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Cohomology of orthosymplectic contactomorphisms acting on λ-densities on the $1|n$-supercircle

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Abstract: We study the standard contact structure on the supercircle, $S^{1|n}$, and the associated supergroups of contactomorphisms $\mathrm{E}(1|n)$ and $\mathrm{OPs}(n|2)$ corresponding to Euclidean and projective geometries. In this work, we begin by computing the cohomology group of the orthosymplectic Lie group $\mathrm{PC}(n|2)$, taking values in the space of tensor densities, $\mathfrak{F}_\lambda^n$. Second, we determine the $\mathfrak{s}(2|n)$-relative cohomology spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2), \mathfrak{s}(2|n); \mathfrak{F}_\lambda^n)$ with $\mathrm{H}^1_{\mathrm{diff}}$ indicating that only differential cochains are considered; meaning that only cochains given by differential operators are taken into account. We also explicitly determine the 1-cocycles generating these cohomology spaces.

Key words: Group of contactomorphisms, euclidean structures, orthosymplectic superalgebra, supermanifolds, weighted densities.

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1. Introduction

Lie superalgebras, supergroups, and their representation theory over the field have been studied extensively in literature since the classification of finite-dimensional complex simple Lie superalgebras by V. Kac [16]. The various and well-known geometries on the circle S^1, namely the Euclidean, affine and projective geometries (see for example [2, 11, 14]), are defined by the symmetry groups $(\mathbb{R}, +)$, $(\mathrm{Aff}(1), \circ)$ and $(\mathrm{PGL}(2), \times)$ respectively, or, equivalently, by their characteristic invariants i.e. the distance, the distance rotation and the cross-ratio.

The space of $\lambda-$densities (or weighted densities with weight λ) on S^1, denoted by:

$$\mathfrak{F}_\lambda^0 = \{h(dx)^\lambda | h \in C^\infty(S^1)\}, \qquad \lambda \in \mathbb{R},$$

is the space of sections of the line bundle $(T^*\mathbb{R})^{\otimes^\lambda}$. Let $\mathrm{Vec}(S^1)$ be the Lie algebra of all vector fields $X_H = H\frac{d}{dx}$, where $H \in C^\infty(S^1)$. The Lie *derivative*

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L_X along the vector field X makes $\mathfrak{F}_\lambda^0$ a $\mathrm{Vec}(S^1)$-module for any $\lambda \in \mathbb{R}$:

$$L_{X_H}^{\lambda}(h(dx)^{\lambda}) = (Hh' + \lambda H'h)(dx)^{\lambda}.$$

The action of $\mathrm{Diff}(S^1)$ on the space $\mathfrak{F}_\lambda^0$ is given by

$$\Phi^*(h) = h(\Phi)(\Phi')^{\lambda}.$$

The space $\mathfrak{F}_\lambda^0$ coincides with the space of all vector fields, functions and differential 1-forms for $\lambda = -1$, 0 and 1, respectively ([4, 10]). If we restrict ourselves to the Lie subalgebra of $\mathrm{Vec}(S^1)$ generated by $\{\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}\}$, isomorphic to the orthosymplectic Lie algebra $\mathfrak{sl}(2)$. The vector field $\{\frac{d}{dx}\}$, spans a commutative Lie algebra isomorphic to $\mathfrak{so}(2)$, the Lie subalgebra of $\mathrm{Vec}(S^1)$ generated by $\{\frac{d}{dx}, x\frac{d}{dx}\}$ is isomorphic to the affine Lie algebra $\mathfrak{a}(1)$.

Therefore, we obtain the following inclusions of the Lie subalgebras

$$\mathfrak{so}(2) \subset \mathfrak{a}(1) \subset \mathfrak{sl}(2).$$

We require that the Lie algebras, we are dealing with contain $\mathfrak{so}(2)$. We get a family of infinite dimensional $\mathfrak{sl}(2)$-modules still denoted $\mathfrak{F}_\lambda^0$. In [9], Okba Basdouri computed the cohomology spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{a}(1), \mathfrak{F}_\lambda^0)$. In [5], the authors compute the cohomology spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathfrak{F}_\lambda^0)$.

From these invariants we can obtain, using Cartan-like formula, three 1-cocycles of $\mathrm{Diff}_+(S^1)$ with coefficients in some tensorial density modules $\mathfrak{F}_\lambda^0$ with $\lambda \in \mathbb{R}$, see [12]. They are the generators of the three nontrivial cohomology spaces $\mathrm{H}^1(\mathrm{Diff}_+(S^1); \mathcal{F}_\lambda)$, with $\lambda = 0, 1, 2$, as proved in [12, 17].

We now turn to the corresponding supergeometric setting. We consider the supercircle $S^{1|n}$ endowed with its standard contact 1-form α_n, and introduce the superspace $\mathfrak{F}_\lambda^n$ of λ-densities on the superspace $S^{1|n}$. These spaces naturally carry a structure of $\mathfrak{k}(n)$-modules, where $\mathfrak{k}(n)$ is the Lie superalgebra of contact vector fields on $S^{1|n}$.

To the orthosymplectic Lie algebra $\mathfrak{sl}(2)$ corresponds the orthosymplectic Lie superalgebra $\mathfrak{osp}(n|2)$ which is naturally embedded as a subalgebra of $\mathfrak{k}(n)$. Similarly, the Euclidean Lie algebra $\mathfrak{so}(2)$ corresponds to the Euclidean Lie superalgebra $\mathfrak{s}(2|n)$ naturally realized as a subalgebra of $\mathfrak{osp}(n|2)$. Restricting the $\mathfrak{k}(n)$-modules to $\mathfrak{osp}(n|2)$, we obtain $\mathfrak{osp}(n|2)$-modules, which we still denote by $\mathfrak{F}_\lambda^n$. The main objective of this article is to compute the first cohomology groups of the orthosymplectic Lie superalgebra acting on $\mathfrak{F}_\lambda^n$ over the supercircle $S^{1|n}$. The extended geometries of $S^{1|n}$ are described by supergroups of supersymmetries, namely translations $\mathrm{E}(1,n)$, affine transformations $\mathrm{Aff}(1,n)$, and orthosymplectic transformations $\mathrm{OSp}(n|2)$. As in the classical case, these supergroups admit characteristic invariants as well. Each supergroup $\mathrm{E}(1,n)$, $\mathrm{Aff}(1,n)$ and $\mathrm{PC}(n|2)=\mathrm{OSp}(n|2)/\{\pm I\}$ acts on $S^{1|n}$ by elements of $\mathcal{K}(n)$.

In this paper, we first compute the $\mathfrak{s}(2|n)$-relative cohomology spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2),\mathfrak{s}(2|n);\mathfrak{F}^n_\lambda)$ where $\mathrm{H}^1_{\mathrm{diff}}$ denotes differential cohomology, meaning that only cochains defined by differential operators are considered. We show that the non-vanishing cohomology spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2),\mathfrak{s}(2|n);\mathfrak{F}^n_\lambda)$ occur only for a finite number of values λ. More precise, let

$$d^n_\lambda=\dim(\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2),\mathfrak{s}(2|n);\mathfrak{F}^n_\lambda)).$$

We show that

$$d^n_\lambda=\begin{cases}2 & \text{if } n=2 \text{ and } \lambda=0,\\ 1 & \text{if } \begin{cases} n=0 \text{ and } \lambda\in\{0,\,1\},\\ n=1 \text{ and } \lambda\in\{0,\,\frac{1}{2}\},\\ n\geq 3 \text{ and } \lambda=0,\end{cases}\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, we derive explicit formulas for all non-trivial 1-cocycles.

Second, we compute the first cohomology groups of the orthosymplectic Lie group with coefficients in $\mathfrak{F}^n_\lambda$, focusing on the corresponding cohomology spaces

$$\mathrm{H}^1(\mathrm{PC}(n|2),\mathfrak{F}^n_\lambda).$$

We prove that the non-vanishing cohomology spaces $\mathrm{H}^1(\mathrm{PC}(n|2),\mathfrak{F}^n_\lambda)$ arise only for a small number of values of λ. More precisely, let

$$D^n_\lambda=\dim(\mathrm{H}^1(\mathrm{PC}(n|2),\mathfrak{F}^n_\lambda)).$$

We prove that

$$D^n_\lambda=\begin{cases}2 & \text{if } n=2 \text{ and } \lambda=0,\\ 1 & \text{if } \begin{cases} n=0 \text{ and } \lambda\in\{0,\,1\},\\ n=1 \text{ and } \lambda\in\{0,\,\frac{1}{2}\},\\ n\geq 3 \text{ and } \lambda=0,\end{cases}\\ 0 & \text{otherwise.}\end{cases}$$

2. Geometry of the superspace

Let $(x,\xi_1,\,\ldots,\,\xi_n)$, denote local coordinates on $S^{1|n}$ where x is even and $\xi_1,\,\ldots,\,\xi_n$ are odd coordinates with relations $\xi_k^2=0$, $\xi_k\xi_j=-\xi_j\xi_k$.

This superspace carries the standard contact structure defined by the distribution $\langle\overline{D}_1,\,\ldots,\,\overline{D}_n\rangle$ spanned by the vector fields $\overline{D}_i=\partial_{\xi_i}-\xi_i\partial_x$ where $\partial_i=\frac{\partial}{\partial\xi_i}$ and $\partial_x=\frac{\partial}{\partial x}$. Equivalently, this distribution is given as the kernel of the following 1-form:

$$\alpha_n=dx+\sum_{i=1}^{n}\xi_i d\xi_i. \tag{2.1}$$

By definition, a diffeomorphism of $S^{1|n}$ is a parity-preserving algebra automorphism of the function algebra $C^\infty(S^{1|n})$. The group of diffeomorphisms of $S^{1|n}$ is denoted by $\mathrm{Diff}(S^{1|n})$. Let $\xi=(\xi_1,\cdots,\xi_n)$, which allows us to use the shorthand notation $\Phi(x,\xi)$ for a diffeomorphism of $S^{1|n}$ determined by a family $(\varphi,\psi_1,\cdots,\psi_n)$ where φ is even and ψ_i $(i=1,\cdots,n)$ are odd functions. Each $\Phi\in\mathrm{Diff}(S^{1|n})$ induces a diffeomorphism on S^1 and $\mathrm{P}\colon\mathrm{Diff}(S^{1|n})\to\mathrm{Diff}(S^1)$ defines a group homomorphism.

2.1. The Lie superalgebra of contact vector fields on $S^{1|n}$

Consider the superspace $C^\infty(S^{1|n})$ consisting of functions F of the form:

$$F=\sum_{1\leq i_1<\cdots<i_k\leq n} f_{i_1,\ldots,i_k}(x)\xi_{i_1}\cdots\xi_{i_k} \qquad \text{where } f_{i_1,\ldots,i_k}\in C^\infty(S^1).$$

Even (respectively odd) elements of $C^\infty(S^{1|n})$ are those functions given in (**??**) where the summation is restricted to even (respectively odd) integers k. Let $|F|$ denote the parity of a homogeneous function F. We now consider the contact bracket on $C^\infty(S^{1|n})$

$$\{F,G\}=FG'-F'G-\frac{1}{2}\sum_{i=1}^{n}D_i(F)\overline{D}_i(G) \tag{2.2}$$

where $D_i=\frac{\partial}{\partial\xi_i}+\xi_i\frac{\partial}{\partial x}$.

Let $\mathrm{Vec}(S^{1|n})$ be the superspace of vector fields on $S^{1|n}$:

$$\mathrm{Vec}(S^{1|n})=\left\{F_0\partial_x+\sum_{i=1}^{n}F_i\partial_i\ :\ F_i\in C^\infty(S^{1|n})\ \text{ for all } i\right\}.$$

Consider the superspace $\mathfrak{k}(n)$ of contact vector fields on $S^{1|n}$. Equivalently, $\mathfrak{k}(n)$ consists of all vector fields on $S^{1|n}$ preserving the contact distribution $\langle \overline{D}_1, \ldots, \overline{D}_n \rangle$:

$$\mathfrak{k}(n) = \left\{ X \in \mathrm{Vec}(S^{1|n}) \,:\, [X, \overline{D}_i] = F_X \overline{D}_i \text{ for some } F_X \in C^\infty(S^{1|n}) \right\}.$$

The Lie superalgebra $\mathfrak{k}(n)$ is spanned by vector fields of the form:

$$X_F = F\partial_x - \frac{1}{2}\sum_{i=1}^{n} D_i(F)\overline{D}_i, \qquad \text{where } F \in C^\infty(S^{1|n}).$$

The vector field X_F has the same parity as F. The Lie bracket in $\mathfrak{k}(n)$ can be written as follows:

$$[X_F, X_G] = X_{\{F,G\}}.$$

2.2. THE SUBSUPERALGEBRA OF $\mathfrak{k}(n)$ The Euclidean Lie algebra $\mathfrak{s}(1|n)$ is isomorphic to the Lie subalgebra of $\mathfrak{k}(n)$ generated by

$$\mathfrak{s}(1|n) = \mathrm{Span}(X_1, X_{\xi_i}, \ldots, X_{\xi_i\xi_j}, \ldots), \qquad \text{where } 1 \le i, j \le n.$$

The orthosymplectic Lie algebra $\mathfrak{osp}(2|n)$ is isomorphic to the Lie subalgebra of $\mathfrak{k}(n)$ generated by

$$\mathfrak{osp}(n|2) = \mathrm{Span}(X_1, X_x, X_{x^2}, X_{\xi_i}, X_{x\xi_i}, \ldots, X_{\xi_i\xi_j}, \ldots), \text{ where } 1 \le i, j \le n.$$

2.3. CONTACTOMORPHISMS The standard contact structure on $S^{1|n}$ is defined by the conformal class of the 1-form (2.1), satisfying $\alpha_i \wedge d\alpha_i \neq 0$. Equivalently, it is given by the kernel of α_n, which is spanned by the odd vector field

$$D_i = \partial_{\xi_i} + \xi_i\partial_x, \qquad \text{for } i = 1, \ldots, n \tag{2.3}$$

whose square $D_i^2 = \frac{1}{2}[D_i, D_i] = \partial_x$ is the Reeb vector field of the structure. Then D_i and ∂x set up a basis of the $C^\infty(S^{1|n})$ left-module $\mathrm{Vec}(S^{1|n})$ while α_i and $\beta_i = d\xi_i$ constitute the dual basis, with $d\alpha_i = \beta_i \wedge \beta_i$.

Let $\Phi \in \mathrm{Diff}(S^{1|n})$ be a diffeomorphism satisfying

$$\Phi^*\alpha_n = E_\Phi \alpha_n \tag{2.4}$$

for some superfunction E_Φ and is called a contactomorphism of $S^{1|n}$ (cf. [12, 18]). It follows that Φ is a contactomorphism if and only if

$$D_i\varphi - \psi.D_i\psi = 0, \qquad \text{for } i = 1, \ldots, n. \tag{2.5}$$

Consequently, we obtain

$$E_\Phi = \varphi' + \psi.\psi' = D_i\psi D_i\psi, \qquad \text{for all } i. \tag{2.6}$$

The subgroup $\mathcal{K}(n) \subset \mathrm{Diff}(S^{1|n})$ preserving the contact structure is denoted by $\mathcal{K}(n)$; its elements are called contactomorphisms.

$\mathcal{K}(n) = \{\Phi \in \mathrm{Diff}(S^{1|n}) \mid \Phi^*\alpha_n = E_\Phi\alpha_n\}$ is the group of contactomorphisms of $S^{1|n}$ where E_Φ denotes a function on $S^{1|n}$ depending on the diffeomorphism Φ.

2.4. Subgroups of Contactomorphisms: Euclidean, affine and projective invariants The orthosymplectic group acts by contactomorphisms through a homomorphism $\mathrm{OSp}(n|2) \to \mathcal{K}(n)$, given by the following projective action on $S^{1|n}$ ([12, 15]), namely

$$\widehat{g}(x,\xi) = \left(\frac{ax+b+\gamma\xi}{cx+d+\epsilon\xi}, \frac{\alpha x+\beta+e\xi}{cx+d+\epsilon\xi}\right) \tag{2.7}$$

where $g \in \mathrm{OSp}(n|2)$, the entries a, b, c, d are even elements, β, α are odd column vectors of size n, while γ, ϵ are odd row vectors of size n. The block e is an even $n \times n$ matrix and ξ is regarded as a column vector.

The kernel of this action is $\{\mathrm{Id}, -\mathrm{Id}\}$. Therefore, the induced action is effective for the quotient supergroup $\mathrm{OSp(n|2)}/\{\pm\mathrm{Id}\} = \mathrm{PC}(n|2)$, the supergroup of conformal projective transformations. When n is odd, this supergroup coincides with the special orthosymplectic group $\mathrm{OSp}_+(n|2)$, i.e. the subgroup of $\mathrm{OSp}(n|2)$ with Berezinian 1. One can still define the Euclidean and affine subgroups of $\mathrm{OSp}(n|2)$, whose elements are

$$h = \begin{pmatrix} a & ab & -a\beta^t \\ 0 & a^{-1} & 0 \\ 0 & \beta & 1 \end{pmatrix} \tag{2.8}$$

where $(a, b, \beta) \in \mathbb{R}^{2|n}$, $a > 0$ defining $\mathrm{Aff}_+(1|n)$ and $a = 1$ defining $\mathrm{E}_+(1|n)$.

The group $\mathrm{Aff}_+(1|n)$ is defined as the subgroup of $\widehat{g} \in \mathcal{K}(n)$ preserving the direction of each β_i, i.e. satisfying $\widehat{g}^*\beta_i = \beta_i F_i$, for some superfunction F_i, with $i = 1, \ldots, n$. The subgroup $\mathrm{E}_+(1|n)$ is characterized by the condition of preserving α.

2.5. Weighted densities Now, consider the action of the Lie superalgebra $\mathfrak{k}(n)$ on $C^\infty(S^{1|n})$ defined by

$$\mathfrak{L}^\lambda_{X_F} = X_F + \lambda F'.$$

We denote by $\mathfrak{F}^n_\lambda$ the $\mathfrak{k}(n)$-module of the space of weighted densities on $S^{1|n}$ of weight λ:

$$\mathfrak{F}^n_\lambda = \left\{ F\alpha_n^\lambda \,:\, F \in C^\infty(S^{1|n}) \right\}.$$

It is clear that the adjoint $\mathfrak{k}(n)$-module, is isomorphic to $\mathfrak{F}^n_{-1}$.

The group of $\mathrm{Diff}(S^{1|n})$ acts on the space $\mathfrak{F}^0_\lambda$ via

$$\Phi^*_\lambda(f\alpha_n^\lambda) = ((E_{\Phi^{-1}})^\lambda f(\Phi^{-1}))\alpha_n^\lambda.$$

Now, consider the one-parameter group action on $C^\infty(S^{1|n})$ generated by $\mathfrak{k}(n)$ given by:

$$\mathfrak{L}^\lambda_{X_F} = X_F + \lambda F'.$$

We denote this $\mathfrak{k}(n)$-module by $\mathfrak{F}^n_\lambda$, the space of all weighted densities on $S^{1|n}$ of weight λ:

$$\mathfrak{F}^n_\lambda = \left\{ F\alpha_n^\lambda \,:\, F \in C^\infty(S^{1|n}) \right\}.$$

It is clear that the adjoint $\mathfrak{k}(n)$-module, is isomorphic to $\mathfrak{F}^n_{-1}$.

We restrict the action to the orthosymplectic Lie (super)algebra $\mathfrak{osp}(n|2)$ and regard the spaces $\mathfrak{F}^n_\lambda$ as $\mathfrak{osp}(n|2)$-modules. Of course, the case $n = 0$ we recover the classical setting: $\mathfrak{k}(0) = \mathrm{Vec}(S^1)$ and the corresponding orthosymplectic Lie algebra $\mathfrak{osp}(0|2)$ reduces to the classical Lie algebra $\mathfrak{sl}(2)$ which is isomorphic to the Lie subalgebra of $\mathrm{Vec}(S^1)$. Clearly, $\mathfrak{osp}(n-1|2)$ can be considered as a subalgebra of $\mathfrak{osp}(n|2)$. Hence, the spaces of weighted densities $\mathfrak{F}^n_\lambda$ carry a natural structure of $\mathfrak{osp}(n-1|2)$-modules. It was established in [1, 5, 6, 9, 11] that, as $\mathfrak{osp}(n-1|2)$-module, we have

$$\mathfrak{F}^n_\lambda \simeq \mathfrak{F}^{n-1}_\lambda \oplus \Pi\left(\mathfrak{F}^{n-1}_{\lambda+\frac{1}{2}}\right) \tag{2.9}$$

where Π is the change of parity operator.

In this section, we briefly present the geometry of the supercircle $S^{1|n}$, required for our purposes, including the basic concepts of super differential geometry and the standard contact structure on $S^{1|n}$.

2.6. Cohomology of Lie superalgebra We begin by recalling some fundamental concepts from cohomology theory (see, e.g., [2, 3, 4, 6, 7, 8, 11]). Let $\mathfrak{A} = \mathfrak{A}_{\bar{0}} \oplus \mathfrak{A}_{\bar{1}}$ be a Lie superalgebra acting on a superspace $\mathcal{N} = \mathcal{N}_{\bar{0}} \oplus \mathcal{N}_{\bar{1}}$ and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{A}$. (If $\mathfrak{h}$ is omitted it is assumed to be $\{0\}$). The space of $\mathfrak{A}$-relative n-cochains of $\mathfrak{k}$ with values in $\mathcal{N}$ is the $\mathfrak{k}$-module

$$C^n(\mathfrak{A}, \mathfrak{h}; \mathcal{N}) := \mathrm{Hom}_{\mathfrak{A}}(\Lambda^n(\mathfrak{A}/\mathfrak{h}); \mathcal{N}).$$

The *coboundary operator* $\delta_n : C^n(\mathfrak{A}, \mathfrak{h}; \mathcal{N}) \to C^{n+1}(\mathfrak{A}, \mathfrak{h}; \mathcal{N})$ is a $\mathfrak{k}$-map satisfying $\delta_n \circ \delta_{n-1} = 0$. The kernel of δ_n, denoted by $Z^n(\mathfrak{A}, \mathfrak{h}; \mathcal{N})$, is the space of $\mathfrak{h}$-relative *n-cocycles*. Among them, the elements in the range of δ_{n-1} are called $\mathfrak{h}$-relative *n-coboundaries*. We denote by $B^n(\mathfrak{A}, \mathfrak{h}; \mathcal{N})$ the space of n-coboundaries.

By definition, the n^{th} $\mathfrak{h}$-relative cohomolgy space is defined as the quotient space

$$\mathrm{H}^n(\mathfrak{A}, \mathfrak{h}; \mathcal{N}) = Z^n(\mathfrak{A}, \mathfrak{h}; \mathcal{N})/B^n(\mathfrak{A}, \mathfrak{h}; \mathcal{N}).$$

In what follows, we only require the formula for δ_n (hereafter denoted by δ) in degrees 0 and 1: for $v \in C^0(\mathfrak{A}, \mathfrak{h}; \mathcal{N}) = \mathcal{N}^{\mathfrak{h}}$, $\delta v(g) := (-1)^{|g||v|} g \cdot v$, where

$$\mathcal{N}^{\mathfrak{h}} = \{v \in \mathcal{N} \mid h \cdot v = 0 \quad \text{for all } h \in \mathfrak{h}\},$$

and for $\Lambda \in C^1(\mathfrak{A}, \mathfrak{h}; \mathcal{N})$,

$$\delta(\Delta)(g, h) := (-1)^{|g||\Delta|} g \cdot \Delta(h) - (-1)^{|h|(|g|+|\Delta|)} h \cdot \Delta(g) - \Delta([g, h]). \quad (2.10)$$

for any $g, h \in \mathfrak{A}$.

We consider the action of the Lie superalgebra $\mathfrak{osp}(n|2)$ on $\mathfrak{F}_\lambda^n$ and compute the first $\mathfrak{s}(2|n)$-relative cohomology space $\mathfrak{osp}(n|2)$ with coefficients in $\mathfrak{F}_\lambda^n$. In this paper, we investigate the differential cohomology spaces

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2), \mathfrak{s}(2|n); \mathfrak{F}_\lambda^n).$$

3. The main results

The principal results of this paper are stated in the following two theorems:

Theorem 3.1. *The structure of the space* $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2), \mathfrak{s}(2|n); \mathfrak{F}_\lambda^n)$ *is as* *follows:*

$$d_\lambda^n = \dim(\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2), \mathfrak{s}(2|n); \mathfrak{F}_\lambda^n)) = \begin{cases} 2 & \textit{if } (n, \lambda) = (2, 0), \\ 1 & \textit{if } \begin{cases} n = 0 \textit{ and } \lambda \in \{0,\ 1\}, \\ n = 1 \textit{ and } \lambda \in \{0,\ \frac{1}{2}\}, \\ n \geq 3 \textit{ and } \lambda = 0, \end{cases} \\ 0 & \textit{otherwise.} \end{cases}$$

The nontrivial space $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2), \mathfrak{s}(2|n); \mathfrak{F}_\lambda^n)$ *admits a basis consisting of the cohomology classes of the* 1*-cocycles collected in Table* 1*.*

$(n,\ \lambda)$	1-cocycles
$(n,\ 0)$	$\Delta_\lambda^n(X_F) = F'$
$(0,\ 1)$	$\Delta_1^0(X_F) = F'' dx^1$
$(1,\ \frac{1}{2})$	$\Delta_{\frac{1}{2}}^1(X_F) = \bar{D}_1(F')\alpha_1^{\frac{1}{2}}$
$(2,\ 0)$	$\overline{\Delta}_0^2(X_F) = \bar{D}_1\bar{D}_2(F)$

Table 1

Theorem 3.2. *The space* $\mathrm{H}^1(\mathrm{PC}(n \mid 2); \mathfrak{F}_\lambda^n)$ *has the following structure:*

$$D_\lambda^n = \dim(\mathrm{H}^1(\mathrm{PC}(n \mid 2); \mathfrak{F}_\lambda^n)) = \begin{cases} 2 & \textit{if } n = 2 \textit{ and } \lambda = 0, \\ 1 & \textit{if } \begin{cases} n = 0 \textit{ and } \lambda \in \{0,\ 1\}, \\ n = 1 \textit{ and } \lambda \in \{0,\ \frac{1}{2}\}, \\ n \geq 3 \textit{ and } \lambda = 0, \end{cases} \\ 0 & \textit{otherwise.} \end{cases}$$

The nontrivial space $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{PC}(n \mid 2); \mathfrak{F}_\lambda^n)$ *admits a basis consisting of the cohomology classes represented by the* 1*-cocycles collected in Table* 2*, where the last identity is valid for any* $i = 1, \ldots, n$*.*

(n,λ)	1-cocycles: $\Xi_1^0(\Phi)\alpha_n^\lambda$
$(0,1)$	$A_1^0(\Phi)dx = \frac{\Phi''}{\Phi'}dx$
$(1,\frac{1}{2})$	$A^1_{\frac{1}{2}}(\Phi)\alpha_1^{\frac{1}{2}} = \frac{DE_\Phi}{E_\Phi}\alpha_1^{\frac{1}{2}}$
$(2,0)$	$A_0^2(\Phi)\alpha_2^0 = \frac{DE_\Phi}{E_\Phi}$
$(n,0)$	$\mathcal{E}_0^n(\Phi) = \mathrm{Log}(E_\Phi) = \mathrm{Log}(D_i\psi)^2$

Table 2

4. Relationship between $\mathrm{H}^1(\mathfrak{osp}(n-1|2), \mathfrak{s}(2|n-1); \mathfrak{F}_\lambda^n)$ and $\mathrm{H}^1(\mathfrak{osp}(n|2), \mathfrak{s}(2|n); \mathfrak{F}_\lambda^n)$

Consider a Lie superalgebra $\mathfrak{A} = \mathfrak{p} \oplus \mathfrak{q}$ where $\mathfrak{p}$ is a subalgebra and $\mathfrak{q}$ is a module over $\mathfrak{p}$ satisfying $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{p}$. Let $\Delta \in \mathrm{Z}^1(\mathfrak{A}, V)$, where V is a $\mathfrak{A}$-module. The cocycle relation is expressed as

$$\Delta([g,h]) - (-1)^{|g||\Delta|} g\cdot\Delta(h) + (-1)^{|h|(|g|+|\Delta|)} h\cdot\Delta(g) = 0, \quad g,h\in\mathfrak{A}. \tag{4.1}$$

Denote $\Delta_\mathfrak{p} = \Delta_{|\mathfrak{p}}$ and $\Delta_\mathfrak{q} = \Delta_{|\mathfrak{q}}$. Obviously, $\Delta_\mathfrak{p}$ is a 1-coycle over $\mathfrak{p}$ and if $\Delta_\mathfrak{p} = 0$ then $\Delta_\mathfrak{q}$ is $\mathfrak{p}$-invariant. Thus, the space $\mathrm{H}^1(\mathfrak{A}, V)$ is closely related to the space $\mathrm{H}^1(\mathfrak{p}, V)$. Moreover, $\Delta_\mathfrak{p}$ and $\Delta_\mathfrak{q}$ are constrained by the following equations:

$$\Delta_\mathfrak{q}([p,q]) - (-1)^{|p||\Delta|} p\cdot\Delta_\mathfrak{q}(q) + (-1)^{|q|(|p|+|\Delta|)} q\cdot\Delta_\mathfrak{p}(p) = 0 \quad \text{for any } p\in\mathfrak{p},\ q\in\mathfrak{q}, \tag{4.2}$$

$$\Delta_\mathfrak{p}([q,q']) - (-1)^{|q||\Delta|} q\cdot\Delta_\mathfrak{p}(p') + (-1)^{|q'|(|q|+|\Delta|)} q'\cdot\Delta_\mathfrak{q}(q) = 0 \quad \text{for any } q,\ q'\in\mathfrak{q} \tag{4.3}$$

In the present setting, we take $\mathfrak{A} = \mathfrak{osp}(n|2)$, $\mathfrak{p} = \mathfrak{osp}(n-1|2)$, $\mathfrak{q} = \Pi(\mathcal{N}_{n-1})$ and $V = \mathfrak{F}_\lambda^n$. We recall that the adjoint $\mathfrak{k}(n)$-module, is isomorphic to $\mathfrak{F}_{-1}^n$. Hence, the isomorphism (2.9) of $\mathfrak{osp}(n-1|2)$-modules induces the following $\mathfrak{osp}(n-1|2)$-isomorphism:

$$\mathfrak{osp}(n|2) \simeq \mathfrak{osp}(n-1|2) \oplus \Pi(\mathcal{N}_{n-1}),$$

where $\mathrm{H} \subset \mathfrak{F}^{n-1}_{-\frac{1}{2}}$. To be more precise, every element X_F admits a decomposition of the form $X_F = X_{F_1} + X_{F_2\xi_n}$ where $\partial_n F_1 = \partial_n F_2 = 0$, and then

$X_{F_1} \in \mathfrak{osp}(n-1|2)$ and $X_{F_2\xi_n}$ is identified with $\Pi(F_2\alpha_{n-1}^{-\frac{1}{2}}) \in \Pi(\mathcal{N}_{n-1})$. Moreover, it follows immediately that

$$[\mathfrak{osp}(n-1|2), \Pi(\mathcal{N}_{n-1})] \subset \Pi(\mathcal{N}_{n-1}) \quad \text{and} \quad [\Pi(\mathcal{N}_{n-1}), \Pi(\mathcal{N}_{n-1})] \subset \mathfrak{osp}(n-1|2).$$

To begin the proof of Theorem 3.1, we first compute

$$\mathrm{H}^1(\mathfrak{osp}(n-1|2), \mathfrak{so}(2|n-1), \mathfrak{F}_\lambda^n).$$

Using the isomorphism (2.9), we observe that the knowledge of

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2), \mathfrak{so}(2|n), \mathfrak{F}_\lambda^n)$$

makes it possible to compute $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2), \mathfrak{so}(2|n); \mathfrak{F}_\lambda^n)$:

$$\begin{aligned} \mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n-1|2), \mathfrak{so}(2|n-1), \mathfrak{F}_\lambda^n) &\simeq \mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n-1|2), \mathfrak{so}(2|n-1); \mathfrak{F}_\lambda^{n-1}) \\ &\oplus \mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n-1|2), \mathfrak{so}(2|n-1); \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^{n-1})). \end{aligned} \tag{4.4}$$

It follows that we can determine the structure of

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n-1|2), \mathfrak{so}(2|n-1), \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^{n-1}))$$

from $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n-1|2), \mathfrak{so}(2|n-1), \mathfrak{F}_\lambda^{n-1})$. In fact, for any

$$\daleth \in Z^1_{\mathrm{diff}}(\mathfrak{osp}(n-1|2), \mathfrak{so}(2|n-1), \mathfrak{F}_\lambda^{n-1})$$

corresponds $\widetilde{\daleth} \in Z^1_{\mathrm{diff}}(\mathfrak{osp}(n-1|2), \mathfrak{so}(2|n-1), \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^{n-1}))$ whith $\widetilde{\daleth}(X_G) = \Pi\left(\daleth(X_G)\right)$. It follows that $\daleth$ is a coboundary if and only if $\widetilde{\daleth}$ is a coboundary.

4.1. The space $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathfrak{so}(2), \mathcal{F}_\lambda)$

We determine the space

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathfrak{so}(2), \mathcal{F}_\lambda),$$

and obtain the following result:

Theorem 4.1.

$$\dim(\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathfrak{so}(2), \mathcal{F}_\lambda)) = \begin{cases} 1 & \text{if } \lambda \in \{0, 1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

The nontrivial spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathfrak{so}(2), \mathcal{F}_\lambda))$ *admit as generators the cohomology classes of the* 1*-cocycles* Δ_λ^0 *given by:*

$$\Delta_0^0(X_f) = f' \qquad \text{and} \qquad \Delta_1^0(X_f) = f'' dx.$$

Proof. Every 1-cocycle $\Delta_\lambda^0 \in \mathrm{Z}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{F}_\lambda)$ is represented by a 1-cochain $\Delta_\lambda^0(X_h) = \sum_{1\leq i\leq 2}\gamma_i h^{(i)}(dx)^\lambda$, where γ_i are, a priori, arbitrary functions; however, from the relation $\delta(\Delta_\lambda^0)(X_1, X_{x^2}) = 0$, it follows immediately that $\frac{d}{dx}\gamma_i(x) = 0$. Thus, from equation (4.1), we obtain

$$\Delta_\lambda^0(X_f) = \begin{cases} (\gamma_1 f' + \gamma_2 f'')dx, & \text{if } \lambda = 1, \\ \gamma_1 f' dx^\lambda, & \text{if } \lambda \neq 1. \end{cases}$$

where γ_0, γ_1 and γ_2 are constants.

We now consider the 1-cocycles

$$\begin{array}{rcl} \Delta_\lambda^0 : \mathfrak{sl}(2) & \longrightarrow & \mathcal{F}_\lambda \\ X_f & \longmapsto & f' dx^\lambda, \end{array} \qquad \begin{array}{rcl} \Delta_1^0 : \mathfrak{sl}(2) & \longrightarrow & \mathcal{F}_1 \\ X_f & \longmapsto & f'' dx. \end{array}$$

We obtain that

$$\Delta_\lambda^0(X_f) = \delta(g dx^\lambda) \qquad \text{if } \lambda \neq 0$$

where $g(x) = 1$. Furthermore, we establish that Δ_1^0 is a nontrivial 1-cocycle, while Δ_0^0 is nontrivial when $\lambda = 0$. ∎

4.2. The space $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(1|2), \mathfrak{s}(2|1); \mathfrak{F}_\lambda^1)$ The main result in this subsection is the following:

Theorem 4.2.

$$\dim(\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(1|2), \mathfrak{s}(2|1); \mathfrak{F}_\lambda^1)) = \begin{cases} 1 & \text{if } \lambda \in \{0, \frac{1}{2}\}, \\ 0 & \text{otherwise.} \end{cases}$$

The nontrivial cohomology spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(1|2), \mathfrak{s}(2|1); \mathfrak{F}_\lambda^1)$ *admit the following* 1*-cocycles as a spanning set:*

$$\Delta_0^1(X_F) = F' \qquad \text{and} \qquad \Delta_{\frac{1}{2}}^1(X_F) = D(F')\alpha^{\frac{1}{2}}.$$

Proof. With respect to the $\mathbb{Z}_2$-grading, any 1-cocycle

$$\Delta_\lambda^1 \in \mathrm{Z}^1_{\mathrm{diff}}(\mathfrak{osp}(1|2), \mathfrak{s}(2|1); \mathfrak{F}_\lambda^1)$$

admits a decomposition $\Delta_\lambda^1 = \Delta_{\bar{0}} + \Delta_{\bar{1}}$ where $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$ denote its even and odd parts, respectively. Clearly, Δ_λ^1 is a 1-cocycle if and only if both components $\Delta_{\bar{0}}^1$ and $\Delta_{\bar{1}}^1$ is a 1-cocycle. Accordingly, we distinguish between the two cases: Δ_λ^1 is even or Δ_λ^1 is odd.

1) Suppose that Δ_λ^1 is even, and write

$$\Delta_\lambda^1(X_F) = \left(\sum_{1\leq i\leq 2} \beta_i f_0^{(i)} + \xi \sum_{0\leq i\leq 1} \gamma_i f_1^{(i)}\right)\alpha_1^\lambda$$

where $f_0,\ f_1 \in C^\infty(\mathbb{R})$ and $F = f_0+\xi f_1$. The coefficients β_i and γ_i are initially arbitrary functions; however, as in the previous case, they are necessarily constant. A direct computation then shows that equation (4.1) holds only for $\lambda = 0$. Furthermore, modulo coboundaries and scalar multiples, Δ_λ^1 is determined by $\Delta_0^1(X_F) = F'$. We then prove that Δ_0^1 is nontrivial.

2) Suppose that Δ_λ^1 is odd. Arguing as in the first case, we prove that it defines a 1-cocycle only for $\lambda \in \{-\frac{1}{2}, \frac{1}{2}\}$. Furthermore, modulo coboundaries and scalar multiples, we obtain

$$\Delta_{\frac{1}{2}}^1(X_F) = D(F')\alpha^{\frac{1}{2}} \qquad \text{and} \qquad \Delta_{-\frac{1}{2}}^1(X_F) = (1-(-1)^{p(F)})D(F)\alpha^{-\frac{1}{2}}.$$

We show that $\Delta_{-\frac{1}{2}}^1(X_F) = \delta(\xi\alpha^{-\frac{1}{2}})$ and $\Delta_{\frac{1}{2}}^1(X_F)$ is nontrivial. ∎

4.3. The space $\mathrm{H}_{\mathrm{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{s}(2|2); \mathfrak{F}_\lambda^2)$ We now state the main result of this subsection:

Proposition 4.1.

$$\dim(\mathrm{H}^1(\mathfrak{osp}(2|2), \mathfrak{s}(2|2); \mathfrak{F}_\lambda^2)) = \begin{cases} 2 & \text{if } \lambda = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

The space $\mathrm{H}_{\mathrm{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{s}(2|2); \mathfrak{F}_0^2)$ *is generated by the cohomology classes represented by the* 1*-cocycles* Δ_0^2 *and* $\bar{\Delta}_0^2$ *defined as follows:*

$$\Delta_0^2(X_F) = F' \qquad \text{and} \qquad \bar{\Delta}_0^2(X_F) = (-1)^{p(F)} D_1 D_2(F).$$

The proof of Proposition 4.1 is presented in Subsection 4.3.2. As a preliminary step, we first describe the spaces $\mathrm{H}^1(\mathfrak{osp}(1|2), \mathfrak{s}(2|1); \mathfrak{F}_\lambda^2)$.

4.3.1. Relationship between $\mathrm{H}_{\mathrm{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{s}(2|1); \mathfrak{F}_\lambda^2)$ and $\mathrm{H}_{\mathrm{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{s}(2|2); \mathfrak{F}_\lambda^2)$ In view of (2.9), it follows that the first cohomology space $\mathrm{H}^1(\mathfrak{osp}(1|2), \mathfrak{s}(2|1); \mathfrak{F}_\lambda^2)$ is closely related to $\mathrm{H}^1(\mathfrak{osp}(1|2), \mathfrak{s}(2|1); \mathfrak{F}_\lambda^1)$:

$$\mathrm{H}^1(\mathfrak{osp}(1|2), \mathfrak{s}(2|1); \mathfrak{F}_\lambda^2) \simeq \\ \simeq \mathrm{H}^1(\mathfrak{osp}(1|2), \mathfrak{s}(2|1); \mathfrak{F}_\lambda^1) \oplus \mathrm{H}^1\left(\mathfrak{osp}(1|2), \mathfrak{s}(2|1); \Pi\left(\mathfrak{F}_{\lambda+\frac{1}{2}}^1\right)\right). \tag{4.7}$$

Thus, we obtain the following result:

Proposition 4.2.

$$\dim(\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(1|2),\mathfrak{s}(2|1);\mathfrak{F}^2_\lambda))=\begin{cases}2 & \text{if } \lambda=0,\\ 1 & \text{if } \lambda\in\{-\frac{1}{2},\frac{1}{2}\},\\ 0 & \text{otherwise.}\end{cases}$$

The following 1-cochains define nontrivial 1-cocycles that generate the corresponding spaces:

$$\Lambda_0(X_{F_1})=F_1',\qquad \overline{\Lambda}_0(X_{F_1})=D_1(F_1)\xi_2,$$
$$\Lambda_{-\frac{1}{2}}(X_{F_1})=F_1'\xi_2\alpha^{-\frac{1}{2}},\qquad \Lambda_{\frac{1}{2}}(X_{F_1})=D_1^3(F_1)\alpha^{\frac{1}{2}}.$$

We recall that the adjoint $\mathfrak{k}(n)$-module, is isomorphic to $\mathfrak{F}^n_{-1}$. Consequently, (2.9) gives rise to the following $\mathfrak{osp}(1|2)$-isomorphism:

$$\mathfrak{osp}(2|2)\simeq\mathfrak{osp}(1|2)\oplus\Pi(\mathcal{N}_1),$$

where $\mathcal{N}_1\subset\mathfrak{F}^1_{-\frac{1}{2}}$ denotes the subspace spanned by $\{\xi_1\alpha_1^{-\frac{1}{2}},\, x\alpha_1^{-\frac{1}{2}},\, \alpha_1^{-\frac{1}{2}}\}$. More precisely, any element X_F admits a decomposition $X_F=X_{F_1}+X_{F_2\xi_2}$ where $\partial_2F_1=\partial_2F_2=0$, and then $X_{F_1}\in\mathfrak{osp}(1|2)$ and $X_{F_2\xi_2}$ is identified with $\Pi(F_2\alpha_1^{-\frac{1}{2}})\in\Pi(\mathcal{N}_1)$. Moreover, one readily sees that

$$[\mathfrak{osp}(1|2),\Pi(\mathcal{N}_1)]\subset\Pi(\mathcal{N}_1)\qquad\text{and}\qquad[\Pi(\mathcal{N}_1),\Pi(\mathcal{N}_1)]\subset\mathfrak{osp}(1|2).\tag{4.8}$$

Lemma 4.1. *A* 1*-cocycle* $\Delta^n_\lambda\in\mathrm{Z}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2),\mathfrak{s}(2|n);\mathfrak{F}^n_\lambda)$ *is a coboundary precisely when its restriction to* $\mathfrak{osp}(n-1|2)$ *is a coboundary.*

Proof. If Δ^n_λ is a coboundary in $\mathfrak{osp}(n|2)$, then its restriction to $\mathfrak{osp}(n-1|2)$ remains a coboundary in $\mathfrak{osp}(n-1|2)$. Hence, assume that $\Delta^n_\lambda|_{\mathfrak{osp}(n-1|2)}$ is a coboundary, meaning that there exists a cochain $\Psi(x,\xi)\alpha_n^\lambda\in\mathfrak{F}^n_\lambda$ such that

$$\Delta^n_\lambda(X_{F_1})-\delta(\Psi(x,\xi)\alpha_n^\lambda)=0\qquad\text{for all } X_{F_1}\in\mathfrak{osp}(n-1|2).$$

By replacing Δ^n_λ by $\Delta^n_\lambda-\delta(\Psi)$, we can assume, without loss of generality, that its restriction to Δ^n_λ is zero. But in this case, by (4.8), the 1-cocycle condition takes the form:

$$\begin{aligned}\Delta^n_\lambda([X_{F_1},X_{H\xi_n}])&=\mathfrak{L}^\lambda_{X_{F_1}}\Delta^n_\lambda(X_{H\xi_n}),\\ (-1)^{|H_1\xi_n||H_2\xi_n|}\mathfrak{L}^\lambda_{X_{H_2\xi_n}}\Delta^n_\lambda(X_{H_2\xi_n})&=\mathfrak{L}^\lambda_{X_{H_1\xi_n}}\Delta^n_\lambda(X_{H_2\xi_n}),\end{aligned}\tag{4.9}$$

where X_{F_1}, $X_{F_2} \in \mathfrak{osp}(n-1|2)$ and $H\alpha_{n-1}^{-\frac{1}{2}}$, $H_1\alpha_{n-1}^{-\frac{1}{2}}$, $H_2\alpha_{n-1}^{-\frac{1}{2}} \in \mathcal{N}_1$. From the first equation in (4.9), it follows that

$$\Delta_\lambda^n(X_{H\xi_n}) = \begin{cases} \varepsilon_1 H\xi_n\alpha_n^{-1} & \text{if } \lambda = -1, \\ \varepsilon_2 H\alpha_n^{-\frac{1}{2}} & \text{if } \lambda = -\frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ and $H\alpha_{n-1}^{-\frac{1}{2}} \in \mathrm{H}$. From the second part of (4.9), it follows that $\varepsilon_1 = 0$, and hence $\Delta_{-1}^n \equiv 0$.

However, in the case $\lambda = -\frac{1}{2}$, we prove that

$$\Delta_{-\frac{1}{2}}^n(X_F) = \delta(\xi_n).$$

∎

The result follows from the following lemma, which completes the proof of Proposition 4.1.

Lemma 4.2. *Every 1-cocycle $\Delta \in Z^1_{\mathrm{diff}}(\mathfrak{osp}(n|2), \mathfrak{s}(2|n); \mathfrak{F}_\lambda^n)$ up to a coboundary, can be written in the following general form:*

$$\Delta_\lambda^n(X_F) = \sum \varepsilon_{k_1\cdots k_3} \bar{D}_1^{k_1} \cdots \bar{D}_n^{k_n}(F)\alpha_n^\lambda,$$

where the coefficients $\varepsilon_{k_1\cdots k_3}$ depend only on the variables ξ_i, and are independent of x.

4.3.2. Proof of Proposition 4.1 Let $\Delta_\lambda^2 \in \mathrm{Z}^1(\mathfrak{osp}(2|2), \mathfrak{s}(2|2); \mathfrak{F}_\lambda^2)$ be a homogeneous 1-cocycle.

1) Assume that Δ_λ^2 is even. By Proposition 4.2 and Lemma 4.2, the 1-cocycle Δ_λ^2 is a coboundary whenever $\lambda \neq 0$. In the case $\lambda = 0$ it follows that, up to a coboundary, we have

$$\Delta_0^2|_{\mathfrak{osp}(1|2)} = \gamma_1\Lambda_0 + \gamma_2\overline{\Lambda}_0, \quad \gamma_1, \gamma_2 \in \mathbb{R}.$$

To determine Δ_0^2, completely, it suffices to compute $\Delta_0^2(X_{H\xi_2})$, where $H\alpha_1^{-\frac{1}{2}} \in \mathcal{N}_{n-1}$. However, we have

$$\Delta_0^2[X_{F_1}, X_{H\xi_2}] - (-1)^{p(F)p(H\xi_2)}\mathfrak{L}^\lambda_{X_{H\xi_2}}\Delta_0^2(X_{F_1}) = \mathfrak{L}^\lambda_{F_1}\Delta_0^2(X_{H\xi_2}),$$

$$\Delta_0^2[X_{H_1\xi_2}, X_{H_2\xi_2}] - (-1)^{p(H_1\xi_2)p(H_2\xi_2)}\mathfrak{L}^\lambda_{X_{H_2\xi_2}}\Delta_0^2(X_{H_2\xi_2}) = \mathfrak{L}^\lambda_{X_{H_1\xi_2}}\Delta_0^2(X_{H_2\xi_2}),$$

where $X_{F_1} \in \mathfrak{osp}(1|2)$ and $H\alpha_1^{-\frac{1}{2}}$, $H_1\alpha_1^{-\frac{1}{2}}$, $H_2\alpha_1^{-\frac{1}{2}} \in \mathcal{N}_{n-1}$. Moreover, Lemma 4.2 gives the general form of Δ_0^2, therefore, we get

$$\Delta_0^2(X_F) = \gamma_1 F' + \gamma_2(-1)^{p(F)} D_1 D_2(F),$$

where $\gamma_1,\, \gamma_2 \in \mathbb{R}$.

2) Now, assume that Δ_λ^2 is a non-trivial odd 1-cocycle. As in the previous case, we only need to consider $\lambda \in \{-\frac{1}{2}, \frac{1}{2}\}$. By the same arguments, we show that, in each case, $\Delta_\lambda^2 = 0$.

Corollary 4.1.

$$\dim(\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(2|2), \mathfrak{s}(2|2); \mathfrak{F}^3_\lambda)) = \begin{cases} 2 & \text{if } \lambda \in \{-\frac{1}{2}, 0\}, \\ 0 & \text{otherwise.} \end{cases}$$

The spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(2|2), \mathfrak{s}(2|2); \mathfrak{F}^3_\lambda)$ *are generated by the cohomology classes of the following 1-cocycles:*

$$\Omega_{-\frac{1}{2}}(X_F) = F'\xi_3\alpha_3^{-\frac{1}{2}}, \qquad \overline{\Omega}_{-\frac{1}{2}}(X_F) = \bar{D}_1\bar{D}_2(F)\xi_3\alpha_3^{-\frac{1}{2}},$$
$$\Omega_0(X_F) = F', \qquad \overline{\Omega}_0(X_F) = \bar{D}_1\bar{D}_2(F).$$

Proof. The isomorphism (2.9) induces the following isomorphism between cohomology spaces:

$$\mathrm{H}^1_{\mathrm{diff}}\left(\mathfrak{osp}(2|2), \mathfrak{s}(2|2); \mathfrak{F}^3_\lambda\right) \simeq$$
$$\simeq \mathrm{H}^1_{\mathrm{diff}}\left(\mathfrak{osp}(2|2), \mathfrak{s}(2|2); \mathfrak{F}^2_\lambda\right) \oplus \mathrm{H}^1_{\mathrm{diff}}\left(\mathfrak{osp}(2|2), \mathfrak{s}(2|2); \Pi(\mathfrak{F}^2_{\lambda+\frac{1}{2}})\right).$$

Thus, we deduce the structure of $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(2|2), \mathfrak{s}(2|2); \mathfrak{F}^3_\lambda)$. ∎

5. Proof of main Theorem 3.1

Consider a 1-cocycle $\Delta_\lambda^n \in \mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2), \mathfrak{s}(2|n); \mathfrak{F}^n_\lambda)$. If the restriction $\Delta_\lambda^n|_{\mathfrak{osp}(n-1|2)}$ is a trivial 1-cocycle, its form is completely determined by Lemma 4.1.

Now assume that the restriction $\Delta_\lambda^n|_{\mathfrak{osp}(n-1|2)}$, is non-trivial. We then distinguish the following cases:

1) The case $n \leq 2$, has already been treated in Section 4.

2) The case $n \geq 3$. Of course, up to coboundary, the general form of $\Delta_\lambda^n|_{\mathfrak{osp}(n-1|2)}$, is determined by $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n-1|2), \mathfrak{s}(2|n-1); \mathfrak{F}^{n-1}_\lambda)$, together

with the isomorphism (4.4). Moreover, $\Delta^n_\lambda|_{\Pi(\mathcal{N}_{n-1})}$, can be essentially descibed by equations (4.2), (4.3) and using arguments similar to those of the proof of Lis essentially described by equations (4.2) and (4.3), by arguments analogous to those used in the proof of Lemma 4.2. Hence, we distinguish the following cases:

(i) The case $n=3$.

By Corollary 4.1, the restriction, $\Delta^3_\lambda|_{\mathfrak{osp}(2|2)}$, is nontrivial for $\lambda\in\{-\frac{1}{2},0\}$. It follows that, up to a coboundary, the non-zero restrictions of the cocycle Δ^3_λ on $\mathfrak{osp}(2|2)$ are given by:

$$\Delta^3_\lambda|_{\mathfrak{osp}(2|2)}(X_{F_1})=\begin{cases}\gamma_1\Omega_{-\frac{1}{2}}(X_{F_1})+\gamma_2\overline{\Omega}_{-\frac{1}{2}}(X_{F_1}) & \text{if } \lambda=-\frac{1}{2},\\ \gamma_1\Omega_0(X_{F_1})+\gamma_2\overline{\Omega}_0(X_{F_1}) & \text{if } \lambda=0,\end{cases}$$

where the coefficients γ_1,γ_2 are constants. Now, applying Lemma 4.2, we obtain

$$\Delta^3_\lambda(X_F)=\sum_{k_1+k_2+k_3\le 6}\varepsilon_{k_1k_2k_3}\bar{D}_1^{k_1}\bar{D}_2^{k_2}\bar{D}_3^{k_3}(F)\alpha_3^\lambda.$$

In each case, we solve equations (4.2) and (4.3) with respect to γ_i, $\varepsilon_{k_1k_2k_3}$.

We obtain

a) For $\lambda=-\frac{1}{2}$, the coefficient γ_i vanishes; so, by lemma 4.1, $\Delta^3_{-\frac{1}{2}}$ is a coboundary. Hence $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(3|2),\mathfrak{s}(2|3);\mathfrak{F}^3_{-\frac{1}{2}})=0$.

b) For $\lambda=0$, the coefficients $\gamma_1\neq 0$, $\gamma_2=0$ and, up to a coboundary, Δ^3_0 is equal γ_1F', see Theorem 3.1. Hence $\dim(\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(3|2),\mathfrak{s}(2|3);\mathfrak{F}^3_0))=1$.

(ii) We proceed by induction on n. Similarly to (i), the result holds for $n=4$. We now assume that it holds for some $n\geq 4$. Once again, by the same arguments as in Lemmas 4.1 and 4.2, combined with the induction hypothesis, we obtain that $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2),\mathfrak{s}(2|n);\mathfrak{F}^{n+1}_\lambda)=0$ if $\lambda\notin\{-\frac{1}{2},0\}$. It remains to consider only the cases $\lambda\in\{-\frac{1}{2},0\}$. As in (i), we deduce the result for $n+1$. Finally, $\dim(\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2),\mathfrak{s}(2|n);\mathfrak{F}^n_\lambda))=1$, for $n\geq 3$ and span by $\Delta^n_0(X_F)=F'$. ∎

6. Proof of main Theorem (3.2)

The van Est cohomology ring for Lie group is defined via the follwing isomorphism (cf. [13]):

$$H^*(\mathrm{Diff}_+(S^1),\mathfrak{F}^0_\lambda)\simeq H^*(\mathrm{Vec}(S^1),\mathfrak{so}(2);\mathfrak{F}^0_\lambda),$$

where $\mathfrak{so}(2)\subset\mathrm{Vec}(S^1)$ is subalgebra of $\mathrm{Vec}(S^1)$ and $\mathrm{Lie}(\mathfrak{so}(2))$ is the maximal compact subgroup of rotations of S^1. According to Fuchs, we have $d^n_\lambda = D^n_\lambda$. In order to compute the cohomology space $\mathrm{H}^1(\mathrm{PC}(n|2);\mathfrak{F}^n_\lambda)$, it suffices to determine the cohomology space $\mathrm{H}^1(\mathfrak{osp}(n|2),\mathfrak{s}(2|n);\mathfrak{F}^n_\lambda)$.

In Section 3, the author determines the space $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2),\mathfrak{s}(2|n);\mathfrak{F}^n_\lambda)$. The result is as follows:

$$d^n_\lambda = \dim(\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2),\mathfrak{s}(2|n);\mathfrak{F}^n_\lambda)).$$

It follows that

$$D^n_\lambda = \begin{cases} 2 & \text{if } n=2 \text{ and } \lambda=0,\\ 1 & \text{if } \begin{cases} n=0 \text{ and } \lambda\in\{0,\,1\},\\ n=1 \text{ and } \lambda\in\{0,\,\frac12\},\\ n\geq 3 \text{ and } \lambda=0,\end{cases}\\ 0 & \text{otherwise.}\end{cases}$$

For $n=0$, $D^0_\lambda = 2$, it is generated by the nontrivial cocycles: $\mathcal{E}^0_0(\Phi) = \mathrm{Log}(\Phi')$ and $A^0_1(\Phi)dx = \frac{\Phi''}{\Phi'}dx$. It is due to Fuks (see [13]). For, $n=1$, $D^1_\lambda = 2$, using the paper of Duval et al. [12], the authors proved that $\mathcal{E}^1_0(\Phi) = \mathcal{E}(\Phi)$ is a nontrivial 1-cocycle of supergroup $\mathrm{PC}(2|1)$ in $\mathfrak{F}^1_0$ and $A^1_{\frac12}(\Phi)\alpha_1^{\frac12} = \frac{DE_\Phi}{E_\Phi}\alpha_1^{\frac12}$ is a nontrivial 1-cocycle of supergroup $\mathrm{PC}(2|1)$ in $\mathfrak{F}^1_{\frac12}$. Similarly, for $n=2$, we obtain $D^2_\lambda = 2$, which is generated by $\mathcal{E}^2_0$ and A^2_0. In the same way for $n\geq 3$, we obtain $D^n_\lambda = 1$ it is generated $\mathcal{E}^n_0$.

Thus, a base of the cohomology ring $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{PC}(n\mid 2);\mathbb{F}^n_\lambda)$ is given by the cohomology classes of the 1-cocycles which are collected in Table 3, where the above equality is satisfied for every $i=1,\dots,N$. ∎

(n,λ)	1-cocycles: $\Xi^0_1(\Phi)\alpha^\lambda_n$
$(0,1)$	$A^0_1(\Phi)dx = \frac{\Phi''}{\Phi'}dx$
$(1,\frac12)$	$A^1_{\frac12}(\Phi)\alpha_1^{\frac12} = \frac{DE_\Phi}{E_\Phi}\alpha_1^{\frac12}$
$(2,0)$	$A^2_0(\Phi)\alpha^0_2 = \frac{DE_\Phi}{E_\Phi}$
$(n,0)$	$\mathcal{E}^n_0(\Phi) = \mathrm{Log}(E_\Phi) = \mathrm{Log}(D_i\psi)^2,$

Table 3

7. Conclusion and Open Problems

We study the standard contact structure on the supercircle $S^{1|n}$, and the associated supergroup of contactomorphisms.

7.1. Conclusion The principal result of this paper consists in computing the first cohomology groups of the orthosymplectic Lie group $\mathrm{PC}(n|2)$ with values in the module $\mathfrak{F}_\lambda^n$. We prove that the nontrivial spaces $\mathrm{H}^1(\mathrm{PC}(n|2), \mathfrak{F}_\lambda^n)$ arise only for specific values of λ. More precisely, let

$$D_\lambda^n = \dim(\mathrm{H}^1(\mathrm{PC}(n|2), \mathfrak{F}_\lambda^n)).$$

We establish that

$$D_\lambda^n = \begin{cases} 2 & \text{if } n = 2 \text{ and } \lambda = 0, \\ 1 & \text{if } \begin{cases} n = 0 \text{ and } \lambda \in \{0, 1\}, \\ n = 1 \text{ and } \lambda \in \{0, \frac{1}{2}\}, \\ n \geq 3 \text{ and } \lambda = 0, \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

7.2. Open Problems and research directions

- In [12], Duval Christin et al. compute the first cohomology of the Lie supergroups $\mathcal{K}(n)$ on the $(1, n)$-dimensional real superspace with coefficients in the superspace of weighted densities for $n \leq 2$. For $n > 2$, characterize the space $\mathrm{H}^1(\mathcal{K}(n); \mathfrak{F}_\lambda^n)$.
- In [17], V. Ovsienko, C. Roger calculated the space $\mathrm{H}^2(\mathrm{Diff}_+(S^1); \mathfrak{F}_\lambda^0)$. This space will be interpreted as an extension of Virasoro group by modules of tensor densities on S^1.
- Determine the spaces $\mathrm{H}^2(\mathcal{K}(n); \mathfrak{F}_\lambda^n)$ and $\mathrm{H}^2(\mathrm{PC}(n|2); \mathfrak{F}_\lambda^n)$.

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