho)$ respectively. Using Lemma 5.6 and Lemma 5.7, we deduce that $(V,\diamond)$ is a Zinbiel algebra and $(V,\{\cdot,\cdot,\cdot\})$ is a 3-pre-Lie algebra. Put, for any $u,v,w\in V$,

$$\begin{aligned}
[u,v,w]_T &:= \circlearrowleft_{u,v,w} \{u,v,w\} = \rho(Tu,Tv)w+\rho(Tv,Tw)u+\rho(Tw,Tu)v,\\
u\cdot_T v &:= u\diamond v+v\diamond u=\mu(Tu)v+\mu(Tv)u.
\end{aligned}$$

By these facts and (4.4), for $v_1,v_2,v_3,v_4,v_5\in V$, one has

$$\begin{aligned}
&F_1(v_1\cdot_T v_2,v_3,v_4,v_5)-v_1\diamond F_1(v_2,v_3,v_4,v_5)-v_2\diamond F_1(v_1,v_3,v_4,v_5)\\
&=\{v_1\cdot_T v_2,v_3,v_4\diamond v_5\}-v_4\diamond\{v_1\cdot v_2,v_3,v_5\}-[v_1\cdot v_2,v_3,v_4]_T\diamond v_5\\
&\quad -v_1\diamond\{v_2,v_3,v_4\diamond v_5\}-v_1\diamond(v_4\diamond\{v_2,v_3,v_5\})-v_1\diamond([v_2,v_3,v_4]_T\diamond v_5)\\
&\quad -v_2\diamond\{v_1,v_3,v_4\diamond v_5\}-v_2\diamond(v_4\diamond\{v_1,v_3,v_5\})-v_2\diamond([v_1,v_3,v_4]_T\diamond v_5)\\
&=\rho(T(v_1\cdot_T v_2),Tv_3)\mu(Tv_4)v_5-\mu(Tv_4)\rho(T(v_1\cdot v_2),Tv_3)v_5\\
&\quad -\mu(T[v_1\cdot v_2,v_3,v_4]_T)v_5-\mu(Tv_1)\rho(Tv_2,Tv_3)\mu(Tv_4)v_5\\
&\quad -\mu(Tv_1)\mu(Tv_4)\rho(Tv_2,Tv_3)v_5-\mu(Tv_1)\mu(T[v_2,v_3,v_4]_T)v_5\\
&\quad -\mu(Tv_2)\rho(Tv_1,Tv_3)\mu(Tv_4)v_5-\mu(Tv_2)\mu(Tv_4)\rho(Tv_1,Tv_3)v_5\\
&\quad -\mu(Tv_2)(\mu(T[v_1,v_3,v_4]_T)v_5\\
&=\mathcal{L}_1(Tv_1\cdot_T Tv_2,v_3,v_4,v_5)-\mu(Tv_1)\mathcal{L}_1(Tv_2,Tv_3,Tv_4,v_5)\\
&\quad -\mu(Tv_2\mathcal{L}_1(Tv_1,Tv_3,Tv_4,v_5)\;=\;0.
\end{aligned}$$

Then, we obtain (5.1). Using a similar computation we can obtain (5.2) and (5.3). Therefore, $(V,\diamond,\{\cdot,\cdot,\cdot\})$ is a ternary pre-Nambu F-algebra. On the other hand, for all $u,v,w\in V$ we have

$$\begin{aligned}
[Tu,Tv,Tw)] &= T\big(\rho(Tu,Tv)w+\rho(Tv,Tw)u+\rho(Tw,Tu)v\big)\\
&= T\big([u,v,w]_T\big)
\end{aligned}$$

and $T(u)\cdot T(v)=T(u\cdot_T v)$. Then, T is a homomorphism from V^c to $(A,\cdot,[\cdot,\cdot,\cdot])$. ∎

Corollary 5.10. *Let $(A,\cdot,[\cdot,\cdot,\cdot])$ be a ternary Nambu F-algebra and $T:V\to A$ an relative Rota Baxter on A with respect to the representation*

$(V;\rho,\mu)$. Then $T(V)=\{T(v)\mid v\in V\}\subset A$ is a subalgebra of A and there is an induced ternary pre-Nambu F-algebra structure on $T(V)$ given by

$$Tu\diamond Tv=T(u\diamond v),\quad \{Tu,Tv,Tw\}=T\{u,v,w\},\qquad \forall\ u,v,w\in V.$$

COROLLARY 5.11. *Let $(A,\cdot,[\cdot,\cdot,\cdot])$ be a ternary Nambu F-algebra and $\mathcal{R}:A\to A$ a Rota-Baxter operator of weight 0. Define two operations on A by*

$$x\diamond y=\mathcal{R}(x)\cdot y,\qquad \{x,y,z\}=[\mathcal{R}(x),\mathcal{R}(y),z].$$

Then $(A,\diamond,\{\cdot,\cdot,\cdot\})$ is a ternary pre-Nambu F-algebra and $\mathcal{R}$ is a homomorphism from the sub-adjacent ternary Nambu F-algebra $(A,\cdot_{\mathcal{R}},[\cdot,\cdot,\cdot]_{\mathcal{R}})$ to $(A,\cdot,[\cdot,\cdot,\cdot])$, where

$$x\cdot_{\mathcal{R}}y=x\diamond y+y\diamond x\qquad\text{and}\qquad [x,y,z]_{\mathcal{R}}=\circlearrowleft_{x,y,z}\{x,y,z\}$$

for all $x,y,z\in A$.

The next result provides the necessary and sufficient condition for a ternary Nambu F-algebra to carry a ternary pre-Nambu F-algebra structure.

PROPOSITION 5.12. *Let $(A,\cdot,[\cdot,\cdot,\cdot])$ be a ternary Nambu F-algebra. There exists a ternary pre-Nambu F-algebra structure on A such that its sub-adjacent ternary Nambu F-algebra is exactly $(A,\cdot,[\cdot,\cdot,\cdot])$ if and only if there exists an invertible relative Rota-Baxter operator on $(A,\cdot,[\cdot,\cdot,\cdot])$.*

Proof. Suppose $T:V\to A$ is an invertible relative Rota-Baxter operator on A with respect to the representation $(V;\rho,\mu)$, then the compatible ternary pre-Nambu F-algebra structure on A is given by

$$x\diamond y=T(\mu(x)(T^{-1}(y))),\qquad \{x,y,z\}=T(\rho(x,y)(T^{-1}z))$$

for all $x,y,y\in A$. Since T is a relative Rota-baxter operator on A, we have

$$\begin{aligned}
x\diamond y+y\diamond x&=T\big(\mu(x)\big(T^{-1}(y)\big)\big)+T\big(\mu(y)\big(T^{-1}(x)\big)\big)\\
&=T\Big(\mu\big(TT^{-1}(x)\big)\big(T^{-1}(y)\big)+\mu\big(TT^{-1}(y)\big)\big(T^{-1}(x)\big)\Big)\\
&=TT^{-1}(x)\cdot TT^{-1}(y)=x\cdot y,\\
\circlearrowleft_{x,y,z}\{x,y,z\}&=T\Big(\rho(x,y)(T^{-1}z)+\rho(y,z)(T^{-1}x)+\rho(z,x)(T^{-1}y)\Big)\\
&=\big[TT^{-1}(x),TT^{-1}(y),TT^{-1}(z)\big]=[x,y,z].
\end{aligned}$$

Conversely, let $(A, \diamond, \{\cdot,\cdot,\cdot\})$ be a ternary pre-Nambu F-algebra and $(A, \cdot, [\cdot,\cdot,\cdot])$ the sub-adjacent ternary Nambu F-algebra. Then the identity map Id is an relative Rota-Baxter operator on A with respect to the representation $(A; \mathbb{L}, L)$. ∎

Let $(A, \cdot, [\cdot,\cdot,\cdot])$ be a ternary Nambu F-algebra and $\mathcal{B} \in \wedge^2 A^*$. We say that $\mathcal{B}$ is a cyclic 2-cocycle in the sense of Connes on the commutative associative algebra $(A, \cdot)$ if

$$\mathcal{B}(x \cdot y, z) + \mathcal{B}(y \cdot z, x) + \mathcal{B}(z \cdot x, y) = 0 \tag{5.8}$$

and $\mathcal{B}$ is said to be a symplectic structure on the 3-Lie algebra $(A, [\cdot,\cdot,\cdot])$ if it satisfies

$$\mathcal{B}([x,y,z],u) - \mathcal{B}([x,y,u],z) + \mathcal{B}([x,z,u],y) - \mathcal{B}([y,z,u],x) = 0, \tag{5.9}$$

for any $x, y, z, u \in A$.

Proposition 5.13. *Let $(A, \cdot, [\cdot,\cdot,\cdot])$ be a coherent ternary Nambu F-algebra and $\mathcal{B}$ be a skew-symmetric bilinear form on A satisfying identities* (5.8) *and* (5.9). *Then there is a compatible ternary-pre-Nambu F-algebra structure on A given by*

$$\mathcal{B}(x \diamond y, z) = \mathcal{B}(y, x \cdot z), \qquad \mathcal{B}(\{x,y,z\},u) = -\mathcal{B}(z, [x,y,u]).$$

Proof. Since $(A, \cdot, [\cdot,\cdot,\cdot])$ is a coherent ternary Nambu F-algebra, then $(A^*; \mathrm{ad}^*, -\mathcal{L}^*)$ is a representation of A. By the fact that $\mathcal{B}$ is a symplectic structure, we deduce that the musical map $\mathcal{B}^\sharp : A \to A^*$ defined by $\langle \mathcal{B}^\sharp(x), y\rangle = \mathcal{B}(x,y)$ is invertible and $(\mathcal{B}^\sharp)^{-1}$ is a relative Rota-Baxter operator on the commutative associative algebra $(A, \cdot)$ with respect to the representation $(A^*; -L^*)$.

In addition, $(\mathcal{B}^\sharp)^{-1}$ is a relative Rota-Baxter operator on the 3-Lie algebra $(A, [\cdot,\cdot,\cdot])$ with respect to the representation $(A^*; \mathrm{ad}^*)$. Then, $(\mathcal{B}^\sharp)^{-1}$ is a relative Rota-Baxter operator on the coherent ternary Nambu F-algebra $(A, \cdot, [\cdot,\cdot,\cdot])$ with respect to the representation $(A^*; \mathrm{ad}^*, -L^*)$. By Proposition 5.12, there is a compatible ternary pre-Nambu F-algebra structure on A given as above. ∎

Next, let T be a relative Rota-Baxter on a ternary Nambu F-algebra $(A, \cdot, [\cdot,\cdot,\cdot])$ with respect to a representation $(V; \rho, \mu)$. According to Theorem 5.9, $(V, \cdot_T, [\cdot,\cdot,\cdot]_T)$ is a ternary Nambu F-algebra. Is there a representation of

V on the vector space A? The answer is affirmative. In fact, define two linear maps $\rho_T : \wedge^2 V \to \mathrm{End}(A)$ and $\mu_T : V \to \mathrm{End}(A)$ by

$$\rho_T(u,v)(x) = [Tu, Tv, x] - T(\rho(Tv,x)u + \rho(x,Tu)v), \tag{5.10}$$
$$\mu_T(u)(x) = Tu \cdot x - T(\mu(x)(u)) \tag{5.11}$$

for all $x \in A, u, v \in V$.

THEOREM 5.14. *With the above notations, the triple* $(A;\rho_T,\mu_T)$ *is a representation of* $(V, \cdot_T, [\cdot,\cdot,\cdot]_T)$.

We may prove Theorem 5.14 by checking that $(A;\rho_T,\mu_T)$ satisfies conditions of Definition 4.1, but here we will prove it by another way which is more elegant. In order to do this, we should introduce and go back to some notions. Recall that a Nijenhuis operator on an associative commutative algebra $(A,\cdot)$ is a linear map $N : A \to A$ obeying to the following integrability condition:

$$Nx \cdot Ny = N(Nx \cdot y + x \cdot Ny - N(x \cdot y)), \qquad \forall\ x, y, \in A. \tag{5.12}$$

On the other hand, a linear map N on a 3-Lie algebra $(A,[\cdot,\cdot,\cdot])$ is called a Nijenhuis operator if it satisfies, for any $x, y, z \in A$,

$$\begin{aligned}[Nx,&Ny,Nz] = N\big([Nx,Ny,z] + [Nx,y,Nz] + [x,Ny,Nz] \\ &- N[Nx,y,z] - N[x,Ny,z] - N[x,y,Nz] + N^2[x,y,z]\big).\end{aligned} \tag{5.13}$$

DEFINITION 5.15. Let $(A,\cdot,[\cdot,\cdot,\cdot])$ be a ternary Nambu F-algebra. A Nijenhuis operator on A is a linear map $N : A \to A$ satisfying (5.12) and (5.13).

Define two deformed products $\cdot_N$ and $[\cdot,\cdot,\cdot]_N$ on A by, for all $x, y, z \in A$,

$$\begin{aligned} x \cdot_N y &= Nx \cdot y + x \cdot Ny - N(x \cdot y), \\ [x,y,z]_N &= [Nx,Ny,z] + [Nx,y,Nz] + [x,Ny,Nz] \\ &\qquad - N[Nx,y,z] - N[x,Ny,z] - N[x,y,Nz] + N^2[x,y,z]. \end{aligned}$$

The following two lemmas are straightforward, so we omit the proofs.

LEMMA 5.16. *With the above notations, the tuple* $(A,\cdot_N,[\cdot,\cdot,\cdot]_N)$ *is a ternary Nambu* F*-algebra.*

LEMMA 5.17. *Let $T : V \to A$ be a relative Rota-Baxter operator on a ternary Nambu F-algebra $(A, \cdot, [\cdot, \cdot, \cdot])$ with respect to a representation $(V; \rho, \mu)$. Then the operator $N_T = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ is a Nijenhuis operator on the semi-direct product ternary Nambu F-algebra $A \oplus V$.*

Proof of Theorem 5.14. According to Lemma 5.17, N_T is a Nijenhuis operator on the semi-direct product ternary Nambu F-algebra $A \oplus V$. Then, we deduce that there is a ternary F-manifold structure on $V \oplus A \cong A \oplus V$ given by, for all $x, y, z \in A,\ u, v, w \in V$,

$$\begin{aligned}
&(x+u) \cdot_{N_T} (y+v) \\
&\quad = N_T(x+u) \cdot_\mu (y+v) + (x+u) \cdot_\mu N_T(y+v) - N_T((x+u) \cdot_\mu (y+v)) \\
&\quad = Tu \cdot_\mu (y+v) + (x+u) \cdot_\mu Tv - N_T(x \cdot y + \mu(x)v + \mu(y)u) \\
&\quad = Tu \cdot y + \mu(Tu)v + x \cdot Tv + \mu(Tv)u - T(\mu(x)v + \mu(y)u) \\
&\quad = u \cdot_T v + \mu_T(u)y + \mu_T(v)x
\end{aligned}$$

and (since $N_T^2 = 0$), we have

$$\begin{aligned}
&[x+u, y+v, z+w]_{N_T} \\
&\quad = [N_T(x+u), N_T(y+v), z+w]_\rho + [N_T(x+u), y+v, N_T(z+w)]_\rho \\
&\qquad + [x+u, N_T(y+v), N_T(z+w)]_\rho - N_T([N_T(x+u), y+v, z+w] \\
&\qquad + [x+u, N_T(y+v), z+w] + [x+u, y+v, N_T(z+w)]_\rho) \\
&\quad = [Tu, Tv, z+w]_\rho + [Tu, y+v, Tw]_\rho + [x+u, Tv, Tw]_\rho \\
&\qquad - N_T([Tu, y+v, z+w]_\rho + [x+u, Tv, z+w]_\rho + [x+u, y+v, Tw]_\rho) \\
&\quad = [Tu, Tv, z] + \rho(Tu, Tv)w + [Tu, y, Tw] + \rho(Tw, Tu)v \\
&\qquad + [x, Tu, Tv] + \rho(Tv, Tw)u - T\big(\rho(Tu, y)w + \rho(z, Tu)v \\
&\qquad + \rho(x, Tv)w + \rho(Tv, z)u + \rho(y, Tw)u + \rho(Tw, x)v\big) \\
&\quad = [u, v, w]_T + \rho_T(u, v)z + \rho_T(v, w)x + \rho_T(w, u)y,
\end{aligned}$$

which implies that $(A; \rho_T, \mu_T)$ is a representation of the ternary Nambu F-algebra $(V, \cdot_T, [\cdot, \cdot, \cdot]_T)$. This finishes the proof. ∎

Remark 5.18. In conclusion, Theorem 5.3 and Theorem 5.9 establish the existence of a bijective functor

$$F : \mathbf{NFA} \longrightarrow \mathbf{pre\text{-}NFA},$$

from the category **NFA** of Nambu F-algebras to the category **pre-NFA** of pre-Nambu F-algebras.

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