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Constructing g-fusion frames for operators

Salah Eddine Oustani [1,@], Choonkil Park [2]

[1] *Laboratory of Analysis, Geometry and Applications, Faculty of Science University of Ibn Tofail, B.P. 133, Kenitra, Morocco*

[2] *Research Institute for Convergence of Basic Science, Intercollege Hanyang University, Seoul 04763, Korea*

salaheddine.oustani.1975@gmail.com, *baak@hanyang.ac.kr*

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Abstract: Fusion frames are a very active area of research because of a myriad of applications in pure mathematics, applied mathematics, image processing and related areas. In the current paper, we mainly establish an operator theoretic approach for constructing K-g-fusion frames in Hilbert spaces from operators satisfying suitable properties.

Key words: K-g-fusion frame, regular operator, EP operator.

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1. Introduction and preliminaries

The theory of frames has become a useful tool in signal processing, allowing uniform description of many linear but non-orthogonal transform techniques. Many years after the advent of frame theory, its importance and applications in various scientific fields have become clear to everyone. The aim of this theory, developed by Duffin and Schaeffer [9], was to solve some problems related to the nonharmonic Fourier series. In fact, the frame theory developed Gabor's studies in signal analysis in a pure form, which was extensively studied by the fundamental paper of Daubechies, Grossmann, and Meyer [7].

A frame as well as an orthonormal basis allows each element in the underlying Hilbert space to be written as an unconditionally convergent infinite linear combination of the frame elements. As is fundamentally appreciated in harmonic analysis, the utility of frames often surpasses that of strict orthonormal bases. While an orthonormal basis guarantees perfect, non-redundant representation, the necessity for diversity and redundancy in real-world, noisy physical spaces mandates the use of frames. Consequently, the study and identification of sequences that exhibit frame-like reconstruction properties, yet offer novel diversity, is a crucial area of research.

[@] Corresponding author

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Over the last decade, frame theory has attracted steady interest in recent applied mathematics research because it is used in various areas, such as signal processing [2] and sampling theory [22]. With the deep development of frame theory, several generalizations of frames were given by some authors, such as frames of subspaces [3], g-frames [29], frames for operators [12]. Also, frames and their generalization in Hilbert C^*-modules and locally compact abelian groups have been studied in [11].

Fusion frames, which were introduced by Casazza and Kutyniok in [3], can be regarded as frames of subspaces. They provide the link between local frames and global ones.

Finding the operator perturbation of frames and its extensions is an important aspect of operator theory. The main purpose of the present paper is to study the invariance of K-g-fusion frames in Hilbert spaces under operators with specific properties.

Throughout this paper, we indicate by $\mathcal{H}$ a separable Hilbert space, by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, and by $\mathcal{CR}(\mathcal{H})$ the set of all closed range operators on $\mathcal{H}$. For an operator $A \in \mathcal{L}(\mathcal{H})$, we will indicate by A^* its adjoint, by $R(A)$ and $N(A)$ the range and the kernel subspaces of A respectively and by $I \in \mathcal{L}(\mathcal{H})$ is the identity operator. We also write $\mathcal{R}^\infty(A) = \bigcap_{n\geq 0} R(A^n)$ for the hyper-range of A. As usual, the orthogonal projection from $\mathcal{H}$ onto $\mathcal{W}_j$ is denoted by $\pi_{\mathcal{W}_j}$, where $\{\mathcal{W}_j\}_{j\in\mathbb{J}}$ is a sequence of closed subspaces of $\mathcal{H}$, where $\mathbb{J}$ is a subset of $\mathbb{N}$.

In the sequel, we aim to review some basic definitions about frames and operator theory that will be needed later. For a comprehensive survey on fusion frame theory and its applications, one can refer to [5, 6, 20, 24].

Definition 1.1. ([4]) Let $\{\mathcal{W}_j\}_{j\in\mathbb{J}}$ be a collection of closed subspaces of $\mathcal{H}$ and $\{\nu_j\}_{j\in\mathbb{J}}$ be a family of weights, i.e., $\nu_j > 0$ for all j $\in \mathbb{J}$. The family $\{(\mathcal{W}_j, \nu_j)\}_{j\in\mathbb{J}}$ is called a fusion frame for $\mathcal{H}$ if there exist constants $0 < \alpha \leq \beta < +\infty$ such that

$$\alpha\|x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\pi_{\mathcal{W}_j} x\|^2 \leq \beta\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

Definition 1.2. ([26]) Let $\{\mathcal{W}_j\}_{j\in\mathbb{J}}$ be a collection of closed subspaces of $\mathcal{H}$, $\{\nu_j\}_{j\in\mathbb{J}}$ be a family of weights, i.e., $\nu_j > 0$ and $\Lambda_j \in \mathcal{L}(\mathcal{H}, \mathcal{W}_j)$ for each $j \in \mathbb{J}$. The family $\{(\mathcal{W}_j, \Lambda_j, \nu_j)\}_{j\in\mathbb{J}}$ is called a g-fusion frame for $\mathcal{H}$ if there exist constants $0 < \alpha \leq \beta < +\infty$ such that

$$\alpha\|x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j} x\|^2 \leq \beta\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

DEFINITION 1.3. ([25]) Let $\{\mathcal{W}_j\}_{j\in\mathbb{J}}$ be a collection of closed subspaces of $\mathcal{H}$, $\{\nu_j\}_{j\in\mathbb{J}}$ be a family of weights, i.e., $\nu_j > 0$ and $\Lambda_j \in \mathcal{L}(\mathcal{H}, \mathcal{W}_j)$ for all j $\in \mathbb{J}$. The family $\{(\mathcal{W}_j, \Lambda_j, \nu_j)\}_{j\in\mathbb{J}}$ is called a K-g-fusion frame for $\mathcal{H}$ if there exist constants $0 < \alpha \leq \beta < +\infty$ such that

$$\alpha\|K^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j} x\|^2 \leq \beta\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

We call α, β lower and upper frame bounds of a K-g-fusion frame, respectively.

The following lemmas are key tools for the proofs of our main results.

LEMMA 1.4. ([13]) *Let $\mathcal{W}$ be a closed subspace of $\mathcal{H}$ and $A \in \mathcal{L}(\mathcal{H})$. Then*

$$\pi_{\mathcal{W}} A^* = \pi_{\mathcal{W}} A^* \pi_{\overline{A\mathcal{W}}}.$$

LEMMA 1.5. ([8]) *Let $A, B \in \mathcal{L}(\mathcal{H})$. Then the following statements are equivalent:*

(1) $R(A) \subseteq R(B)$;
(2) $\|A^*x\|^2 \leq c\|B^*x\|^2$ *for all* $x \in \mathcal{H}$ *and for some* $c \geq 0$;
(3) *there exists* $C \in \mathcal{L}(\mathcal{H})$ *such that* $A = BC$.

In particular, there exists a unique $T \in \mathcal{L}(\mathcal{H})$ such that $A = BT$ with

$$\|T\| = \inf\left\{c > 0 \,:\, \|A^*x\|^2 \leq c\|B^*x\|^2,\ \forall x \in \mathcal{H}\right\}.$$

Remark 1.6. If $A \neq 0$, then then there exists $T \in \mathcal{L}(\mathcal{H})$ such that $A = BT$ with $\|T\| = c \neq 0$.

In [14], recall that the Moore-Penrose inverse of an operator $A \in \mathcal{L}(\mathcal{H})$ with closed range is defined as the unique operator $A^\dagger \in \mathcal{L}(\mathcal{H})$ satisfying:

$$AA^\dagger = \pi_{R(A)} \qquad \text{and} \qquad A^\dagger A = \pi_{R(A^*)}$$

EXAMPLE 1.7. ([21]) Let $A \in \mathcal{L}(\mathcal{H})$ such that $A^2 = A$. Then

$$A^\dagger = \pi_{R(A^*)} \pi_{R(A)}.$$

Now, we list below some useful properties related to Moore-Penrose inverses.

PROPOSITION 1.8. *Let $A \in \mathcal{CR}(\mathcal{H})$. Then*

(1) $R(A^\dagger) = R(A^*) = N(A)^\perp$;
(2) $N(A^\dagger) = N(A^*) = R(A)^\perp$.

The next proposition will be useful in our work.

PROPOSITION 1.9. ([16]) *Let $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{CR}(\mathcal{H})$ be an arbitrary operator which commutes with A. Then A commutes with $B^\dagger$.*

In mathematics, an EP matrix (or range-Hermitian matrix) is a square matrix A whose range is equal to the range of its conjugate transpose A^* [27]. EP matrices, as an extension of normal matrices, has been extended by Campbell and Meyer [28] to operators with closed range on a Hilbert space.

DEFINITION 1.10. ([28]) $A \in \mathcal{L}(\mathcal{H})$ is called an EP operator if $R(A)$ is closed and $R(A) = R(A^*)$.

EXAMPLE 1.11. Let $A \in \mathcal{L}(l^2\mathbb{C}))$ be defined as follows:

$$A\left((x_j)_{j\geq 1}\right) = ((y_j)_{j\geq 1}),$$

where

$$y_j = \begin{cases} x_1 - x_3 & \text{if } j = 1, \\ 0 & \text{if } j = 2, \\ x_j & \text{if } j \geq 3. \end{cases}$$

By some straightforward computations, we obtain that A is an EP operator.

PROPOSITION 1.12. ([15]) *Let $A \in \mathcal{CR}(\mathcal{H})$ be normal. Then A is an EP operator.*

Additionally, the closedness of range of operators is an attractive problem which appears in operator theory, especially, in the theory of Fredholm operators. The following quantity is closely connected with the closeness of the range (see [23])

$$\gamma(A) := \inf\left\{\|Ax\| \,:\, x \in \mathcal{H},\ \operatorname{dist}(x, N(A)) = 1\right\}.$$

Formally, we set $\gamma(0) := \infty$. Clearly, $\gamma(A) > 0$ if and only if $R(A)$ is closed.

EXAMPLE 1.13. Let $A \in \mathcal{L}(\mathbb{C}^2)$ be defined as follows:

$$\begin{array}{rcl} A : \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ (x_1, x_2) & \longmapsto & (x_1, x_1). \end{array}$$

For $x = (x_1, x_2) \in \mathbb{C}^2$, we have

$$\|Ax\| = \sqrt{2}|x_1|$$

and

$$\mathrm{dist}(x, N(A)) = |x_1|.$$

Then

$$\gamma(A) = \sqrt{2}.$$

Following Mbekhta [18], an operator $A \in \mathcal{CR}(\mathcal{H})$ is said to be *regular* if $N(A) \subset \mathcal{R}^\infty(A)$.

EXAMPLE 1.14. All injective operators with closed range are regular. Some other examples may be found in [17].

PROPOSITION 1.15. ([19]) *Let A, B, C, D be mutually commuting operators such that $AC + BD = I$. Then*

$$\mathcal{R}^\infty(AB) = \mathcal{R}^\infty(A) \cap \mathcal{R}^\infty(B).$$

Moreover, AB is regular if and only if both A and B are regular.

A subspace $\mathcal{E} \subset \mathcal{H}$ is called a *reducing subspace for $A \in \mathcal{L}(\mathcal{H})$* if $A(\mathcal{E}) \subset \mathcal{E}$ and $A^*(\mathcal{E}) \subset \mathcal{E}$.

THEOREM 1.16. ([10]) *Let A be regular such that $\mathcal{R}^\infty(A)$ reduces A. Then $A^\dagger$ is regular.*

Next, we collect below some useful properties related to regular operators.

PROPOSITION 1.17. ([1]) *Let $A \in \mathcal{L}(\mathcal{H})$ be regular. Then*

(1) *$\mathcal{R}^\infty(A)$ is closed;*
(2) *A^* is regular;*
(3) *$\mathcal{R}^\infty(A - zI) = \mathcal{R}^\infty(A) = A(\mathcal{R}^\infty(A))$, $\forall |z| < \gamma(A)$.*

2. K-g-fusion frames via operators

In this section, consider $K \in \mathcal{L}(\mathcal{H})$ and $A \in \mathcal{CR}(\mathcal{H})$ and we denote by

$$\zeta(A) = A^{\dagger *}.$$

LEMMA 2.1. *Let $A \in \mathcal{CR}(\mathcal{H})$. Then*

$$R(\zeta(A)) = R(A).$$

Proof. By Proposition 1.8, we have

$$R(\zeta(A)) = N(A^{\dagger})^{\perp} = (N(A^*))^{\perp} = R(A).$$

This completes the proof. ∎

THEOREM 2.2. *Let $\Lambda = \{(\mathcal{W}_j, \Lambda_j, \nu_j)\}_{j\in\mathbb{J}}$ be a K-g-fusion frame for $\mathcal{H}$ with $K \neq 0$ and A be an EP operator such that $KA^* = A^*K$ and $R(K^*) \subseteq R(A)$. Then $\left\{\left(\overline{\zeta(A)\mathcal{W}_j}, \Lambda_j \pi_{\mathcal{W}_j} A^{\dagger}, \nu_j\right)\right\}_{j\in\mathbb{J}}$ is a K-g-fusion frame for $\mathcal{H}$.*

Proof. Let $r \in R(A)$. By Lemma 2.1, we have

$$r = \zeta(A)\zeta(A)^{\dagger} r.$$

Since

$$\mathcal{H} = R(A) \oplus R(A)^{\perp},$$

if $x \in \mathcal{H}$, there exist $r \in R(A)$ and $s \in R(A)^{\perp}$ such that

$$Kx = Kr + Ks.$$

Thus we get

$$Kx = K\zeta(A)(A^* r) + Ks.$$

It follows from Proposition 1.9 that $\zeta(A)K = K\zeta(A)$.

From $R(K^*) \subseteq R(A)$, we get $R(A)^{\perp} \subset R(K^*)^{\perp} = N(K)$, and hence

$$Kx = \zeta(A)K(A^* r + s).$$

Since A is EP, we have

$$A^* x \in R(A).$$

By Lemma 1.5, there exists $c > 0$ such that

$$c\|K^*x\|^2 \le \|(\zeta(A)K)^*x\|^2, \qquad \forall x \in \mathcal{H}.$$

Since $\mathit{\Lambda}$ is a K-g-fusion frame, we have

$$\alpha\|(\zeta(A)K)^*x\|^2 \le \sum_{j\in\mathbb{J}} \nu_j^2 \|\mathit{\Lambda}_j \pi_{\mathcal{W}_j} \zeta(A)^* x\|^2 \le \beta\|\zeta(A)^*x\|^2, \qquad \forall x \in \mathcal{H}.$$

This yields

$$c\alpha\|K^*x\|^2 \le \sum_{j\in\mathbb{J}} \nu_j^2 \|\mathit{\Lambda}_j \pi_{\mathcal{W}_j} \zeta(A)^* x\|^2 \le \beta\|A^\dagger\|^2\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

By Lemma 2.1, we get

$$\pi_{\mathcal{W}_j} \zeta(A)^* \pi_{\overline{\zeta(A)\mathcal{W}_j}} = \pi_{\mathcal{W}_j} \zeta(A)^*.$$

Therefore,

$$c\alpha\|K^*x\|^2 \le \sum_{j\in\mathbb{J}} \nu_j^2 \|\mathit{\Lambda}_j \pi_{\mathcal{W}_j} A^\dagger \pi_{\overline{\zeta(A)\mathcal{W}_j}} x\|^2 \le \beta\|A^\dagger\|^2\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

This completes the proof. ∎

Corollary 2.3. *Let $\{(\mathcal{W}_j, \mathit{\Lambda}_j, \nu_j)\}_{j\in\mathbb{J}}$ be a K-g-fusion frame for $\mathcal{H}$ and $A \in \mathcal{CR}(\mathcal{H})$ be normal such that $KA^* = A^*K$. Then*

$$\left\{ \left(\overline{\zeta(A)\mathcal{W}_j}, \mathit{\Lambda}_j \pi_{\mathcal{W}_j} A^\dagger, \nu_j \right) \right\}_{j\in\mathbb{J}}$$

is a K-g-fusion frame for $\mathcal{H}$.

Proof. The proof follows from Proposition 1.12 and Theorem 2.2. ∎

Lemma 2.4. *Let A be regular such that $\mathcal{R}^\infty(A)$ reduces A. Then $\zeta(A)$ is regular.*

Proof. The proof follows from Theorem 1.16 and Proposition 1.17. ∎

Theorem 2.5. *Let $\mathit{\Lambda} = \{(\mathcal{W}_j, \mathit{\Lambda}_j, \nu_j)\}_{j\in\mathbb{J}}$ be a K-g-fusion frame for $\mathcal{H}$ with $K \ne 0$ and A be regular such that $KA^* = A^*K$. Then*

$$\left\{ \left(\overline{\zeta(A)\mathcal{W}_j}, \Lambda_j \pi_{\mathcal{W}_j} A^\dagger, \nu_j \right) \right\}_{j\in\mathbb{J}}$$

is a K-g-fusion frame for $R^\infty(\zeta(A))$.

Proof. Let $x \in \mathcal{R}^{\infty}(\zeta(A))$. Then there exists $y \in \mathcal{R}^{\infty}(\zeta(A))$ such that

$$x = \zeta(A)\, y,$$

and hence

$$Kx = K\zeta(A)y.$$

Using Proposition 1.9, we get

$$K\zeta(A) = \zeta(A)\, K.$$

By Lemma 1.5, there exists $c > 0$ such that

$$c\|K^*x\|^2 \leq \|(\zeta(A)K)^*x\|^2, \qquad \forall x \in \mathcal{R}^{\infty}(\zeta(A)).$$

Since Λ is a K-g-fusion frame, for all $x \in \mathcal{R}^{\infty}(\zeta(A))$, we have

$$\alpha\|(\zeta(A)K)^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j} \zeta(A)^*x\|^2 \leq \beta\|\zeta(A)^*x\|^2.$$

Thus

$$c\alpha\|K^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j} \zeta(A)^*x\|^2 \leq \beta\|A^{\dagger}\|^2\|x\|^2.$$

By Lemma 1.4, we have

$$\pi_{\mathcal{W}_j}\zeta(A)^*\pi_{\overline{\zeta(A)\mathcal{W}_j}} = \pi_{\mathcal{W}_j}\zeta(A)^*.$$

Hence

$$c\alpha\|K^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j} A^{\dagger}\pi_{\overline{\zeta(A)\mathcal{W}_j}} x\|^2 \leq \beta\|A^{\dagger}\|^2\|x\|^2.$$

This completes the proof. ∎

For any scalar z with $|z| < \gamma(A)$, set

$$p(A) = A^2 - zA.$$

Proposition 2.6. *Let A be regular. Then $p(A)$ is a regular operator. Moreover, we have*

$$\mathcal{R}^{\infty}(p(A)) = \mathcal{R}^{\infty}(A).$$

Proof. It follows from Proposition 1.15 that $p(A)$ is regular. Moreover, we have

$$\mathcal{R}^\infty(p(A)) = \mathcal{R}^\infty(A) \cap \mathcal{R}^\infty(A - zI).$$

By Proposition 1.17, we get

$$\mathcal{R}^\infty\left(p\left(A\right)\right) = \mathcal{R}^\infty(A).$$

This completes the proof. ∎

THEOREM 2.7. *Let $\Lambda = \{(\mathcal{W}_j, \Lambda_j, \nu_j)\}_{j\in\mathbb{J}}$ be a K-g-fusion frame for $\mathcal{H}$ with $K \neq 0$ and A be regular such that $KA = AK$. Then*

$$\left\{\left(\overline{p(A)\mathcal{W}_j}, \Lambda_j \pi_{\mathcal{W}_j}(p(A)^*, \nu_j\right)\right\}_{j\in\mathbb{J}}$$

is a K-g-fusion frame for $\mathcal{R}^\infty(A)$.

Proof. Let $x \in \mathcal{R}^\infty(A)$. By Proposition 2.6, there exists $y \in \mathcal{R}^\infty(A)$ such that

$$x = p(A)y,$$

and hence

$$Kx = p(A)Ky.$$

By Lemma 1.5, there exists $c > 0$ such that

$$c\|K^*x\|^2 \leq \|(p(A)K)^*x\|^2.$$

Since Λ is a K-g-fusion frame, for all $x \in \mathcal{R}^\infty(A)$, we have

$$\alpha\|(p(A)K)^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j} p(A)^* x\|^2 \leq \beta\|p(A)^*x\|^2.$$

By Lemma 1.4, we have

$$\pi^*_{\mathcal{W}_j} \pi_{\overline{p(A)\mathcal{W}_j}} = \pi_{\mathcal{W}_j} p(A)^*.$$

Thus

$$\alpha c\|K^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j} p(A)^* \pi_{\overline{p(A)\mathcal{W}_j}} x\|^2 \leq \beta\|p(A)\|^2\|x\|^2.$$

This completes the proof. ∎

In the following, we aim to construct some K-g-fusion frames induced by some projections.

THEOREM 2.8. *Let $L \in \mathcal{L}(\mathcal{H})$ such that $KLK = K$ and $\left\{(\mathcal{W}_j, \Lambda_j, \nu_j)\right\}_{j\in\mathbb{J}}$ be a K-g-fusion frame for $\mathcal{H}$. Then $\left\{(\overline{(KL)\mathcal{W}_j}, \Lambda_j \pi_{\mathcal{W}_j}(KL)^*, \nu_j)\right\}_{j\in\mathbb{J}}$ is a K-g-fusion frame for $\mathcal{H}$.*

Proof. By assumption, there exist $\alpha, \beta > 0$ such that

$$\alpha\|K^*(KL)^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j}(KL)^*x\|^2 \leq \beta\|(KL)^*x\|^2, \qquad \forall x \in \mathcal{H}.$$

Since $KLK = K$,

$$\alpha\|K^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j}(KL)^*x\|^2 \leq \beta\|KL\|^2\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

By Lemma 1.4, we have

$$\pi_{\mathcal{W}_j}(KL)^*\pi_{\overline{(KL)\mathcal{W}_j}} = \pi_{\mathcal{W}_j}(KL)^*.$$

Consequently,

$$\alpha\|K^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j}(KL)^*\pi_{\overline{(KL)\mathcal{W}_j}}x\|^2 \leq \beta\|KL\|^2\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

This completes the proof. ∎

From now on, we consider $K \in \mathcal{CR}(\mathcal{H})$ such that $K \neq 0$.

LEMMA 2.9. *Let $K \in \mathcal{CR}(\mathcal{H})$. Then there exists $c > 0$ such that*

$$c\|\zeta(K)^*x\|^2 \leq \|K^*x\|^2, \qquad \forall x \in \mathcal{H}.$$

Proof. The proof follows from Lemma 2.1 and Lemma 1.5. ∎

LEMMA 2.10. *Let U be a unitary operator. Then*

$$\zeta(UKU^*) = U\zeta(K)U^*.$$

Proof. Direct computations show that

$$(UK^*U^*)^\dagger = UK^{*\dagger}U^* = UK^{\dagger *}U^*.$$

This implies that

$$\zeta(UKU^*) = (UKU^*)^{*\dagger} = U\zeta(K)U^*.$$

This completes the proof. ∎

THEOREM 2.11. *Let U be unitary and $\{(\mathcal{W}_j, \Lambda_j, \nu_j)\}_{j\in\mathbb{J}}$ be a K-g-fusion frame for $\mathcal{H}$. Then $\left\{\left(U\mathcal{W}_j, \Lambda_j\pi_{\mathcal{W}_j}U^{-1}, \nu_j\right)\right\}_{j\in\mathbb{J}}$ is a $U\zeta(K)U^{-1}$-g-fusion frame for $\mathcal{H}$.*

Proof. By assumption, there exist $0 < \alpha \leq \beta < +\infty$ such that

$$\alpha\|K^*U^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2\|\Lambda_j\pi_{\mathcal{W}_j}U^*x\|^2 \leq \beta\|U^*x\|^2, \qquad \forall x \in \mathcal{H}.$$

By Lemma 2.9, there exists $c > 0$ such that

$$c\|(\zeta(UKU^*))^*x\|^2 \leq \|(UKU^*)^*x\| \leq \|K^*U^*x\|, \qquad \forall x \in \mathcal{H}.$$

Thus

$$\alpha c\|(\zeta(UKU^*))^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2\|\Lambda_j\pi_{\mathcal{W}_j}U^*x\|^2 \leq \beta\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

By Lemma 1.4, we have

$$\pi_{\mathcal{W}_j}U^*\pi_{U\mathcal{W}_j} = \pi_{\mathcal{W}_j}U^*.$$

It follows from Lemma 2.10 that

$$\alpha c\|(U\zeta(K)U^{-1})^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2\|\Lambda_j\pi_{\mathcal{W}_j}U^{-1}\pi_{U\mathcal{W}_j}x\|^2 \leq \beta\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

This completes the proof. ∎

Next, new g-fusion frames for some idempotent operators are established.

THEOREM 2.12. *Let $K \in \mathcal{CR}(\mathcal{H})$ and $\{(\mathcal{W}_j, \Lambda_j, \nu_j)\}_{j\in\mathbb{J}}$ be a K-g-fusion frame for $\mathcal{H}$. Then $\{\mathcal{W}_j, \Lambda_j, \nu_j\}_{j\in\mathbb{J}}$ is a $\pi_{R(K)}$-g-fusion frame for $\mathcal{H}$.*

Proof. By assumption, there exist $0 < \alpha \leq \beta < +\infty$ such that

$$\alpha\|K^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j} x\|^2 \leq \beta\|x\|^2.$$

Using the fact that

$$\zeta(K)K^* = (KK^\dagger)^* = (\pi_{R(A)})^* = \pi_{R(K)},$$

we obtain

$$\|(\pi_{R(K)})^*x\|^2 \leq \|K\|^2\|\zeta(K)^*x\|^2, \qquad \forall x \in \mathcal{H}.$$

By Lemma 2.9, there exists $c > 0$ such that

$$\alpha c\|\zeta(K)^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j} x\|^2 \leq \beta\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

Then

$$c\alpha\|K^{-2}\|(\pi_{R(K)})^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j} x\|^2 \leq \beta\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

This completes the proof. ∎

THEOREM 2.13. *Let K be EP and $\Lambda = \{(\mathcal{W}_j, \Lambda_j, \nu_j)\}_{j\in\mathbb{J}}$ be a K-g-fusion frame for $\mathcal{H}$. Then $\{\mathcal{W}_j, \Lambda_j, \nu_j\}_{j\in\mathbb{J}}$ is a K^*-g-fusion frame for $\mathcal{H}$.*

Proof. Since K is EP, we have

$$R(K^*) = R(K).$$

By Lemma 1.5, there exists $c > 0$ such that

$$c\|Kx\|^2 \leq \|K^*x\|^2.$$

Since Λ is a K-g-fusion frame for $\mathcal{H}$, for all $x \in \mathcal{H}$, we have

$$\alpha\|K^*x\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j} x\|^2 \leq \beta\|\zeta(A)^*x\|^2.$$

This implies that

$$c\alpha\|Kx\|^2 \leq \sum_{j\in\mathbb{J}} \nu_j^2 \|\Lambda_j \pi_{\mathcal{W}_j} x\|^2 \leq \beta\|A^\dagger\|^2\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

This completes the proof. ∎

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