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Some properties of (k, ψ)-Hadamard fractional integral

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Abstract: In this paper, we study the boundedness properties of the (k, ψ)-Hadamard fractional integral in various functional spaces, including $C[a, b]$, $C^1[a, b]$, and the weighted Lebesgue spaces $L^p_\omega(a, b)$. In addition, we establish several fundamental analytical results associated with this operator, such as a Mean Value Theorem and an Integration by Parts Theorem. These findings enhance the theoretical understanding of (k, ψ)-Hadamard type operators and provide a rigorous foundation for their application in the study of differential and integral equations exhibiting nonlocal behavior and variable-scaling effects.

Key words: (k, ψ)–Hadamard fractional integrals, weighted Lebesgue space, fractional integration by parts.

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1. Introduction

Fractional calculus, which deals with integrals and derivatives of arbitrary (non-integer) order, has become an increasingly active field of research due to its wide range of applications in mathematics, physics, engineering, and applied sciences, see for instance [11, 13, 15, 21, 30, 8, 22, 20, 23, 24, 1, 25, 19, 29, 7] and the references therein.

Among the various approaches to fractional integration, the *Hadamard fractional integral*, introduced by Jacques Hadamard in 1892 [12], occupies a distinguished place in the theory of fractional calculus. Unlike the classical Riemann-Liouville and Caputo formulations, the Hadamard integral is defined through a logarithmic kernel, which makes it particularly suitable for problems posed on unbounded domains or involving multiplicative scaling properties.

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While the Riemann-Liouville fractional derivative can be formally regarded as a fractional power $(\frac{d}{dx})^\alpha$ of the differentiation operator and is invariant under translation along the real axis, Hadamard [12] proposed a fractional power of the form $(x\,\frac{d}{dx})^\alpha$, which is invariant under dilation. This dilation invariance provides a natural framework for modeling phenomena where the underlying dynamics exhibit scale invariance rather than translational symmetry.

The Hadamard approach to fractional integration originates from a logarithmic generalization of the classical n-fold integral, given by

$$\begin{aligned} I^n_{a+}u(x) &= \int_a^x \frac{d\sigma_1}{\sigma_1}\int_a^{\sigma_1}\frac{d\sigma_2}{\sigma_2}\cdots\int_a^{\sigma_{n-1}} u(\sigma_n)\frac{d\sigma_n}{\sigma_n} \\ &= \frac{1}{(n-1)!}\int_a^x \left(\log\frac{x}{t}\right)^{n-1} u(t)\,\frac{dt}{t}. \end{aligned}$$

Motivated by this representation, Hadamard introduced the *left-sided fractional integral* of order $\alpha > 0$ for a function $u : [a,b] \to \mathbb{R}$, with $0 < a < b$, defined by

$$_H I^\alpha_{a+}u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{t}\right)^{\alpha-1}\frac{u(t)}{t}\,dt, \qquad x \in (a,b].$$

This operator generalizes the classical concept of integration to a logarithmic scale, thereby capturing the intrinsic multiplicative structure of many natural and physical processes. Consequently, the Hadamard fractional integral has proven particularly effective in describing phenomena governed by exponential growth, scale invariance, and self-similar dynamics across diverse scientific and engineering disciplines. For further developments and applications, see [4, 5, 6, 16, 17, 18, 26].

On the other hand, in 2015, Farid and Habibullah [10], inspired by the generalized k-Gamma and k-Beta functions introduced by Díaz and Pariguan [9], proposed a broader version of the Hadamard fractional integral. They defined the *k-Hadamard fractional integral* of order $\alpha > 0$ as follows:

$$^k_H I^\alpha_{a+}u(x) = \frac{1}{\Gamma_k(\alpha)}\int_a^x \left(\log\frac{x}{t}\right)^{\frac{\alpha}{k}-1}\frac{u(t)}{t}\,dt, \qquad x > a,$$

where $\Gamma_k(\cdot)$ denotes the k-Gamma function, given by

$$\Gamma_k(z) = \int_0^1 s^{z-1}e^{-\frac{s^k}{k}}\,ds.$$

In their work, the authors established several fundamental properties of this new operator, including the semigroup and commutativity laws, as well as boundedness results in appropriate function spaces, ensuring that the operator is mathematically well-posed and suitable for further analytical developments.

This extension provides a unifying framework that connects classical Hadamard fractional calculus with k-deformed structures, allowing a richer modeling of systems exhibiting non-standard growth, non-uniform scaling, or fractal-like dynamics. Consequently, the k-Hadamard fractional integral opens new possibilities for applications in physics, biology, and engineering, where generalized fractional models are increasingly relevant.

As is well known, fractional calculus has become an indispensable tool in the modeling of processes exhibiting non-locality, memory effects, and scale-invariant behavior. In the classical Hadamard framework, fractional integrals and derivatives are defined through logarithmic kernels, which ensure dilation invariance and make them particularly suitable for multiplicative-type dynamics. Nevertheless, many real-world problems require a broader formulation –for instance, when the underlying variable undergoes non-uniform scaling or when the kernel must adapt according to an auxiliary function. Motivated by these considerations, Balachandran et al. [3] recently introduced a new class of Hadamard-type fractional integrals with respect to another function, defined by

$${}_{H}I_{a+}^{\alpha;\psi}u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{\psi(x)}{\psi(t)}\right)^{\alpha-1}\frac{\psi'(t)}{\psi(t)}\,u(t)\,dt,$$

where $\psi \in C^1[a,b]$ is a strictly increasing function and u is a suitable function. The authors derive and rigorously prove several fundamental properties of these new operators, including the semigroup (composition) law, boundedness results in appropriate function spaces, and consistency with the classical Hadamard operator when the auxiliary function is the identity, i.e., when $\psi(x) = x$. This generalization provides greater flexibility for modeling phenomena characterized by variable scaling behaviors or non-uniform geometries.

Motivated by previous works and by the desire to introduce an additional layer of flexibility into the ψ-Hadamard fractional integral –specifically through a deformation parameter k that modifies the kernel's exponent structure– Khathem and Geem [14] recently proposed a novel k-fractional extension of the ψ-Hadamard fractional integral. For a suitable function ϕ and a parameter $k \in (0,\infty)$, the left (k,ψ)-Hadamard fractional integral of order $\alpha > 0$ is defined as

$$ {}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t) = \frac{1}{k\,\Gamma_k(\alpha)} \int_a^t \frac{\psi'(s)}{\psi(s)} \left(\log \frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} \phi(s)\,ds. $$

The main motivation of this generalization is to capture phenomena governed by more intricate scaling laws and to unify a broader class of fractional operators involving both function-dependent kernels and deformation parameters. The authors establish several analytical properties of these new operators, including semigroup (composition) laws, commutativity relations, and boundedness results in suitable function spaces, thus ensuring their well-posedness and analytical consistency. Furthermore, they provide illustrative examples showing how the proposed definitions reduce to well-known cases and how the deformation parameter k affects the operator's behavior, thereby demonstrating the modeling versatility of this extended framework.

Motivated by the aforementioned works, the main objective of this paper is to investigate the boundedness properties of the (k,ψ)-Hadamard fractional integral in various functional spaces such as $C[a,b]$, $C^1[a,b]$, and the weighted Lebesgue spaces $L^p_\omega(a,b)$. Furthermore, we aim to establish fundamental analytical results associated with this operator, including a Mean Value Theorem and an Integration by Parts Theorem. These results contribute to a deeper understanding of the analytical structure of (k,ψ)-Hadamard type operators and provide a theoretical foundation for their potential applications in differential and integral equations involving nonlocal and variable-scaling effects.

2. Well-posedness of the (k,ψ)-Hadamard fractional integral in classical function spaces such as $C[a,b]$, $AC[a,b]$, and $C^1[a,b]$

In this section, we investigate the regularizing properties of the (k,ψ)-Hadamard fractional integral when acting on functions with low regularity. More precisely, we prove that the operator is invariant on the space $AC[a,b]$ and exhibits a nonlocal smoothing effect on continuous functions, yielding fractional Hölder regularity.

These results highlight two fundamental features of the operator: its stability on classical function spaces such as $AC[a,b]$, and its intrinsic smoothing mechanism, quantitatively described through explicit Hölder-type estimates. Such properties play a central role in the qualitative analysis of the associated fractional models and provide a robust functional framework for the study of related fractional differential equations.

Before stating our main results, we recall the definitions of the k-Gamma and k-Beta functions. For $z \in \mathbb{C}$ with $\Re(z) > 0$ and $k > 0$, the k-Gamma function is defined by (see [9])

$$\Gamma_k(z) = \int_0^\infty s^{z-1} e^{-\frac{s^k}{k}}\, ds.$$

Similarly, for $z, w \in \mathbb{C}$ with $\Re(z) > 0$ and $\Re(w) > 0$, the k-Beta function is defined as

$$\beta_k(z,w) = \frac{1}{k}\int_0^1 s^{\frac{z}{k}-1}(1-s)^{\frac{w}{k}-1}\, ds.$$

The k-Gamma and k-Beta functions satisfy the following fundamental properties:

Proposition 2.1. *For all admissible values of $z, w \in \mathbb{C}$, the following hold:*

(i) $\Gamma_k(z+k) = z\,\Gamma_k(z)$;

(ii) $\Gamma_k(k) = 1$;

(iii) $\Gamma_k(z) = k^{\frac{z}{k}-1}\,\Gamma\big(\frac{z}{k}\big)$;

(iv) $\beta_k(z,w) = \frac{1}{k}\,\beta\big(\frac{z}{k}, \frac{w}{k}\big)$;

(v) $\beta_k(z,w) = \dfrac{\Gamma_k(z)\,\Gamma_k(w)}{\Gamma_k(z+w)}$.

Throughout this section, we assume the following hypothesis:

(H) Let $0 \le a < b \le \infty$ and $\psi \in C^1[a,b]$ be a strictly increasing function such that $\psi(t) > 0$ for all $t \in [a,b]$.

For a suitable function ϕ and $k \in (0,\infty)$, the left- and right-sided (k,ψ)-Hadamard fractional integrals of order $\alpha > 0$ are defined, respectively, by

$${}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t) = \frac{1}{k\,\Gamma_k(\alpha)} \int_a^t \frac{\psi'(s)}{\psi(s)} \left(\log \frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} \phi(s)\, ds$$

and

$${}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{b^-}\phi(t) = \frac{1}{k\,\Gamma_k(\alpha)} \int_t^b \frac{\psi'(s)}{\psi(s)} \left(\log \frac{\psi(s)}{\psi(t)}\right)^{\frac{\alpha}{k}-1} \phi(s)\, ds.$$

In this section, we establish several important properties of the (k,ψ)-Hadamard fractional integrals ${}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}$ and ${}_H^k\mathbf{J}_{b^-}^{\alpha;\psi}$, including well-posedness, boundedness, and an integration by parts formula, among others. For simplicity, the proofs are presented only for the left-sided operator ${}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}$, as analogous arguments apply to the right-sided case.

THEOREM 2.2. *Suppose that* $(\mathbf{H})$ *holds and let* $\alpha, k > 0$. *If* $\phi \in AC[a,b]$, *then* ${}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi \in AC[a,b]$. *Moreover*

$$\begin{aligned} {}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t) &= \frac{\phi(a)}{\Gamma_k(\alpha+k)}\left(\log\frac{\psi(t)}{\psi(a)}\right)^{\frac{\alpha}{k}} \\ &\quad + \frac{1}{\Gamma_k(\alpha+k)}\int_a^t \left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}} \phi'(s)\,ds. \end{aligned} \tag{2.1}$$

Proof. Since $\phi \in AC[a,b]$, it admits the representation

$$\phi(s) = \phi(a) + \int_a^s \phi'(\sigma)\,d\sigma, \qquad \text{with } \phi' \in L^1(a,b).$$

Substituting this expression into the definition of the fractional integral, we obtain

$$\begin{aligned} {}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t) &= \frac{1}{k\Gamma_k(\alpha)}\int_a^t \frac{\psi'(s)}{\psi(s)}\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\left(\phi(a) + \int_a^s \phi'(\sigma)\,d\sigma\right)ds \\ &= \frac{\phi(a)}{k\Gamma_k(\alpha)}\int_a^t \frac{\psi'(s)}{\psi(s)}\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} ds \\ &\quad + \frac{1}{k\Gamma_k(\alpha)}\int_a^t\int_a^s \frac{\psi'(s)}{\psi(s)}\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\phi'(\sigma)\,d\sigma\,ds. \end{aligned} \tag{2.2}$$

For the first integral, we perform the change of variables

$$\xi = \log\frac{\psi(t)}{\psi(s)}, \qquad d\xi = -\frac{\psi'(s)}{\psi(s)}ds,$$

which yields

$$\int_a^t \frac{\psi'(s)}{\psi(s)}\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} ds = \frac{k}{\alpha}\left(\log\frac{\psi(t)}{\psi(a)}\right)^{\frac{\alpha}{k}}.$$

For the second term, by applying Fubini's theorem, we interchange the order of integration to obtain

$$\begin{aligned}\int_a^t \int_a^s \frac{\psi'(s)}{\psi(s)} \left(\log \frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} \phi'(\sigma)\, d\sigma\, ds \\ &= \int_a^t \left(\int_\sigma^t \frac{\psi'(s)}{\psi(s)} \left(\log \frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} ds \right) \phi'(\sigma)\, d\sigma \\ &= \frac{k}{\alpha} \int_a^t \left(\log \frac{\psi(t)}{\psi(\sigma)}\right)^{\frac{\alpha}{k}} \phi'(\sigma)\, d\sigma.\end{aligned}$$

Substituting the above identities into (2.2), we obtain the desired representation (2.1).

Finally, observe that

$$\left(\log \frac{\psi(t)}{\psi(a)}\right)^{\frac{\alpha}{k}} = \frac{\alpha}{k} \int_a^t \frac{\psi'(s)}{\psi(s)} \left(\log \frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} ds,$$

which shows that the mapping

$$t \mapsto \left(\log \frac{\psi(t)}{\psi(a)}\right)^{\frac{\alpha}{k}}$$

belongs to $AC[a,b]$. Moreover, since $\phi' \in L^1(a,b)$ and the kernel

$$\sigma \mapsto \left(\log \frac{\psi(t)}{\psi(\sigma)}\right)^{\frac{\alpha}{k}}$$

is locally integrable, it follows that

$$\left(\log \frac{\psi(t)}{\psi(\cdot)}\right)^{\frac{\alpha}{k}} \phi'(\cdot) \in L^1(a,b).$$

Hence, the second term in (2.1) defines an absolutely continuous function as the primitive of an L^1-function. Consequently,

$${}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi \in AC[a,b].$$

∎

Remark 2.3. It should be noted that if ϕ is continuously differentiable that is $\phi \in C^1[a,b]$, then Theorem 2.2 holds.

THEOREM 2.4. *Suppose that* (**H**) *holds. Let* $\alpha \in (0,1)$, $k \in (1,\infty)$. *If* $\phi \in C[a,b]$, *then* ${}^{k}_{H}\mathbf{J}^{\alpha,\psi}_{a^+}\phi \in H^{\frac{\alpha}{k}}[a,b]$. *Furthermore, we have*

$$\left\| {}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi \right\|_{H^{\frac{\alpha}{k}}[a,b]} \le \mathscr{K}\|\phi\|_\infty, \tag{2.3}$$

where

$$\mathscr{K} = \frac{1}{\Gamma_k(\alpha+k)}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}}\left((b-a)^{\frac{\alpha}{k}} + 2^{\frac{\alpha}{k}} + \frac{\alpha}{k} + 1\right).$$

Proof. Taking $t_1, t_2 \in (a,b]$ such that $t_1 < t_2$, then

$$\left| {}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_2) - {}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_1) \right| \le \Pi_1 + \Pi_2, \tag{2.4}$$

where

$$\Pi_1 = \frac{1}{k\Gamma_k(\alpha)}\int_a^{t_1}\frac{\psi'(s)}{\psi(s)}\left|\left(\log\frac{\psi(t_2)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} - \left(\log\frac{\psi(t_1)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\right||\phi(s)|\,ds,$$
$$\Pi_2 = \frac{1}{k\Gamma_k(\alpha)}\int_{t_1}^{t_2}\frac{\psi'(s)}{\psi(s)}\left(\log\frac{\psi(t_2)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}|\phi(s)|\,ds.$$

To Π_2 the change of variable $\xi = \log\frac{\psi(t_2)}{\psi(s)}$ yields that

$$\begin{aligned}\Pi_2 &\le \frac{\|\phi\|_\infty}{k\Gamma_k(\alpha)}\int_{t_1}^{t_2}\frac{\psi'(s)}{\psi(s)}\left(\log\frac{\psi(t_2)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}ds \\ &\le \frac{\|\phi\|_\infty}{\Gamma_k(\alpha+k)}\left|\log\psi(t_2) - \log\psi(t_1)\right|^{\frac{\alpha}{k}}.\end{aligned}$$

Now, by the mean value theorem there is $\sigma \in (t_1, t_2)$ such that

$$|\log\psi(t_2) - \log\psi(t_1)| = \left|\frac{\psi'(\sigma)}{\psi(\sigma)}(t_2 - t_1)\right| \le \frac{|\psi'\|_\infty}{\psi(a)}|t_2 - t_1|. \tag{2.5}$$

Consequently,

$$\Pi_2 \le \frac{1}{\Gamma_k(\alpha+k)}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}}\|\phi\|_\infty|t_2 - t_1|^{\frac{\alpha}{k}}. \tag{2.6}$$

On the other hand, to Π_1 the change of variable $\xi = \log\frac{\psi(t_1)}{\psi(s)}/\log\frac{\psi(t_2)}{\psi(t_1)}$ and (2.5) yield that

$$\Pi_1 \leq \frac{\|\phi\|_\infty\,|\log\psi(t_2)-\log\psi(t_1)|^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^{\log\frac{\psi(t_1)}{\psi(a)}/\log\frac{\psi(t_2)}{\psi(t_1)}}\left|(1+\xi)^{\frac{\alpha}{k}-1}-\xi^{\frac{\alpha}{k}-1}\right|d\xi$$

$$\leq \frac{1}{k\Gamma_k(\alpha)}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}}\|\phi\|_\infty|t_2-t_1|^{\frac{\alpha}{k}}\int_0^{\log\frac{\psi(t_1)}{\psi(a)}/\log\frac{\psi(t_2)}{\psi(t_1)}}\left|(1+\xi)^{\frac{\alpha}{k}-1}-\xi^{\frac{\alpha}{k}-1}\right|d\xi.$$

If $\frac{\psi(t_1)}{\psi(a)} \leq \frac{\psi(t_2)}{\psi(t_1)}$, by simple computations we get

$$\int_0^{\log\frac{\psi(t_1)}{\psi(a)}/\log\frac{\psi(t_2)}{\psi(t_1)}}\left|(1+\xi)^{\frac{\alpha}{k}-1}-\xi^{\frac{\alpha}{k}-1}\right|d\xi \leq \frac{k}{\alpha}2^{\frac{\alpha}{k}}.$$

Otherwise, by taking $\beta = \frac{\alpha}{k} < 1$ in the inequality

$$|x^\beta - y^\beta| \leq |\beta|(x-y)y^{\beta-1} \qquad (x \geq y > 0 \;\text{ and }\; \beta \leq 1),$$

we are able to show that

$$\int_0^{\log\frac{\psi(t_1)}{\psi(a)}/\log\frac{\psi(t_2)}{\psi(t_1)}}\left|(1+\xi)^{\frac{\alpha}{k}-1}-\xi^{\frac{\alpha}{k}-1}\right|d\xi \leq \frac{k}{\alpha}2^{\frac{\alpha}{k}}+1.$$

Therefore

$$\Pi_1 \leq \frac{2^{\frac{\alpha}{k}}+\frac{\alpha}{k}}{\Gamma_k(\alpha+k)}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}}\|\phi\|_\infty|t_2-t_1|^{\frac{\alpha}{k}}. \tag{2.7}$$

So, combining (2.6), (2.7) with (2.4) we obtain

$$\begin{aligned}&\left|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_2) - {}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_1)\right| \\ &\qquad \leq \left(2^{\frac{\alpha}{k}}+\frac{\alpha}{k}+1\right)\frac{1}{\Gamma_k(\alpha+k)}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}}\|\phi\|_\infty|t_2-t_1|^{\frac{\alpha}{k}},\end{aligned} \tag{2.8}$$

which implies that ${}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi$ is a $\frac{\alpha}{k}$-Hölder continuous function in $(a,b]$ and hence continuous in $(a,b]$.

On the other hand, note that for every $t > a$, the change of variable

$\xi = \log \frac{\psi(t)}{\psi(s)}$ and (2.5) yield that

$$\begin{aligned}\left|{}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t)\right| &\le \frac{\|\phi\|_\infty}{k\Gamma_k(\alpha)}\int_a^t \frac{\psi'(s)}{\psi(s)}\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} ds \\ &\le \frac{\|\phi\|_\infty}{\Gamma_k(\alpha+k)}\left|\log\psi(t)-\log\psi(a)\right|^{\frac{\alpha}{k}} \\ &\le \frac{\|\phi\|_\infty}{\Gamma_k(\alpha+k)}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}}|t-a|^{\frac{\alpha}{k}}.\end{aligned}$$

Hence

$$\lim_{t\to a^+} {}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t) = 0 \tag{2.9}$$

and we can define ${}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi(a^+) = 0$. Consequently, if $a = t_1 < t_2$ by the previous computation we get

$$\left|{}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t_2) - {}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi(a)\right| = \left|{}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t_2)\right| \le \frac{\|\phi\|_\infty}{\Gamma_k(\alpha+k)}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}}|t_2-a|^{\frac{\alpha}{k}}.$$

Thus, ${}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi \in H^{\frac{\alpha}{k}}[a,b]$.

Finally, we have that

$$\left\|{}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi\right\|_\infty \le \frac{1}{\Gamma_k(\alpha+k)}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}}(b-a)^{\frac{\alpha}{k}}\|\phi\|_\infty, \tag{2.10}$$

and so

$$\begin{aligned}\left\|{}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi\right\|_{H^{\frac{\alpha}{k}}[a,b]} &= \left\|{}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi\right\|_\infty + \sup_{\substack{x_1,x_2\in[a,b]\\ x_1\neq x_2}}\frac{\left|{}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t_2) - {}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t_1)\right|}{|t_2-t_1|^{\frac{\alpha}{k}}} \\ &\le \mathscr{K}\|u\|_\infty.\end{aligned}$$

∎

3. Well-posedness of the (k,ψ)-Hadamard fractional integral in the weighted Lebesgue space $L^p_\omega(a,b)$

In this section, we present a detailed analysis of the mapping properties of the (k,ψ)-Hadamard fractional integral in weighted Lebesgue and Hölder-type spaces, with the aim of establishing sharp boundedness results –together with explicit constants– covering the subcritical, critical, and supercritical regimes determined by the parameters α, k, and p.

To this end, we begin by recalling that a *weight* is a nonnegative, locally integrable function on $\mathbb{R}$ which is strictly positive almost everywhere. Under assumption ($\mathbf{H}$), the function

$$\omega(t) = \frac{\psi'(t)}{\psi(t)}, \qquad t \in [a,b],$$

defines a weight, which naturally induces the weighted Lebesgue spaces $L^p_\omega(a,b)$ for $p \in [1,\infty)$, namely

$$L^p_\omega(a,b) = \left\{ \phi : (a,b) \to \mathbb{R} : \int_a^b |\phi(t)|^p \omega(t)\, dt < \infty \right\}.$$

Equipped with the norm

$$\|\phi\|_{L^p_\omega(a,b)} = \left(\int_a^b |\phi(t)|^p \omega(t)\, dt \right)^{1/p},$$

this space is a Banach space.

Within this functional framework, and by adapting and refining the techniques developed in [27, 28], we establish a collection of boundedness results for the operator ${}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}$ acting on $L^p_\omega(a,b)$. We first prove that the operator is well-defined and bounded on $L^p_\omega(a,b)$ for all $p \in [1,\infty]$, thereby ensuring its stability in the underlying weighted setting. We then turn to the subcritical regime $\alpha \in \left(0, \frac{k}{p}\right)$, where we derive sharp L^p–L^q estimates, showing that ${}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}$ defines a continuous mapping from $L^p_\omega(a,b)$ into $L^q_\omega(a,b)$ for all $q \in \left[p, \frac{kp}{k-\alpha p}\right)$. These bounds are obtained through refined applications of Hölder's inequality combined with a precise analysis of the associated logarithmic kernel.

The critical case $p = \frac{k}{\alpha}$ requires a separate treatment; in this limiting situation, we prove boundedness into $L^q_\omega(a,b)$ for every finite $q \geq p$, thereby revealing a threshold phenomenon in the operator's mapping properties. In contrast, in the supercritical regime $\alpha > \frac{k}{p}$, we establish a genuine regularizing effect, showing that ${}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}$ maps $L^p_\omega(a,b)$ continuously into the Hölder space $H^{\frac{\alpha}{k}-\frac{1}{p}}[a,b]$, which yields a quantitative gain in regularity.

Collectively, these results provide a unified and sharp characterization of both the boundedness and smoothing effects of the (k,ψ)-Hadamard fractional integral. The endpoint case $q = \frac{kp}{k-\alpha p}$, however, remains unresolved and appears to be closely related to optimal embedding inequalities, thus representing a natural direction for future investigation.

Theorem 3.1. *Suppose that* (**H**) *holds and let* $\alpha, k > 0$. *If* $p \in [1, \infty)$ *and* $\phi \in L^p_\omega(a,b)$, *then*

$$\int_a^t \left(\log \frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} |\phi(s)|\omega(s)\,ds < \infty \qquad \text{for a.e. } t \in (a,b).$$

Moreover, ${}^k_H\mathbf{J}^{\alpha;\psi}_{a+}\phi \in L^p_\omega(a,b)$ *with*

$$\left\|{}^k_H\mathbf{J}^{\alpha;\psi}_{a+}\phi\right\|_{L^p_\omega(a,b)} \leq \frac{\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\|\phi\|_{L^p_\omega(a,b)}. \tag{3.1}$$

If $p = \infty$ *and* $\phi \in L^\infty(a,b)$, *then* ${}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi \in L^\infty(a,b)$ *and*

$$\left\|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi\right\|_{L^\infty(a,b)} \leq \frac{\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\|\phi\|_{L^\infty(a,b)}. \tag{3.2}$$

Proof. Case $p=1$: Let $\phi \in L^1_\omega(a,b)$. By Fubini's theorem and the positivity of the kernel, we can interchange the order of integration to obtain

$$\int_a^b \left|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)\right|\omega(t)\,dt \leq \frac{1}{k\Gamma_k(\alpha)}\int_a^b |\phi(s)|\omega(s)\int_s^b \left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\frac{\psi'(t)}{\psi(t)}\,dt\,ds.$$

Performing the change of variables $u = \log\frac{\psi(t)}{\psi(s)}$, we obtain

$$\int_s^b \left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\frac{\psi'(t)}{\psi(t)}\,dt = \int_0^{\log\frac{\psi(b)}{\psi(s)}} u^{\frac{\alpha}{k}-1}\,du = \frac{k}{\alpha}\left(\log\frac{\psi(b)}{\psi(s)}\right)^{\frac{\alpha}{k}}.$$

Hence,

$$\int_a^b \left|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)\right|\omega(t)\,dt \leq \frac{1}{k\Gamma_k(\alpha)}\cdot\frac{k}{\alpha}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}}\int_a^b|\phi(s)|\omega(s)\,ds.$$

Using the identity $\Gamma_k(\alpha+k) = \frac{\alpha}{k}\Gamma_k(\alpha)$, we conclude

$$\left\|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi\right\|_{L^1_\omega(a,b)} \leq \frac{\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\|\phi\|_{L^1_\omega(a,b)}.$$

Case $1<p<\infty$: Let $\phi\in L^p_\omega(a,b)$ and let $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. By Hölder's inequality,

$$\begin{aligned}\left|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)\right| &\le \frac{1}{k\Gamma_k(\alpha)}\int_a^t\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}|\phi(s)|\omega(s)\,ds\\ &\le \frac{1}{k\Gamma_k(\alpha)}\left(\int_a^t\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\omega(s)\,ds\right)^{\frac{1}{q}}\\ &\quad\times\left(\int_a^t\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}|\phi(s)|^p\omega(s)\,ds\right)^{\frac{1}{p}}.\end{aligned}$$

Arguing as before,

$$\int_a^t\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\omega(s)\,ds\le\frac{k}{\alpha}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}},$$

and therefore

$$\begin{aligned}&\left|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)\right|\\ &\le\frac{1}{k\Gamma_k(\alpha)}\left[\frac{k}{\alpha}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}}\right]^{\frac{1}{q}}\left(\int_a^t\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}|\phi(s)|^p\omega(s)\,ds\right)^{\frac{1}{p}}.\end{aligned}$$

Raising to the power p, integrating over (a,b), and applying again Fubini's theorem, we get

$$\begin{aligned}\int_a^b\left|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)\right|^p\omega(t)\,dt &\le\frac{1}{[k\Gamma_k(\alpha)]^p}\left[\frac{k}{\alpha}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}}\right]^{\frac{p}{q}}\\ &\quad\times\int_a^b|\phi(s)|^p\omega(s)\int_s^b\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\omega(t)\,dt\,ds.\end{aligned}$$

Estimating the inner integral as before, we obtain

$$\int_a^b\left|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)\right|^p\omega(t)\,dt\le\frac{1}{[k\Gamma_k(\alpha)]^p}\left[\frac{k}{\alpha}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}}\right]^{\frac{p}{q}+1}\|\phi\|^p_{L^p_\omega(a,b)}.$$

Using again $\Gamma_k(\alpha+k)=\frac{\alpha}{k}\Gamma_k(\alpha)$, we conclude

$$\left\|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi\right\|_{L^p_\omega(a,b)}\le\frac{\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\|\phi\|_{L^p_\omega(a,b)}.$$

Case $p=\infty$: Let $\phi \in L^{\infty}(a,b)$. Then,

$$\begin{aligned} \left| {}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)\right| &\leq \frac{1}{k\Gamma_k(\alpha)}\|\phi\|_{L^\infty(a,b)}\int_a^t \left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\omega(s)\,ds \\ &\leq \frac{1}{k\Gamma_k(\alpha)}\|\phi\|_{L^\infty(a,b)}\cdot\frac{k}{\alpha}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}}. \end{aligned}$$

Hence,

$$\left\|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi\right\|_{L^\infty(a,b)} \leq \frac{\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\|\phi\|_{L^\infty(a,b)}.$$

∎

Remark 3.2. Arguing as in the proof of Theorem 2.4, one can show that the fractional integral operator

$${}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+} : L^\infty(a,b) \longrightarrow C[a,b]$$

is well-defined and bounded. In particular, there exists a constant $C_\infty > 0$, depending only on α, k, and ψ, such that

$$\left\|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi\right\|_{C[a,b]} \leq C_\infty\|\phi\|_{L^\infty(a,b)}, \qquad \forall\,\phi\in L^\infty(a,b).$$

Moreover, let $1\leq q<p<\infty$. Then the (k,ψ)-Hadamard fractional integral

$${}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+} : L^p_\omega(a,b) \longrightarrow L^q_\omega(a,b)$$

defines a bounded linear operator. Indeed, by Hölder's inequality, we have the continuous embedding

$$\|\phi\|_{L^q_\omega(a,b)} \leq \left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{1}{q}-\frac{1}{p}}\|\phi\|_{L^p_\omega(a,b)}, \qquad \forall\,\phi\in L^p_\omega(a,b).$$

Combining this estimate with Theorem 3.1, which ensures that

$$\left\|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi\right\|_{L^p_\omega(a,b)} \leq \frac{1}{\Gamma_k(\alpha+k)}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}}\|\phi\|_{L^p_\omega(a,b)},$$

we obtain

$$\left\|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi\right\|_{L^q_\omega(a,b)} \frac{1}{\Gamma_k(\alpha+k)}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}+\frac{1}{q}-\frac{1}{p}}\|\phi\|_{L^p_\omega(a,b)}.$$

This proves the boundedness of ${}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}$ from $L^p_\omega(a,b)$ into $L^q_\omega(a,b)$.

THEOREM 3.3. *Suppose that* (**H**) *holds. Let* $p\in[1,\infty)$, $k\in(1,\infty)$ *and* $\alpha\in\big(0,\frac{k}{p}\big)$. *If* $q\in\big[p,\frac{kp}{k-\alpha p}\big)$, *then* ${}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}:L^p_\omega(a,b)\to L^q_\omega(a,b)$ *is a bounded operator and, for every* $u\in L^p_\omega(a,b)$, *it holds that*

$$\left\|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi\right\|_{L^q_\omega(a,b)}\le\mathscr{C}\|\phi\|_{L^p_\omega(a,b)}, \tag{3.3}$$

where

$$\mathscr{C}=\begin{cases}\frac{1}{k\Gamma_k(\alpha)[q\left(\frac{\alpha}{k}-1\right)+1]^{1/q}}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}+\frac{1}{q}-1}, & \text{if } p=1,\\ \frac{1}{k\Gamma_k(\alpha)}\left[\frac{1}{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\right]^{1-\frac{1}{p}+\frac{1}{q}}, & \text{if } p\in\left(1,\frac{k}{\alpha}\right).\end{cases}$$

Proof. Case $p=1$: Let $q\in\big[1,\frac{k}{k-\alpha}\big)$ and assume first that $q\in\big(1,\frac{k}{k-\alpha}\big)$. The endpoint case $q=1$ follows directly from Theorem 3.1. Let $\sigma>1$ be the conjugate exponent of q, i.e.,

$$\frac{1}{q}+\frac{1}{\sigma}=1.$$

By Hölder's inequality, for every $t\in(a,b)$ we obtain

$$\begin{aligned}\left|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)\right|&\le\frac{1}{k\Gamma_k(\alpha)}\int_a^t\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}|\phi(s)|\omega(s)\,ds\\&\le\frac{1}{k\Gamma_k(\alpha)}\left(\int_a^t\left(\log\frac{\psi(t)}{\psi(s)}\right)^{q\left(\frac{\alpha}{k}-1\right)}|\phi(s)|\omega(s)\,ds\right)^{\frac{1}{q}}\left(\int_a^t\omega(s)\,ds\right)^{\frac{1}{\sigma}}.\end{aligned}$$

Since

$$\int_a^t\omega(s)\,ds=\log\frac{\psi(t)}{\psi(a)}\le\log\frac{\psi(b)}{\psi(a)},$$

it follows that

$$\left|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)\right|\le\frac{1}{k\Gamma_k(\alpha)}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{1}{\sigma}}\left(\int_a^t\left(\log\frac{\psi(t)}{\psi(s)}\right)^{q\left(\frac{\alpha}{k}-1\right)}|\phi(s)|\omega(s)\,ds\right)^{\frac{1}{q}}.$$

Since $q<\frac{k}{k-\alpha}$, we have

$$q\left(\frac{\alpha}{k}-1\right)+1>0.$$

Raising to the power q, integrating over (a,b), and applying Fubini's theorem, we obtain

$$\begin{aligned}&\left\|{}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi\right\|_{L^q_\omega(a,b)}^q\\ &\quad\leq \frac{1}{[k\Gamma_k(\alpha)]^q}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{q}{\sigma}}\int_a^b\int_s^b\left(\log\frac{\psi(t)}{\psi(s)}\right)^{q\left(\frac{\alpha}{k}-1\right)}\omega(t)\,dt\,|\phi(s)|\omega(s)\,ds.\end{aligned}$$

Evaluating the inner integral,

$$\int_s^b\left(\log\frac{\psi(t)}{\psi(s)}\right)^{q\left(\frac{\alpha}{k}-1\right)}\omega(t)\,dt=\frac{1}{q\left(\frac{\alpha}{k}-1\right)+1}\left(\log\frac{\psi(b)}{\psi(s)}\right)^{q\left(\frac{\alpha}{k}-1\right)+1},$$

and using the monotonicity of ψ, we deduce

$$\left\|{}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi\right\|_{L^q_\omega(a,b)}^q\leq\frac{1}{[k\Gamma_k(\alpha)]^q}\frac{1}{q\left(\frac{\alpha}{k}-1\right)+1}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{q\left(\frac{\alpha}{k}-1\right)+1+\frac{q}{\sigma}}\|\phi\|_{L^1_\omega(a,b)}.$$

Taking the q-th root, we conclude

$$\left\|{}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi\right\|_{L^q_\omega(a,b)}\leq\frac{1}{k\Gamma_k(\alpha)}\frac{1}{\left[q\left(\frac{\alpha}{k}-1\right)+1\right]^{1/q}}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\frac{\alpha}{k}+\frac{1}{q}-1}\|\phi\|_{L^1_\omega(a,b)}.$$

Case $1<p<\frac{k}{\alpha}$: Let $q\in\left[p,\frac{kp}{k-\alpha p}\right)$. The case $q=p$ follows from Theorem 3.1, hence we assume $q>p$. Define

$$\Theta_1:=\frac{1}{p}-\frac{1}{q}\in(0,1),\qquad \Theta_2:=\frac{1}{1-\Theta_1}.$$

Then $\Theta_2>1$ and $\Theta_2<q$. Applying Hölder's inequality twice, we obtain

$$\begin{aligned}\left|{}_{H}^{k}\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t)\right|\leq\frac{1}{k\Gamma_k(\alpha)}&\left(\int_a^t\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)}|\phi(s)|^p\omega(s)\,ds\right)^{\frac{1}{q}}\\ &\times\left(\int_a^t\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)}\omega(s)\,ds\right)^{\frac{q-\Theta_2}{q\Theta_2}}\\ &\times\left(\int_a^t|\phi(s)|^p\omega(s)\,ds\right)^{\frac{\Theta_2-1}{\Theta_2}}.\end{aligned}$$

Estimating the kernel as before, we obtain

$$\int_a^t \left(\log\frac{\psi(t)}{\psi(s)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)} \omega(s)\,ds \le \frac{1}{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}.$$

Therefore,

$$\left|{}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t)\right| \le \frac{1}{k\Gamma_k(\alpha)}\left[\frac{1}{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\right]^{1-\frac{1}{p}}$$
$$\times \|\phi\|_{L^p_\omega(a,b)}^{1-\frac{p}{q}}\left(\int_a^t\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)}|\phi(s)|^p\omega(s)\,ds\right)^{\frac{1}{q}}.$$

Proceeding as before, raising to the power q, integrating over (a,b), and applying Fubini's theorem, we deduce

$$\left\|{}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi\right\|_{L^q_\omega(a,b)}^q$$
$$\le \frac{1}{[k\Gamma_k(\alpha)]^q}\left[\frac{1}{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\right]^{q\left(1-\frac{1}{p}+\frac{1}{q}\right)}\|\phi\|^q_{L^p_\omega(a,b)}.$$

Taking the q-th root yields the estimate

$$\left\|{}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi\right\|_{L^q_\omega(a,b)}$$
$$\le \frac{1}{k\Gamma_k(\alpha)}\left[\frac{1}{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\right]^{1-\frac{1}{p}+\frac{1}{q}}\|\phi\|_{L^p_\omega(a,b)}.$$

This completes the proof. ∎

THEOREM 3.4. *Suppose that* (**H**) *holds. Let* $k\in(1,\infty)$, $\alpha\in(0,1)$ *and* $p=\frac{k}{\alpha}$. *If* $q\in[p,\infty)$, *then* ${}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}: L_\omega^{\frac{k}{\alpha}}(a,b)\to L^q_\omega(a,b)$ *is a bounded operator and, for every* $u\in L_\omega^{\frac{k}{\alpha}}(a,b)$, *it holds that*

$$\left\|{}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi\right\|_{L^q_\omega(a,b)}$$
$$\le \frac{1}{k\Gamma_k(\alpha)}\left[\frac{1}{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\right]^{1-\frac{1}{p}+\frac{1}{q}}\|\phi\|_{L^p_\omega(a,b)}.$$

Proof. The case $q = p$ follows directly from Theorem 3.1. Hence, we restrict ourselves to the case $q \in (p, \infty)$. Define

$$\Theta_1 := \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{k} - \frac{1}{q}, \qquad \Theta_2 := \frac{1}{1 - \Theta_1}.$$

It is immediate that

$$0 < \Theta_1 < \frac{\alpha}{k} < 1, \qquad 1 < \Theta_2 < \frac{k}{k - \alpha}.$$

Proceeding as in the proof of Theorem 3.3, we apply Hölder's inequality in a suitable factorized form. For every $t \in (a, b)$, we obtain

$$\begin{aligned}\left|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)\right| &\le \frac{1}{k\Gamma_k(\alpha)} \int_a^t \left(\log \frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} |\phi(s)|\omega(s)\, ds \\ &\le \frac{1}{k\Gamma_k(\alpha)} \left(\int_a^t \left(\log \frac{\psi(t)}{\psi(s)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)} |\phi(s)|^p \omega(s)\, ds\right)^{\frac{1}{q}} \\ &\quad \times \left(\int_a^t \left(\log \frac{\psi(t)}{\psi(s)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)} \omega(s)\, ds\right)^{\frac{p-1}{p}} \|\phi\|^{\frac{q-p}{q}}_{L^p_\omega(a,b)}.\end{aligned}$$

The kernel estimate yields

$$\int_a^t \left(\log \frac{\psi(t)}{\psi(s)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)} \omega(s)\, ds \le \frac{1}{\Theta_2\left(\frac{\alpha}{k} - 1\right) + 1} \left(\log \frac{\psi(b)}{\psi(a)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}.$$

Therefore,

$$\begin{aligned}\left|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)\right| &\le \frac{1}{k\Gamma_k(\alpha)} \left[\frac{1}{\Theta_2\left(\frac{\alpha}{k} - 1\right) + 1} \left(\log \frac{\psi(b)}{\psi(a)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\right]^{\frac{p-1}{p}} \\ &\quad \times \|\phi\|^{\frac{q-p}{q}}_{L^p_\omega(a,b)} \left(\int_a^t \left(\log \frac{\psi(t)}{\psi(s)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)} |\phi(s)|^p \omega(s)\, ds\right)^{\frac{1}{q}}.\end{aligned}$$

Raising to the power q, integrating over (a, b), and applying Fubini's theorem, we obtain

$$\begin{aligned}&\int_a^b \left|{}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t)\right|^q \omega(t)\,dt\\ &\quad\le \frac{1}{[k\Gamma_k(\alpha)]^q}\left[\frac{1}{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\right]^{\frac{q(p-1)}{p}}\\ &\qquad\times\|\phi\|_{L^p_\omega(a,b)}^{q-p}\int_a^b\int_s^b\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)}\omega(t)\,dt\,|\phi(s)|^p\omega(s)\,ds.\end{aligned}$$

Estimating the inner integral as before, we deduce

$$\begin{aligned}&\int_a^b \left|{}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t)\right|^q \omega(t)\,dt\\ &\le \frac{1}{[k\Gamma_k(\alpha)]^q}\left[\frac{1}{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\right]^{q\left(1-\frac{1}{p}+\frac{1}{q}\right)}\times\|\phi\|_{L^p_\omega(a,b)}^{q}.\end{aligned}$$

Taking the q-th root, we conclude that

$$\begin{aligned}&\left\|{}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi\right\|_{L^q_\omega(a,b)}\\ &\quad\le \frac{1}{k\Gamma_k(\alpha)}\left[\frac{1}{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{\Theta_2\left(\frac{\alpha}{k}-1\right)+1}\right]^{1-\frac{1}{p}+\frac{1}{q}}\|\phi\|_{L^p_\omega(a,b)}.\end{aligned}$$

This completes the proof. ∎

Theorem 3.5. *Suppose that* (**H**) *holds. Let* $p\in(1,\infty)$, $k\in(1,\infty)$ *and* $\alpha\in\left(\frac{k}{p},\frac{k}{p}+k\right)$. *Then* ${}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}: L^p_\omega(a,b)\to H^{\frac{\alpha}{k}-\frac{1}{p}}[a,b]$ *is a bounded operator and, for any* $\phi\in L^p_\psi(a,b)$, *it holds that*

$$\left\|{}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi\right\|_{H^{\frac{\alpha}{k}-\frac{1}{p}}[a,b]}\le\mathscr{H}\|u\|_{L^p_\omega(a,b)} \tag{3.4}$$

where

$$\mathscr{H}=\frac{1}{k\Gamma_k(\alpha)}\frac{1}{[q\left(\frac{\alpha}{k}-1\right)+1]^{1/q}}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}-\frac{1}{p}}|b-a|^{\frac{\alpha}{k}-\frac{1}{p}}+\mathcal{H}$$

and

$$\mathcal{H} := \begin{cases} \Big[\left|\frac{\alpha}{k} - 1\right| \frac{kp}{[q(2-\frac{\alpha}{k})-1]^{1/q}(\alpha p - k)} \\ \qquad + \frac{1}{[q(\frac{\alpha}{k}-1)+1]^{1/q}} \Big] \left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}-\frac{1}{p}}, & \textit{if } t_1, t_2 \in (a,b], \textit{ with } t_1 \neq t_2, \\ \frac{1}{k\Gamma_k(\alpha)} \frac{1}{[q(\frac{\alpha}{k}-1)+1]^{1/q}} \left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}-\frac{1}{p}}, & \textit{if } t_1 = a \textit{ and } t_2 \in (a,b]. \end{cases}$$

Proof. Note that, if $\phi \in L^p_\omega(a,b)$, then ${}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)$ is well defined for almost every $t \in (a,b)$, because by Hölder inequality we have

$$\begin{aligned} \left|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)\right| &\leq \frac{1}{k\Gamma_k(\alpha)} \left(\int_a^t \left(\log\frac{\psi(t)}{\psi(s)}\right)^{q\left(\frac{\alpha}{k}-1\right)} \omega(s)\, ds\right)^{1/q} \\ &\quad \times \left(\int_a^t |\phi(s)|^p \omega(s)\, ds\right)^{1/p} \\ &\leq \frac{1}{k\Gamma_k(\alpha)} \left[\frac{1}{q\left(\frac{\alpha}{k}-1\right)+1}\left(\log\frac{\psi(t)}{\psi(a)}\right)^{q\left(\frac{\alpha}{k}-1\right)+1}\right]^{1/q} \|\phi\|_{L^p_\omega(a,b)}. \end{aligned} \tag{3.5}$$

On the other hand, for any $t_1, t_2 \in (a,b]$ with $t_1 < t_2$ we derive

$$\begin{aligned} &\left|{}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_2) - {}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_1)\right| \\ &\quad \leq \frac{1}{k\Gamma_k(\alpha)} \int_a^{t_1} \left|\left(\log\frac{\psi(t_2)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} - \left(\log\frac{\psi(t_1)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\right| \\ &\qquad\qquad \times |\phi(s)|\omega(s)\, ds \\ &\quad + \frac{1}{k\Gamma_k(\alpha)} \int_{t_1}^{t_2} \left(\log\frac{\psi(t_2)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} |\phi(s)|\omega(s)\, ds. \end{aligned} \tag{3.6}$$

In what follows, by using the Hölder inequality we estimates the two terms of the right hand side. In fact, by (2.5) we get

$$
\begin{aligned}
&\int_{t_1}^{t_2}\left(\log\frac{\psi(t_2)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}|\phi(s)|\omega(s)\,ds\\
&\quad\le\left(\int_{t_1}^{t_2}\left(\log\frac{\psi(t_2)}{\psi(s)}\right)^{q\left(\frac{\alpha}{k}-1\right)}\omega(s)\,ds\right)^{1/q}\left(\int_{t_1}^{t_2}|\phi(s)|^p\omega(s)\,ds\right)^{1/p}\\
&\quad\le\left(\frac{1}{q\left(\frac{\alpha}{k}-1\right)+1}\left(\log\frac{\psi(t_2)}{\psi(t_1)}\right)^{q\left(\frac{\alpha}{k}-1\right)+1}\right)^{1/q}\|\phi\|_{L^p_\omega(a,b)}\\
&\quad\le\left(\frac{1}{q\left(\frac{\alpha}{k}-1\right)+1}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{q\left(\frac{\alpha}{k}-1\right)+1}\right)^{1/q}|t_2-t_1|^{\frac{\alpha}{k}-\frac{1}{p}}\|\phi\|_{L^p_\omega(a,b)}.
\end{aligned}
\tag{3.7}
$$

On the other hand, by the fundamental theorem of calculus we get

$$
\begin{aligned}
&\int_a^{t_1}\left|\left(\log\frac{\psi(t_2)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}-\left(\log\frac{\psi(t_1)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\right||\phi(s)|\omega(s)\,ds\\
&\quad=\int_a^{t_1}\left|\int_{t_1}^{t_2}\frac{d}{dt}\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}dt\right||\phi(s)|\omega(s)\,ds\\
&\quad=\int_a^{t_1}\left|\int_{t_1}^{t_2}\left(\frac{\alpha}{k}-1\right)\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-2}\omega(t)dt\right||\phi(s)|\omega(s)\,ds\\
&\quad\le\left|\frac{\alpha}{k}-1\right|\int_a^{t_1}\int_{t_1}^{t_2}\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-2}\omega(t)dt|\phi(s)|\omega(s)\,ds.
\end{aligned}
$$

Hence, Hölder and generalized Minkowski inequalities yield that

$$
\begin{aligned}
&\int_a^{t_1}\left|\left(\log\frac{\psi(t_2)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}-\left(\log\frac{\psi(t_1)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\right||\phi(s)|\omega(s)\,ds\\
&\quad\le\left|\frac{\alpha}{k}-1\right|\left(\int_a^{t_1}\left(\int_{t_1}^{t_2}\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-2}\omega(t)\,dt\right)^q\omega(s)\,ds\right)^{\frac{1}{q}}\\
&\qquad\times\left(\int_a^{t_1}|\phi(s)|^p\omega(s)\,ds\right)^{\frac{1}{p}}\\
&\quad\le\left|\frac{\alpha}{k}-1\right|\int_{t_1}^{t_2}\left(\int_a^{t_1}\left(\log\frac{\psi(t)}{\psi(s)}\right)^{q\left(\frac{\alpha}{k}-2\right)}\omega(s)ds\right)^{\frac{1}{q}}\omega(t)\,dt\,\|\phi\|_{L^p_\omega(a,b)}.
\end{aligned}
$$

Now, since $\alpha \in \left(\frac{k}{p}, \frac{k}{p}+k\right)$, then $q\left(2-\frac{\alpha}{k}\right)-1>0$, so doing the change of variable $\xi = \log\frac{\psi(t)}{\psi(s)}$ we get

$$\begin{aligned}
&\int_a^{t_1}\left(\log\frac{\psi(t)}{\psi(s)}\right)^{q\left(\frac{\alpha}{k}-2\right)}\omega(s)\,ds\\
&\quad=\frac{1}{q\left(2-\frac{\alpha}{k}\right)-1}\left[\left(\log\frac{\psi(t)}{\psi(t_1)}\right)^{q\left(\frac{\alpha}{k}-2\right)+1}-\left(\log\frac{\psi(t)}{\psi(a)}\right)^{q\left(\frac{\alpha}{k}-2\right)+1}\right]\\
&\quad\leq\frac{1}{q\left(2-\frac{\alpha}{k}\right)-1}\left(\log\frac{\psi(t)}{\psi(t_1)}\right)^{q\left(\frac{\alpha}{k}-2\right)+1}.
\end{aligned}$$

Combining this inequality with the previous one and with (2.5) we get

$$\begin{aligned}
&\int_a^{t_1}\left|\left(\log\frac{\psi(t_2)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}-\left(\log\frac{\psi(t_1)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\right||\phi(s)|\omega(s)\,ds\\
&\quad\leq\left|\frac{\alpha}{k}-1\right|\frac{1}{\left[q\left(2-\frac{\alpha}{k}\right)-1\right]^{1/q}}\int_{t_1}^{t_2}\left(\log\frac{\psi(t)}{\psi(t_1)}\right)^{\frac{\alpha}{k}-\frac{1}{p}-1}\omega(t)\,dt\,\|\phi\|_{L^p_\omega(a,b)}\\
&\quad=\left|\frac{\alpha}{k}-1\right|\frac{1}{[q(2-\frac{\alpha}{k})-1]^{1/q}}\frac{1}{\frac{\alpha}{k}-\frac{1}{p}}\left(\log\frac{\psi(t)}{\psi(t_1)}\right)^{\frac{\alpha}{k}-\frac{1}{p}}\|\phi\|_{L^p_\omega(a,b)} \qquad (3.8)\\
&\quad\leq\left|\frac{\alpha}{k}-1\right|\frac{kp}{[q(2-\frac{\alpha}{k})-1]^{1/q}(\alpha p-k)}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}-\frac{1}{p}}\|\phi\|_{L^p_\omega(a,b)}|t_2-t_1|^{\frac{\alpha}{k}-\frac{1}{p}}.
\end{aligned}$$

Consequently, (3.7)-(3.8) combined with (3.6) yield that

$$\begin{aligned}
&\left|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_2)-{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_1)\right|\\
&\qquad\qquad\leq\mathcal{C}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}-\frac{1}{p}}\|\phi\|_{L^p_\omega(a,b)}|t_2-t_1|^{\frac{\alpha}{k}-\frac{1}{p}},
\end{aligned} \qquad (3.9)$$

where

$$\mathcal{C}=\left[\left|\frac{\alpha}{k}-1\right|\frac{kp}{[q(2-\frac{\alpha}{k})-1]^{1/q}(\alpha p-k)}+\frac{1}{[q\left(\frac{\alpha}{k}-1\right)+1]^{1/q}}\right].$$

Furthermore, the continuity of $\log(\cdot)$ and $\psi(\cdot)$ combined with (3.5) to get

$$\lim_{t\to a^+}{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t)=0,$$

hence ${}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}$ can be extended continuously by 0 at $t=a$. Moreover, if $a=t_1<t_2$, then

$$\begin{aligned}
&\left|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_2)-{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_1)\right| \\
&\quad=|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_2)| \\
&\quad\le\frac{1}{k\Gamma_k(\alpha)}\frac{1}{\left[q\left(\frac{\alpha}{k}-1\right)+1\right]^{1/q}}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}-\frac{1}{p}}\|\phi\|_{L^p_\omega(a,b)}|t_2-a|^{\frac{\alpha}{k}-\frac{1}{p}}.
\end{aligned}\tag{3.10}$$

Therefore,

$$\left|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_2)-{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_1)\right|\le\mathcal{H}\|\phi\|_{L^p_\omega(a,b)}|t_2-t_1|^{\frac{\alpha}{k}-\frac{1}{p}}.$$

Thus, ${}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi$ be a Hölder continuous function for all $t\in[a,b]$.

Finally, by (3.5) we obtain that

$$\left\|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi\right\|_\infty\le\frac{1}{k\Gamma_k(\alpha)}\frac{1}{\left[q\left(\frac{\alpha}{k}-1\right)+1\right]^{1/q}}\left[\frac{\|\psi'\|_\infty}{\psi(a)}\right]^{\frac{\alpha}{k}-\frac{1}{p}}\|\phi\|_{L^p_\omega(a,b)}|b-a|^{\frac{\alpha}{k}-\frac{1}{p}}.$$

Hence,

$$\begin{aligned}
\left\|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi\right\|_{H^{\frac{\alpha}{k}-\frac{1}{p}}[a,b]}&=\left\|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi\right\|_\infty+\sup_{\substack{t_1,t_2\in[a,b]\\ t_1\neq t_2}}\frac{\left|{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_1)-{}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}\phi(t_2)\right|}{|t_1-t_2|^{\frac{\alpha}{k}-\frac{1}{p}}} \\
&\le\mathscr{H}\|\phi\|_{L^p_\omega(a,b)}.
\end{aligned}$$

As we desired. The proof is completed. ∎

Open problem. It is worth noting that, under the assumptions of Theorem 3.3 –namely (**H**), $p\in[1,\infty)$, $k\in(1,\infty)$, and $\alpha\in\left(0,\frac{k}{p}\right)$– we have shown that the (k,ψ)-Hadamard fractional integral

$${}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}:L^p_\omega(a,b)\longrightarrow L^q_\omega(a,b)$$

is a bounded operator for every $q\in\left[p,\frac{kp}{k-\alpha p}\right)$. However, the critical case remains an open question, that is, when $q=\frac{kp}{k-\alpha p}$. In other words, the boundedness of ${}^{k}_{H}\mathbf{J}^{\alpha;\psi}_{a^+}$ from $L^p_\omega(a,b)$ into $L^{\frac{kp}{k-\alpha p}}_\omega(a,b)$ is still unresolved.

4. Some applications

In this section, as a direct application of the preceding results, we derive several fundamental properties of the (k, ψ)-Hadamard fractional integral. First, we establish a mean value theorem adapted to this framework, which characterizes the averaged behavior of the operator. We then prove a corresponding law of exponents, describing the composition structure of (k, ψ)-Hadamard fractional integrals and highlighting their algebraic properties. Finally, we present the associated integration by parts formula, a fundamental identity in the nonlocal setting, which is instrumental in the formulation of weak solutions, the development of variational methods, and the analysis of boundary value problems.

Theorem 4.1. *Suppose that* $(\mathbf{H})$ *holds. Let* $\alpha \in (0,1)$, $k \in (1,\infty)$ *and* $\phi \in C[a,b]$. *Then, for every* $t \in (a,b)$, *there is* $\sigma \in (a,t)$ *such that*

$$ {}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t) = \frac{1}{\alpha\Gamma_k(\alpha)}\left(\log\frac{\psi(t)}{\psi(a)}\right)^{\frac{\alpha}{k}}\phi(\sigma). $$

Proof. For every $t \in (a,b]$ and $s \in (a,t)$, let us define the function

$$ \Psi_t(s) = \frac{1}{\alpha\Gamma_k(\alpha)}\left[\left(\log\frac{\psi(t)}{\psi(a)}\right)^{\frac{\alpha}{k}} - \left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}}\right]. $$

Since

$$ \frac{d}{ds}\Psi_t(s) = \frac{1}{k\Gamma_k(\alpha)}\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\omega(s) > 0, $$

it follows that $\Psi_t(s)$ is a positive and strictly increasing function. Consequently, the (k, ψ)-Hadamard fractional integral can be represented as a Riemann-Stieltjes integral, namely

$$ {}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t) = \frac{1}{k\Gamma_k(\alpha)}\int_a^t\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\phi(s)\omega(s)\,ds = \int_a^t\phi(s)\,d\Psi_t(s). $$

Applying [2, Theorem 3.1], there exists $\sigma \in (a,t)$ such that

$$ {}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t) = (\Psi_t(t) - \Psi_t(a))\,\phi(\sigma) = \frac{1}{\alpha\Gamma_k(\alpha)}\left(\log\frac{\psi(t)}{\psi(a)}\right)^{\frac{\alpha}{k}}\phi(\sigma). $$

∎

LEMMA 4.2. *Let* $\phi \in L^1_\omega[a,b]$, $\alpha,\beta \in (0,+\infty)$ *and* $k \in (0,+\infty)$. *Then*

$$ {}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\,{}^k_H\mathbf{J}^{\beta;\psi}_{a^+}\phi(t) = {}^k_H\mathbf{J}^{\alpha+\beta;\psi}_{a^+}\phi(t). $$

Proof. By definition, Proposition 2.1 and doing the change of variable $\xi = \frac{\log\frac{\psi(s)}{\psi(\sigma)}}{\log\frac{\psi(t)}{\psi(\sigma)}}$ we obtain

$$ {}^k_H\mathbf{J}^{\alpha;\psi}_{a^+}\,{}^k_H\mathbf{J}^{\beta;\psi}_{a^+}\phi(t) = \frac{1}{k\Gamma_k(\alpha)}\int_a^t \frac{\psi'(s)}{\psi(s)}\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1} {}^k\mathbf{I}^{\beta;\psi}_{a^+}\phi(s)\,ds $$
$$ = \frac{1}{k\Gamma_k(\alpha)}\frac{1}{k\Gamma_k(\beta)}\int_a^t u(\sigma)\frac{\psi'(\sigma)}{\psi(\sigma)}\int_\sigma^t \frac{\psi'(s)}{\psi(s)}\left(\log\frac{\psi(t)}{\psi(s)}\right)^{\frac{\alpha}{k}-1}\left(\log\frac{\psi(s)}{\psi(\sigma)}\right)^{\frac{\beta}{k}-1} ds\,d\sigma $$
$$ = \frac{1}{k\Gamma_k(\alpha)}\frac{1}{k\Gamma_k(\beta)}\int_a^t \frac{\psi'(\sigma)}{\psi(\sigma)}\left(\log\frac{\psi(t)}{\psi(\sigma)}\right)^{\frac{\alpha}{k}+\frac{\beta}{k}-1} u(\sigma)\int_0^1 (1-\xi)^{\frac{\alpha}{k}-1}\xi^{\frac{\beta}{k}-1}\,d\xi $$
$$ = {}^k_H\mathbf{J}^{\alpha+\beta;\psi}_{a^+}\phi(t). $$

∎

Finally, we consider the k-fractional integration by parts formula associated with the (k,ψ)-Hadamard fractional integral. This identity plays a central role in the weak analysis of fractional problems, as it allows the transfer of (k,ψ)-fractional derivatives between test functions within integral formulations. Consequently, it provides the appropriate framework for defining weak solutions, working in weighted or ψ-adapted fractional Sobolev-type spaces, and applying variational methods.

Moreover, for nonlocal operators induced by the (k,ψ)-Hadamard structure, the integration by parts formula yields symmetric energy identities of the form

$$ \int_\Omega u\,\mathcal{L}_{k,\psi}v\,dx = \int_\Omega v\,\mathcal{L}_{k,\psi}u\,dx, $$

which are fundamental for deriving energy estimates, variational principles, and uniqueness results.

Finally, due to the nonlocal nature of the operator, boundary conditions must be interpreted in a non-classical sense. In this context, the integration by parts formula allows one to identify the effective boundary contributions, typically expressed as weighted or nonlocal terms involving the kernel ψ.

Theorem 4.3. *Suppose that* (**H**) *holds. Let* $\alpha \in (0,1)$, $k \in (1,\infty)$ *and* $p, q \in (1,\infty)$ *be such that*

$$\frac{1}{p} + \frac{1}{q} \leq 1 + \frac{\alpha}{k}.$$

If $\phi \in L^p_\omega(a,b)$ *and* $\varphi \in L^q_\omega(a,b)$, *then*

$$\int_a^b {}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t)\varphi(t)\omega(t)\, dt = \int_a^b \phi(t) {}_H^k\mathbf{J}_{b^-}^{\alpha;\psi}\varphi(t)\omega(t)\, dt. \tag{4.1}$$

Proof. First, assume that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\alpha}{k}.$$

Hence

$$\frac{1}{p} = \frac{\alpha}{k} + 1 - \frac{1}{q} > \frac{\alpha}{k} \quad \Longrightarrow \quad p \in \left(1, \frac{k}{\alpha}\right).$$

Let $\varrho > 0$ be such that $\frac{1}{q} + \frac{1}{\varrho} = 1$. Then

$$\varrho = \frac{kp}{k - \alpha p}.$$

Therefore, by Hölder's inequality we obtain

$$\begin{aligned}
&\int_a^b {}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t)\varphi(t)\omega(t)\, dt \\
&\quad \leq \left(\int_a^b \left|{}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t)\right|^\varrho \omega(t)\, dt\right)^{\frac{1}{\varrho}} \left(\int_a^b |\varphi(t)|^q \omega(t)\, dt\right)^{\frac{1}{q}} \\
&\quad \leq \mathscr{C}_\varrho \|\phi\|_{L^p_\omega(a,b)} \|\varphi\|_{L^q_\omega(a,b)}.
\end{aligned}$$

Thus, the right-hand side of (4.1) is well defined.

Now, suppose that

$$\frac{1}{p} + \frac{1}{q} < 1 + \frac{\alpha}{k}.$$

In this case, we need to distinguish the following situations:

$$p > \frac{k}{\alpha}, \quad p < \frac{k}{\alpha} \qquad \text{and} \qquad p = \frac{k}{\alpha}.$$

If $p > \frac{k}{\alpha}$, then Theorem 3.5 together with Hölder's inequality yields

$$\begin{aligned}\int_a^b {}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi(t)\varphi(t)\,dt &\\ &\le \left\| {}_H^k\mathbf{J}_{a^+}^{\alpha;\psi}\phi \right\|_\infty \int_a^b \varphi(t)\omega(t)\,dt \\ &\le \frac{1}{k\Gamma_k(\alpha)[\tilde{p}\left(\frac{\alpha}{k}-1\right)+1]^{1/\tilde{p}}}\left(\log\frac{\psi(b)}{\psi(a)}\right)^{1+\frac{\alpha}{k}-\frac{1}{p}-\frac{1}{q}}\|\phi\|_{L^p_\omega(a,b)}\|\varphi\|_{L^q_\omega(a,b)},\end{aligned}$$

where $\tilde{p}$ denotes the conjugate exponent of p. Again the right-hand side of (4.1) is well defined in this case.

If $p < \frac{k}{\alpha}$, then there is $\tilde{q} \in (1,q)$ such that

$$\frac{1}{p}+\frac{1}{\tilde{q}} = 1+\frac{\alpha}{k}.$$

Thus, $\varphi \in L^q_\omega(a,b) \subset L^{\tilde{q}}_\omega(a,b)$. Therefore, arguing as in the first case with q replaced by $\tilde{q}$, the integral on the right-hand side of (4.1) is well-defined.

To finish our analysis, let's assume that $p = \frac{k}{\alpha}$. Then $u \in L^{\tilde{p}}_\omega(a,b)$ for any $\tilde{p} \in (1,p)$. Therefore, we can simply repeat the analysis from the previous case.

Finally, straightforward computations show that the relation (4.1) is valid. ∎

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