EXTRACTA MATHEMATICAE

Article in press
Available online April 27, 2026

The dark side of categorical Banach space theory

Jesús M.F. Castillo

Instituto de Matemáticas Imuex, Universidad de Extremadura
Avenida de Elvas, 06006 *Badajoz, Spain*

castillo@unex.es

Received March 7, 2026
Accepted March 17, 2026

Presented by A. Martsinkovsky

Abstract: This paper could be considered the third in the series *The Hitchhiker Guide to Categorical Banach Space Theory* [24, 25]. We explore (quasi) Banach space formulations and applications for advanced categorical topics, such as relative homology (with either respect to size or relative to an operator ideal), Buchsbaum's satellites (and homological derivation in quasi-Banach spaces), coherent functors and sheaves, Quillen's adjunction theorem, Auslander's formula, the Heart of the categories of Banach and quasi-Banach spaces, the vanishing of some Ext^2 functors, Martsinkovsky-Russell stabilizations, nets of Banach spaces or 3-space problems.

Key words: Categorical ideas in Banach space theory; quasi-Banach spaces; homological derivation; Operator ideals and relative homology; functor categories and the heart of Banach spaces; coherent and effaceable functors in Banach spaces; extension of compact operators; tensors and Tor in Banach spaces; Rochberg spaces; Quillen's theorem; Serre subcategories and 3-space problems.

MSC (2020): 46Mxx: 46M15,46M18,46M10,46M40, 46M05.

Contents

ISSN: 0213-8743 (print), 2605-5686 (online)

22. Speak to me

Let's speak once more about categorical ideas applied to Banach space theory. Once again, because this paper can be considered the third of the *The Hitchhiker Guide to Categorical Banach Space Theory* series [24, 25] and it comes after all what has been said and done in the book *Homological methods in Banach space theory* [13]. We will quite often refer to those works, even if we have made every effort to make this paper readable and self contained [1]. In this regard, the reader is addressed to Section 24 *On the run* for most unexplained terms and notation, even for those that could have been used in the preceding sections.

Why are categorical ideas so intimidating for Banach spacers? Well, as Justin Hsu puts it [2] "Category theory is sprawling". Moreover, a high degree

[1] In particular, and as far as possible, references to previous Banach/Quasi Banach space result have been addressed to [13].

[2] Entry "Teaching Category Theory to Computer Scientists" (but we could well

of super-abstraction pervades everything. Has it to be so? Has all that jibber-jabber some tangible meaning? Indeed, it has, but for reasons hard to swallow for a Banach spacer: categorical people work nonchalantly in non-concrete categories (a category is termed *concrete* when its objects are sets; this does not mean that its arrows are necessarily maps — for instance, consider the category whose objects are the natural numbers and with a unique arrow $n \to m$ whenever $n \leq m$. But we may momentarily accept the simplified intuition). A nice discussion in https://math.stackexchange.com/questions/3345586/is-there-a-name-for-categories-whose-objects-are-sets concludes with an enlightening remark from *Arthur*: "Category theory doesn't really care what its objects are per se". However, THEM, the Categorical people, must work with highly abstract stuff in non-concrete categories, and that forces them to walk the abstract way. And since – fair warning – we ourselves will eventually need to work with non-concrete categories, to some extent we too must walk that abstract path. The inherent danger here, namely falling into nonsense nonsense, has never been described better than by Austin [3] [3] (see also [48]). We have done our best to avoid that.

So, the setting becomes now: US are almost ready to accept that we need some extra abstraction. How much? And, can we do that at a reasonable cost? Let us calmly sit under a weirwood to gain a three-eyed-raven panoramic vision [4], until we realize that the argument works both ways: Does the theory of Rochberg spaces, viewed as higher-order derived spaces from complex interpolation scales, contain anything interesting to THEM? No. Does the theory of Ext^n spaces, viewed as higher-order derived spaces from Hom, contain anything interesting to US? No.

But they do.

As MacLane says it with unparalleled clarity, "Category theory starts with the observation that many properties of mathematical systems can be unified and simplified by a presentation with diagrams and arrow" [67]. Therefore, categorical thinking requires formulating ideas diagrammatically – yet another complication, since we are not entirely convinced that Banach spacers are

write "to Banach spacers") in SIGPLAN `blog.sigplan.org/2023/04/04/teaching-category-theory-to-computer-scientists`.

[3] Excerpt: "… and conjectured that the total number of Woffles would be at least as great as the number so far known to exist. They asked if this conjecture was the strongest possible."

[4] You can learn whatever you need about weirwoods and three-eyed ravens by sitting under a weirwood to read the five volumes of A Song of Ice and Fire by George R.R. Martin, and by watching – as a three-eyed raven would – the TV series Game of Thrones.

particularly fond of diagrams. Let us provide a perfectly classical example: for Banach spacers the kernel of an operator $\tau : X \to Y$ is $\ker \tau = \{x \in X : \tau(x) = 0\}$, while the categorical kernel of the arrow τ is another arrow κ such that $\tau\kappa = 0$ and having the universal property with respect to this diagram; namely, whenever ν is another arrow such that $\tau\nu = 0$, then ν factorizes through κ; i.e., $\nu = \kappa\eta$ for some arrow η. The core of categorical notions is the *universal* part of the definition. The reader might enjoy checking that the canonical inclusion $\imath : \ker \tau \to X$ is the categorical kernel κ (and, more or less vice versa, $\ker \tau = \mathrm{Im}\kappa$). Another feature of category theory is that any notion comes accompanied with its *dual* (i.e., reversing arrows) notion. Thus, the *cokernel* notion is lurking behind the words: the cokernel of the arrow τ is another arrow ς such that $\varsigma\tau = 0$ and having the universal property with respect to this diagram; namely, whenever ν is another arrow such that $\nu\tau = 0$ then ν factorizes through ς; i.e., $\nu = \eta\kappa$ for some arrow η. The reader might enjoy (maybe not) finding the Banach space cokernel. In [24, Section 3 and Section 4] and [13, Chapter 2: The language of homology] one can find the categorical translation of those and other basic mathematical ideas.

Our declared targets in this paper are the categories **Ban** of Banach spaces and bounded linear maps, and the category **QBan** of quasi-Banach spaces and bounded linear maps. Why bring **QBan** into the picture? Because, as we shall see, one naturally needs to move to **QBan** when studying exact sequences in **Ban** . Why bring even more abstract, non-concrete categories into the picture? Because – as categorical people know– the right places in which to develop categorical work are Abelian categories: categories where, give or take an axiom, the morphism spaces Hom are Abelian groups, kernels and cokernels exist, and everything behaves in a perfectly reasonable way (an arrow is the cokernel of its kernel, etc). It takes little time to suspect, and then to verify (see [24, 25, 23, 13]), that neither **Ban** nor **QBan** are Abelian categories. Indeed, the ultimate reason behind the misbehaviour of **Ban** and **QBan** is that Banach and quasi-Banach spaces intertwine a vector-space structure with a metric structure – something that existing categorical notions do not manage effortlessly. Still, categorical constructions can be used in Banach space theory and often do work, even if we do not always know exactly why, *as if* **Ban** were an Abelian category.

And with this we close the argumentation circle. Moving Banach space ideas into categorical lands, or bringing categorical ideas into Banach lands, pays off. Thus, US should care about why **Ban** is not an Abelian category and how that might be amended, just as THEM should care about using **Ban** and

QBan as laboratory categories where ideas and techniques can be tested in a not-purely-abstract yet still controlled environment.

23. Breathe

Don't be afraid to read. Let us see instead why categorical ideas are necessary and how they enter, or at least should enter, in our common toolkit. For all unexplained terms or notation, the reader is addressed to Section 24.

When one begins the study of a category the first type of objects to consider are the *projective* and *injective* ones. To get the idea we need to set first the notions of *monic* and *epic* arrow, heritage of the set theoretic notions of injective and surjective map. An arrow m is monic if $mf = mg \Longrightarrow f = g$, for whatever two arrows f, g. An arrow e is epic if $fe = ge \Longrightarrow f = g$, for whatever two arrows f, g. Incredulous readers will have the time of their lives checking that monic arrows in **Ban** are the injective operators, while epic arrows in **Ban** are the dense range (not the surjective) operators! Funny? Wait and see.

23.1. Projective and injective spaces.

An object $\mathcal{P}$ in a category is called projective when given an epic arrow $Z \to X$ and an arrow $\mathcal{P} \to X$ there is an arrow $\mathcal{P} \dashrightarrow Z$ making the diagram

$$\begin{tikzcd} Z \arrow[r] & X \\ & \mathcal{P} \arrow[u] \arrow[ul, dashed] \end{tikzcd}$$

commute. An object $\mathcal{I}$ is called injective when given a monic arrow $Y \to Z$ and an arrow $Y \to \mathcal{I}$ there is an arrow $Z \dashrightarrow \mathcal{I}$ making commutative the whole diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z . \arrow[dl, dashed] \\ \mathcal{I} & \end{tikzcd}$$

First problem: **Ban** does not have projective or injective spaces [78], and the reason is that an operator such as the canonical inclusion $\ell_1 \to c_0$ is both monic and epic, while it is impossible to lift the operator $\mathbb{R} \to c_0$ sending 1 to $(1/n)_n$ and it is impossible as well to extend the functional $\ell_1 \to \mathbb{R}$

given by $f(x) = \sum x(n)$. Thus, $\mathbb{R}$ is neither projective nor injective. And since the Hahn-Banach theorem implies that any operator $\mathbb{R} \to E$ can be extended to any superspace of $\mathbb{R}$, the conclusion is at hand. But we *need* projective/injective objects in a category to do ... well, the things we do in Section 23.2.

To circumvent this difficulty, common sense suggests relaxing monic and epic maps to their concrete original notions (injective and surjective maps), with a topological twist: having closed range. Thus, we will restrict ourselves to working with *embeddings* (injective operators with closed range) and *quotient* maps (surjective operators – which, of course, have closed range). This is a clever move, because it automatically produces a rich theory of injective and projective Banach spaces, but at the same time it leads us into the unexplored territory of relative homology (see Section 25). Indeed, with these new definitions, projective and injective spaces in **Ban** do exist. There are, however, striking differences between them:

- Injective spaces are a huge mystery: $L_\infty(\mu)$ spaces are injective because of the Hahn-Banach theorem, but there are others; see [4, Chapter 1: A primer on injective spaces].
- Projective spaces are simple. All $\ell_1(\Gamma)$ spaces are projective since, after all, the identity $\mathfrak{L}(\ell_1(\Gamma), X) = \ell_\infty(\Gamma, X)$ means that it is extremely simple to construct operators $\ell_1(\Gamma) \to X$: just provide a bounded family $\{x_\gamma : \gamma \in \Gamma\}$ of elements of X and set $e_\gamma \to x_\gamma$. At this point is when things become interesting: the $\ell_1(\Gamma)$ are *all* projective Banach spaces there are. Precisely

Proposition 23.1. *A projective Banach space is isomorphic to some* $\ell_1(\Gamma)$.

Everything is contained in the following result of Köthe [46]. Recall that a subspace $\imath : E \to X$ of a Banach space is said to be *complemented* if there is a *projection* (i.e., a retraction or left inverse) $\pi : X \to E$. We have:

Proposition 23.2. *A complemented subspace of* $\ell_1(\Gamma)$ *is isomorphic to some* $\ell_1(\Gamma')$.

Let us explain why: Projective and injective spaces of **Ban** share a property: there are *enough* of them, which means:

- Given a Banach space X, there is some index set Γ for which there is a quotient map $\pi : \ell_1(\Gamma) \to X$.

- Given a Banach space X, there is some index set Γ for which there is an embedding $\imath : X \to \ell_\infty(\Gamma)$.

The injective case is simple: pick $\Gamma = B_{X^*}$, the unit ball of its dual space $X^* = \mathfrak{L}(X, \mathbb{R})$ and set $\imath(x)(x^*) = \langle x^*, x\rangle$. The projective case is trickier: pick as Γ a "half" of the unit sphere of X (in order to make the forthcoming maps homogeneous) and set $\pi((\lambda_\gamma)_\gamma) = \sum \lambda_\gamma x_\gamma$. Choose "the half" as you please. To present these ideas properly, we recall from [13, Chapter 2] one of the foundational rules of homological thinking: diagrams must be complete — i.e., beginning with 0 and ending with 0. Acting this way, embeddings $\imath$ and quotient maps π can be described more cleanly by a single object: an exact sequence

$$0 \longrightarrow Y \xrightarrow{\ \imath\ } Z \xrightarrow{\ \pi\ } X \longrightarrow 0\,.$$

The sequence is said to be *trivial*, or to *split*, when the subspace $\imath : Y \to Z$ is complemented. The middle space Z in an exact sequence as above is called sometimes a *twisted sum* of Y and X (in contrast with the *direct sum* $Y \oplus X$, which is the vector product space $Y \times X$ endowed with the product norm, which appears precisely when the exact sequence is trivial). The existence of enough projectives or injectives can now be reformulated as:

- Every Banach space X admits an exact sequence

$$0 \longrightarrow \ker \pi \longrightarrow \ell_1(\Gamma) \xrightarrow{\ \pi\ } X \longrightarrow 0$$

  usually called a *projective presentation* of X.
- Every Banach space X admits an exact sequence

$$0 \longrightarrow X \xrightarrow{\ \imath\ } \ell_\infty(\Gamma) \longrightarrow \operatorname{coker} \imath \longrightarrow 0$$

  usually called an *injective presentation* of X.

Being clear that a projective presentation of an already projective space must split, projective spaces coincide with the complemented subspaces of $\ell_1(\Gamma)$. It is also true that injective presentations of injective spaces must split and, therefore, injective spaces coincide with the complemented subspaces of some $\ell_\infty(\Gamma)$, but the obnoxious point here is that no known classification is nowadays available for complemented subspaces of $\ell_\infty(\Gamma)$ when Γ is uncountable (the infinite-dimensional complemented subspaces of $\ell_\infty(\mathbb{N})$ are just isomorphic to $\ell_\infty(\mathbb{N})$; see, e.g., [4]).

Now, if you think we are already acting categorically, let me say we are not. Not yet. The reason is that exact sequences of Banach spaces have entered the game, and exact sequences of Banach spaces come with an intrinsic drawback: there exist nontrivial exact sequences of the form

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathsf{R} \longrightarrow \ell_1 \longrightarrow 0$$

[81] whose twisted sum space R cannot be therefore a Banach space by the Hahn-Banach theorem. If R is not a Banach space, what is it? It is a quasi-Banach space which, intuitively speaking, is *like* a Banach space except that its closed unit ball is a bounded set but not necessarily convex. A beautiful result [13, Proposition 2.3.4] shows that, nevertheless, the middle space Z in any exact sequence

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

in which both Y, X are quasi-Banach spaces must itself be a quasi-Banach space. Moreover, quasi-Banach spaces admit a gradation: p-Banach spaces. Thus, "between" the categories of Banach and quasi-Banach spaces we encounter the category **pBan** of p-Banach spaces. Unfortunately, it is false that the middle space Z in an exact sequence

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

in which both Y, X are p-Banach spaces must itself be a p-Banach space [13, Proposition 2.3.4]. In functional analytic terms, all of this can be summarized as follows: being p-Banach space is not a 3-space property, $0 < p \leq 1$, while being a quasi-Banach space is a 3-space property. See Problem 30.24. As an inevitable consequence, any attempt to study exact sequences of Banach spaces must involve quasi-Banach spaces.

Since we have entered into the world of exact sequences, we need to set an identity for them. Two exact sequences $0 \to Y \to Z \to X \to 0$ and $0 \to Y \to Z' \to X \to 0$ are said to be equivalent if and only if there exists a bounded linear operator $T : Z \to Z'$ making a commutative diagram

$$\begin{tikzcd}
 & & Z \arrow[dd, "T"] \arrow[dr] & & \\
0 \arrow[r] & Y \arrow[ur] \arrow[dr] & & X \arrow[r] & 0\,. \\
 & & Z' \arrow[ur] & &
\end{tikzcd}$$

It is customary to denote by $\mathrm{Ext}(X,Y)$ the vector space of exact sequences

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

modulo the equivalence relation. The previous observations that **Ban** and **pBan** lack the 3-space property in **QBan** make necessary to introduce some notation:

- $\mathrm{Ext}_{\mathbf{Q}}(X,Y)$ is the vector space of all exact sequences $0 \to Y \to Z \to X \to 0$ of quasi-Banach spaces, modulo the standard equivalence relation.
- $\mathrm{Ext}_{\mathbf{B}}(X,Y)$ is the vector space of all exact sequences $0 \to Y \to Z \to X \to 0$ (modulo the standard equivalence relation) in which the three entries Y, Z, X are Banach spaces.
- $\mathrm{Ext}_{\mathbf{p}}(X,Y)$ is the vector space of all exact sequences $0 \to Y \to Z \to X \to 0$ (modulo the standard equivalence relation) in which the three entries Y, Z, X are p-Banach spaces.

In some occasions (for instance, when X is superreflexive [60, 13] or an $\mathcal{L}_\infty$-space [64] then $\mathrm{Ext}_{\mathbf{Q}}(X,Y) = \mathrm{Ext}_{\mathbf{p}}(X,Y) = \mathrm{Ext}_{\mathbf{B}}(X,Y)$ and we will just write $\mathrm{Ext}(X,Y)$. Thus, in real (mathematical) life, it is probably a good idea that whenever one is thinking about saying something about the category **Ban** , think also how to say it about **QBan** and **pBan** . So, let us say something about projective spaces in **QBan** or **pBan** .

The category **pBan** is very much like that of Banach spaces (case $p = 1$) and, thus, it is no surprise that every p-Banach space is a quotient of some $\ell_p(\Gamma)$ (same proof, same difficulties as in the case $p = 1$), and that a complemented subspace of $\ell_p(\Gamma)$ is isomorphic to some $\ell_p(\Gamma')$, for $0 < p \leq 1$. This last part is Ortyński's theorem [74]. A complete detailed account, uncovering a few dramatic surprises can be found in [19]. The inexorable consequence of all this is that $\ell_p(\Gamma)$ are the only projective spaces in **pBan** [74] and, consequently, since $\ell_p(\Gamma)$ is no longer projective in the category of q-Banach spaces when $q < p < 1$, we get:

Proposition 23.3. *The category* **QBan** *does not have infinite-dimensional projective objects.*

These bad news quickly turn into very bad news:

Proposition 23.4. *The category* **QBan** *does not have nonzero injective objects.*

In fact, what we will actually show is that the category **pBan** does not have, for $p < 1$, nonzero injective objects. The proof, that can be followed in detail in [13, 1.8.3 and 2.9.1], can be sketched as:

Proof. Let $\mathfrak{I}$ be a possible injective space. Consider the exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow L_p \longrightarrow L_p/\mathbb{R} \longrightarrow 0$$

for $p < 1$ and apply homology [13, Chapter 4] to get

$$\begin{array}{ccccccccc}
\mathfrak{L}(L_p, \mathfrak{I}) & \longrightarrow & \mathfrak{L}(\mathbb{R}, \mathfrak{I}) & \longrightarrow & \operatorname{Ext}(L_p/\mathbb{R}, \mathfrak{I}) & \longrightarrow & \operatorname{Ext}(L_p, \mathfrak{I}) & \longrightarrow & \cdots \\
\| & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathfrak{I} & \longrightarrow & \operatorname{Ext}(L_p/\mathbb{R}, \mathfrak{I}) & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

where the first identity $L_p^* = \mathfrak{L}(L_p, \mathbb{R}) = 0$ is a classical result of Day (see [13, 1.1.5]) while $\operatorname{Ext}(L_p, \mathbb{R}) = 0$ is a result of Kalton (see [13, Theorem 3.7.8]. Hence $\mathfrak{I} = \operatorname{Ext}(L_p/\mathbb{R}, \mathfrak{I})$. Here it is when the nontrivial part of the proof comes: [13, Corollary of 1.8.3] shows that $\mathfrak{L}(L_p, \mathfrak{I}) = 0$ for $0 < p < 1$. Now, if $\mathfrak{I} \neq 0$ there is a non-zero operator $\mathbb{R} \to \mathfrak{I}$ that cannot be extended to L_p, and $\mathfrak{I}$ cannot be injective. ∎

A résumé of all this is:

- the category **Ban** has enough injective or projective objects;
- the category **pBan** has enough projective objects but not injective objects;
- the category **QBan** does not have projective or injective objects.

23.2. Projective and injective presentations. We now enter into serious business. The core idea is that mathematical objects never come alone. A Banach space X is not just a vector space plus a norm, but it rather contains a lot of structure, relationships and connections, even if some of them might be "invisible" depending on the approach one adopts. For instance, consider the Hilbert space. A milestone theorem says that all Hilbert spaces of the same dimension are isometric, which can be read as "there is only one Hilbert space of a given dimension". But that is not all. The Hilbert spaces $\ell_2, L_2(0,1), L_2(\mathbb{R})$ are isometric but they are not "the same". A way to observe the difference is to realize that "the structure" of ℓ_2 involves all the other ℓ_p spaces, including the quasi-Banach spaces $0 < p < 1$ while the structure of

$L_2(0,1)$ involves all the other spaces $L_p(0,1)$. And, freely quoting some lines in [26]: *by the virtues of classical Riesz interpolation theorem, when a linear operator $\ell_\infty \to \ell_\infty$ also acts continuously from ℓ_1 to ℓ_1 it automatically acts continuously from ℓ_p to ℓ_p*. While it is not true that an operator sending L_1 to L_1 and L_∞ to L_∞ also sends ℓ_2 to ℓ_2 (whatever that could mean). And there are much worse Hilbert spaces around.

Homology focuses on considering as part of the structure of X its projective presentations $0 \to \kappa \to \mathcal{P} \to X \to 0$ as well as its injective presentations $0 \to X \to \mathcal{I} \to c\kappa \to 0$. As you see, we have started to denote with κ (sometimes $\kappa(X)$ when several spaces are involved) a generic kernel of a projective presentation of X and with $c\kappa$ (sometimes $c\kappa(X)$ when several spaces are involved) a generic cokernel of an injective presentation of X. An issue to deal with here is that there are many non-equivalent (as exact sequences) projective presentations of the same object: for instance, if X is a separable Banach space, two exact sequences $0 \to \kappa \to \ell_1 \to X \to 0$ and $0 \to \ell_1(\Gamma)\oplus\kappa \to \ell_1(\Gamma)\oplus\ell_1 \to X \to 0$ define, for uncountable Γ, non-equivalent projective presentations of X. In which sense, all projective presentations are "the same" ? A first answer comes from the diagonal principles [13, 2.11], which establish that two projective presentations of X in **Ban** or **pBan** are connected as follows: there are isomorphisms α, β making a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{P}' \oplus \kappa \arrow[r] \arrow[d, "\alpha"'] & \mathcal{P}' \oplus \mathcal{P} \arrow[r] \arrow[d, "\beta"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{P} \oplus \kappa' \arrow[r] & \mathcal{P} \oplus \mathcal{P}' \arrow[r] & X \arrow[r] & 0\,. \end{tikzcd}$$

Therefore, kernels of projective presentations are unique "up to a product with a projective space". A way to manage that is introducing a new category to make all projective presentations isomorphic in that category. This line was pursued in [73] (see also [39]). We will just keep here record of that fact while threading another categorical track: can kernels of projective presentations be chosen functorially? After all, all you touch and all you see is all your mathematical life will ever be. Is it possible to find a Banach functor κ: **Ban** $\to$ **Ban** such that $\kappa(X)$ "is" (the word that encompasses all the problems) the kernel of a projective presentation of X? The answer is mostly yes [13, Theorem 3.10.2], as we will explain it soon.

To understand what we are looking for, let us observe what a projective presentation of X does: it provides a representation for every exact sequence $0 \to Y \to Z \to X \to 0$ in the pushout form

$$\begin{tikzcd} 0 \arrow[r] & \ker\pi \arrow[r] \arrow[d, "\phi"'] & \mathcal{P} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] & 0\,, \end{tikzcd}$$

so that exact sequences $0 \to Y \to Z \to X \to 0$ correspond to operators ϕ. This correspondence can be carried to a natural equivalence: set the equivalence relation for operators $\kappa(X) \to Y$ given by $\tau \sim \tau'$ if and only if $\tau - \tau'$ can be extended to an operator $\mathcal{P} \to Y$ to obtain $\mathrm{Ext}_{\mathbf{B}}(X,Y) = \mathfrak{L}(\kappa(X),Y)/\sim$ and $\mathrm{Ext}_{\mathbf{p}}(X,Y) = \mathfrak{L}(\kappa(X),Y)/\sim$. No analogous description exists in **QBan** .

23.3. Quasilinear maps. The reader can review what we already said about this topic in Parts I and II [24, 25]. We will however sketch the exposition in [13].

Definition 23.5. Let X and Y be quasinormed spaces. A map $\Phi : X \to Y$ is quasilinear if it is homogeneous and there is a constant Q such that

$$\|\Phi(x+y) - \Phi(x) - \Phi(y)\| \leq Q\left(\|x\| + \|y\|\right)$$

for every $x, y \in X$. The quasilinearity constant of Φ, denoted $Q(\Phi)$, is the infimum of the numbers Q above.

The functional $\|(y,x)\|_\Phi = \|y - \Phi(x)\| + \|x\|$ is a quasinorm on $Y \times X$, and if we denote by $Y \oplus_\Phi X$ the corresponding quasinormed space then $0 \to Y \xrightarrow{\imath} Y \oplus_\Phi X \xrightarrow{\pi} X \to 0$ with $\imath(y) = (y,0)$ and $\pi(y,x) = x$ is an exact sequence in which $\imath$ is an into isometry and π a quotient map sending the unit ball of $Y \oplus_\Phi X$ onto the unit ball of X.

Definition 23.6. A homogeneous map $\Phi : X \to Y$ is said to be p-quasilinear if there is $C > 0$ such that for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in X$ one has

$$\left\| \Phi\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} \Phi(x_i) \right\| \leq C \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}. \tag{23.1}$$

The least possible constant C above shall be referred to as the p-quasilinearity constant of Φ and denoted by $Q^{(p)}(\Phi)$. Each p-quasilinear map is quasilinear and, indeed, p-quasilinearity can be seen as a stronger form of quasilinearity involving an arbitrary number of variables instead of two. We will denote $\mathsf{Q}^{(p)}$ the space of p-quasilinear maps between two given spaces.

The connection between the quality of the twisted sum and the properties of Φ is given by the combination of [13, Proposition 3.3.7]and [13, Proposition 3.6.7]:

PROPOSITION 23.7. *For every exact sequence* $0 \to Y \to Z \to X \to 0$ *there is a quasilinear map* $\Phi : X \to Y$ *and a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "\imath"] \arrow[d, equal] & Y \oplus_\Phi X \arrow[r, "\pi"] \arrow[d] & X \arrow[r] \arrow[d, equal] \arrow[ll, dashed, bend right, "\Phi"'] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] & 0\,. \end{tikzcd} \tag{23.2}$$

Moreover, if X, Y *are* p*-Banach spaces then* Φ *is* p*-quasilinear if and only if* $Y \oplus_\Phi X$ *is isomorphic to a* p*-Banach space.*

We can thus obtain natural equivalences for the functors $\mathrm{Ext}_{\mathbf{Q}}$, $\mathrm{Ext}_{\mathbf{B}}$ and $\mathrm{Ext}_{\mathbf{p}}$ as spaces of quasilinear maps. To do so we just need to translate the equivalence relation of exact sequences to the nonlinear world [13, Lemma 3.3.2]: the exact sequence induced by the quasilinear map Φ is trivial (splits) if and only if Φ is the sum of a bounded plus a linear map. Therefore,

DEFINITION 23.8. Two quasilinear maps Ω, Φ will be called *equivalent*, denoted $\Omega \equiv \Phi$, when their difference $\Omega - \Phi$ can be written as the sum of a bounded plus a linear map.

We will need the following spaces of functions:

- B, the space of homogeneous bounded maps;
- L, the space of linear maps;
- Q, the space of quasilinear maps;
- $\mathsf{Q}^{(p)}$, the space of p-quasilinear maps, $0 < p \leq 1$;
- $\mathfrak{L}$ the space of bounded operators.

One has, for $0 < r < p \leq 1$, the containments:

$$\begin{tikzcd} & \mathsf{B} \arrow[dr] & & & \\ \mathfrak{L} = \mathsf{B} \cap \mathsf{L} \arrow[ur] \arrow[dr] & & \mathsf{Q}^{(p)} \arrow[r] & \mathsf{Q}^{(r)} \arrow[r] & \mathsf{Q}\,. \\ & \mathsf{L} \arrow[ur] & & & \end{tikzcd}$$

To overhaul from there the identity $\mathrm{Ext}(X,Y)=\mathcal{Q}(X,Y)$ we need to also consider the spaces:

- $\mathsf{Q_L}=\mathsf{Q}/\mathsf{L}$, the space of quasilinear maps modulo linear perturbations;
- $\mathsf{Q_B}=\mathsf{Q}/\mathsf{B}$, the space of quasilinear maps modulo homogeneous bounded perturbations;
- $\mathcal{Q}=\mathsf{Q}/(\mathsf{B}+\mathsf{L})$.

There are analogous versions for p-quasilinear maps $\mathsf{Q}^{(p)}$, $\mathsf{Q}_{\mathsf{L}}^{(p)}$, $\mathsf{Q}_{\mathsf{B}}^{(p)}$ and $\mathcal{Q}^{(p)}$. Since, as the caterpillar advices [5], identity is a tricky business, those objects are different. An element $\Omega\in\mathsf{Q}(X,Y)$ corresponds to a precise exact sequence $0\to Y\to Y\oplus_\Omega X\to X\to 0$. An element in $\mathsf{Q_L}(X,Y)$ corresponds to the class of an exact sequence $0\to Y\to Z\to X\to 0$ modulo the relation: there is a linear isometry (instead of a mere isomorphism) T making the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Y\oplus_\Omega X \arrow[r] \arrow[d, "T"'] & X \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

commute. The space $\mathsf{Q_L}(X,Y)$ endowed with the quasilinearity constant is a quasi-Banach space, and the corresponding p-version $\mathsf{Q}_{\mathsf{L}}^{(p)}(X,Y)$ acting between p-Banach spaces is a p-Banach space [13, Section 3.6 and Corollary 3.6.4]. The spaces $\mathsf{Q}(X,Y)$ and $\mathcal{Q}(X,Y)$ are complete but not necessarily Hausdorff as well as $\mathsf{Q_B}(X,Y)$. A non-obvious exhibition of its non-Hausdorff character can be found in [13, Proposition 4.5.5].

All that work yields the desired natural equivalences [13, Section 4.4]:

- $\mathrm{Ext}_{\mathbf{Q}}=\mathcal{Q}$;
- $\mathrm{Ext}_{\mathbf{p}}=\mathcal{Q}^{(p)}$;
- $\mathrm{Ext}_{\mathbf{B}}=\mathcal{Q}^{(1)}$.

23.4. The z-dual. The spaces $\mathsf{Q_L}$ are important now. Given a Banach space X, the z-dual X^z of X is defined as $\mathsf{Q}_L^{(1)}(X,\mathbb{R})$ the z-dual. It can be identified with the space of 1-quasilinear maps $X\to\mathbb{R}$ vanishing on a prefixed

[5] "Advice from a caterpillar", from the musical Wonderland.

Hamel basis $\mathcal{H}$ of X and it is a Banach space. Different choices of bases $\mathcal{H}$ yield different but isometric z-duals. Consider the closed subspace

$$\mathrm{co}^{(1)}(X) = \overline{\mathrm{span}\Big\{\delta_x \in (X^z)^* : x \in X\Big\}} \subset (X^z)^*.$$

There is a natural 1-quasilinear map $\mho : X \to \mathrm{co}^{(1)}(X)$ given by $\mho(x) = \delta_x$ that vanishes on $\mathcal{H}$ and generates an exact sequence

$$0 \longrightarrow \mathrm{co}^{(1)}(X) \longrightarrow \mathrm{co}^{(1)}(X) \oplus_{\mho} X \longrightarrow X \longrightarrow 0\,.$$

This sequence has an extraordinary property [13, Theorem 3.10.2]: given an exact sequence $0 \to A \to B \to X \to 0$ defined by a 1-quasilinear map $\Psi : X \to A$ that vanishes on $\mathcal{H}$ there is an operator $\phi_\Psi : \mathrm{co}^{(1)}(X) \to A$ such that $\phi_\Psi \mho = \Psi$; which, in particular, yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{co}^{(1)}(X) \arrow[r] \arrow[d, "\phi_\Psi"'] & \mathrm{co}^{(1)}(X) \oplus_{\mho} X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & X \arrow[r] & 0\,.
\end{tikzcd}$$

What we have said means slightly more than what we have said: $\mho$ behaves like a projective presentation of X, even if we are unable to decide whether $\mathrm{co}^{(1)}(X) \oplus_{\mho} X$ is isomorphic to a projective Banach space. A slightly more delicate construction $\mathrm{co}^{(p)}(\cdot)$ (see [13]) can be performed for p-Banach spaces. In all cases, the correspondence $X \to \mathrm{co}^{(p)}(X)$ is functorial in the following sense: consider the category **pBanH** whose objects are pairs $(X, \mathcal{H})$ formed by a p-Banach space X and a Hamel basis $\mathcal{H}$ for X; and its morphisms $\phi : (X, \mathcal{H}) \to (Y, \mathcal{H}')$ are operators $\phi : X \to Y$ sending $\mathcal{H}$ into $\mathcal{H}'$. Then,

PROPOSITION 23.9. $\mathrm{co}^{(p)} : \mathbf{pBanH} \to \mathbf{pBanH}$ *is a functor.*

The forthcoming exposition for injective presentations is formally a dual-verbatim repetition of the previous one. But conceptually it is not. Homology also focuses on considering as part of the structure of Y the injective presentations $0 \to Y \to \mathcal{I} \to c\kappa \to 0$ of Y, where $\mathcal{I}$ is an injective object. The issue that there are many non-equivalent (as exact sequences) injective presentations of the same object can be treated as before: the diagonal principles [13, 2.11] establish that two injective presentations of Y in **Ban** are connected: there are isomorphisms β, γ making a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & \mathcal{I} \oplus \mathcal{I}' \arrow[r] \arrow[d, "\beta"] & c\kappa \oplus \mathcal{I}' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & Y \arrow[r] & \mathcal{I}' \oplus \mathcal{I} \arrow[r] & c\kappa' \oplus \mathcal{I} \arrow[r] & 0\,.
\end{tikzcd}$$

Therefore, cokernels of injective presentations are unique "up to a product with an injective space". Can cokernels of injective presentations be chosen functorially? Well, here is when things become interesting, in a funny way. On one hand, injective presentations provide a representation for every exact sequence $0 \to Y \to Z \to X \to 0$ in the pullback form

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & \mathcal{I} \arrow[r] & c\kappa(Y) \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & Z \arrow[r] \arrow[u] & X \arrow[r] \arrow[u] & 0\,,
\end{tikzcd}$$

so that exact sequences $0 \to Y \to Z \to X \to 0$ correspond to operators $\phi : X \to c\kappa(Y)$. This correspondence can be carried to a natural equivalence: set the equivalence relation for operators $X \to c\kappa(Y)$ given by $\tau \sim \tau'$ if and only if $\tau - \tau'$ can be lifted to an operator $X \to \mathcal{I}$ to obtain $\operatorname{Ext}_{\mathbf{Ban}}(X,Y) = \mathfrak{L}(X, c\kappa(Y))/\sim$ (see [13, Section 2.6]). No analogous description exists in either **pBan** or **QBan** .

On the other hand, the analogue of $\mathrm{co}^{(1)}(\cdot)$ does not exist. And if it exists we do not know it. But the z-dual can take its place sometimes. Indeed, if we consider a projective presentation of X,

$$0 \longrightarrow \kappa \longrightarrow \mathcal{P} \longrightarrow X \longrightarrow 0\,,$$

since the dual $\mathcal{P}^*$ of a projective space is injective we obtain the following injective presentation for X^*:

$$0 \longrightarrow X^* \longrightarrow \mathcal{P}^* \longrightarrow \kappa^* \longrightarrow 0\,.$$

Observe that the projective presentation above and the sequence generated by $\mho$ are semi-equivalent (in the sense of [13]), which means that one is pushout from the other

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{co}^{(1)}(X) \arrow[r] \arrow[d, leftrightarrow] & \mathrm{co}^{(1)}(X) \oplus_{\mho} X \arrow[r] \arrow[d, leftrightarrow] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \kappa \arrow[r] & \mathcal{P} \arrow[r] & X \arrow[r] & 0\,,
\end{tikzcd}$$

therefore, by dualization, we get

$$\begin{tikzcd}
0 \arrow[r] & X^* \arrow[r] \arrow[d, equal] & \left(\mathrm{co}^{(1)}(X) \oplus_{\mho} X\right)^* \arrow[r] \arrow[d, leftrightarrow] & X^z \arrow[r] \arrow[d, leftrightarrow] & 0 \\
0 \arrow[r] & X^* \arrow[r] & \mathcal{P}^* \arrow[r] & \mathcal{P}^*/X \arrow[r] & 0\,.
\end{tikzcd}$$

Therefore, it is quite tempting to explore to what extent this peculiar type of injective presentations, observed by Auslander and studied by Martsinkovsky in [69] in connection with the Auslander-Reiten formula, may take the form (with all due precautions)

$$0 \longrightarrow X^* \longrightarrow \mathcal{P}^* \longrightarrow X^z \longrightarrow 0\,.$$

And once this is the case, the "innocuous-looking" (as Martsinkovsky describes it) construction by Auslander of the so-called transpose of X

$$0 \longrightarrow X^* \longrightarrow \mathcal{P}_1^* \longrightarrow \mathcal{P}_2^* \longrightarrow \mathrm{Tr}(X) \longrightarrow 0$$

would then appear as if $\mathrm{Tr}(X)$ were the z-dual of the cokernel of X^*. We will not pursue this direction here. We merely remark that, if so, in the same way that the "projective" presentation using $\mathrm{co}^{(1)}$ yields a neat description of the kernel, the "injective" presentation above provides a neat representation of the quotient space as the space of 1-quasilinear maps vanishing on $\mathcal{H}$. Finally,

Proposition 23.10. *The correspondence $X \to X^z$ generates a functor* **BanH** $\to$ **BanH**.

Proof. Indeed, given $\tau : (X, \mathcal{H}) \to (Y, \mathcal{H}')$ and $[\omega] \in \mathcal{Q}_L^{(1)}(Y, \mathbb{R})$ then $[\omega\tau] \in \mathcal{Q}_L^{(1)}(X, \mathbb{R})$. ∎

To conclude this section, observe that the representations of exact sequences as either pushouts from projective presentation or pullbacks of injective presentation immediately yield a couple of characterization of injective and projective spaces that will be exploited later:

Proposition 23.11. *Let E be a Banach space:*

(i) *E is injective if and only if* $\mathrm{Ext}_{\mathbf{B}}(\cdot, E) = 0$.
(ii) *E is projective if and only if* $\mathrm{Ext}_{\mathbf{B}}(E, \cdot) = 0$.

A p-Banach space E is projective if and only if $\mathrm{Ext}_{\mathbf{p}}(E,\cdot)=0$.

The characterization of $\mathrm{Ext}_{\mathbf{B}}(\cdot,E)$ as the derived functor of $\mathfrak{L}(\cdot,E)$ or, if one prefers, the outcome of applying the functor $\mathfrak{L}(\cdot,E)$ to a projective presentation of X, yields the exact sequence:

$$0 \longrightarrow \mathfrak{L}(X,E) \longrightarrow \mathfrak{L}(\mathcal{P},E) \longrightarrow \mathfrak{L}(\kappa,E) \longrightarrow \mathrm{Ext}_{\mathbf{B}}(X,E) \longrightarrow 0\,.$$

Dually, the characterization of $\mathrm{Ext}_{\mathbf{B}}(E,\cdot)$ as the derived functor of $\mathfrak{L}(E,\cdot)$ or, if one prefers, the outcome of applying the functor $\mathfrak{L}(E,\cdot)$ to the injective presentation of X yields the exact sequence

$$0 \longrightarrow \mathfrak{L}(E,X) \longrightarrow \mathfrak{L}(E,\mathcal{I}) \longrightarrow \mathfrak{L}(E,c\kappa) \longrightarrow \mathrm{Ext}_{\mathbf{B}}(E,X) \longrightarrow 0\,.$$

Definition 23.12. A functor is called *exact* if it transforms exact sequences into exact sequences.

Observe that exact functors must be additive, since they must preserve the zero object (that is, trivial sequences). The same applies to *left*-exact or *right* exact functors (whose formal definition will appear later, though their meaning can easily be anticipated). It is therefore clear that a Banach (resp. p-Banach) space E is projective if and only if $\mathfrak{L}(E,\cdot)$ is exact, and that it is injective if and only if $\mathfrak{L}(\cdot,E)$ is exact. Once we have accepted that all (homological) things we perform in Banach and quasi-Banach spaces are relative, it remains only to choose the meaning of such relativization. We will focus on two interesting ways of doing so: relativization of objects by considering spaces of a given size, and relativization of morphisms using operator ideals.

23.5. Relative homology in size. The idea behind relativization in size requires, first of all, fixing a measure for the size of a space. Cardinality is usually not a good choice: after all, c_0 and ℓ_∞ have the same cardinality, yet the former is separable and the latter is not – and separability is important. Thus, since separability matters, let us measure instead the cardinal of a dense subset; this measure is usually called the density character of the space. But again, the density character of $\mathbb{R}$ is the same as that of c_0. A better choice seems to be the *dimension* (cf. [13]): the smallest cardinality of a set spanning a dense subspace. In this form, everything fits: the dimension of $\mathbb{R}^n$ is n, separable spaces are precisely those having dimension $\aleph_0$ and, from that point on, dimension and cardinal coincide. We will write $\dim X$ to mean the dimension of X and use the prefix *separably* to describe spaces of dimension $\aleph_0$.

Observe now a crucial bifurcation: separably injective and separably projective spaces. The definitions seem canonical: A Banach, p-Banach or quasi-Banach space $\mathcal{SI}$ is called separably injective (SI for short) when, given Y, Z separable spaces in the same category, an into isomorphism $Y \to Z$ and an operator $Y \to \mathcal{SI}$, there is an operator $Z \to \mathcal{SI}$ making the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[dl, dashed] \\ \mathcal{SI} & \end{tikzcd}$$

commute. That is $\mathcal{SI}$-valued operators can be extended from subspaces of separable spaces to the whole space. Equivalently, when the functor $\mathfrak{L}(\cdot, \mathcal{SI})$ is exact when acting in the separable spaces of the category. Or, yet another equivalent formulation, when $\mathrm{Ext}(S, \mathcal{SI}) = 0$ whenever S is separable. A space $\mathcal{SP}$ is called separably projective (SP for short) when operators from $\mathcal{SP}$ to a quotient of a separable space can be lifted to the whole space making the diagram

$$\begin{tikzcd} Z \arrow[r] & X \\ & \mathcal{SP} \arrow[u] \arrow[ul, dashed] \end{tikzcd}$$

commute. Equivalently, the functor $\mathfrak{L}(\mathcal{SP}, \cdot)$ is exact when acting on the separable spaces of the category.

Separably injective spaces turned out to have a rich theory behind (displayed in [4] and [13]), together with a large supply of examples living in nature –something that the theory of injective spaces notably lacks. Moreover, it is also natural to consider the notion of *universal separable injectivity*: a space $\mathcal{USI}$ is said to be universally separably injective when $\mathcal{USI}$-valued operators from separable spaces can be extended anywhere (in the category). To see how natural these notions are, observe that in the fundamental exact sequence

$$0 \longrightarrow c_0 \longrightarrow \ell_\infty \longrightarrow \ell_\infty/c_0 \longrightarrow 0$$

the space c_0 is separably injective, ℓ_∞ is injective and ℓ_∞/c_0 is universally separably injective [4]. Oter examples of USI spaces include ultrapowers (via good ultrafilters) of $\mathcal{L}_{\infty,\lambda}$-spaces.

But –be careful [6]– c_0 is the only separable separably injective Banach space,

[6] with that axe, Eugene

by Zippin's theorem (see [13, 1.7.3]). This implies that the category of separable Banach spaces has injective objects, but not enough of them. Furthermore, no separable infinite dimensional injective space can exist, since c_0 is not injective. The reader is invited to check that there are no separably injective spaces in **QBan** either.

If we turn our attention to separably projective Banach spaces, a break of symmetry appears: there exists a unique separable SP space ℓ_1, but this already means that the category of separable Banach spaces admits enough projective spaces. Much worse than that is that we do not have answers to the following

Problem 23.13. Do there exist separably projective Banach spaces other than $\ell_1(\Gamma)$?

This apparently simple question is deeper than it seems, because there is no straightforward way to parse what happens with separably projective spaces. Observe that we have left aside the third characterization: we will say that a Banach space E is SPE if $\mathrm{Ext}(E, S) = 0$ for every separable space S. It is clear that SPE implies SP, but it is by no means clear whether SP implies SPE: Why? Because, as we said above, any injective presentation $0 \to X \to \ell_\infty \to \ell_\infty/X \to 0$ of X involves non-separable spaces. A third notion of separable projectivity can be introduced following the idea of universal separable injectivity: A Banach space E is universally separably projective (USP in short) if, given a quotient map $Z \to S$ with S separable, every operator $\tau : E \to S$ can be lifted to any Z. Obviously USP implies SP, hence SPE.

Now, SP spaces must be $\mathcal{L}_1$-spaces by [13, Proposition 5.2.10]. The next step could be proving that an SP space enjoys some of the classical properties of $\ell_1(\Gamma)$ spaces: the Schur property (weakly null sequences are norm null), or the Separable Complementation Property (SCP for short): every separable subspace is contained in a separable complemented subspace. For instance, $\ell_\infty(\Gamma)$ does not have SCP, while reflexive Banach spaces do.

In any reasonable world –wish this were one– one should be able to prove that SP spaces enjoy the Schur property; and this should follow from Kalton's theorem [13, Proposition 8.6.4]: If E is separable and $\mathrm{Ext}_{\mathbf{B}}(E, C[0,1]) = 0$ then E has the Schur property – assuming a nonseparable version of the theorem were true. See [41, Proposition 9 and Proposition 10] for some related results. But it seems unlikely that this world is reasonable. Anyway:

Problem 23.14. Does $\mathrm{Ext}_{\mathbf{B}}(E, C[0,1]) = 0$ imply that E has the Schur property?

Another guess (suggested by Félix Cabello) which might shed some light on our vision obscured by the clouds, would be to show that the kernel of a projective presentation of a separably projective space is projective. The SCP could be a good fulcrum for that problem:

Proposition 23.15. *The kernel of a projective presentation of a separably projective space has SCP.*

Proof. Let $0 \to \kappa \to \mathcal{P} \to X \to 0$ be a projective presentation of X and $S \subset \kappa$ a separable subspace. Since S is a separable subspace of $\mathcal{P} \simeq \ell_1(\Gamma)$ there is a separable complemented subspace $S' \simeq \ell_1$ of $\ell_1(\Gamma)$ containing S. Let $\pi : \mathcal{P} \to S'$ the projection and consider S'' the closure of $\pi[\kappa]$. Since X is separably projective, the operator $\pi|_\kappa : \kappa \to S''$ extends to an operator $\pi' : \mathcal{P} \to S''$. The operator $\pi'|_\kappa : \kappa \to S''$ is a projection since $\pi'(\pi(k)) = \pi(\pi(k)) = \pi(k)$ for every $k \in \kappa$. ∎

The reader is addressed to [19] for old and new results concerning the SCP.

24. On the run

As its name clearly indicates, this section attempts to put the reader on the right track by providing the Banach, categorical, and homological elements that may be needed.

After [24, 25] and [13], it is probably too late to explain what a category, a functor or a natural transformation are. Or what a Banach space is, for that matter. But perhaps not a quasi-Banach space. Just in case: A quasinorm on a real or complex vector space X is a map $\|\cdot\| : X \to \mathbb{R}^+$ that satisfies

- $\|x\| = 0 \implies x = 0$,
- $\|\lambda x\| = |\lambda|\|x\|$,
- $\|x + y\| \leq \Delta(\|x\| + \|y\|)$,

for all $x, y \in X$, all scalars λ and some constant $\Delta \geq 1$. If $\Delta = 1$, then $\|\cdot\|$ is a norm. A quasinormed space is a vector space equipped with a quasinorm; when the space is moreover complete is called a quasi-Banach space. If the quasinorm is a norm then it will be called a Banach space.

An operator is a linear continuous map. A linear map $u : X \to Y$ acting between quasinormed spaces is continuous if and only if it is bounded, with the meaning $\|u\| = \sup_{\|x\|\leq 1} \|u(x)\| < \infty$. The space of all operators from X to Y is denoted by $\mathfrak{L}(X, Y)$. The space $\mathfrak{L}(X, \mathbb{K})$ is the *dual* space of X, usually denoted by X^*, and is always a Banach space. A p-norm, $0 < p \leq 1$, is a quasinorm satisfying $\|x + y\|^p \leq \|x\|^p + \|y\|^p$. Obviously p-norms are q-norms for $0 < q < p$. The Aoki-Rolewicz Theorem establishes that each quasinorm is equivalent to some p-norm.

Short exact sequences of quasi-Banach or p-Banach spaces have already appeared: they are diagrams

$$0 \longrightarrow Y \xrightarrow{\imath} Z \xrightarrow{\pi} X \longrightarrow 0$$

formed by objects in the category (i.e., quasi-Banach or p-Banach spaces) and bounded linear operators, with the property that the kernel of each arrow is the image of the preceding one. The sequence is said to be trivial, or to *split*, when the subspace $\imath : Y \to Z$ is complemented. Once exact sequences have been isolated, the categories **Ban** and **QBan** are *exact* categories [12]. Pullback and pushout constructions (arbitrary in **pBan** and finite in **QBan**) exist in those categories; details can be found in [13] and [24, 25]. Still, if at any moment some additional fact is required, we will make a short, sharp, shock statement.

To grasp the idea behind homological derivation, think of functors as functions: just as the derivative of a function measures how far it is from being constant, the derivative of a functor measures how far it is from being exact. How does homological derivation achieves this? The reader may learn everything from, say, [58]; or from the shortened but still sufficiently detailed account in [13, 4.6.2]; or may simply keep reading.

The first thing to tidy up is that functors can be either covariant or contravariant. And the derivation process can be carried out using either injective or projective presentations. Derivation of a covariant functor via projective elements is usually called left derivation, while derivation using injective objects is called right derivation. However, when the functor is contravariant, derivation via projective elements is called right derivation since it is right derivation of the associated covariant opposite functor; and derivation using injective objects is called left derivation. And this is a mess. Following the path we initiated at [31], we attempt to a more intuitive description that highlights only how the derivation is obtained. Thus, we will simply use the terms *projective* derivation and *injective* derivation, regardless of whether the functor is covariant or contravariant.

Now, there is a striking difference between classical and homological derivation: homological derivation of a functor $\mathcal{F}$ produces not just one derived functor, but rather a whole sequence of derived functors $\mathcal{F}^{(n)}$. And it is usually false that the first derived functor of the first derived functor of $\mathcal{F}$ coincides with the second derived functor of $\mathcal{F}$. See Section 26.

Consider the standard Hom functors: the covariant $\mathsf{Hom}(A,\cdot)$ and the contravariant $\mathsf{Hom}(\cdot,A)$. In functor categories these are called the Yoneda functors $\mathcal{Y}_A$ and $\mathcal{Y}^A$, respectively. In the Banach/quasi-Banach context the covariant functor $\mathfrak{L}(A,\cdot)$ will be denoted $\mathfrak{L}_A$; and the contravariant functor $\mathfrak{L}(\cdot,A)$ will be denoted $\mathfrak{L}^A$. The classical foundational result (see [58], [13, Chapter 4] or [24, 25]) is that in a category of modules projective derivation of $\mathsf{Hom}(\cdot,Y)$ at X agrees with injective derivation of $\mathsf{Hom}(X,\cdot)$ at Y. More precisely, there are natural isomorphisms

$$\mathsf{Hom}(X,\cdot)^{(n)}(Y) = \mathsf{Hom}(\cdot,Y)^{(n)}(X)\,.$$

This space is usually denoted $\operatorname{Ext}^n(X,Y)$, and admits several useful representations:

- The Yoneda representation [58, 16] that does not require injective or projective objects: elements of $\operatorname{Ext}^n(X,Y)$ are equivalence classes of exact sequences

  $$0 \longrightarrow Y \longrightarrow Z_n \longrightarrow \;\cdots\; \longrightarrow Z_1 \longrightarrow X \longrightarrow 0$$

  with respect to the equivalence relation: given $\mathcal{E},\mathcal{E}' \in \operatorname{Ext}^n(X,Y)$ we will write $\mathcal{E}\to\mathcal{E}'$ to mean the existence of a commutative diagram

  $$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z_n \arrow[r] \arrow[d] & \cdots \arrow[r] & Z_1 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z_n' \arrow[r] & \cdots \arrow[r] & Z_1' \arrow[r] & X \arrow[r] & 0\,, \end{tikzcd}$$

  while $\mathcal{E}\leftarrow\mathcal{E}'$ means the existence of a commutative diagram

  $$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z_n \arrow[r] & \cdots \arrow[r] & Z_1 \arrow[r] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z_n' \arrow[r] \arrow[u] & \cdots \arrow[r] & Z_1' \arrow[r] \arrow[u] & X \arrow[r] & 0\,. \end{tikzcd}$$

Then $\mathcal{E} \equiv \mathcal{E}'$ if and only if there exists $\mathcal{F}$ such that either $\mathcal{E} \to \mathcal{F} \leftarrow \mathcal{E}'$ or $\mathcal{E} \leftarrow \mathcal{F} \to \mathcal{E}'$

- When there are enough projectives then [50, 16]

$$\mathrm{Ext}^n(X,Y) = \mathsf{Hom}(\kappa_n(X), Y)/\sim$$

where $T \sim S$ if and only if $T - S$ can be extended to a morphism $P_n \to Y$.

- When there are enough injectives then [50, 16]

$$\mathrm{Ext}^n(X,Y) = \mathsf{Hom}(X, c\kappa_n(Y))/\sim$$

where $T \sim S$ if and only if $T - S$ can be lifted to a morphism $X \to I_n$.

Let's move ahead and provide a description of how derivation goes. First of all, let us fix the notation we will use for a *cochain complex*

$$0 \longrightarrow C_0 \xrightarrow{\ \delta^0\ } C_1 \xrightarrow{\ \delta^1\ } C_2 \xrightarrow{\ \delta^2\ } \cdots$$

and its *cohomology groups*:

$$H^n(X) = \frac{\ker \delta^n}{\mathrm{Im}\delta^{n-1}}$$

with the usually overlooked case $H^0(X) = \ker \delta^0$. Also, for a *chain complex*

$$\cdots \longrightarrow C_2 \xrightarrow{\ \delta_2\ } C_1 \xrightarrow{\ \delta_1\ } C_2 \xrightarrow{\ \delta_0\ } 0$$

(watch out that now $\partial_0 = 0$) and its *homology groups*:

$$H_n(X) = \frac{\ker \partial_n}{\mathrm{Im}\partial_{n+1}}$$

with no singular case except, when one gets interested in, coker δ_1.

$\heartsuit$ Injective covariant derivation (usually called Right derivation). It requires starting with an injective resolution

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dr] & & \mathcal{J}_0 \arrow[rr, "\delta^0"] \arrow[dr, "\varpi_0"'] & & \mathcal{J}_1 \arrow[rr, "\delta^1"] \arrow[dr, "\varpi_1"'] & & \mathcal{J}_2 \arrow[rr, "\delta^2"] & & \cdots \\
& X \arrow[ur, "\jmath_0"] & & c\kappa_0 \arrow[ur, "\jmath_1"'] & & c\kappa_1 \arrow[ur, "\jmath_2"'] & & &
\end{tikzcd} \tag{24.1}$$

apply $\mathcal{F}$, obtain

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dr] & & \mathcal{F}(\mathfrak{I}_0) \arrow[rr, "\mathcal{F}(\delta^0)"] \arrow[dr, "\mathcal{F}(\varpi_0)"'] & & \mathcal{F}(\mathfrak{I}_1) \arrow[rr, "\mathcal{F}(\delta^1)"] \arrow[dr, "\mathcal{F}(\varpi_1)"'] & & \mathcal{F}(\mathfrak{I}_2) \arrow[r, "\mathcal{F}(\delta^2)"] & \cdots, \\
& \mathcal{F}(X) \arrow[ur, "\mathcal{F}(\jmath_0)"'] & & \mathcal{F}(c\kappa_0) \arrow[ur, "\mathcal{F}(\jmath_1)"'] & & \mathcal{F}(c\kappa_1) \arrow[ur, "\mathcal{F}(\jmath_2)"'] & &
\end{tikzcd}$$

to then define

$$R^{(n)}\mathcal{F}(X) = H^n(X) = \frac{\ker \mathcal{F}(\delta^n)}{\mathrm{Im}\mathcal{F}(\delta^{n-1})}\,.$$

The 0^{th} case is also interesting and will be more interesting soon: $R^{(0)}\mathcal{F}(X) = H^0(X) = \ker \mathcal{F}(\delta^0)$ generates a natural transformation $\rho_{\mathcal{F}} : \mathcal{F} \to R^{(0)}\mathcal{F}$, which becomes a natural equivalence when $\mathcal{F}$ is left exact, in which case $\mathcal{F}$ is the (right) 0-derived of $\mathcal{F}$. In this and forthcoming cases we have attempted that the notation $\rho^{\mathcal{F}}$ and $\rho_{\mathcal{F}}$ match the notation in [69] as much as possible.

Moreover, given a short exact sequence $0 \to Y \to Z \to X \to 0$ there is a long exact sequence formed by all right derived functors (see [13])

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(X) \arrow[r] & \mathcal{F}(Z) \arrow[r] & \mathcal{F}(Y) \arrow[r] & \mathcal{F}^{(1)}(X) \arrow[r] & \mathcal{F}^{(1)}(Z) \arrow[r] & \mathcal{F}^{(1)}(Y) \arrow[dlllll, dashed] \\
& \mathcal{F}^{(2)}(X) \arrow[r] & \mathcal{F}^{(2)}(Z) \arrow[r] & \mathcal{F}^{(2)}(Y) \arrow[r] & \mathcal{F}^{(3)}(X) \arrow[r] & \mathcal{F}^{(3)}(Z) \arrow[r] & \mathcal{F}^{(3)}(Y) \arrow[dlllll, dashed] \\
& \cdots & & & & &
\end{tikzcd}$$

In particular, injective derivation of $\mathfrak{L}_E$ in **Ban** yields

$$\begin{aligned}
\mathfrak{L}_E^{(n)}(X) = H^n(X) &= \ker \mathfrak{L}_E(\delta^n)/\mathrm{Im}\mathfrak{L}_E(\delta^{n-1}) \\
&= \mathfrak{L}(E, c\kappa_{n-1}/\sim^{n-1} = \mathrm{Ext}^n_{\mathbf{B}}(E, X)\,,
\end{aligned}$$

where $\phi \sim^{n-1} \psi \Longleftrightarrow \exists \tau : E \to \mathfrak{I}_{n-1} : \varpi_{n-1}\tau = \phi - \psi$. And, of course,

$$\mathrm{Ext}^1_{\mathbf{B}}(E, X) = \mathrm{Ext}_{\mathbf{B}}(E, X)\,.$$

The long cohomology sequence would in this case be

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{L}(E,Y) \arrow[r] & \mathfrak{L}(E,Z) \arrow[r] & \mathfrak{L}(E,X) \arrow[dll] & \\
& \mathrm{Ext}^1_{\mathbf{B}}(E,Y) \arrow[r] & \mathrm{Ext}^1_{\mathbf{B}}(E,Z) \arrow[r] & \mathrm{Ext}^1_{\mathbf{B}}(E,X) \arrow[dll] & \\
& \mathrm{Ext}^2_{\mathbf{B}}(E,Y) \arrow[r] & \mathrm{Ext}^2_{\mathbf{B}}(E,Z) \arrow[r] & \mathrm{Ext}^2_{\mathbf{B}}(E,X) \arrow[r] & \cdots .
\end{tikzcd}$$

♢ Projective covariant derivation (usually called Left derivation) requires to pick a projective resolution

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{P}_2 \arrow[rr, "\partial_2"] \arrow[dr, "\pi_2"] & & \mathcal{P}_1 \arrow[rr, "\partial_1"] \arrow[dr, "\pi_1"] & & \mathcal{P}_0 \arrow[rr, "\partial_0"] \arrow[dr, "\pi_0"] & & 0 \\
& & \kappa_1 \arrow[ur, "\imath_1"'] & & \kappa_0 \arrow[ur, "\imath_0"'] & & X \arrow[ur] &
\end{tikzcd} \tag{24.2}$$

of X, apply $\mathcal{F}$ to get

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{F}(\mathcal{P}_2) \arrow[rr, "\mathcal{F}(\partial_2)"] \arrow[dr] & & \mathcal{F}(\mathcal{P}_1) \arrow[rr, "\mathcal{F}(\partial_1)"] \arrow[dr] & & \mathcal{F}(\mathcal{P}_0) \arrow[rr, "\mathcal{F}(\partial_0)"] \arrow[dr, "\mathcal{F}(\pi_0)"'] & & 0 \\
& & \mathcal{F}(\kappa_1) \arrow[ur] & & \mathcal{F}(\kappa_0) \arrow[ur] & & \mathcal{F}(X) \arrow[ur] &
\end{tikzcd}$$

and define

$$L^{(n)}\mathcal{F}(X) = H_n(X) = \frac{\ker \mathcal{F}(\partial_n)}{\mathrm{Im}\mathcal{F}(\partial_{n+1})}.$$

About the 0^{th} case, we have a natural transformation $\rho^{\mathcal{F}} : L^{(0)}\mathcal{F} \to \mathcal{F}$, which becomes a natural equivalence when $\mathcal{F}$ is right exact. I.e., when $\mathcal{F}$ is right-exact then $\mathcal{F} = L^{(0)}\mathcal{F}$.

♠ Projective contravariant derivation (usually called Right) consists in applying $\mathcal{F}$ to a projective resolution (24.2) to get

$$\begin{tikzcd}
0 \arrow[rr, "\mathcal{F}(\partial_0)"] \arrow[dr] & & \mathcal{F}(\mathcal{P}_0) \arrow[r, "\mathcal{F}(\partial_1)"] & \mathcal{F}(\mathcal{P}_1) \arrow[r, "\mathcal{F}(\partial_2)"] & \mathcal{F}(\mathcal{P}_2) \ \cdots \\
& \mathcal{F}(X) \arrow[ur, "\mathcal{F}(\pi_0)"'] & & &
\end{tikzcd}$$

beware now that, as a cochain complex, $\mathcal{F}(\partial_1)$ is "the corresponding" $\overline{\partial}_0$ –and, in general, $\mathcal{F}(\partial_n)$ is the new $\overline{\partial}_{n-1}$ – and therefore

$$R^{(n)}\mathcal{F}(X) = H_n(X) = \frac{\ker \overline{\partial}_n}{\mathrm{Im}\overline{\partial}_{n-1}} = \frac{\ker \mathcal{F}(\partial_{n+1})}{\mathrm{Im}\mathcal{F}(\partial_n)}\,.$$

In addition, we have a natural transformation $\rho^{\mathcal{F}} : \mathcal{F} \to R^{(0)}\mathcal{F}$. When $\mathcal{F}$ is moreover right exact then $\rho^{\mathcal{F}}$ is a natural equivalence, namely, $\mathcal{F} = R^{(0)}\mathcal{F}$.

This derivation above provides, when $\mathcal{F}$ is right exact and given a short exact sequence $0 \to Y \to Z \to X \to 0$, a long exact sequence formed by all derived functors (see [13])

$$0 \to \mathcal{F}(Y) \to \mathcal{F}(Z) \to \mathcal{F}(X) \to \mathcal{F}^{(1)}(Y) \to \mathcal{F}^{(1)}(Z) \to \mathcal{F}^{(1)}(X) \to \cdots\,.$$

In particular, projective derivation of $\mathfrak{L}^E$ in **Ban** yields

$$\begin{aligned}\mathfrak{L}^{E^{(n)}}(X) = H_n(X) &= \ker \mathfrak{L}^E(\partial_n)/\mathrm{Im}\mathfrak{L}^E(\partial_{n-1})\\ &= \mathfrak{L}(\kappa_{n-1}, E)/\sim_{n-1} = \mathrm{Ext}^n_{\mathbf{B}}(X,E)\,,\end{aligned}$$

where $\phi \sim_{n-1} \psi \iff \exists \tau : \mathcal{P}_{n-1} \to Y : \tau|_{\kappa_{n-1}} = \phi - \psi$. The long homology sequence would then be

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{L}(X,E) \arrow[r] & \mathfrak{L}(Z,E) \arrow[r] & \mathfrak{L}(Y,E) \arrow[dll] \\ & \mathrm{Ext}^1(X,E) \arrow[r] & \mathrm{Ext}^1(Z,E) \arrow[r] & \mathrm{Ext}^1(Y,E) \arrow[dll] \\ & \mathrm{Ext}^2(X,E) \arrow[r] & \mathrm{Ext}^2(Z,E) \arrow[r] & \mathrm{Ext}^2(Y,E) \arrow[r] & \cdots\,. \end{tikzcd}$$

♣ Injective contravariant derivation (usually called Left) requires starting with an injective resolution

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] & & \mathcal{I}_0 \arrow[rr, "\delta^0"] \arrow[dr, "\varpi_0"'] & & \mathcal{I}_1 \arrow[rr, "\delta^1"] \arrow[dr, "\varpi_1"'] & & \mathcal{I}_2 \arrow[rr, "\delta^2"] & & \cdots \\ & X \arrow[ur, "\jmath_0"] & & c\kappa_0 \arrow[ur, "\jmath_1"'] & & c\kappa_1 \arrow[ur, "\jmath_2"'] & & & \end{tikzcd} \tag{24.3}$$

and apply $\mathcal{F}$ to get

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \mathcal{F}(\mathfrak{I}_2) \arrow[rr, "\mathcal{F}(\delta^1)"] \arrow[dr] & & \mathcal{F}(\mathfrak{I}_1) \arrow[rr, "\mathcal{F}(\delta^0)"] \arrow[dr] & & \mathcal{F}(\mathfrak{I}_0) \arrow[r, "\mathcal{F}(\jmath_0)"] \arrow[dr] & \mathcal{F}(X) \arrow[r] & 0 \\
& & \mathcal{F}(c\kappa_1) \arrow[ur] & & \mathcal{F}(c\kappa_0) \arrow[ur, "\mathcal{F}(\varpi_0)"'] & & \operatorname{coker} \mathcal{F}(\delta^0) \arrow[u, dashed] &
\end{tikzcd}$$

to then define $L^{(0)}\mathcal{F}(X) = H^0(X) = \operatorname{coker} \mathcal{F}(\delta^0)$ and

$$L^{(n+1)}\mathcal{F}(X) = H^{n+1}(X) = \frac{\ker \mathcal{F}(\delta^n)}{\mathrm{Im}\mathcal{F}(\delta^{n+1})}\,.$$

Again, there is a natural transformation $\rho_{\mathcal{F}} : L^{(0)}\mathcal{F} \to \mathcal{F}$, that becomes a natural equivalence when $\mathcal{F}$ is moreover left exact, shortly described as: $\mathcal{F}$ is the (left) 0-derived of $\mathcal{F}$.

Now, the Yoneda description of Ext^n –in any of the three categories **Ban** , **QBan** or **pBan** , for $n \geq 1$ can be carried out in quasilinear terms "analogously" to the case $n = 1$ explained in Section 23.3 as follows. Let us treat the case of **QBan** : elements of $\operatorname{Ext}^n_{\mathbf{Q}}(X_0, X_n)$ correspond to concatenations $\Omega_n \cdots \Omega_1$ of n quasilinear maps $\Omega_j : X_{j-1} \to X_j$ taken modulo a certain equivalence relation. This relation is natural and easy to describe for $n = 1$ (Ω_1 is the sum of a bounded plus a linear map) and for $n = 2$. In that case we have a commutative diagram (from now on we will omit the $0's$ to keep diagrams cleaner)

$$\begin{tikzcd}
X_2 \arrow[r, equal] \arrow[d] & X_2 \arrow[d] & \\
X_2 \oplus_{\Omega_2} X_1 \arrow[r] \arrow[d] & \square \arrow[r] \arrow[d] & X_0 \arrow[d, equal] \\
X_1 \arrow[r] \arrow[uu, dashed, bend left=60, "\Omega_2"] & X_1 \oplus_{\Omega_1} X_0 \arrow[r] & X_0 \arrow[ll, dashed, bend left=30, "\Omega_1"]
\end{tikzcd} \tag{24.4}$$

which can be comfortably described as follws: Ω_2 can be extended to $X_1 \oplus_{\Omega_1} X_0$ or, equivalently, Ω_1 can be lifted to $X_2 \oplus_{\Omega_2} X_1$. See for instance [73] or [15, Lemma 3.1]. There is no available (manageable) description of the equivalence relation for $n \geq 3$ in quasilinear terms.

25. Time

This is a section for Banach spacers, tired of lying in the abstract sunshine and willing to run after something sound to take home. So, let us now face the abstract derivation process and see what we can get from it.

25.1. Operator ideals and homology. The idea behind this relativization is to replace the operator ideal $\mathfrak{L}$ of all operators (the usual Hom) by another operator ideal $\mathfrak{A}$. As Bühler says [12, Section 2.2]: "There are of course many other [monoidal structures, see Section 27.2] and the corresponding ... Hom-functors are treated in the literature under the name of operator ideals". An operator ideal is an assignment $(X, Y) \to \mathfrak{A}(X, Y) \subset \mathfrak{L}(X, Y)$ with the properties:

(i) the ideal $\mathfrak{F}$ of finite rank operators is contained in $\mathfrak{A}$;

(ii) $\mathfrak{A} + \mathfrak{A} \subset \mathfrak{A}$;

(iii) $\mathfrak{L}\mathfrak{A}\mathfrak{L} \subset \mathfrak{A}$.

The basic operator ideals one should be familiar with (see [77]) are:

1. $\mathfrak{L}$: all linear bounded operators;
2. $\mathfrak{F}$: finite rank operators, which are those having its image lying in a finite dimensional subspace;
3. $\mathfrak{K}$: compact operators, those for which the image of the unit ball is a relatively compact set;
4. $\mathfrak{W}$: weakly compact operators, those for which the image of the unit ball is a relatively compact set in the weak topology (the p-variation of this notion is the ideal $\mathfrak{W}_p$ of weakly p-compact operators, those transforming bounded sequences into sequences admitting a weakly p-converging subsequence);
5. $\mathfrak{X}$: separable operators, those having separable range;
6. $\mathfrak{U}$: unconditionally converging operators, those that are not an isomorphism on any copy of c_0 –equivalently, transforming weakly 1-summable sequences into norm null sequences (the p-variation if this notion is the ideal $\mathfrak{C}_p$ of p-converging operators, those transforming weakly p-summable sequences into norm null sequences);
7. $\mathfrak{S}$: strictly singular operators, those that are not an isomorphism on any infinite dimensional subspace;

8. $\mathfrak{V}$: completely continuous operators, those transforming weakly convergent sequences into convergent sequences;
9. $\mathfrak{CS}$: strictly cosingular operators, those for which no composition QT with a quotient map Q is surjective
10. $\mathfrak{N}$: nuclear operators, those factorizing through a diagonal map $\ell_\infty \to \ell_1$ –equivalently, an operator τ is nuclear if and only if $\tau = \sum \sigma_n x_n^* \otimes y_n$ with $\sigma = (\sigma_n) \in \ell_1$ and $\sup \|x_n^*\|, \|y_n\| < \infty$;
11. Π_p : absolutely p-summing operators, those transforming weakly p-summable sequences into absolutely p-summable sequence –equivalently those factorizing through some canonical inclusion $C(K) \to L_p(K)$;
12. $\mathfrak{G}$: approximable operators, those that are limits, in the operator norm, of a sequence of finite-rank operators.

Returning to our business:

Definition 25.1. Given a Banach space E and an operator ideal $\mathfrak{A}$, we will say that E is

(a) $\mathfrak{A}$-injective if $\mathfrak{A}(-, E)$ is exact;

(b) $\mathfrak{A}$-projective if $\mathfrak{A}(E, -)$ is exact.

In the classical Banach space language, being $\mathfrak{K}$ (resp. $\mathfrak{W}$) -injective receives the name "having the compact (resp. weakly compact) extension property". Recall that an infinite dimensional Banach space X is said to be an $\mathcal{L}_p$-space, $p = 1, \infty$, if there is $C > 0$ such that every finite dimensional subspace of X is contained in a finite dimensional subspace of X that is C-isomorphic to the corresponding ℓ_p^n-space. $\mathcal{L}_\infty$-spaces can be considered the local version of $C(K)$-spaces and $\mathcal{L}_1$-spaces the local version of $L_1(\mu)$-spaces. A Banach space is called an *ultrasummand* if it is complemented in its bidual. $\mathcal{L}_1$-spaces can be characterized (see [13]) as follows: a Banach space X is an $\mathcal{L}_1$-space if and only if $\mathrm{Ext}_{\mathbf{B}}(X, U) = 0$ for every ultrasummand U.

We have:

Proposition 25.2. *Let E be a Banach space.*

(i) *E is $\mathfrak{K}$-injective if and only if it is an $\mathcal{L}_\infty$-space* [7, Proposition 1.33].

(ii) *E is $\mathfrak{K}$-projective if and only if it is an $\mathcal{L}_1$-space* [34].

(iii) *E is $\mathfrak{W}$-injective if and only if it is an $\mathcal{L}_\infty$-space with the Schur property* [7, Proposition 1.35].

(iv) *Every Banach space is $\mathfrak{N}$-injective and $\mathfrak{N}$-projective as well as $\mathfrak{F}$-injective and $\mathfrak{F}$-projective.*

There is no current characterization of $\mathfrak{W}$-projective spaces.

25.2. Derivation is the same, in a relative way. Consider this section an expanded version of [25, Section 14.1]. After all, we have time to kill today and we may as well tackle the idea of picking an operator ideal $\mathfrak{A}$ and proceed with the relative derivation of the functors $\mathfrak{A}^A$ and $\mathfrak{A}_A$, all of which should lead to a kind of $\mathrm{Ext}_{\mathfrak{A}}$ functor.

Initial steps in this direction were taken in [34, 20, 33]. The first of those papers, [34], sets reasonable conditions on an operator ideal $\mathfrak{A}$ to simplify derivation – conditions such as being injective (if $ab \in \mathfrak{A}$ and a is injective with closed range then $b \in \mathfrak{A}$) and surjective (if $ab \in \mathfrak{A}$ and b is surjective then $a \in \mathfrak{A}$) – see [77]–. The ideals $\mathfrak{F}, \mathfrak{K}, \mathfrak{W}, \mathfrak{U}, \mathfrak{X}$ are both injective and surjective (once again, see [77]). The paper does not delve into the results of relative derivation; instead, it focuses on an unexpected fact: to ensure, as it occurs with the operator ideal $\mathfrak{L}$, that projective and injective derivation give the same outcome, one additional property on the ideal $\mathfrak{A}$ is needed: to be *balanced.*

Definition 25.3. An operator ideal $\mathfrak{A}$ will be called balanced when given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \kappa \arrow[r] \arrow[d, "\phi"'] & \mathcal{P} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & Y \arrow[r] & \mathcal{J} \arrow[r] & c\kappa \arrow[r] & 0
\end{tikzcd}$$

the operator ϕ can be chosen in $\mathfrak{A}$ if and only if the operator ψ can be chosen in $\mathfrak{A}$.

To explain the meaning of what we are doing, recall that projective derivation of $\mathfrak{L}^X$ at Y yields

$$\mathfrak{L}(\kappa(X), Y)/\mathfrak{L}(\mathcal{P}, Y)$$

a space which, thanks to the many things we know about exact sequences and pushouts agrees with $\operatorname{Ext}(X,Y)$, while injective derivation of $\mathfrak{L}_Y$ at X yields

$$\mathfrak{L}(X, c\kappa(Y))/\mathfrak{L}(X, \mathcal{I})$$

a space which, thanks to the many things we know about exact sequences and pullbacks agrees with $\operatorname{Ext}(X,Y)$. Replacing $\mathfrak{L}$ by another ideal $\mathfrak{A}$ yields that projective derivation of $\mathfrak{U}^X$ at Y produces

$$\operatorname{Ext}^{\mathfrak{U}}(X,Y) = \mathfrak{U}(\kappa(X), Y)/\mathfrak{U}(\mathcal{P}, Y)$$

while injective derivation of $\mathfrak{U}_Y$ at X yields

$$\operatorname{Ext}_{\mathfrak{U}}(X,Y) = \mathfrak{U}(X, c\kappa(Y))/\mathfrak{U}(X, \mathcal{I}).$$

If the ideal $\mathfrak{U}$ is balanced then $\operatorname{Ext}_{\mathfrak{U}}(X,Y) = \operatorname{Ext}^{\mathfrak{U}}(X,Y)$.

The role of injectivity/surjectivity is perfectly displayed when considering the ideal $\mathfrak{N}$, which is neither injective nor surjective [77, 8.4.5 and 8.5.5]. Since nuclear operators factor through ℓ_∞ and ℓ_1 [77, 6.3.3]), they can be extended and lifted anywhere, which might tempt the reader into thinking that both injective and projective derivation of $\mathfrak{N}$ should vanish. However, applying $\mathfrak{N}(\cdot, E)$ to a projective presentation $0 \to \kappa_1 \to \mathcal{P} \to X \to 0$ of X yields the exact sequence

$$0 \longrightarrow \mathfrak{N}^{surj}(X,E) \longrightarrow \mathfrak{N}(\mathcal{P}, E) \longrightarrow \mathfrak{N}(\kappa, E) \longrightarrow 0\,,$$

where $\mathfrak{N}^{inj}$ is the smallest injective ideal containing $\mathfrak{N}$; while applying $\mathfrak{N}(E, \cdot)$ to an injective presentation $0 \to Y \to \mathcal{I} \to \rho \to 0$ yields the exact sequence

$$0 \longrightarrow \mathfrak{N}^{inj}(E,Y) \longrightarrow \mathfrak{N}(E, \mathcal{I}) \longrightarrow \mathfrak{N}(E, \rho) \longrightarrow 0\,,$$

where $\mathfrak{N}^{surj}$ is the smallest surjective ideal containing $\mathfrak{N}$.

However, [34] shows that only $\mathfrak{K}$ among the ideals listed is balanced. That is why [20] focuses on the relative derivation of the (functors associated to) to the ideal $\mathfrak{K}$ of compact operators. The ideal $\mathfrak{K}$ is injective, surjective and balanced, and therefore forms the ideal bridgehead protecting our landing on operator ideal derivation. The abstract outcome of the derivation process is already known: $\mathfrak{K}^{(1)}(X,Y) = \operatorname{Ext}_{\mathfrak{K}}(X,Y)$ and therefore, the meaning of $\mathfrak{K}^{(1)}(X,Y) = 0$ is clear: given a projective presentation $0 \to \kappa(X) \to \mathcal{P} \to X \to 0 \to$ of X, all compact operators $\kappa(X) \to Y$ admit compact extensions to $\mathcal{P}$; equivalently, given an injective presentation $0 \to Y \to \mathcal{I} \to c\kappa(Y) \to 0 \to$ of Y, all compact operators $X \to c\kappa(Y)$ admit a compact lifting to $\mathcal{I}$. The unexpected appears when [20, Proposition 4.4] shows that if E, X are Banach spaces with E an ultrasummand having the BAP then they are equivalent:

1. $\mathrm{Ext}_{\mathfrak{K}}(X, E) = 0$;
2. $\mathrm{Ext}_{\mathbf{B}}(X, E) = 0$;
3. $\mathrm{Ext}_{\mathbf{B}}(X, E) = 0$ is Hausdorff in its natural vector space topology compatible with the homology sequence.

The equivalence between (1) and (2) reveals the surprising fact that, under the additional conditions imposed, being able to extend all compact operators to compact oprators already guarantees to be able to extend all operators. The equivalence between (2) and (3) connects the vanishing of either the derivative of $\mathfrak{L}$ or $\mathfrak{K}$ with the Hausdorff character of $\mathrm{Ext}_{\mathbf{B}}$ spaces. We will return to this topic in Section 30.2.

Finally, [33] sets caution aside and deals with the injective/projective derivations of the other natural non-balanced operator ideals above.

26. The great kerfuffle in the sky

If homological derivation depends, when made by the book, on either injective or projective presentations, what if the category does not have enough injective or projective objects? The question is important to us since **QBan** has neither projective nor injective objects as shownwe proved in Section 23.1. However, for the same reason that no researcher (especially L. Schwartz) would have ever said: *since differentiable functions must be continuous, discontinuous functions cannot have derivatives*; we cannot stick to *functors on categories without projective/injective objects cannot be derived.* A. Buchsbaum made that point clear in the series of papers [8, 9, 10] written, for all we know, independently of Grothendieck's Tohoku paper [53] [7].

Pick a category without projective or injective spaces and a functor to derive in it. We choose here **QBan** as the category and the contravariant functor $\mathfrak{L}^X$ to be derived at A. What we will do in absence of a projective presentation for X is to consider all exact sequences $0 \to M \to Z \to A \to 0$.

[7] See also pp. 18-20 in [54] or the entry `math.stackexchange.com/questions/4192906/was-grothendieck-familiar-with-buchsbaums-exact-categories-while-writing-tohoku` in StackExchange

Since exact sequences of quasi-Banach spaces are generated by quasilinear maps, let us call ω both the sequence and the quasilinear map. We introduce the following order: given another exact sequence $\omega' = 0 \to M' \to Z' \to A \to 0$ we say $\omega \geq \omega'$ if ω' is a pushout of ω; namely, there is an operator $g : M \to M'$ such that $g\omega = \omega'$. A map f spontaneously appears making the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\imath_\omega} & Z & \longrightarrow & A & \longrightarrow & 0 \\
 & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & \| & & \\
0 & \longrightarrow & M' & \xrightarrow{\imath_{\omega'}} & Z' & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

commute. When we apply $\mathfrak{L}^X$ to this diagram we get a commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{L}(Z', X) & \longrightarrow & \mathfrak{L}(M', X) & \longrightarrow & \operatorname{coker} \mathfrak{L}^X(\imath_{\omega'}) & \longrightarrow & 0 \\
{\scriptstyle f^*}\downarrow & & {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle u_{\omega',\omega}} & & \\
\mathfrak{L}(Z, X) & \longrightarrow & \mathfrak{L}(M, X) & \longrightarrow & \operatorname{coker} \mathfrak{L}^X(\imath_\omega) & \longrightarrow & 0\,,
\end{array}$$

where the cokernels are taken the category of vector spaces (or Abelian groups) or, as in our case, in the category of topological vector spaces (non-necessarily Hausdorff) and $u_{\omega',\omega}$ is the induced map. Actually, this map is independent of both f, g as soon as the equality $g\omega = \omega'$ is replaced by standard equivalence of exact sequences (or quasilinear maps) $g\omega \equiv \omega'$. Since the order is filtering (in the increasing sense ω), we have a net $\operatorname{coker} \mathfrak{L}^X(\imath_\omega), u_{\omega',\omega}\}$ and Buchsbaum defines the sequence of *satellites* –that will play the role of the sequence of derived functors– in the form

$$S^1(\mathfrak{L}^X)(A) = \lim_{\rightarrow}\{\operatorname{coker} \mathfrak{L}^X(\imath_\omega)\}$$

namely

$$\bigoplus \operatorname{coker} \mathfrak{L}^X(\imath_\omega) \quad /H\,,$$

where $H = \operatorname{span}\Big[z_\omega - u_{\omega',\omega} z_{\omega'} : z_\omega \in \operatorname{coker} \mathfrak{L}^X(\imath_\omega);\ z_{\omega'} \in \operatorname{coker} \mathfrak{L}^X(\imath_{\omega'})\Big]$, since this is how direct limits are usually obtained in vector space categories. After this, in a very bold move, Buchsbaum defines

$$S^{n+1}(\mathfrak{L}^X) = S^1\Big(S^n(\mathfrak{L}^X)\Big),$$

which obviously requires taking for granted that the definition of S^1 applies to any contravariant additive functor acting between two categories $\mathbf{A} \to \mathbf{B}$ with

$\mathbf{B}$ admitting limits. In particular, to $S^1(\mathfrak{L}^X)$. The work that needs to be done for covariant functors such as $\mathfrak{L}_X$ acting between two categories $\mathbf{A} \to \mathbf{B}$ such that $\mathbf{B}$ has colimits is completely transparent. What makes BuchsbaumÆs move bold is the definition of the second *satellite* derivative as the satellite derivative of the satellite derivative. This is something that fails spectacularly for usual derivation: observe that the existence of enough projectives plus the fact that a derived functor $F^{(1)}$ must vanish on projectives mean that the (usual) derived functor of $\mathrm{Ext}_{\mathbf{B}}(X, A)$ must be 0[8], which makes impossible for Ext^2 to be the derived functor of Ext since there are examples of Banach spaces X, Y for which $\mathrm{Ext}^2_{\mathbf{B}}(X, Y) \neq 0$. One, however, has (:

PROPOSITION 26.1.

- $S^1(\mathfrak{L}^X)(A) = \mathrm{Ext}_{\mathbf{Q}}(A, X)$ *in* $\mathbf{QBan}$ *;*
- $S^1(\mathfrak{L}^X)(A) = \mathrm{Ext}_{\mathbf{B}}(A, X)$ *in* $\mathbf{Ban}$ *;*
- $S^1\Big(\mathrm{Ext}_{\mathbf{Q}}(\cdot, X)\Big)(A) = \mathrm{Ext}^2_{\mathbf{Q}}(A, X)$*;*
- $S^1\Big(\mathrm{Ext}^n_{\mathbf{B}}(\cdot, X)\Big)(A) = \mathrm{Ext}^{n+1}_{\mathbf{B}}(A, X)$*;*
- $S^1\Big(\mathrm{Ext}^n_{\mathbf{p}}(\cdot, X)\Big)(A) = \mathrm{Ext}^{n+1}_{\mathbf{p}}(A, X)$*.*

Proof. In our case, the presence of quasilinear maps is essential, as is the fact that $\operatorname{coker} \mathfrak{L}^X(\imath_\omega)$ corresponds to operators $M \to X$ modulo extension to Z as we know well, and the only difference with the Banach or p-Banach case is that here we do not have an "initial" (i.e., final in the inductive limit) element –namely, the projective presentation of A– from which every other sequence could be obtained as a pushout. But this absence changes nothing in order to get

$$\varinjlim\{\operatorname{coker} \mathfrak{L}^X(\imath_\omega)\} = \mathcal{Q}(A, X)\,.$$

The arrow $\varinjlim\{\operatorname{coker} \mathfrak{L}^X(\imath_\omega)\} \to \mathcal{Q}(A, X)$ is the one induced by $g \to g\omega$. The reverse arrow $\mathcal{Q}(A, X) \to \varinjlim\{\operatorname{coker} \mathfrak{L}^X(\imath_\omega)\} \to$ appears once we notice that the sequence generated by any $\Omega \in \mathcal{Q}(A, X)$ is itself one of the exact sequences (whose cokernel after applying $\mathfrak{L}^X$) sooner or later shows up in the filtering family.

The proof for Ext proceeds in the same spirit, with a twist: now coker $\mathrm{Ext}(\cdot, X)(\imath_\omega)$ correspond to quasilinear maps $M \to X$ modulo extension to Z,

[8] We acknowledge Nazaret Trejo for this observation.

and these are precisely the elements of $\mathrm{Ext}^2(\cdot, X)$ as illustrated by diagram (24.4). Everything else works as before.

The proof for $\mathrm{Ext}^n_{\mathbf{B}}$ is similar but contains one extra ingredient. Let us visualize the case $n = 2$: If we call $\omega_M : X \to M$ the quasilinear map that defines the sequence $0 \to M \xrightarrow{\imath_\omega} Z \to A \to 0$ then $\operatorname{coker} \mathrm{Ext}^2(\cdot, X)(\imath_\omega)$ corresponds to concatenations $\Phi\Psi$ of quasilinear maps $M \to X$ modulo the of $\Phi\Psi$ extension to Z through $\imath_\omega$. In **Ban** or **pBan** the sequence ω is a pushout of a projective presentation of A so we can well assume that ω is $0 \to M \xrightarrow{\imath_\omega} \mathcal{P} \to A \to 0$ with $\mathcal{P}$ projective. In this setting, see [16, Lemma 2.1 (b)], $\Omega\Psi\omega \equiv 0$ if and only if $\Omega\Psi \equiv 0$ and, thus, $\operatorname{coker} \mathrm{Ext}^2(\cdot, X)(\imath_\omega) = \mathrm{Ext}^3(A, X)$ with the identification $\Omega\Psi \to \Omega\Psi\omega$. In the general case, the identification becomes $\Omega_n \cdots \Omega_1 \to \Omega_n \cdots \Omega_1\omega$. ∎

It is by no means clear to us whether even $S^1\Big(\mathrm{Ext}^2_{\mathbf{Q}}(\cdot, X)\Big)(A) = \mathrm{Ext}^3_{\mathbf{Q}}(A, X)$ because now $\operatorname{coker} \mathrm{Ext}^2(\cdot, X)(\imath_\omega)$ corresponds to concatenations $\Phi\Psi$ of quasilinear maps $M \to X$ modulo extension to Z. But this does not (necessarily) match he equivalence relation in Ext^3 for concatenations $\Omega_3\Omega_2\Omega_1$.

27. Monoids

Like the song, the architecture of this section is just a sustained topic on top of which we layer a few still open lines of research. The topic is that the existence of a single “multiplication operation” in a category already suffices to build a reasonable cohomology theory, hence a reasonable substitute for homological derivation.

27.1. Coboundaries and cocycles. We begin with the classical complex in cohomology of groups:

$$0 \longrightarrow Y \xrightarrow{\ d^1\ } \mathsf{F}(X, Y) \xrightarrow{\ d^2\ } \mathsf{F}(X^2, Y) \xrightarrow{\ d^3\ } \mathsf{F}(X^3, Y) \longrightarrow \cdots, \tag{27.1}$$

where $\mathsf{F}(A, B)$ denotes the set of all mappings from A to B: thus, $Y = \mathsf{F}(X^0, Y)$ since $X^0 = \{e\}$ and

$$\begin{aligned}(d^n f)(x_1, \dots, x_n) = f(x_2, \dots, x_n) + \sum_{1\le i\le n-1} &(-1)^i f(x_1, \dots, x_i + x_{i+1}, \dots, x_n)\\ &+ (-1)^n f(x_1, \dots, x_{n-1})\end{aligned}$$

so that $d^1 = 0$. We will from now on omit the superscript in d. The elements of $\mathsf{F}(X^n, Y)$ are called cochains:

- an n- cocycle is a cochain $f : X^n \to Y$ such that $df = 0$;
- an n- coboundary is a cochain $f : X^n \to Y$ such that $f = dg$ for some cochain $g : X^{n-1} \to Y$.

The associated cohomology groups H^n are

$$H^n(X,Y) = \frac{\ker\left(d^{n+1} : F(X^n,Y) \to F(X^{n+1},Y)\right)}{\mathrm{Im}\left(d^n : F(X^{n-1},Y) \to F(X^n,Y)\right)} = \frac{n\text{-cocycles}}{n\text{-coboundaries}}\,.$$

In our Banach/quasi-Banach context groups are vector spaces and the elements of $\mathsf{F}(A, B)$ are (bounded) homogeneous maps. Let's focus on $n = 1, 2$.

- A 1-coboundary is 0.
- A 1-cocycle is a map homogeneous $f : X \to Y$ such that
$$df(x, y) = f(x + y) - f(x) - f(y) = 0\,,$$
i.e., a linear map.
- A 2-coboundary is a homogeneous function $f : X^2 \to Y$ such that $f = d^2 g$ for some $g : X \to Y$; i.e.,
$$f(x_1, x_2) = g(x_1) + g(x_2) - g(x_1 + x_2)\,.$$
- A 2-cocycle is a function $f : X^2 \to Y$ satisfying
$$df(x_1, x_2, x_3) = f(x_2, x_3) - f(x_1 + x_2, x_3) + f(x_1, x_2 + x_3) - f(x_1, x_2) = 0\,.$$

It is easy to check that if $\Omega : X \to Y$ is a homogeneous map, in particular a quasilinear map, then its Cauchy differences

$$[+, \Omega](x, y) = \Omega(x + y) - \Omega(x) - \Omega(y)$$

are cocycles; i.e., $d[+, \Omega] = 0$ (the notation comes from the usual commutator convention: $[a, b] = ab - ba$). When the norm enters the game and all maps are required to be bounded, the sequence (27.1) takes the form

$$0 \longrightarrow Y \xrightarrow{d^1} \mathsf{B}(X,Y) \xrightarrow{d^2} (\mathsf{B} \cap \mathsf{F})(X^2,Y) \xrightarrow{d^3} (\mathsf{B} \cap \mathsf{F})(X^3,Y) \longrightarrow \cdots\,.$$

Observe the existence of two natural maps. The first of them is the identity $\mathfrak{L}(X,Y) \to H^1(X,Y) = \ker d^2$. The second, $\mathcal{Q}(X,Y) \to H^2(X,Y)$, is induced by the correspondence $\mathsf{Q}(X,Y) \to \ker d^3$ given by $\Omega \to [+,\Omega]$. Indeed, Ω is trivial (that is, $\Omega = B + L$ if and only if its associated 2-cocycle is a 2-coboundary: clearly, if $[+,\Omega] = [+,B]$ then $\Omega - B$ is linear. It is true that the correspondence is surjective (i.e., every 2-cocycle f is $f = [+,\Omega]$ for some quasilinear map), but not as easy to prove as it sounds. The next level becomes a headache: there is a well-behaved correspondence $\mathrm{Ext}^2(X,Y) \to H^3(X,Y)$, such is the content of [15, Lemma 3.3], but the surjectivity of this map is not guaranteed. What occurs in higher levels is a living hell (a fascinating mystery, in the words of Cabello [9]). There exists a natural transformation $\mathrm{Ext}^n \to H^{n+1}$, a consequence from the fact that $H^\bullet$ is a δ-functor and $\mathrm{Ext}^\bullet$ is a universal δ-functor. The real question is whether this transformation is injective for $n > 2$. A passing glance to [13, 3.13.2] should cause no permanent harm.

27.2. Monoidal categories. The reason to bring this material here is that, at a price, cohomology has been reconstructed using essentially a single operation (sums) —plus homogeneity (when working with vector spaces), plus boundedness (when working with Banach spaces). And, as everybody who knows it knows, cohomology can be performed in categories in which there is an operation $\square : X \times X \to X$. Objects endowed with a *suitable* operation as above (it could be the group operation, the sum operation, etc) are called *monoids* and thus categories $\mathbf{C}$ possessing a compatible (functorial) operation $\mathbf{C}\times\mathbf{C} \to \mathbf{C}$ (such as direct product, tensor product, etc) satisfying the appropriate axioms are called *monoidal* categories. The series [76, 67, 75, 5] provides a certain perspective on this construction.

Passing to Banach spaces, the standard symmetric and associative monoidal structure in **Ban** is the one induced by the projective tensor product [12, Section 2.2] and its Hom spaces are precisely the spaces of bounded linear operators endowed with the operator norm. We bring here the quote of Bühler [12, Section 2.2] again: "There are of course many other monoidal structures and the corresponding ... Hom-functors are treated in the literature under the name of operator ideals". Pursuing this line would mean to delve into the question:

[9] to whom the author acknowledges for allowing him to poke around his notes *Cohomology for dummies*

PROBLEM 27.1. Let$\mathfrak{U}$ be an operator ideal. Does $\mathfrak{U}_X$ admit a left adjoint?

See Section 29 and Section 30. Two sluggishly formulated additional questions could be:

PROBLEM 27.2. Is **Ban** a monoidal category in such a way that the resulting cohomology sequence is the cocycle-coboundary sequence described above?

The monoidal structure provided by the projective tensor product does not seem to be enough to solve this problem.

PROBLEM 27.3. Is **QBan** a monoidal category?

And, if so, (read back the first sentence of this section).

28. US AND THEM

US, of course, are the Banach spacers and THEM the categorical people. But, as Jules Winnfield [10] would say, "Or, it could mean something different". US could be the category we are working in, and THEM the "right" category we ought to be working with. As mentioned before, the canonical correct environment is that of Abelian categories. What if the category $\mathbf{C}$ we are actually working in is not Abelian? There are two reasonable paths to follow:

- One may isolate some of the properties that our category does enjoy and, since we need something More to obtain the desired results, call, say, Cymbaline categories to categories with those properties and then translate (or obtain, or reconstruct) the necessary machinery from the Abelian setting to the Cymbaline one.
- Alernatively, one may search for a larger Abelian category $\mathbf{C}^\heartsuit$ into which $\mathbf{C}$ embeds as a full subcategory. The category $\mathbf{C}^\heartsuit$ is then called a Heart of $\mathbf{C}$ [11]. The reader is invited to peruse [25, Section 19] for an entertaining monologue on the meaning of heart category and on the possibilities for constructing a heart for Banach spaces.

Since **Ban** is not Abelian, both strategies are available. Following the first approach, **Ban** is an *exact* category, in the sense that there is a clear meaning

[10] The character played by Samuel L. Jackson in Pulp Fiction.

[11] to be true, *an* Abelianization of $\mathbf{C}$, keeping the name "heart" for a more specific Abelianization attributed to Deligne; see [6, Ex. 1.3.22].

for the term *short exact sequence*. The same holds for **QBan** . In general, an exact category is one endowed with a distinguished class of morphisms, called *admissible*,from which one forms a well-natured collection of so-called *short exact sequences*. Descending to a definition (see [25, Section 19]), although there are various ways to do that, we adhere to Quillen's approach [79, 80]; see [11] for a very detailed account and systematic study of exact categories. Indeed, the notion of exact category has proved remarkably successful and a substantial amount of work has been devoted to translating results from the Abelian to the exact setting.

28.1. Functor categories. Functor categories are what their name say upfront: a category $\mathbf{A}^{\mathbf{C}}$ whose objects are functors $\mathbf{C} \to \mathbf{A}$ and whose morphisms are natural transformations between them. Given two functors F, G we denote by $\blacktriangleleft F, G \blacktriangleright$ the Hom object of natural transformations between them. When F, G are Banach functors then $\blacktriangleleft F, G \blacktriangleright$ is a Banach space (see [24, 3.5]). The topic of functor categories is, indeed, fearsome because of its level of abstraction. Yet many of the arguments in [24] reveal that these are the *real* categories, which is what makes them worth studying. Moreover, the glaring truth is that things often become simpler in functor categories ... provided one can breathe there. A good example is the proof we briefly tiptoed into in [24, Proposition 5.4], asserting that left (resp. right) adjoints preserve direct (resp. inverse) limits. As we remarked there, "the proof is very simple with a bit of extra abstraction".

To understand how simplicity in the ideas and complexity of execution mingle, consider the topic treated with some care in [24] concerning how to represent objects of a category $\mathbf{C}$ as functors of $\mathbf{A}^{\mathbf{C}}$. When $\mathbf{C}$ is a small category, the Yoneda representation functor $\mathcal{Y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}}$, which can be either $\mathcal{Y}(B) = \mathcal{Y}_B = \mathsf{Hom}(B, \cdot)$ when you prefer using covariant functors, or $\mathcal{Y}(B) = \mathcal{Y}^B = \mathsf{Hom}(\cdot, B)$ if you prefer to work with contravariant functors is one of the nicest. Since $\mathsf{Hom}(\cdot, B)$ is contravariant, sometimes one pretends working in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ instead of $\mathbf{Set}^{\mathbf{C}}$ to make it covariant. The main advantage of using the contravariant Yoneda representation is that $\blacktriangleleft \mathcal{Y}^A, \mathcal{Y}^B \blacktriangleright = \mathbf{C}(A, B)$, respecting the natural order, while $\blacktriangleleft \mathcal{Y}_A, \mathcal{Y}_B \blacktriangleright = \mathbf{C}(B, A)$. We shall therefore use the contravariant Yoneda whenever the identity $\blacktriangleleft \mathcal{Y}^A, \mathcal{Y}^B \blacktriangleright = \mathbf{C}(A, B)$ is required and the covariant version when it is not. It is likely that the next result catches the reader unprepared [52, p. 21]:

Proposition 28.1. *Every element of* $\mathbf{Set}^{\mathbf{C}}$ *is a direct limit of Yoneda functors.*

The proof, in the language and notation of [25, Section 10], goes as follows: given $F \in \mathbf{Set}^{\mathbf{C}}$, let $\mathbf{F}$ be the sometimes called *comma category*, whose objects are pairs $(\text{よ}(C), f)$, being $f : \text{よ}(C) \to F$ an arrow; and whose arrows are the obvious ones (see [25]). If

$$\Box : \mathbf{F} \to \mathbf{Set}^{\mathbf{C}^{op}}$$

is the forgetful functor,

$$F = \lim_{\to} \Box(\text{よ}(C), f) = \lim_{\to} \text{よ}(C)\,.$$

This echoes the proof [25, Proposition 10.2], showing that every Banach space is a direct limit of finite dimensional projective spaces. In the present context, as we will show in Proposition 28.5, the projective objects of $\mathbf{Set}^{\mathbf{C}}$ are precisely the functors $\text{よ}(C)$.

As Gabriel (Peter, not Pierre) says, a left-adjoint for よ is something to observe[12], something we shall soon do in Section 28.2. A general result (see [52, Proposition 1.3]) is the following:

Proposition 28.2. *Let* $\mathbf{C}$ *be a category with direct limits. A functor* $G : \mathbf{Set}^{\mathbf{D}} \to \mathbf{C}$ *is a left adjoint if and only if it preserves limits.*

Proof. Since two categories $\mathbf{C}$ and $\mathbf{D}$ are involved, we will distinguish their Yoneda functors by writting $\text{よ}^{\mathbf{C}}$ and $\text{よ}^{\mathbf{D}}$. So, to give an insight of what is happening, let us draw a map:

$$\begin{tikzcd}
\mathbf{Set}^{\mathbf{D}} \arrow[r, "G"] & \mathbf{C} \arrow[d, "\text{よ}^{\mathbf{C}}"] \\
\mathbf{D} \arrow[u, "\text{よ}^{\mathbf{D}}"] & \mathbf{Set}^{\mathbf{C}}\,.
\end{tikzcd}$$

The right adjoint we are looking for, R, is not the obvious composition (who, after all, would land in the wrong category), but the functor

$$\mathcal{R}(c)(d) = \text{よ}^{\mathbf{C}}(c)\left(G\left(\text{よ}^{\mathbf{D}}\right)(d)\right) = \mathsf{Hom}\left(G\left(\text{よ}^{\mathbf{D}}\right)(d), c\right).$$

[12] Solsbury Hill

We have to check the identity

$$\mathsf{Hom}\,(G(f), C) \;= \blacktriangleleft f, \mathcal{D}(c) \blacktriangleright .$$

Observe that, in general, the correspondence associated to the arrow η is the natural transformation $\overline{\eta}$ defined by

$$\overline{\eta}_d(\lambda) = \eta G(\lambda)\,, \qquad \lambda \in f(d) = \blacktriangleleft \text{よ}(d), f \blacktriangleright .$$

and d an object of $\mathbf{D}$. When $f = \text{よ}^{\mathbf{D}}(d_0)$, the natural transformation $\overline{\eta}$ takes the form

$$G(\text{よ}^D(d)) \xrightarrow{G(\lambda)} G(\text{よ}^D(d_0)) \xrightarrow{\eta} c\,.$$

Thus, for $f = \text{よ}^D(d_0)$ every

$$\varepsilon \;\in \blacktriangleleft \text{よ}^D(d_0), \mathcal{D}(c) \blacktriangleright = \; D(c)(d_0) = \mathsf{Hom}(G\left(\text{よ}^D(d_0)\right), c)$$

clearly determines an $\eta \in \mathsf{Hom}(G\left(\text{よ}^D(d_0)\right), c)$ such that $\overline{\eta} = \epsilon$.

We complete the proof: left adjoints preserve direct limits, and that is the easy half. If, on the other hand, G preserves direct limits, since every element of $\mathbf{Set}^{\mathbf{D}}$ is the direct limit of Yoneda functors, it is enough to verify the adjunction on Yoneda functors; which is what we just did. ∎

Observe that the functor $G \to G\text{よ}$ induces an equivalence between the full subcategory formed by the functors that commute with direct limits and the functor category $\mathbf{D}^{\mathbf{C}}$.

28.2. Banach heart mother. Regarding the second path toward "Abelianizing" a category –namely, to find a heart for it– we now show that exact categories fully embed into Abelian functor categories:

Theorem 28.3. *Given a small exact category* $\mathbf{A}$, *the contravariant Yoneda representation* $A \to \mathsf{Hom}(\cdot, A)$ *embeds* $\mathbf{A}$ *into the Abelian category* $\mathbf{Ab}^{\mathbf{A}^{op}}$ *of left-exact functors* $\mathbf{A}^{op} \to \mathbf{Ab}$.

Bühler provides in [11, Appendix A] an extraordinarily detailed account of this result, its would-be proofs and its actual proof. Let us present the yummy parts of its Banach space analogue. As before, let

$$\text{よ} : \mathbf{Ban} \to \mathbf{Ab}^{\mathbf{Ban}^{op}}$$

be the Yoneda functor $\mathcal{Y}(A) = \mathfrak{L}^A$. Consider the subcategory ¥ consisting of its left-exact functors. In the standard terminology, functors of $\mathbf{Ab}^{\mathbf{Ban}^{op}}$ are called *pre-sheaves*, while left exact ones are called *sheaves*. Let us, following the notation of Bühler [11], denote by $\jmath_* : ¥ \to \mathbf{Ab}^{\mathbf{Ban}^{op}}$ the canonical "inclusion" functor. This is one of the moments where all the complexity, jargon and machinery of category theory become indispensable. For although $\jmath_*$ might looks very much like an inclusion, it is not necessarily an exact functor. The reason is that it does not necessarily transform epic arrows into epic arrows. This is hard to digest when one is used to work with set-shaped categories in which epic means surjective (if an arrow $A \to B$ is surjective between, say the vector spaces A, B, it remains surjective between the sets A, B) or, as it is the case of **Ban** or **QBan** , when epic means "having dense range". But an epic arrow f in ¥ means that given two natural transformations η, η' between sheaves such that $\eta f = \eta' f$ then $\eta = \eta'$. BY contrast, to be epic in $\mathbf{Ab}^{\mathbf{Ban}^{op}}$ one requires the same property but for all natural transformations η, η' between all presheaves, not only for sheaves. See the comments at the end of the following proof if in need.

Theorem 28.4. *The categories ¥ and* $\mathbf{Ab}^{\mathbf{Ban}^{op}}$ *are Abelian. The functor*

$$\mathcal{Y} : \mathbf{Ban} \longrightarrow ¥$$

is exact although the functor $\jmath_*\mathcal{Y} : \mathbf{Ban} \to \mathbf{Ab}^{\mathbf{Ban}^{op}}$ *is not (necessarily) exact.*

Proof. To prove the exactness of ¥ one has to show that given an exact sequence $0 \to Y \xrightarrow{\imath} Z \xrightarrow{\pi} X \to 0$ the induced sequence

$$0 \longrightarrow \mathfrak{L}^Y \xrightarrow{\eta_\imath} \mathfrak{L}^Z \xrightarrow{\eta_\pi} \mathfrak{L}^X \longrightarrow 0$$

is exact in ¥. Since all other parts of the proof are routine, the key point is to verify that the arrow $\mathfrak{L}^Z \xrightarrow{\eta_\pi} \mathfrak{L}^X$ is epic. To that end, let

$$\mathfrak{L}^Z \xrightarrow{\eta_\pi} \mathfrak{L}^X \xrightarrow{\epsilon} G_{¥} \longrightarrow 0$$

be the cokernel of η_π in ¥. We aim to show that $G_{¥} = 0$.

Working temporarily in the full functor category. Let us pretend we make the proof in the functor category $\mathbf{Ban} \to \mathbf{Ab}^{\mathbf{Ban}^{op}}$ and denote by G the cokernel in this larger category. Fix a Banach space A. Evaluating at A gives an exact sequence

$$\mathfrak{L}(A, Z) \xrightarrow{\eta_\pi(A)} \mathfrak{L}(A, X) \xrightarrow{\epsilon(A)} G(A) \longrightarrow 0\,.$$

Pick $x \in G(A)$. By surjectivity, there is $f \in \mathfrak{L}(A, X)$ such that $\epsilon(A)(f) = x$. Next, form the pullback of f along π and apply G to obtain

$$\begin{tikzcd} Z \arrow[r, "\pi"] & X & & G(Z) \arrow[d, "G(f')"'] & G(X) \arrow[l, "G(\pi)"'] \arrow[d, "G(f)"] \\ \mathrm{PB} \arrow[u, "f'"] \arrow[r, "\pi'"'] & A \arrow[u, "f"'] \arrow[r, dashed] & {} & G(\mathrm{PB}) & G(A) \arrow[l, "G(\pi')"] \, . \end{tikzcd}$$

Since $\blacktriangleleft \mathfrak{L}^X, G \blacktriangleright = G(X)$, we may regard ϵ as an element of $G(X)$ satisfying $G(f)(\epsilon) = x$. Therefore,

$$G(\pi')(x) = G(\pi')G(f)(\epsilon) = G(f\pi')(\epsilon) = G(\pi f')(\epsilon) = 0 \, .$$

Since π is surjective, π' is surjective, and this would be almost the end: because if G were left-exact $G(\pi')$ would be injective, $x = 0$ and we are done. But G is not. Only $G_{¥}$ is left-exact.

The real difficulty when returning to ¥. But working in ¥ introduces other problems: contrarily to what occurs in the functor category $\mathbf{Ab}^{\mathbf{Ban}^{op}}$ where exactness of a sequence means the exactness of its evaluations, the exactness of $0 \to \mathfrak{L}^Y \xrightarrow{\eta_\imath} \mathfrak{L}^Z \xrightarrow{\eta_\pi} \mathfrak{L}^X \to 0$ in ¥ does not imply the exactness of its evaluations $0 \to \mathfrak{L}(A, Y) \xrightarrow{\eta_\imath(A)} \mathfrak{L}^{(}A, Z) \xrightarrow{\eta_\pi(A)} \mathfrak{L}(A, X) \to 0$ in **Ban** [13]. Said bluntly, evaluation is not an exact functor on ¥ and that is why we cannot conclude from the above computation that $x = 0$. Working in ¥ thus requires a more delicate argument, which comes next.

There is a way to understand the meaning of epic arrows in ¥ but it requires a different approach to left-exact functors –as sheaves via coverings– a viewpoint that we mostly defer to Part IV of The Hitchhiker Guide. In our setting, a covering for the Banach space X is simply a quotient map $\pi : C \to X$. Of course, considering all coverings would be useless: the identity $X \to X$ is itself a covering. So let us refine coverings: a covering $\pi' : C' \to X$

[13] This topic is discussed in `math.stackexchange.com/questions/1083994/yoneda-embedding-into-left-exact-functors` and, as Qiaochu Yuan neatly says "You've implicitly assumed that evaluation at A is exact, which need not be the case". After that, Zhen Lin's punch line says "You mean, limits and colimits in [the functor category] can be computed pointwise. But anything can happen in a subcategory".

is said to be a refinement of $\pi : C \to X$ if π' factorizes through π; i.e., $\pi' = \pi\phi$ as in the diagram

$$\begin{tikzcd} C \arrow[r, "\pi"] & X \\ & C' . \arrow[ul, "\phi"] \arrow[u, "\pi'"'] \end{tikzcd}$$

We feel more comfortable now since the identity is the less refined covering there is. Through pullbacks the refinement induces a filtering order on coverings, and the point of all this is that the exactness of a sequence of sheaves –as well as the computation of other limits in the category– can be tested by evaluation on suitable cofinal families of (spaces C yielding the) coverings. In our case, this means that the previous argument goes through after replacing the arbitrary Banach space A by a covering C. Once this replacement is made, the proof proceeds exactly as before: since $G_{¥}$ is now left exact on this cofinal system of coverings, injectivity of $G_{¥}(\pi')$ folows and therefore we conclude that $G_{¥} = 0$. ∎

What has been going on here? [14]. In fact, that a combination of what Freyd elegantly says [49], "the proof that ¥ is Abelian is not formal" and Quillen sentence [80] (see [11, p. 55] for details) "The details will be omitted, as they are not really important for the sequel" justify that we skipped most details. What we actually needed and take for granted is that cokernels exist both in ¥ and in $\mathbf{Ab}^{\mathbf{Ban}^{op}}$, although they might be different. The following issues deserve consideration, though:

- Exactness of よ versus non-exactness of $\jmath_*$よ. Since よ $: \mathbf{Ban} \to$ ¥ is exact, given an exact sequence $0 \to Y \xrightarrow{\imath} Z \xrightarrow{\pi} X \to 0$ the induced sequence $0 \to \mathfrak{L}^Y \xrightarrow{\eta_\imath} \mathfrak{L}^Z \xrightarrow{\eta_\pi} \mathfrak{L}^X \to 0$ is exact in ¥ but not in $\mathbf{Ab}^{\mathbf{Ban}^{op}}$.
- Pointwise exactness in the presheaf category. The exactness of a sequence in $\mathbf{Ab}^{\mathbf{Ban}^{op}}$ means the exactness of its evaluations. Consequently, since we know that $\mathfrak{L}^A$ is just left-exact and usually not exact, the sequence $0 \to \mathfrak{L}^Y \xrightarrow{\eta_\imath} \mathfrak{L}^Z \xrightarrow{\eta_\pi} \mathfrak{L}^X \to 0$ cannot be exact in $\mathbf{Ab}^{\mathbf{Ban}^{op}}$. In other (categorical) words, the functor $\jmath_*$ is not exact.
- Non-splitting and lack of projectivity. The exactness and non-splitting of $0 \to \mathfrak{L}^Y \xrightarrow{\eta_\imath} \mathfrak{L}^Z \xrightarrow{\eta_\pi} \mathfrak{L}^X \to 0$ in ¥ implies that $\mathfrak{L}^X$ is not projective in ¥, in sharp contrast with what happens in the full presheaf functor category:

[14] Theorem 28.4 can be summarized, in the words of Freyd [49], as *Relative homological algebra made absolute.*

PROPOSITION 28.5. *For every Banach space X the functor $\mathfrak{L}^X$ is projective in $\mathbf{Ab}^{\mathbf{Ban}^{op}}$.*

Proof. Let $F \xrightarrow{u} G \to 0$ be exact, hence $F(X) \xrightarrow{u(X)} G(X) \to 0$ is also exact. Let $\eta : \mathfrak{L}^X \to G$. Since $\blacktriangleleft \mathfrak{L}^X, G \blacktriangleright = G(X)$, we assume again that $\eta \in G(X)$ and therefore $\eta = u(X)(f)$ for some $f \in F(X) = \blacktriangleleft \mathfrak{L}^X, F \blacktriangleright$. Therefore $uf = \eta$ and f is the desired lifting attesting that $\mathfrak{L}^X$ is projective. ∎

- Why projectivity does not descend to ¥. The result is once again bedazzling because one easily could think that since ¥ is a full subcategory of $\mathbf{Ab}^{\mathbf{Ban}^{op}}$, a projective element of $\mathbf{Ab}^{\mathbf{Ban}^{op}}$ will be a projective element of ¥. But it is not: a diagram in ¥

$$\begin{tikzcd} Z \arrow[r, "\epsilon"] & X \\ & \mathfrak{L}^X \arrow[u] \end{tikzcd}$$

  with ϵ epic is not a diagram in $\mathbf{Ab}^{\mathbf{Ban}^{op}}$ with ϵ epic. In fact, Weibel remarks in [87, Remark p. 29] "cokernels in ¥ are different from cokernels in $\mathbf{Ab}^{\mathbf{Ban}^{op}}$".
- A remark of José Navarro. We thank José Navarro for the following enlightening remark (among others): Given an exact sequence $0 \to Y \xrightarrow{\imath} Z \xrightarrow{\pi} X \to 0$, the functor $\mathfrak{L}^X$ is the cokernel in ¥ of $\mathfrak{L}^Y \xrightarrow{\eta_\imath} \mathfrak{L}^Z$. We already know that it is not the cokernel in $\mathbf{Ab}^{\mathbf{Ban}^{op}}$ and it is simple that it is in the "category of Yoneda functors $\mathcal{Y}(C)$". So, what is needed for a functor $\mathcal{F}$ to "accept" $\mathfrak{L}^X$ as the cokernel is that every arrow $\alpha : \mathfrak{L}^Z \to \mathcal{F}$ that vanishes when composed with $\eta_\imath$ factorizes as $\alpha = \beta\eta_\pi$ through a unique arrow $\beta : \mathfrak{L}^X \to \mathcal{F}$. But that turns out as saying that $0 \to \mathcal{F}(X) \to \mathcal{F}(Z) \to \mathcal{F}(X)$ is exact. This additionally means that ¥ is the largest subcategory of $\mathbf{Ab}^{\mathbf{Ban}^{op}}$ reflecting the exactness of $\mathbf{Ban}$.
- Non-exactness of $\jmath_*$ and sheafification. Regarding the non-exactness of $\jmath_*$, what actually occurs is that $\jmath_*$ is the right-adjoint of a functor usually called *sheafification*

$$\begin{tikzcd} ¥ \arrow[r, shift left, "\jmath_*"] & \mathbf{Ab}^{\mathbf{C}} \arrow[l, shift left, dashed, "sheaf"] \end{tikzcd}.$$

  Here "sheaf" is the sheafification functor; the reader will understand that we will just denote it by $\jmath^*$. Thus we have the adjunction $\jmath^* \dashv \jmath_*$, which automatically implies that $\jmath_*$ is left-exact and $\jmath^*$ is right exact. It can be proved that sheafification is an exact process (see [11, Lemma A.18]).

28.3. Meddle. As we have seen, ascending to functor categories provides additional regularity. In this section we present Auslander's theory of *coherent* functors [2]; see [56] for an extraordinarily limpid exposition .

Definition 28.6. A contravariant (resp. covariant) functor F is said to be coherent if it is the cokernel in an exact sequence

$$\mathfrak{L}^Z \longrightarrow \mathfrak{L}^X \longrightarrow F \longrightarrow 0$$

(resp. $\mathfrak{L}_Z \to \mathfrak{L}_Y \to F \to 0$).

The projective character of $\mathfrak{L}^A$ and $\mathfrak{L}_A$ immediately yields

Proposition 28.7. *To be coherent is a 3-space property; i.e., given an exact sequence*

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

in which F and H are coherent then G is coherent.

Slightly more demanding is the proof that given a morphism $\eta : F \to G$ between coherent functors in an Abelian category both $\ker \eta$ and $\operatorname{coker} \eta$ are coherent functors, see [11, Theorem 1.1. (a)]. Two obvious examples come to mind:

- The functors $\mathfrak{L}^A$ and $\mathfrak{L}_A$ are coherent.
- The functors $\mathrm{Ext}(\cdot, A)$ and $\mathrm{Ext}(A, \cdot)$ are coherent.

The obvious choices $\mathfrak{L}^0 \to \mathfrak{L}^A = \mathfrak{L}^A \to 0$ (resp. $\mathfrak{L}_0 \to \mathfrak{L}_A = \mathfrak{L}_A \to 0$) prove the first assertion. The second follows because Ext is the derived functor of $\mathfrak{L}$.

Coherent functors in an Abelian category $\mathbf{A}$ form a full Abelian subcategory of $\mathbf{Ab}^{\mathbf{A}^{op}}$ usually denoted $\mathrm{mod}(\mathbf{A})$. But, alas! **Ban** is only an exact category, so mod(**Ban**) is not Abelian. Coherent functors can however be defined on exact categories with only a bit of additional fuss: one must restrict to *admissible* arrows. In particular, we need that the transformations $\mathfrak{L}^Z \to \mathfrak{L}^X$ come induced by admissible arrows (something that the identity $\blacktriangleleft \mathfrak{L}^Z, \mathfrak{L}^X \blacktriangleright = \mathsf{Hom}(Z, X)$ does not guarantee) automatically). This extra condition allows us to safely manage the exact sequences $\mathfrak{L}^Z \to \mathfrak{L}^X \to F \to 0$. None of this worries us too much in practice: in our context, all that is required is that the operator $Z \to X$ have closed range. Let us readily verify that $\mathrm{Ext}_{\mathbf{B}}$ functors are still coherent:

- If we start with an injective presentation $0 \to A \to \mathcal{J} \to c\kappa \to 0$ of A we get that $\mathrm{Ext}_{\mathbf{B}}(\cdot, A)$ is coherent because

$$\mathfrak{L}^{\mathcal{J}} \longrightarrow \mathfrak{L}^{c\kappa} \longrightarrow \mathrm{Ext}(\cdot, A) \longrightarrow 0\,.$$

- If we start with a projective presentation $0 \to \kappa \to \mathcal{P} \to A \to 0$ of A we get that $\mathrm{Ext}_{\mathbf{B}}(A, \cdot)$ is coherent because

$$\mathfrak{L}_{\mathcal{P}} \longrightarrow \mathfrak{L}_{\kappa} \longrightarrow \mathrm{Ext}_{\mathbf{B}}(A, \cdot) \longrightarrow 0\,.$$

This leads naturally to the notion of *effaceable* functor: in the contravariant case, a coherent functor is said to be effaceable when the transformation $\mathfrak{L}^Z \to \mathfrak{L}^X$ is induced by an epic map $f : Z \to X$. In the covariant case, the transformation $\mathfrak{L}_Z \to \mathfrak{L}_Y$ must come from a monic map $f : Y \to Z$. Attentive readers will concur with us in the:

Obvious claim. *The functors* $\mathrm{Ext}_{\mathbf{B}}(\cdot, A)$, $\mathrm{Ext}_{\mathbf{B}}(A, \cdot)$ *and* $\mathrm{Ext}_{\mathbf{p}}(A, \cdot)$ *are effaceable.*

Observe moreover that the exact sequence $\mathfrak{L}^Z \xrightarrow{\mathcal{Y}(f)} \mathfrak{L}^X \to F \to 0$ leads to the commutative diagram

$$\begin{tikzcd}
\mathfrak{L}^Z \arrow[rr, "\mathcal{Y}(f)"] & & \mathfrak{L}^X \arrow[r] \arrow[dr] & F \arrow[r] \arrow[d] & 0\,. \\
& & & \mathfrak{L}^{\operatorname{coker} f} &
\end{tikzcd}$$

Therefore, taking $F = \mathfrak{L}^A$ yields $A = \operatorname{coker} f$. Hence, if f is epic then $F = \mathfrak{L}^0 = 0$. The covariant case is analogous, replacing $\operatorname{coker} f$ with $\ker\ f$. Consequently:

Another simple claim. *The functors* $\mathfrak{L}_A$ *and* $\mathfrak{L}^A$ *are not effaceable*

Which can now be followed by a far less obvious claim.

PROBLEM 28.8. Are the functors $\mathrm{Ext}_{\mathbf{Q}}(\cdot, A)$ and $\mathrm{Ext}_{\mathbf{Q}}(A, \cdot)$ effaceable?

Beware that the notion of effaceability we are using is not Grothendieck's original one [53, p. 141]: A functor F is G-effaceable (resp. G-coeffaceable) if for all C there is a monic $u : C \to C'$ (resp. an epic $u : C' \to C$) such that $F(u) = 0$. The first notion should apply in the covariant case and the second in the contravariant case; for instance, $\mathrm{Ext}_{\mathbf{B}}(A, \cdot)$ will be G-effaceable just taking an embedding $u : C \to \mathcal{I}$ while $\mathrm{Ext}_{\mathbf{B}}(\cdot, A)$ will be G-coeffaceable just taking a quotient $u : \mathcal{P} \to C$.

Is $\mathrm{Ext}_{\mathbf{Q}}(A, \cdot)$ G-effaceable now that we cannot afford an injective embedding? Well, if amalgams exist for objects in the target category, as it happens in $\mathbf{Ab}^{\mathbf{QBan}}$ then G-effaceable becomes *weakly effaceable* (cf. [57, Definition 3.19]) with the meaning: for all C and all $x \in F(C)$ there is a monic $u : C \to C'$ (resp. an epic $u : C' \to C$) such that $F(u)(x) = 0$. Let us show that we have:

A partial but technically acceptable answer. *The functors* $\mathrm{Ext}_{\mathbf{Q}}(\cdot, A)$ *and* $\mathrm{Ext}_{\mathbf{Q}}(A, \cdot)$ *are weakly effaceable.*

Funnily, the proof relies on two hard-core results of quasi-Banach space theory:

- The Aoki-Rolewicz theorem [13, 1.1.1]: every quasi-Banach space is a p-Banach space for some $p \leq 1$.
- Kalton's theorem [13, 3.11], which in particular implies: a twisted sum of a p-Banach and a q-Banach space is a $\frac{1}{2}\min\{p, q\}$-Banach space.

The proof for $\mathrm{Ext}_{\mathbf{Q}}(\cdot, A)$ is now simple since, once C has been fixed, every element of $\mathrm{Ext}_{\mathbf{Q}}(C, A)$ lives in some category **pBan**, where projective objects exist. Let us make the proof for $F = \mathrm{Ext}_{\mathbf{Q}}(A, \cdot)$: all action involving the second quantifier (namely, $\forall a \in \mathrm{Ext}_{\mathbf{Q}}(A, C)$) takes place in **pBan** for some $p < 1$. Therefore if –by a slight abuse of notation– a denotes the exact sequence $0 \to C \to a \to A \to 0$ and C' is the pushout (in the appropriate category **pBan**; see [13, Chapter 2]) of the family of embeddings $\{C \to a\}$ then the canonical embedding $u : C \to C'$ satisfies the condition of weak effaceability.

At this point, categorical readers would be thinking about the previous down-to-earth quasi-Banach argument: "God only knows it's not what we would choose to do" [15]. Ok then, let us present a categorical proof (see [57,

[15] See the title of this Section 28.

Proposition 3.20]) that, additionally, sets the connection between the two notions of effaceability:

PROPOSITION 28.9. *A functor of* $\mathbf{Ab}^{\mathbf{QBan}^{op}}$ *is effaceable if and only if it is weakly effaceable and coherent.*

Proof. Let $\mathfrak{L}^Z \xrightarrow{\mathcal{Y}(f)} \mathfrak{L}^X \xrightarrow{\epsilon} F \to 0$ represent the effaceable functor F; namely, f is surjective. Pick C. Evaluation at C yields $\mathfrak{L}(C,Z) \xrightarrow{\mathcal{Y}(f)} \mathfrak{L}(C,X) \xrightarrow{\epsilon_C} F(C) \to 0$. Since ϵ_C is surjective, given $x \in F(C)$ there is $g \in \mathfrak{L}(C,X)$ such that $\epsilon_C(g) = x$. Form the pullback diagram

$$\begin{tikzcd} Z \arrow[r, "f"] & X \\ C' \arrow[u, "g'"] \arrow[r, "u"'] & C \arrow[u, "g"'] \end{tikzcd}$$

and recall that since f is surjective so is u and this is the map we were searching for. Let us check that $F(u)(x) = 0$. Observe the commutative diagram

$$\begin{tikzcd} \mathfrak{L}(C,Z) \arrow[r, "\mathcal{Y}(f)"] \arrow[d, "u^*"'] & \mathfrak{L}(C,X) \arrow[r, "\epsilon_C"] \arrow[d, "u^*"'] & F(C) \arrow[r] \arrow[d, "F(u)"] & 0 \\ \mathfrak{L}(C',Z) \arrow[r, "\mathcal{Y}(f)"] & \mathfrak{L}(C',X) \arrow[r, "\epsilon_{C'}"] & F(C') \arrow[r] & 0 \end{tikzcd}$$

that yields

$$\begin{aligned} F(u)(x) &= F(u)\epsilon_C(g) = \epsilon_{C'}u^*(g) \\ &= \epsilon_{C'}(gu) = \epsilon_{C'}(fg') = \epsilon_{C'}\mathcal{Y}(f)(g') = 0\,. \end{aligned}$$

The other implication is simpler. ∎

We still have a modified version of the far less obvious claim in Problem 28.8:

PROBLEM 28.10. Are the functors $\mathrm{Ext}_{\mathbf{Q}}(\cdot, A)$ and $\mathrm{Ext}_{\mathbf{Q}}(A, \cdot)$ coherent?

We are now ready for *Auslander's formula*. If we denote $\mathrm{mod}(\mathbf{C})$ the category of coherent functors and by $\mathrm{eff}(\mathbf{C})$ its full subcategory of the effaceable functors then

$$\mathbf{C} = \mathrm{mod}(\mathbf{C})/\mathrm{eff}(\mathbf{C})$$

with the meaning that an Abelian category can be represented as the quotient category of coherent functors modulo effaceable functors. Let us follow the succinct proof of [65], both for its conceptual elegance and for the interesting technical issues it displays. Consider the following chart

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\text{よ}"] \arrow[d, equal] & ¥ \arrow[rr, shift left, "\jmath_*"] & & \mathbf{Ab}^{\mathbf{C}} \arrow[ll, shift left, dashed, "sheaf=\jmath^*"] \\
\mathbf{C} \arrow[r, shift left, "\text{よ}"] & \mathrm{mod}(\mathbf{C}) \arrow[l, shift left, dashed, "L"] & & \mathrm{eff}(\mathbf{C}) \arrow[ll]
\end{tikzcd}$$

in which dotted arrows are left adjoints to be defined next. The left adjoint $\jmath^*$ of $\jmath_*$ may at this point give us cold feet, but the left-adjoint $\mathcal{L}$ of the Yoneda functor when taking values in the category $\mathrm{mod}(\mathbf{C})$ of coherent functors does not. Indeed, given a coherent functor F presented as $\mathfrak{L}^Z \xrightarrow{\text{よ}(f)} \mathfrak{L}^X \xrightarrow{\epsilon} F \to 0$ its natural left-adjoint candidate is

$$\mathcal{L}(F) = \operatorname{coker} f.$$

In fact, if we start with the presentation $Z \xrightarrow{f} X \xrightarrow{\pi} \operatorname{coker} f \to 0$ and observe the diagram introduced after the Obvious Claim

$$\begin{tikzcd}
\mathfrak{L}^Z \arrow[r, "\text{よ}(f)"] & \mathfrak{L}^X \arrow[r, "\epsilon"] \arrow[dr, "\text{よ}(\pi)"'] & F \arrow[r] \arrow[d, "v"] & 0 \\
& & \mathfrak{L}^{\operatorname{coker} f} &
\end{tikzcd}$$

then we obtain

$$\mathfrak{L}(\operatorname{coker} f, Y) \ = \blacktriangleleft F, \text{よ}(Y) \blacktriangleright .$$

Quotient categories and Auslander's formula. The next step in the proof of Auslander's formula involves the meaning of *quotient category*, which technically requires work it through the localization process [25]. Since we are determined to stay in an operative mood, let us declare that the quotient of category $\mathbf{C}$ by a reasonable subcategory $\mathbf{B}$ is a category, obviously called $\mathbf{C}/\mathbf{B}$, together with a functor $Q : \mathbf{C} \to \mathbf{C}/\mathbf{B}$ subject to the conditions $Q(B) = 0$ for every object B of $\mathbf{B}$; and with the universal property with respect to that diagram: every *similar* functor $G : \mathbf{C} \to \mathbf{D}$ such that $G(B) = 0$ for every object B of $\mathbf{B}$ factorizes uniquely through Q. The meaning of *similar* is that, depending on the categories we are working with, it could be necessary to impose some conditions to the functors; say, additivity.

The categorical argument. Following the always untwisted categorical ways, consider the quotient category $\mathrm{mod}(\mathbf{C})/\mathrm{eff}(\mathbf{C})$ with quotient functor $Q : \mathrm{mod}(\mathbf{C}) \to \mathrm{mod}(\mathbf{C})/\mathrm{eff}(\mathbf{C})$, whatever they are. Observe that the functor $\mathcal{L} : \mathrm{mod}(\mathbf{C}) \to \mathbf{C}$ behaves as nicely as one could hope: it vanishes on all effaceable functors (those for which $\operatorname{coker} f = 0$). Actually, $\mathcal{L}(F) = 0$ if and only if F is effaceable. Thus $\mathcal{L}$ factorizes through Q in the form $\mathcal{L} = \mathcal{L}'Q$. This forces $\mathcal{L}'$ to be an equivalence: indeed, if $Q(F)$ is any object in the quotient category, $\mathcal{L}'Q(F) = 0$ implies $\mathcal{L}(F) = 0$, hence F is effaceable and $Q(F) = 0$. The abstract proof can be found in [52], in which case we must use that $\mathcal{L}$ preserves limits, as any left-adjoint functor must.

A more concrete identification of the quotient. The previous argument requires to know in advance that the quotient category exists. To know which is which and who is who we attempt a more concrete route showing that **Ban** (resp. **pBan** , **QBan**) is the quotient category in our setting, and that $\mathcal{L}$ is the quotient functor: pick $G : \mathrm{mod}(\mathbf{C}) \to \mathbf{D}$ a functor that respects limits and vanishes on effaceable functors. The obvious factorization of $G = \mathcal{L}'\mathcal{L}$ appears taking $\mathcal{L}' = G\mathcal{Y}$. Let us check it works, i.e., that $G = G\mathcal{Y}\mathcal{L}$. From the presentation $\mathfrak{L}^Z \xrightarrow{\mathcal{Y}(f)} \mathfrak{L}^X \xrightarrow{\epsilon} F \to 0$ we get an arrow ν making the diagram

$$\begin{tikzcd}
G\left(\mathfrak{L}^Z\right) \arrow[r] & G\left(\mathfrak{L}^X\right) \arrow[r] \arrow[dr] & G(F) \arrow[r] \arrow[d, "\nu"] & 0 \\
& & G\left(\mathfrak{L}^{\operatorname{coker} f}\right) &
\end{tikzcd}$$

commutative. On the other hand, applying $\mathcal{Y}$ to $Z \xrightarrow{f} X \xrightarrow{\pi} \operatorname{coker} f \to 0$ we get $\mathfrak{L}^X \to \mathfrak{L}^{\operatorname{coker} f} \to M \to 0$. Since M is effaceable and G respects limits, applying G yields $G\left(\mathfrak{L}^Z\right) \to G\left(\mathfrak{L}^X\right) \to G\left(\mathfrak{L}^{\operatorname{coker} f}\right) \to 0$, so $G\left(\mathfrak{L}^{\operatorname{coker} f}\right)$ is the cokernel, which produces an arrow $G\left(\mathfrak{L}^{\operatorname{coker} f}\right) \to G(F)$ making the diagram commutative, hence an equivalence.

Extension to exact categories. The only caveat is that Auslander's approach is formulated for Abelian categories, while **Ban** or **QBan** are merely exact categories. Henrard, Kvamme and van Roosmalen [57] have thought about that and transplanted Auslander's theorem to the fertile soil of exact categories [57, Theorem 3.11], adapting the notion of coherent functor to "coherent functor obtained from admissible arrows" and quotient category to "quotient exact category" asking the quotient functor to be exact, which is what we did above. In conclusion, they show that if $\mathbf{E}$ is an exact category then $\mathbf{E} = \mathrm{mod}(\mathbf{E})/\mathrm{eff}(\mathbf{E})$. A good Banach résumé for all that work could be:

THEOREM 28.11.

- $\mathbf{Ban} = \mathrm{mod}(\mathbf{Ban})/\mathrm{eff}(\mathbf{Ban})$;
- $\mathbf{pBan} = \mathrm{mod}(\mathbf{pBan})/\mathrm{eff}(\mathbf{pBan})$;
- $\mathbf{QBan} = \mathrm{mod}(\mathbf{QBan})/\mathrm{eff}(\mathbf{QBan})$.

29. ANY COLOUR YOU LIKE

In [58, Corollary 10.2]) we encounter the identity

$$\blacktriangleleft R^{(1)}\mathsf{Hom}(A,\cdot), F \blacktriangleright = (L^{(1)}F)(A)$$

for right exact covariant functors. In our Banach space context this yields

$$F^{(1)}(A) \ = \blacktriangleleft \mathrm{Ext}_{\mathbf{Q}}(A,\cdot), F \blacktriangleright$$

which makes tingling in our head the idea of defining a new, more adventurous derived functor F'. For clarity, let us change in this section the notation: in the same way that we denoted $\mathsf{Hom}(A,\cdot)$ by $\mathfrak{L}_A$ and $\mathsf{Hom}(\cdot,A)$ by $\mathfrak{L}^A$, we will denote the functors $\mathrm{Ext}(A,\cdot)$ by Ext_A and $\mathrm{Ext}(\cdot,A)$ by Ext^A. One may then contemplate defining a new derived functor via $F'(A) = \blacktriangleleft \mathrm{Ext}_A, F \blacktriangleright$ in analogy with $F(A) = \blacktriangleleft \mathfrak{L}_A, F \blacktriangleright$. However, caution is needed: the identity above works for right exact functors, precisely those whose first derived functor should vanish. Trying to extend that to non exact functors is delusional: for instance, if one picks $F = \mathfrak{L}_B$ the identity

$$\mathfrak{L}'_B(A) = \mathrm{Ext}(B,A) \stackrel{?}{=} \blacktriangleleft \mathrm{Ext}_A, \mathfrak{L}_B \blacktriangleright \tag{29.1}$$

seems false, while if $F = \mathrm{Ext}_B$ the identity

$$\mathrm{Ext}'_B(A) = 0 \stackrel{?}{=} \blacktriangleleft \mathrm{Ext}_A, \mathrm{Ext}_B \blacktriangleright$$

is false because of [58, Proposition 10.3]):

PROPOSITION 29.1. $\blacktriangleleft \mathrm{Ext}_A, \mathrm{Ext}_B \blacktriangleright = \mathfrak{L}(B,A)$.

Proof. It is plain that every operator $\alpha : A \to B$ induces a natural transformation $\overline{\alpha} : \mathrm{Ext}(B,\cdot) \to \mathrm{Ext}(A,\cdot)$ in the form: for given X,

$$\overline{\alpha}_X(\Omega) = \Omega\alpha.$$

Pick, as before, a projective presentation Σ_B of B and apply homology to $\mathfrak{L}_A$ and $\mathfrak{L}_B$. Assuming the existence of a natural transformation $\eta : \mathrm{Ext}(B,\cdot) \to \mathrm{Ext}(A,\cdot)$ we get the commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathfrak{L}(B,B) \arrow[r, "\Sigma_B^*"] & \mathrm{Ext}(B,\kappa(B)) \arrow[r, "\imath_*"] \arrow[d, "\eta_{\kappa(B)}"'] & \mathrm{Ext}(B,\mathcal{P}) \arrow[r] \arrow[d, "\eta_{\mathcal{P}}"] & \cdots \\
\cdots \arrow[r] & \mathfrak{L}(A,B) \arrow[r, "\Sigma_B^*"'] & \mathrm{Ext}(A,\kappa(B)) \arrow[r, "\imath_*"'] & \mathrm{Ext}(A,\mathcal{P}) \arrow[r] & \cdots .
\end{tikzcd}$$

Since $\imath_*\Sigma_B = 0$, it turns out that $\eta_{\kappa(B)}(\Sigma_B) \in \ker \imath_* = \mathfrak{L}(A,B)$. This is the way in which an operator $\alpha \in \mathfrak{L}(A,B)$ such that $\Sigma_B\alpha = \Sigma_B^*(\alpha) = \eta_{\kappa(B)}(\Sigma_B)$ appears. Once it has been obtained that η acts as left composition with α on projective elements, it is simple to check the same happens on all elements. ∎

Observe that the identity in Proposition 29.1 is just identity (29.1) with the roles of $\mathfrak{L}$ and Ext reversed, namely $\mathfrak{L}_B(A) = \blacktriangleleft \mathrm{Ext}_A, \mathfrak{L}'_B \blacktriangleright$. This suggests investigating to what extent relative derivation of other operator ideals would yield analogues of Proposition 29.1.

Problem 29.2. Let $\mathfrak{U}$ be a suitable operator ideal. Is it possible that

$$\mathfrak{U}(A,B) = \blacktriangleleft \mathrm{Ext}_{\mathfrak{U}}(B,\cdot), \mathrm{Ext}_{\mathfrak{U}}(A,\cdot) \blacktriangleright .$$

For instance, it is easy to check that

$$\mathfrak{U}(A,B) \subset \blacktriangleleft \mathrm{Ext}_{\mathfrak{U}}(B,\cdot), \mathrm{Ext}_{\mathfrak{U}}(A,\cdot) \blacktriangleright .$$

Indeed, each operator $\alpha \in \mathfrak{U}(A,B)$ induces $\widehat{\alpha} \in \blacktriangleleft \mathrm{Ext}_{\mathfrak{U}}(B,\cdot), \mathrm{Ext}_{\mathfrak{U}}(A,\cdot) \blacktriangleright$ as follows: Let Σ_B be a projective presentation

$$0 \longrightarrow \kappa(B) \xrightarrow{\ \imath\ } \mathcal{P} \longrightarrow B \longrightarrow 0$$

of B so that each element $\Omega \in \mathrm{Ext}_{\mathfrak{U}}(B,X)$ has the form $\Omega = \phi\Sigma_B$ for some $\phi \in \mathfrak{U}(\kappa, X)$. We define $\widehat{\alpha}_X : \mathrm{Ext}_{\mathfrak{U}}(B,X) \to \mathrm{Ext}_{\mathfrak{U}}(A,X)$ as

$$\widehat{\alpha}_X(\Omega) \ = \ \widehat{\alpha}_X(\phi\Sigma_B) \ = \ \phi\Sigma_B\alpha\,.$$

To check that $\phi\Sigma_B\alpha \in \operatorname{Ext}_{\mathfrak{U}}(A,X)$, observe that $\Sigma_B\alpha = \psi\Sigma_A$ for some $\psi \in \mathfrak{L}(\kappa(A),\kappa(B))$. Therefore $\phi\Sigma_B\alpha = \phi\psi\Sigma_A$ and $\phi\psi \in \mathfrak{U}$ by the ideal property. Therefore, if $\mathfrak{L}(A,B) = \mathfrak{U}(A,B)$ we obtain a correspondence

$$\begin{tikzcd}
\blacktriangleleft \operatorname{Ext}(B,\cdot), \operatorname{Ext}(A,\cdot) \blacktriangleright \arrow[r, "\gimel"] \arrow[d, equal] & \blacktriangleleft \operatorname{Ext}_{\mathfrak{U}}(B,\cdot), \operatorname{Ext}_{\mathfrak{U}}(A,\cdot) \blacktriangleright \\
\mathfrak{L}(A,B) \arrow[r, equal] & \mathfrak{U}(A,B)\,. \arrow[u, dashed]
\end{tikzcd}$$

Identifying natural transformations in $\blacktriangleleft \operatorname{Ext}_{\mathfrak{U}}(B,\cdot), \operatorname{Ext}_{\mathfrak{U}}(A,\cdot) \blacktriangleright$ does not seem simple. Indeed, the proof of Proposition 29.1 presented above does not carry over to $\operatorname{Ext}_{\mathfrak{U}}$ in **Ban** because the projective presentation Σ_B is not an element of $\operatorname{Ext}_{\mathfrak{U}}(B,\kappa(B))$ unless the identity of $\kappa(B)$ belongs to $\mathfrak{U}$.

Reproducing Proposition 29.1 in **pBan** seems reasonable; establishing its validity in **QBan** appears substantially harder. A related and noteworthy result in **pBan** was obtained in [38, Lemma 4.2] and goes as follows: Let $0 < p < q \leq 1$ and consider two subspaces A, B of $L_p = L_p(0,1)$. Consider the functors $\operatorname{Ext}(L_p/A,\cdot)$ and $\operatorname{Ext}(L_p/B,\cdot)$ acting on **QBan** and with values on the category of Vector spaces.

Proposition 29.3. $\mathfrak{L}(B,A) = \blacktriangleleft \operatorname{Ext}(L_p/A,\cdot), \operatorname{Ext}(L_p/B,\cdot) \blacktriangleright$.

Proof. If X is a q-Banach space, $p < q$, then $\mathfrak{L}(L_p, X) = 0$ [13, 1.1.5]. Therefore, applying homology to $0 \to A \to L_p \to L_p/A \to 0$ yields the identity $\operatorname{Ext}(L_p/A, X) = \mathfrak{L}(A,X)$. Consequently, a natural transformation between $\operatorname{Ext}(L_p/A,\cdot)$ and $\operatorname{Ext}(L_p/B,\cdot)$ generates a natural transformation between $\mathfrak{L}_A$ and $\mathfrak{L}_B$. Since $\blacktriangleleft \mathfrak{L}_A, \mathfrak{L}_B \blacktriangleright = \mathfrak{L}(B,A)$, we are done. ∎

Let us now recover some ideas of [69, 70]. To frame them properly, recall from Section 24 (see also [83, Proposition 6.3, (c) or (d)]) that covariant right-exact functors (resp. left-exact) coincide with their 0^{th}-left (resp. right) derived; while a contravariant left-exact (resp. right-exact) functors coincide with their 0^{th}- left (resp. right) derivative. This is not necessarily true for non-additive functors. Here, as always, we have an issue with the names because we adopted the terms *projective* and *injective* derivations instead of the continuously shifting *left, right* in the literature. So, the translation of the above, with some additional detail and fixing of notation is:

- Let $\mathcal{F}$ be a covariant functor, and let $\mathcal{F}^p$ be its projective (i.e., left) derivation and $\mathcal{F}^i$ its injective (i.e., right) derivations. There are natural transformations $\rho^{\mathcal{F}} : \mathcal{F}^p \to \mathcal{F}$ and $\rho_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^i$. If $\mathcal{F}$ is right-exact then $\rho^{\mathcal{F}}$ is a natural equivalence. If $\mathcal{F}$ is left-exact then $\rho_{\mathcal{F}}$ is a natural equivalence.

- Let $\mathcal{F}$ be a contravariant functor, and let $\mathcal{F}^p$ be its projective (i.e., right) derivation and $\mathcal{F}^i$ its injective (i.e., left) derivations. There are natural transformations $\rho_{\mathcal{F}} : \mathcal{F}^i \to \mathcal{F}$ and $\rho^{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^p$. If $\mathcal{F}$ is right-exact then $\rho^{\mathcal{F}}$ is a natural equivalence. If $\mathcal{F}$ is left-exact then $\rho_{\mathcal{F}}$ is a natural equivalence.

In the general case, when $\mathcal{F}$ is simply an additive covariant or contravariant functor, neither $\rho_{\mathcal{F}}$ nor $\rho^{\mathcal{F}}$ need be natural equivalences. Martsinkovsky and Russell [70, 71, 72] consider their *projective* and *injective* stabilizations introduced by Auslander [2] as follows:

If $\mathcal{F}$ is covariant:

- The *projective stabilization* of $\mathcal{F}$ is
$$\underline{\mathcal{F}} = \operatorname{coker}\left(\rho^{\mathcal{F}} : \mathcal{F}^p \to \mathcal{F}\right).$$
- The *injective stabilization* of $\mathcal{F}$ is
$$\overline{\mathcal{F}} = \ker\left(\rho_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^\imath\right).$$

If $\mathcal{F}$ is contravariant:

- The *projective stabilization* of $\mathcal{F}$ is
$$\underline{\mathcal{F}} = \operatorname{coker}\left(\rho_{\mathcal{F}} : \mathcal{F}^\imath \to \mathcal{F}\right).$$
- The *injective stabilization* of $\mathcal{F}$ is
$$\overline{\mathcal{F}} = \ker\left(\rho^{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^p\right).$$

As the authors of [70] point out, the mnemonics for this are: projective stabilization are cokernels of counits and injective stabilization are kernels of units. With a touch of *pentimento* Martsinkovsky later renames in [69] injective stabilization as *sub-stabilization* and projective stabilization as *quot-stabilization*.

Additional stabilizations. In [31] the following two missing notions were introduced for a contravariant functor:

- The *coinjective stabilization* of $\mathcal{F}$ is

$$\overline{\mathcal{F}}^{co} = \mathrm{coker}\left(\rho^{\mathcal{F}} : \mathcal{F} \to \mathcal{F}\right).$$

- The *coprojective stabilization* of $\mathcal{F}$ is

$$\underline{\mathcal{F}}_{co} = \ker\left(\rho_{\mathcal{F}} : \mathcal{F}^{\imath} \to \mathcal{F}\right).$$

Analogously, for the covariant case we set:

- The *coprojective stabilization* of $\mathcal{F}$ is

$$\underline{\mathcal{F}}_{co} = \ker\left(\rho^{\mathcal{F}} : \mathcal{F}^{p} \to \mathcal{F}\right).$$

- The *coinjective stabilization* of $\mathcal{F}$ is

$$\overline{\mathcal{F}}^{co} = \mathrm{coker}\left(\rho_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^{\imath}\right).$$

These notions are perhaps worth consideration while working with non additive functors. Pick for instance the case of a covariant functor $\mathcal{F}$ we obtain the following four term exact sequences:

$$0 \longrightarrow \underline{\mathcal{F}}_{co} \longrightarrow \mathcal{F}^{p} \xrightarrow{\rho_{\mathcal{F}}} \mathcal{F} \longrightarrow \underline{\mathcal{F}} \longrightarrow 0\,,$$

$$0 \longrightarrow \overline{\mathcal{F}} \longrightarrow \mathcal{F} \xrightarrow{\rho^{\mathcal{F}}} \mathcal{F}^{\imath} \longrightarrow \overline{\mathcal{F}}^{co} \longrightarrow 0\,.$$

Further (and longer) exact sequences involving the projective and injective stabilizations, that deserve careful consideration and study, have been obtained in [69].

30. Banach damage

If Rick Wakeman was able to return to the center of the earth [16], we can return to the place the journey started: Banach/quasi-Banach space theory. We revisit next a few relevant topics, perhaps anticipating further studies.

[16] Rick Wakeman's albums, Journey to the center of the earth; and Return to the center of the earth.

30.1. Ext^2 in Banach spaces. The reader is addressed to either [24, 25] or to [16] for standard interpretations of Ext^n spaces in **Ban** . And of course to [50]. Here we want to focus on Ext^2. Recall from the end of Section 24 that elements of $\mathrm{Ext}^2(X, Y)$ can be described as a concatenation $\Omega\Phi$ of two 1-quasilinear maps $\Phi : X \to E$ and $\Omega : E \to Y$ for some Banach space E, subject to the equivalence relation: $\Omega\Phi \equiv 0$ if and only if Ω can be extended or Ψ can be lifted; i.e., the concatenation "splits" in the sense that there is a commutative diagram (24.4); namely,

$$\begin{tikzcd}
Y \arrow[r, equal] \arrow[d] & Y \arrow[d] & \\
Y \oplus_\Omega E \arrow[r] \arrow[d] & \square \arrow[r] \arrow[d] & X \arrow[d, equal] \\
E \arrow[r] \arrow[uu, bend left, dashed, "\Omega"] & E \oplus_\Phi X \arrow[r] & X . \arrow[ll, bend left, dashed, "\overline{\Phi}"]
\end{tikzcd}$$

The extra punch of this description is that it allows us to work with exact sequences, or with elements of Ext^2, described by quasilinear maps enjoying additional properties. Here is one particularly important case due to Kalton [62, 61]:

Definition 30.1. Let X be a Banach space endowed with a Banach L_∞-module structure. A quasilinear map Ω on X is called an *L_∞-centralizer* if

$$\|\xi\Omega(x) - \Omega(\xi x)\| \leq C\|\xi\|_\infty \|x\|_2$$

for some constant $C > 0$, all $\xi \in L_\infty$ and all $x \in X$.

Essentially all quasilinear maps between Hilbert spaces currently known are centralizers (those generated by complex interpolation, say; see next sections). A breakthrough appeared with the following result of [15]:

Proposition 30.2. *There exist two centralizers Ω, Ψ on ℓ_2 such that $\Omega\Psi \neq 0$ in $\mathrm{Ext}^2(\ell_2, \ell_2)$.*

In particular, $\mathrm{Ext}^2(\ell_2, \ell_2) \neq 0$. In this context, the following result from [18] is remarkable:

PROPOSITION 30.3. *If* $\Omega \in \mathrm{Ext}(\ell_2, \ell_2)$ *is a centralizer then* $\Omega\Omega \equiv 0$ *in* $\mathrm{Ext}^2(\ell_2, \ell_2)$.

This suggests

PROBLEM 30.4. Does the identity $\Omega\Omega \equiv 0$ remain true for arbitrary quasi-linear maps?

The determination of those pairs of Banach (or quasi-Banach) spaces X, Y for which $\mathrm{Ext}^2(X, Y) = 0$ is a widely open problem, and almost everything of what is known can be found in [16] and [41]. Let us here consider a particularly interesting case of study, the following long time open problem:

PROBLEM 30.5. Is $\mathrm{Ext}^2(c_0, \ell_1) = 0$?

We need to remark that showing $\mathrm{Ext}(c_0, \ell_1) \neq 0$ is already a tough work [13, Proposition 5.2.20], and still nowadays the result is not entirely understood (see startled comments and further approaches to the problem in [21, 29]). Proving $\mathrm{Ext}^2(c_0, \ell_1) \neq 0$ requires, by the standard reduction technique, showing that $\mathrm{Ext}(c_0, \ell_\infty/\ell_1) \neq 0$. Here is when something wicked this way comes: consider a separable space with a similar $\mathcal{L}_\infty$ structure to that of ℓ_∞, say $C[0,1]$ and let us see what happens when switching from ℓ_∞ to $C[0,1]$. For the same reason that $\mathrm{Ext}(c_0, \ell_\infty) = 0$ while $\mathrm{Ext}(c_0, C[0,1]) \neq 0$ (see [13]), it is thinkable that $\mathrm{Ext}(c_0, \ell_\infty/\ell_1)$ could be 0 even if:

PROPOSITION 30.6. $\mathrm{Ext}(c_0, C[0,1]/\ell_1) \neq 0$.

Proof. Consider any nontrivial exact sequence $0 \to C[0,1] \to D \to c_0 \to 0$ that we will call Ω and form the commutative pushout diagram (the choice of the embedding $\jmath$ is irrelevant):

$$\begin{tikzcd}
\ell_1 \arrow[r, equal] \arrow[d, "\jmath"'] & \ell_1 \arrow[d, equal] & \\
C[0,1] \arrow[r] \arrow[d, "\pi"'] & D \arrow[r] \arrow[d] & c_0 \arrow[d, equal] \\
C[0,1]/\ell_1 \arrow[r] & D/\ell_1 \arrow[r] & c_0 .
\end{tikzcd}$$

The lower sequence $\pi\Omega$ cannot be trivial since, otherwise, we would get a pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & \ell_1 \arrow[r] \arrow[d, "\jmath"'] & Z \arrow[r] \arrow[d] & c_0 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C[0,1] \arrow[r] & D \arrow[r] & c_0 \arrow[r] & 0 \end{tikzcd}$$

whose lower sequence must split since a beautiful result of Kalton [13] establishes that every operator $\ell_1 \to C[0,1]$ can be extended to any separable superspace. ∎

30.2. Intermission: extension of compact operators, one by one. The extension/lifting of compact operators is an important topic in Banach space theory since Lindenstrauss' germinal memoir [66]. Basic questions such as: given an exact sequence $0 \to Y \to Z \to X \to 0$ of Banach spaces and a compact operator $\tau : Y \to E$, does τ admit a (compact) extension to an operator $Z \to E$? are still open. In this section we focus on the most important class of exact sequences: twisted Hilbert sequences and twisted Hilbert paces, namely, Banach spaces that are twisted sums of two Hilbert spaces. Or else, the middle spaces Z in exact sequences $0 \to H \to Z \to H' \to 0$ in which H and H' are Hilbert spaces. For the sake of simplicity we will restrict ourselves to the separable Hilbert space ℓ_2. There exist twisted Hilbert spaces not isomorphic to Hilbert spaces, as it was first proved by Enflo, Lindenstrauss and Pisier [47]. Let us call ELP either this sequence or the quasilinear map that generates it. We do not know whether ELP is equivalent to a centralizer. The notorious Kalton-Peck example [63] that came next provides an exact sequence $0 \to \ell_2 \to Z_2 \to \ell_2 \to 0$ generated by the ℓ_∞-centralizer

$$\mathsf{KP}x = x \log \frac{|x|}{\|x\|}$$

(see [13, 3.12.5]). The quotient map of the Kalton-Peck sequence is a strictly singular operator, which gives an idea of how far is Z_2 from being a Hilbert space. In fact, Z_2 does not contain complemented copies of Hilbert spaces [63]. The Kalton-Peck space Z_2 is one of the central objects in Banach space theory and the reader is addressed to [13, Chapter 10] for a quick display of its basic properties. The following shocking result appeared in [27]:

PROPOSITION 30.7. *Let $\alpha, \gamma \in \mathfrak{L}(\ell_2, \ell_2)$. If the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \end{array}$$

is commutative then $\alpha - \gamma \in \mathfrak{K}(\ell_2, \ell_2)$.

Another still more specific situation is considered in [22]: Recall that an operator on ℓ_2 is called *diagonal* if it is defined by a sequence $\sigma \in \ell_\infty$ in the form $\sigma(x) = (\sigma_n x_n)$. It turns out that a diagonal operator $\sigma : \ell_2 \to \ell_2$ can be extended to an operator $Z_2 \to \ell_2$ if and only if $(\sigma_n^* \log n) \in \ell_\infty$, where σ^* is the decreasing rearrangement of σ. The paper [22] contains other relevant results from the categorical point of view: the authors define the operator ideal (on Hilbert spaces) $\mathfrak{E}$ of extensible operators as those $\tau \in \mathfrak{L}(\ell_2, \ell_2)$ such that for every twisted Hilbert sequence $0 \to \ell_2 \to Z \to \ell_2 \to 0$ the operator τ can be extended to an operator $Z \to \ell_2$. One could have defined the ideal $\mathfrak{Li}$ of liftable operators as those $\tau \in \mathfrak{L}(\ell_2, \ell_2)$ such that for every twisted Hilbert sequence as above the operator τ can be lifted to an operator $\ell_2 \to Z$. The first surprise comes when [22, Corollay 2.2] shows that $\mathfrak{E} = \mathfrak{Li}$. The second, when specific and general results somehow mix in [22, Proposition 3.4]: A diagonal operator σ on ℓ_2 can be extended to Z_2 if and only if it can be extended through *all* twisted Hilbert spaces generated by *centralizers*. A combination of both approaches yields:

PROPOSITION 30.8. *Let $\alpha, \gamma \in \mathfrak{L}(\ell_2, \ell_2)$ be such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \end{array}$$

is commutative. Then

- *The compact operator $\alpha - \gamma$ can be extended to Z_2 if and only if the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \\ & & {\scriptstyle\gamma}\downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \end{array}$$

  is commutative.

- *The compact operator $\alpha-\gamma$ can be lifted to Z_2 if and only if the diagram*

$$\begin{tikzcd} 0 \arrow[r] & \ell_2 \arrow[r] \arrow[d, "\alpha"'] & Z_2 \arrow[r] \arrow[d] & \ell_2 \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & \ell_2 \arrow[r] & Z_2 \arrow[r] & \ell_2 \arrow[r] & 0 \end{tikzcd}$$

is commutative.

Proof. Let us call KP the Kalton-Peck sequence. Recall that the commutativity of a diagram

$$\begin{tikzcd} 0 \arrow[r] & \ell_2 \arrow[r] \arrow[d, "u"'] & Z_2 \arrow[r] \arrow[d] & \ell_2 \arrow[r] \arrow[d, "v"] & 0 \\ 0 \arrow[r] & \ell_2 \arrow[r] & Z_2 \arrow[r] & \ell_2 \arrow[r] & 0 \end{tikzcd}$$

can be reformulated as $u\mathsf{KP} - \mathsf{KP}v \equiv 0$. Moreover, and due to the standard properties of the pullback and pushout, if $\tau \in \mathfrak{L}(\ell_2, \ell_2)$ is an operator, $\tau\mathsf{KP} \equiv 0$ if and only if τ extends to an operator $Z_2 \to \ell_2$ and $\mathsf{KP}\tau \equiv 0$ if and only if τ lifts to an operator $\ell_2 \to Z_2$. Now, the first part is

$$(\alpha - \gamma)\mathsf{KP} \equiv \alpha\mathsf{KP} - \mathsf{KP}\gamma + \mathsf{KP}\gamma - \gamma\mathsf{KP} \equiv \mathsf{KP}\gamma - \gamma\mathsf{KP}$$

and therefore $(\alpha - \gamma)\mathsf{KP} \equiv 0$ if and only if $\gamma\mathsf{KP} - \mathsf{KP}\gamma \equiv 0$. Analogously,

$$\mathsf{KP}(\alpha - \gamma) \equiv \mathsf{KP}\alpha - \alpha\mathsf{KP} + \alpha\mathsf{KP} - \mathsf{KP}\gamma \equiv \mathsf{KP}\alpha - \alpha\mathsf{KP}.$$

Therefore $\mathsf{KP}(\alpha - \gamma) \equiv 0$ if and only if $\alpha\mathsf{KP} - \mathsf{KP}\alpha \equiv 0$. ∎

When α is a diagonal operator then $\alpha\mathsf{KP} \equiv \mathsf{KP}\alpha$ as a consequence of the Commutator Theorem 30.19; see also [37]. Therefore:

Proposition 30.9. *Let α, γ be diagonal operators making the diagram*

$$\begin{tikzcd} 0 \arrow[r] & \ell_2 \arrow[r] \arrow[d, "\alpha"'] & Z_2 \arrow[r] \arrow[d] & \ell_2 \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & \ell_2 \arrow[r] & Z_2 \arrow[r] & \ell_2 \arrow[r] & 0 \end{tikzcd}$$

commute. Then $\alpha - \gamma \in \mathfrak{E}$.

It is mystifying why some results, such as Proposition 30.7, seem to work only for the Kalton-Peck sequence (although recently Côrrea, Dantas and Rodríguez-Vidanes [43, Section 7] obtained intriguing and promising results for other Kalton-Peck-like spaces), other apply generally and other, such as [22, Proposition 3.4], percolate from specific to general. Let us formulate a problem likely capturing ingredients of all these behaviours:

PROBLEM 30.10. Given a twisted Hilbert space $0 \to \ell_2 \to Z \to \ell_2 \to 0$ identify those compact operators on ℓ_2 that can be extended/lifted to Z.

Another intriguing conundrum is the extension of compact operators on quasi-Banach spaces for the simple reason that, like John Snow [17] we know nothing about compact operators on quasi-Banach spaces.

30.3. TENSOR AND BILINEAR DERIVATION. Ignorance is bliss! For very good reasons, the functors *eligible* to be derived are either *covariant* left exact, hence additive, (suitable for right or injective derivation); or *contravariant* left exact (suitable for projective derivation). What about trying non additive (in patricular, non-left exact) functors? Let us make a couple of test cases: the covariant functor of tensorization in Banach spaces and the contravariant functor of taking bilinear forms.

Tensorization. The projective tensor product is an important covariant Banach functor. Its categorical nature is: it defines the object transforming bilinear maps into linear maps. Let us introduce topology: Given two Banach spaces X, Y, $X\widehat{\otimes}_\pi Y$ is the Banach space that transforms bounded bilinear forms $b : X \oplus Y \to E$ on the product into bounded linear forms $\ell : X\widehat{\otimes}_\pi Y \to E$ making commutative the diagram

$$\begin{tikzcd} X \oplus Y \arrow[rr, "\otimes"] \arrow[dr, "b"'] & & X\widehat{\otimes}_\pi Y \arrow[dl, "\ell"] \\ & E & \end{tikzcd}$$

where $\otimes(x, y) = x \otimes y$. Its intrinsic description is: $X\widehat{\otimes}_\pi Y$ is the completion of $X \otimes Y$ with respect to the π-norm, usually called the projective tensor product

$$\|z\|_\pi \;=\; \inf \sum \|x_n\|\|y_n\| : \; z \;=\; \sum x_n \otimes y_n \,.$$

[17] The character from A song of Ice and Fire (Game of thrones, for tv fans) from George R.R. Martin.

The connection with operator ideals is clear:

- The natural tensor product as vector spaces yields $X^* \otimes Y = \mathfrak{F}(X,Y)$. The natural operator norm induces on $X^* \otimes Y$ a tensor norm denoted ϵ, usually called the injective tensor norm ϵ. Hence, $\mathfrak{G}(X,Y) = X^* \widehat{\otimes}_\epsilon Y$, the completion of $X^* \otimes Y$ with respect to ϵ.
- The projective tensor norm π yields however $X^* \widehat{\otimes}_\pi Y = \mathfrak{N}(X,Y)$.

Given a Banach space A we will denote by $\widehat{\otimes}_A$ the covariant functor in that associates to each Banach space X the corresponding tensor product $X \widehat{\otimes}_\pi A$ and to each operator $T : X \to Y$ the (canonical extension of the) tensor product operator $T \otimes \mathrm{id}_A$ (to the completions). In algebraic categories such as modules, the tensor product is a covariant right exact functor. But in Banach spaces the picture is subtler:

- Algebraic tensor product (no completion). As normed spaces, i.e., taking $X \otimes_\pi Y$ without completion, everything is purely algebraic; and since vector spaces are just free $\mathbb{R}$-modules tensorization is an exact functor. Its derivative is therefore 0. In general, the derived functor of $\otimes_A$ is denoted Tor_A. Given a projective presentation $0 \to \kappa_0 \xrightarrow{\imath_0} \mathcal{P}_0 \to X \to 0$ one gets $\mathrm{Tor}(A,X) = \ker(\mathrm{id}_A \otimes \imath_0)$ (see, e.g., [58, Section III.8]).
- Banach tensor product (completed). In Banach spaces, the complete tensorization endowed with the π-topology only preserves surjectivity, namely

  Lemma 30.11. *The tensor $T \otimes S$ of two surjective maps $T : U \to E$ and $S : V \to F$ is surjective.*

  Proof. Pick $z \in E \widehat{\otimes}_\pi F$ with $\|z\| < 1$ so that $\sum \|e_i\|\|f_i\| < 1$ for some finite representation $z = \sum e_i \otimes f_i$. By the surjectivity of T and S, and assuming $B_E \subset \sigma_T T(B_U)$ and $B_F \subset \sigma_S S(B_V)$, $z = \sum T a_i \otimes S b_i$ for $\|a_i\| \leq \sigma_T$ and $\|b_i\| \leq \sigma_S$; hence $z = (T \otimes S)\left(\sum a_i \otimes b_i\right)$ and $\pi\left(\sum a_i \otimes b_i\right) \leq \sigma_T \sigma_S$. ∎

  However, if $0 \to Y \xrightarrow{\imath} Z \xrightarrow{Q} X$ is an exact sequence and A is a Banach space then the sequence

  $$0 \longrightarrow A \widehat{\otimes}_\pi Y \xrightarrow{\mathrm{id}_A \otimes \imath} A \widehat{\otimes}_\pi Z \xrightarrow{\mathrm{id}_A \otimes Q} A \widehat{\otimes}_\pi X \longrightarrow 0$$

  is not necessarily exact at either its middle or its left term.

This means that the still uncharted study of the functor Tor in Banach spaces has to tackle two difficulties: a) the exactness of the tensorized sequence in the middle term (see a counterexample in [86, 3.3.2]); and b) the injectivity of the tensorized injection. Recall that a Banach space A is said to have the Approximation Property if every compact operator is the limit of a sequence of finite rank operators. It turns out [44, 5.8, Corollary 4] that A has the Approximation Property if and only if $\widehat{\otimes}_A$ preserves injectives. With this, we have almost exhausted what we know about Tor in Banach spaces.[18] More about Tor in subsection 30.5.

Adjointness and the tensor-bilinear forms duality. The functors $\widehat{\otimes}_A$ and $\mathfrak{L}_A$ are moreover closely related through adjointness. Adjointness is, in the words of [13, 4.6.1], one of the landmarks of category theory; a phenomenon that pervades modern mathematics but virtually imperceptible outside category theory. Two covariant functors $\mathcal{F} : \mathbf{A} \to \mathbf{B}$ and $\mathcal{G} : \mathbf{B} \to \mathbf{A}$ are said to be adjoints, noted $\mathcal{F} \dashv \mathcal{G}$, if for every two objects A in $\mathbf{A}$ and B in $\mathbf{B}$ there is a natural equivalence

$$\mathsf{Hom}_{\mathbf{B}}(\mathcal{F}X, Y) = \mathsf{Hom}_{\mathbf{A}}(X, \mathcal{G}Y)\,.$$

The functor $\mathcal{F}$ is called the left-adjoint and the functor G is called the right-adjoint. Banach examples, examples and advanced examples can be found in [13, 4.6.1], [24, 5.1 – 5.3] and [25, 11.1 – 11.6]. Two fundamental facts to keep record of are (see [24, 25]): Left adjoints preserve direct limits and right adjoints preserve inverse limits. The very definition of the projective tensor product of Banach spaces yields the identity

$$\mathfrak{L}(X \widehat{\otimes}_\pi A, Y) = \mathfrak{L}(X, \mathfrak{L}(A, Y))\,,$$

namely,

$$\widehat{\otimes}_A \dashv \mathfrak{L}_A\,.$$

Fuks theorem (to which we will return in Section 30.5) asserts that the only two adjoint covariant Banach functors, i.e., functors $\mathbf{Ban} \to \mathbf{Ban}$, are $\widehat{\otimes}_A$ and $\mathfrak{L}_A$. In other words, if $\mathcal{F}, \mathcal{G}$ are two covariant Banach functors then there is a Banach space A for which $\mathcal{F}$ is naturally equivalent to $\widehat{\otimes}_A$ and $\mathcal{G}$ is naturally equivalent to $\mathfrak{L}_A$.

Problem 30.12. Identify the left derived functors $\mathrm{Tor}_n(A, \cdot)$ of $\widehat{\otimes}_A$.

[18] We thank Félix Cabello and Ricardo García for several fruitful conversations showing that this does not exhaust what *they* know about Tor in Banach spaces.

And we cannot leave unmentioned the counterpart of Problem 24.4:

PROBLEM 30.13. Determine whether $\mathrm{Tor}_n(\ell_2, \ell_2) = 0$ for all (some) n.

Derivation of the bilinear form functor. On the other hand, bilinear forms are the elements of the dual of the projective tensor product, so deriving such functor could help. Thus, consider the derivation of the functor $X \to \mathfrak{B}(X)$ that assigns to a Banach space the space of bounded scalar bilinear forms on X. In [30] we obtained a curious formula

$$\mathrm{Ext}^2(\ell_2, \ell_2) = \mathfrak{B}(\kappa_0(\ell_2))/\sim,$$

where $\sim$ is the equivalence relation $B \sim B' \iff B - B'$ extends to a bounded bilinear form on the corresponding projective space $\mathcal{P}_0$. A diagram could help as a visualizer of the identity above:

$$\begin{tikzcd}
0 \arrow[r] & \ell_2 \arrow[r] \arrow[d, equal] & \mathrm{PB} \arrow[r] \arrow[d] & \kappa(\ell_2) \arrow[r] \arrow[d] \arrow[dd, dashed, bend left] & \ell_1 \arrow[r] \arrow[d] & \ell_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ell_2 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & \ell_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ell_2 \arrow[r] & \ell_\infty \arrow[r] & \kappa(\ell_2)^* \arrow[r] & \mathrm{PO} \arrow[r] & \ell_2 \arrow[r] & 0 .
\end{tikzcd}$$

This formula was extended in [31, Proposition 6.1] to:

$$\mathrm{Ext}^{2(n+1)}_{\mathbf{B}}(X, X^*) = \mathfrak{B}(\kappa_n(X))/\sim, \tag{30.1}$$

with the analogous corresponding meaning for the equivalence relation: $B \sim B' \iff B - B'$ extends to a bounded bilinear form on the projective space $\mathcal{P}_n$ appearing in $0 \to \kappa_n(X) \to \mathcal{P}_n \to \kappa_{n-1}(X) \to 0$. Observe that in the general formula we have used the notation of this paper, that begins with $n = 0$. It seems impossible for a homologically trained eye to overlook the resemblance with derivation; more precisely, the isomorphisms above are whispering "$\mathrm{Ext}^{2(n+1)}_{\mathbf{B}}(X, X^*)$ could be the n^{th} derivative of $\mathfrak{B}$". Well, reality is much more complex than that: the reader is invited to find out in [31] the correct derivatives of the functor $\mathfrak{B}$: the projective derivation in [31, Theorem 5.1] and the injective derivation in [31, Section 7]. Beyond that, derivation of non-left or non-right exact functors uncovers an es sential role for the Martsinkovsky-Russell stabilizations [70]. However, what nobody could see it coming was that those stabilizations (described in Section 29) would produce in the Banach space world the categorical context for already important constructions. Let us give a try: recall from [32] that a scalar bilinear form on a Banach space is said to be *extendable* if it can be extended to a scalar bilinear form on any superspace. Let $\mathfrak{B}^E(X)$ denote the space of extendable bilinear forms on X. One has [31, Proposition 8.3]:

- Injective stabilization: $\overline{\mathfrak{B}}(X) = 0$.
- Projective stabilization $\underline{\mathfrak{B}}(X) = \mathfrak{B}(X)/\mathfrak{B}^E(X)$.
- Co-injective stabilization $\overline{\mathfrak{B}}(X)^{co} = \mathsf{Q}_{\mathsf{B}}^{(1)}(X, X^*)$. This is going to be (see Section 30.4) the space of exact sequences modulo bounded equivalence.

Back to Ext^2 problems. The bilinear approach (equation (30.1)) transforms the $\mathrm{Ext}^2(\ell_2, \ell_2) \neq 0$ problem into the existence of bilinear forms on the kernel $\kappa(\ell_2)$ of a projective presentation of ℓ_2 which cannot be extended to bilinear forms on ℓ_1. We do not have an explicit description for such non-extendible bilinear forms. An explicit description could provide information about the possible existence of such bilinear forms on $\kappa_n(\ell_2)$; in other words, further steps towards a solution to:

PROBLEM 30.14. Is $\mathrm{Ext}^n(\ell_2, \ell_2) \neq 0$?

Or, with a different twist:

PROBLEM 30.15. Do there exist symmetric bilinear forms on the kernel $\kappa(\ell_2)$ of a projective presentation of ℓ_2 than cannot be extended to symmetric bilinear forms on ℓ_1?

An attack to the vanishing of $\mathrm{Ext}^2(c_0, \ell_1)$ from the bilinear side is enlightening too. By equation (30.1), $\mathrm{Ext}^2(c_0, \ell_1) \neq 0$ if and only if, given a projective presentation $0 \to \kappa \to \ell_1 \to c_0 \to 0$ there is a scalar bilinear form on κ that cannot be extended to a bilinear form on ℓ_1. How could we produce such bilinear form B? Assume that B has associated operator $\tau : \kappa \to \kappa^*$; i.e., $B(x, y) = \langle \tau x, y \rangle$ and observe the commutative diagram

$$\begin{tikzcd}
\kappa \arrow[r] \arrow[d, equal] & \ell_\infty \arrow[r] \arrow[ddd, dashed, bend left] & \ell_\infty/\kappa \\
\kappa \arrow[r, "\jmath"] \arrow[d, "\tau"'] & \ell_1 \arrow[u] \arrow[r] \arrow[d, dashed] & c_0 \arrow[u] \\
\kappa^* \arrow[d, equal] & \ell_\infty \arrow[l, "\jmath^*"'] & \ell_1 \arrow[l] \\
(\kappa)^* & \ell_\infty^* \arrow[l] \arrow[u] & (\ell_\infty/\kappa)^* . \arrow[l] \arrow[u]
\end{tikzcd}$$

If B is extendable, i.e., if τ can be extended to the dotted curved arrow, then τ must be 2-summing since every operator from an $\mathcal{L}_\infty$ space to an $\mathcal{L}_1$ space must be 2-summing [45, Theorem 3.7]. We know the existence of non-2-summing operators $\kappa \to \kappa^*$: indeed, we know the existence of non-2-summing operators $\kappa \to \ell_2$ because $\mathrm{Ext}(c_0, \ell_2) \neq 0$ and every operator $\ell_1 \to \ell_2$ is 2-summing [45, Theorem 3.1]; pick one, say ϕ, and form the bilinear form $B(x, y) = \langle \phi x, \phi y \rangle$. If this B were extendable then, according to [32, Theorem 2.2], there would be two 2-summing operators $u, v : \kappa \to \ell_2$ such that $B(x, y) = \langle ux, vy \rangle$ and then ϕ would be 2-summing because, given a weakly 2-summable sequence (x_n), we would have

$$\begin{aligned}
\sum \|\phi x_n\|^2 &= \sum \langle \phi x_n, \phi x_n \rangle \\
&= \sum \langle u x_n, v x_n \rangle \\
&\leq \Big(\sum \|u x_n\|^2 \Big)^{1/2} \Big(\sum \|v x_n\|^2 \Big)^{1/2} < \infty .
\end{aligned}$$

Thus, we are sure that there are operators τ that do not extend to curved dotted arrows but, why not to a straight dotted arrow $T : \ell_1 \to \ell_\infty$? The funny thing here is that if such T exists then we would have two further extensions: One, $\overline{T} : \ell_\infty \to \ell_\infty$ by the injectivity of ℓ_∞; Two, a lifting $\underline{T} : \ell_1 \to (\ell_\infty)^*$ of $\jmath^* T$ by the projectivity of ℓ_1. Is this enough? Alternatively:

Problem 30.16. Does there exist a non 2-summing operator $\widehat{\tau} : \ell_1 \to \ell_\infty$ such that $\jmath^* \widehat{\tau} \jmath$ is 2-summing?

30.4. Nets and Rochberg spaces. Pullback and pushout diagrams have been thoroughly studied in [13, Chapter 2]. Combining both yields the classical commutative diagram at the core of the Snake lemma,

$$\begin{tikzcd}
0 \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & 0 ,
\end{tikzcd}$$

studied in [1] and, of course, at [13]. A more complex commutative diagram –in which all rows and columns are exact– will be considered now:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & 0 . \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Its study has has been started in [17], where it was called a *net*. Natural instances of nets arise from taking projective or injective presentations of the spaces of an exact sequence

$$\begin{tikzcd}
\kappa(Y) \arrow[r] \arrow[d] & \kappa(Z) \arrow[r] \arrow[d] & \kappa(X) \arrow[d] \\
\mathcal{P}_Y \arrow[r] \arrow[d] & \mathcal{P}_Z \arrow[r] \arrow[d] & \mathcal{P}_X \arrow[d] \\
Y \arrow[r] & Z \arrow[r] & X ,
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X \arrow[d] \\
\mathcal{I}_Y \arrow[r] \arrow[d] & \mathcal{I}_Z \arrow[r] \arrow[d] & \mathcal{I}_X \arrow[d] \\
c\kappa(Y) \arrow[r] & c\kappa(Z) \arrow[r] & c\kappa(X) .
\end{tikzcd}$$

Another example is provided by the bidual net

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X \arrow[d] \\
Y^{**} \arrow[r] \arrow[d] & Z^{**} \arrow[r] \arrow[d] & X^{**} \arrow[d] \\
Y^{**}/Y \arrow[r] & Z^{**}/Z \arrow[r] & X^{**}/X
\end{tikzcd} \tag{30.2}$$

which, by the way, provides the most natural proof there is (see [88]) for the identity

$$(Z^{**}/Z)/Y^{**}/Y = (Z/Y)^{**}/(Z/Y).$$

It could perfectly happen that the middle horizontal sequence in a net may split while neither the upper or the lower sequences do: pick $Y = c_0$, and $Z = \ell_\infty$ in (30.2).

When seen through quasilinear lens, what we would see is

$$\begin{tikzcd}
& 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] \arrow[ll, dashed, bend right=60, "\Phi"'] & 0 \\
& \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] & \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d] \arrow[uu, dashed, bend left=60, "\Omega"] & \cdot \arrow[r] & \cdot \arrow[r] \arrow[d] \arrow[uu, dashed, bend right=60, "\Upsilon"'] \arrow[ll, dashed, bend left=60, "\Psi"] & 0 \\
& 0 & & 0 &
\end{tikzcd} \tag{30.3}$$

which suggests some connection between the geometry of the net and the possible vanishing of the elements $\Omega\Psi$ and/or $\Phi\Upsilon$ in their respective Ext^2 spaces. Once again, the bidual net (30.2) is lethal): consider

$$\begin{tikzcd}
c_0 \arrow[r] \arrow[d] & \ell_\infty \arrow[r] \arrow[d] & \ell_\infty/c_0 \arrow[d] \arrow[ll, dashed, bend right=40, "\Omega"'] \\
\ell_\infty \arrow[r] \arrow[d] & \ell_\infty^{**} \arrow[r] \arrow[d] & (\ell_\infty/c_0)^{**} \arrow[d] \\
\ell_\infty/c_0 \arrow[r] \arrow[uu, dashed, bend left=60, "\Omega"] & \ell_\infty^{**}/\ell_\infty \arrow[r] & (\ell_\infty/c_0)^{**}/(\ell_\infty/c_0)\,. \arrow[uu, dashed, bend right=60, "\Psi"'] \arrow[ll, dashed, bend left=40, "\Psi"]
\end{tikzcd} \tag{30.4}$$

Here the middle sequence splits, but $\Omega\Psi$ cannot be 0 because ℓ_∞/c_0 is not injective whereas $\ell_\infty^{**}/\ell_\infty$ is.

A richer theory appears when such diagrams are transplanted from the world of exact sequences modulo standard equivalence (i.e, working with $\mathsf{Q}^{(1)}/(\mathsf{B}+\mathsf{L}\ )$ to the less well-known world of $\mathsf{Q}^{(1)}_{\mathsf{B}} = \mathsf{Q}^{(1)}/\mathsf{B}$. This means that an object is now $0 \to Y \to Y \oplus_\omega X \to X \to 0$ where ω can be replaced by a *bounded* perturbation $\omega + B$ with $B : X \to Y$ homogeneous and bounded, but not by a linear perturbation. In other words, if $L : X \to Y$ is a linear unbounded map then $0 \to Y \to Y \oplus_L X \to X \to 0$ is no longer trivial since L is not a bounded perturbation of 0. Two quasi linear maps Ω, Ψ will be called *boundedly equivalent* when their difference $\Omega - \Psi$ is bounded. Compare with definition 23.8.

This shift leads naturally to the notion of *bounded net*: a commutative diagram (without zeros)

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\imath_\Psi"'] & A \oplus_\Omega C \arrow[r] \arrow[d] & C \arrow[d, "\imath_\Upsilon"] \\
A \oplus_\Psi H \arrow[r] \arrow[d, "\pi_\Psi"'] & E \oplus_\Xi G \arrow[r] \arrow[d] & C \oplus_\Upsilon J \arrow[d, "\pi_\Upsilon"] \\
H \arrow[r] & H \oplus_\Phi J \arrow[r] & J ,
\end{tikzcd} \tag{30.5}$$

in which $\imath_\Psi \Omega$ and $\Xi \imath_\Upsilon$ are boundedly equivalent, as well as $\pi_\Psi \Xi$ and $\Phi \pi_\Upsilon$. Bounded nets were introduced in [17] with the purpose of obtaining an interpolation-free proof – the first proof that $\Omega\Omega = 0$ for every centralizer [18] was obtained via complex interpolation theory– for Proposition 30.3. Let us provide a sketch of both proofs since each of them contains potentially relevant categorical elements.

The plot for the diagrammatic proof is to show that, starting with a centralizer Ω defined on ℓ_2, one can construct a *nontrivial* bounded net

$$\begin{tikzcd}
\ell_2 \arrow[r] \arrow[d] & \ell_2 \oplus_\Omega \ell_2 \arrow[r] \arrow[d] & \ell_2 \arrow[d] \\
\ell_2 \oplus_\Omega \ell_2 \arrow[r] \arrow[d] & \square \arrow[r] \arrow[d] & \ell_2 \oplus_\Omega \ell_2 \arrow[d] \\
\ell_2 \arrow[r] & \ell_2 \oplus_\Omega \ell_2 \arrow[r] & \ell_2 ,
\end{tikzcd} \tag{30.6}$$

that is, a bounded net whose middle horizontal exact sequence does not split. Passing from such bounded net to $\Omega\Omega \equiv 0$ requires a few partial ad-hoc answers to the question:

Problem 30.17. Assume one has a bounded net in which the middle horizontal sequence splits. Does either the upper or the lower exact sequences split too?

The ad-hoc answers take the following form:

Lemma 30.18. *Assume one has a bounded net like* (30.3).

(i) *If* Ω *is a bounded pullback of* Φ *then* $\Omega\Psi \equiv 0$.

(ii) *If* Ψ *is bounded pushout of* Υ *then* $\Phi\Upsilon \equiv 0$.

The meaning of the lemma is easy to explain: If a net like (30.3) admits an operator τ

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] \arrow[ll, dashed, bend right=60, "\Phi"'] & \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] & \cdot \arrow[r] \arrow[d] \arrow[ddll, "\tau"'] & 0 \\
 & \cdot \arrow[d] & & \cdot \arrow[d] & \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d] \arrow[uu, dashed, bend left=80, "\Omega"] & \cdot \arrow[r] & \cdot \arrow[r] \arrow[d] \arrow[uu, dashed, bend right=80, "\Upsilon"'] \arrow[ll, dashed, bend left=60, "\Upsilon"] & 0 \\
 & 0 & & 0 &
\end{tikzcd} \tag{30.7}$$

such that $\Omega\tau$ is boundedly equivalent to Φ then $\Omega\Psi \equiv 0$. If the operator τ makes $\tau\Upsilon$ boundedly equivalent to Ψ it is then $\Phi\Upsilon$ who vanishes. The net (30.4) shows that none of the implications above holds in the world of plain equivalence: certainly Ω is a bounded pullback of Ω, and Ψ is a bounded pushout of Ψ but $\Omega\Psi \neq 0$.

Rochberg spaces. We introduce here the Rochberg spaces because they are relevant at this point. On the one hand, they provide a tool to obtain the complex interpolation proof for the identity $\Omega\Omega \equiv 0$ of Proposition 30.3. On the other hand, they offer a unifying framework connecting twisted Hilbert spaces with the higher-order derived spaces arising from complex interpolation. A sound approach to Rochberg spaces can be obtained from [14, 36, 42, 28]. Everything we need at this moment is to accept that Rochberg spaces are simultaneous generalizations of twisted (Hilbert) spaces and of the derived spaces generated by complex interpolation theory. Let us give a brief explanation for this: given a complex interpolation schema (X_0, X_1) then the interpolation space X_z at z is the first Rochberg space $\mathfrak{R}_1(z)$. The interpolation process generates (we do not care now how) a quasilinear map Ω_z. The twisted sum $X_z \oplus_{\Omega_z} X_z$ is the second Rochberg space $\mathfrak{R}_2(z)$. In particular, if one thinks at the interpolation scale (ℓ_∞, ℓ_1) of ℓ_p-spaces, then $\ell_2 = (\ell_\infty, \ell_1)_{1/2}$ and thus Hilbert spaces are the perfect example of Rochberg spaces or order 1 and the Kalton-Peck Z_2 space is the perfect example of a Rochberg space of order 2. In general, consider a complex interpolation scale (X_z) of Banach spaces: if its associated Rochberg space $\mathfrak{R}_2(z)$ of order 2 at z comes defined by an exact sequence $0 \to \mathfrak{R}_1(z) \to \mathfrak{R}_2(z) \to \mathfrak{R}_1(z) \to 0$, its associated higher order Rochberg spaces are defined by exact sequences (we omit from now on the variable z)

$$0 \longrightarrow \mathfrak{R}_n \xrightarrow{\imath_{n,n+m}} \mathfrak{R}_{n+m} \xrightarrow{\pi_{n+m,m}} \mathfrak{R}_m \longrightarrow 0$$

that interlace in commutative diagrams

$$\begin{tikzcd}
\mathfrak{R}_k \arrow[r, equal] \arrow[d, "\imath_{k,n}"'] & \mathfrak{R}_k \arrow[d, "\imath_{k,n+m}"] & \\
\mathfrak{R}_n \arrow[r, "\imath_{n,n+m}"] \arrow[d, "\pi_{n,n-k}"'] & \mathfrak{R}_{n+m} \arrow[r, "\pi_{n+m,m}"] \arrow[d, "\pi_{n+m,n+m-k}"] & \mathfrak{R}_m \arrow[d, equal] \\
\mathfrak{R}_{n-k} \arrow[r, "\imath_{n-k,n-k+m}"'] & \mathfrak{R}_{n-k+m} \arrow[r, "\pi_{n-k+m,m}"'] & \mathfrak{R}_m ,
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{R}_n \arrow[r, "\imath_{n,n+k}"'] \arrow[d, equal] & \mathfrak{R}_{n+k} \arrow[r, "\pi_{n+k,k}"] \arrow[d, "\imath_{n+k,n+m}"'] & \mathfrak{R}_k \arrow[d, "\imath_{k,m}"] \\
\mathfrak{R}_n \arrow[r, "\imath_{n,n+m}"'] & \mathfrak{R}_{n+m} \arrow[r, "\pi_{n+m,m}"] \arrow[d, "\pi_{n+m,m-k}"'] & \mathfrak{R}_m \arrow[d, "\pi_{m,m-k}"] \\
& \mathfrak{R}_{m-k} \arrow[r, equal] & \mathfrak{R}_{m-k} .
\end{tikzcd}$$

In particular, choosing a scale that interpolated at $z = 1/2$ yields $X_{1/2} = \ell_2$

as interpolation space, we get the commutative diagram

$$\begin{tikzcd} \ell_2 \arrow[r, equal] \arrow[d] & \ell_2 \arrow[d] & \\ \mathfrak{R}_2 \arrow[r] \arrow[d] & \mathfrak{R}_3 \arrow[r] \arrow[d] & \ell_2 \arrow[d, equal] \\ \ell_2 \arrow[r] & \mathfrak{R}_2 \arrow[r] & \ell_2 \end{tikzcd}$$

which shows that if Ω is the quasilinear map (actually centralizer) associated to $0 \to \ell_2 \to \mathfrak{R}_2 \to \ell_2 \to 0$ then $\Omega\Omega \equiv 0$. The rest of the proof is just summoning Kalton's beautiful theory of centralizers [62] (that can be seen explained in the references about Rochberg spaces given above): every centralizer is boundedly equivalent to a centralizer generated by complex interpolation.

One might wonder whether the Rochberg spaces themselves can be arranged into a net of the form

$$\begin{tikzcd} \mathfrak{R}_a \arrow[r] \arrow[d] & \mathfrak{R}_{a+b} \arrow[r] \arrow[d] & \mathfrak{R}_b \arrow[d] \\ \mathfrak{R}_n \arrow[r] \arrow[d] & \mathfrak{R}_{n+m} \arrow[r] \arrow[d] & \mathfrak{R}_m \arrow[d] \\ \mathfrak{R}_{n-a} \arrow[r] & \mathfrak{R}_{n+m-a-b} \arrow[r] & \mathfrak{R}_{m-b} \,. \end{tikzcd}$$

The answer is no, at least no in a natural way involving the canonical maps $\imath_{i,j}$ and $\pi_{i,j}$ from earlier diagrams. This is important because it shows that the construction depicted in diagram (30.6) is "something else". More precisely, if in the bounded net there constructed we set $\mathfrak{R}_1 = \ell_2$ and $\mathfrak{R}_2 = Z_2$ then we would get

$$\begin{tikzcd} \ell_2 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & \ell_2 \arrow[d] \\ Z_2 \arrow[r] \arrow[d] & \square \arrow[r] \arrow[d] & Z_2 \arrow[d] \\ \ell_2 \arrow[r] & Z_2 \arrow[r] & \ell_2 \end{tikzcd}$$

and the space $\square$ is not, by far, $\mathfrak{R}_4$. The results in [14, Section 7], more precisely, Proposition 7.3, yield the existence of nets (probably not bounded nets) such as

$$\begin{tikzcd}
\mathfrak{R}_1 \arrow[r] \arrow[d] & \mathfrak{R}_3 \arrow[r] \arrow[d] & \mathfrak{R}_2 \arrow[d] \\
\mathfrak{R}_2 \arrow[r] \arrow[d] & \mathfrak{R}_1 \oplus \mathfrak{R}_4 \arrow[r] \arrow[d] & \mathfrak{R}_3 \arrow[d] \\
\mathfrak{R}_1 \arrow[r] & \mathfrak{R}_2 \arrow[r] & \mathfrak{R}_1 .
\end{tikzcd}$$

Let us describe one further question mark emerging from the twisted sum + complex interpolation nature of Rochberg spaces. Consider an interpolation pair (X_0, X_1) and the Rochberg derived spaces obtained from it. Keep in mind that $\mathfrak{R}_1 = X_z$ is the interpolation space $(X_0, X_1)_z$. Since $\mathfrak{R}_2$ is a twisted sum $\mathfrak{R}_1 \oplus_\Omega \mathfrak{R}_1$ and an operator $A \oplus_\Omega B \to A \oplus_\Omega B$ defined on a twisted sum can be described with a 2×2 matrix $\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ whose entries are in turn operators, an operator U on $\mathfrak{R}_2$ that can be inserted in a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{R}_1 \arrow[r] \arrow[d, "u"'] & \mathfrak{R}_2 \arrow[r] \arrow[d, "U"'] & \mathfrak{R}_1 \arrow[r] \arrow[d, "u"] & 0 \\
0 \arrow[r] & \mathfrak{R}_1 \arrow[r] & \mathfrak{R}_2 \arrow[r] & \mathfrak{R}_1 \arrow[r] & 0
\end{tikzcd}$$

must be upper-triangular, that is $U = \begin{pmatrix} u & d \\ 0 & u \end{pmatrix}$. Who could the operator d be? We must invoke at this point the celebrated Commutator Theorem [82]:

THEOREM 30.19. (COMMUTATOR THEOREM) *If u is an operator acting on a complex interpolation scale (X_z) then $\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$ is a bounded operator on $\mathfrak{R}_2$.*

The meaning of "u acts on the complex interpolation scale (X_z)" is that the same $u : X_z \to X_z$ is a bounded operator for all z.

Now assume that for each z we have an operator $u_z : X_z \to X_z$ and assume that the operators (u_z) are not independent one of the other, rather they form an analytic family, which operatively means that the function $z \to u_z$, once properly defined, is an analytic (Banach space valued) function and therefore we can differentiate with respect to z. When, moreover, each u_z defines an action of a fixed group, say G, on each of the interpolation spaces X_z of the scale, the discussion in [29, Section 7] shows that we get a family of commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_z & \longrightarrow & X_z \oplus_{\Omega_z} X_z & \longrightarrow & X_z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u_z} & & \downarrow{\scriptstyle U_z} & & \downarrow{\scriptstyle u_z} & & \\ 0 & \longrightarrow & X_z & \longrightarrow & X_z \oplus_{\Omega_z} X_z & \longrightarrow & X_z & \longrightarrow & 0\,, \end{array}$$

in which the middle operator $U_z = \begin{pmatrix} u_z & d_z \\ 0 & u_z \end{pmatrix}$ has as upper right entry the operator derivative of the analytic family (u_z), namely

$$d_{z_0} = \frac{d}{dz} u_z|_{z_0}\,.$$

This is the bee's knees: the upper-right entry is the derivative of the diagonal entry. In particular, there is no surprise here, when one realizes that the Commutator Theorem actually says that when $u_z = u$ for all z ("u acts on the complex interpolation scale $(X_z)_z$") then $d_z = 0$ and therefore $U_z = \begin{pmatrix} u_z & 0 \\ 0 & u_z \end{pmatrix}$.

A remarkable fact is that our explorations of the coinjective stabilization of the bilinear functor $\mathfrak{B}$ point in the same direction through the identity $\overline{\mathfrak{B}}^{co}(X) = \mathsf{Q}_{\mathsf{B}}^{(1)}(X, X^*)$. Indeed, if we describe the natural equivalence $\mathfrak{n} : \mathsf{Q}_{\mathsf{B}}^{(1)}(X, X^*) \to \overline{\mathfrak{B}}^{co}(X)$ it turns out that given $\Omega \in \mathsf{Q}_{\mathsf{B}}^{(1)}(X, X^*)$ —i.e., 1-quasilinear maps $X \to X^*$ modulo bounded equivalence then $\mathfrak{n}(\Omega) \in \overline{\mathfrak{B}}^{co}(X)$ can be determined (see [31]) by an operator $X^* \oplus_\Omega X \to X^* \oplus_\Omega X$ making a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^* & \longrightarrow & X^* \oplus_\Omega X & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \mathfrak{n}(\Omega)} & & \downarrow{\scriptstyle v} & & \\ 0 & \longrightarrow & X^* & \longrightarrow & X^* \oplus_\Omega X & \longrightarrow & X & \longrightarrow & 0\,. \end{array}$$

The operator $\mathfrak{n}(\Omega)$ has to have the form

$$\mathfrak{n}(\Omega) = \begin{pmatrix} u & \mathfrak{d}(\Omega) \\ 0 & v \end{pmatrix}$$

and it is impossible not to wonder to what extent the upper right corner entry $\mathfrak{d}(\Omega)$ can be considered the "derivative" of Ω?

There is another analogy with the case of analytic actions described first that points at the "derivative" character of $\mathfrak{n}(\Omega)$. Let us make a complete display of the preceding diagram to get, after choosing a projective presentation $0 \to \kappa(X) \to \kappa(X) \oplus_\Lambda X \to X \to 0$ of X up to bounded equivalence, and setting $(\kappa(X) \oplus_\Lambda X)^* = X^* \oplus_{\Lambda^*} \kappa(X)^*$, up to bounded equivalence,

$$\begin{tikzcd}
0 \arrow[r] & \kappa(X) \arrow[r] \arrow[d, "u"] & \kappa(X) \oplus_\Lambda X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X^* \arrow[r] \arrow[d, equal] & X^* \oplus_\Omega X \arrow[r] \arrow[d, "\mathfrak{n}(\Omega)"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X^* \arrow[r] \arrow[d, equal] & X^* \oplus_\Omega X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "v"] & 0 \\
0 \arrow[r] & X^* \arrow[r] & (\kappa(X) \oplus_\Lambda X)^* \arrow[r] & \kappa(X)^* \arrow[r] & 0\,,
\end{tikzcd}$$

namely, $u\Lambda$ and $\Lambda^* v$ are boundedly equivalent to Ω. Then:

- In the case of analytic actions it can be checked [29] that $-d_z$ is the bounded operator at finite distance from the commutator $[u_z, \Omega_z]$.
- In this case, $-\mathfrak{d}(\Omega)$ is the linear map at finite distance from the commutator-like $u\Gamma - \Gamma^* v$.

30.5. Quillen's theorem. Returning to our homological derivation problems, observe that Tor and Ext seem to maintain concealed connections, which is interesting for us since we know quite a bit about Ext and almost nothing about Tor. Recall Fuks theorem [51, Section 6], see also [24, 25], asserting that if two Banach functors F, G are $F \dashv G$ then there is a Banach space A such that F is naturally equivalent to $\widehat{\otimes}_A$ and G is naturally equivalent to $\mathfrak{L}_A$. In other words, that $\widehat{\otimes}_A \dashv \mathfrak{L}_A$ is the only existing adjunction for Banach functors. The proof is based on four facts

1. Tensors products exist.
2. Every Banach space X is a direct limit $X = \lim_{\to} \ell_1^n$ of projective objects (see an explicit proof in [25, Proposition 10.2]). Check Proposition 28.1 again.
3. Projective objects are direct limits.
4. $\mathfrak{L}_{\mathbb{R}} = id$.

Can Fuks theorem be reproduced in **pBan** ? Assertions (2),(3) and (4) hold more or less without difficulties in **pBan** . The sticking point is (1): it is not clear if tensor products exist in either **QBan** or **pBan** : Hansen [55] approaches the topic for quasi-Banach spaces with separating dual, Turpin [85] shows that the natural candidate to tensor product of p-Banach spaces is only a $(p/2)$-Banach space and an example of Kalton [59] exhibits two p-Banach spaces with 0 as tensor product. Proposition 28.1 provides an analog for (2) in functor categories, where (1) was discussed in [24].

Be as it may, the adjunction $\widehat{\otimes}_A \dashv \mathfrak{L}_A$ suggests some new possible approaches to the derivation problem. By rescuing a few ideas from [40, 8.2], we apply the adjunction above to $A = \mathrm{co}^{(1)}(X)$ and obtain

$$\widehat{\otimes}_{\mathrm{co}^{(1)}(X)} \dashv \mathfrak{L}_{\mathrm{co}^{(1)}(X)} \,.$$

Since $\mathfrak{L}_{\mathrm{co}^{(1)}(X)} = \mathsf{Q}_{\mathsf{L}}^{(1)}(X,\cdot)$, but being aware that $\mathsf{Q}_{\mathsf{L}}^{(1)} = \mathsf{Q}^{(1)}/\mathsf{L}$ is not yet $\mathcal{Q}^{(1)} = \mathsf{Q}^{(1)}/(\mathsf{B}+\mathsf{L}) = \mathrm{Ext}$, it is not hard to believe that a careful translation of the necessary equivalence relations could provide a positive answer to:

Problem 30.20. Is $\mathrm{Tor}(X,\cdot) \dashv \mathrm{Ext}_{\mathbf{B}}(X,\cdot)$?

In support of this line of thinking it comes Quillen's adjunction theorem for derived functors [79], which says that *under certain conditions*, if $F \dashv G$ then also the left-derived of F is adjoint to the right-derived of G, i.e.,

$$L^{(1)}F \dashv R^{(1)}G$$

and recall that $\mathrm{Tor}(X,\cdot) = L^{(1)}\widehat{\otimes}_X$ and $\mathrm{Ext}(X,\cdot) = R^{(1)}\mathfrak{L}_X$. The problem here is that those *certain conditions* are far from the simple checkable stuff Banach spacers are accustomed to. The streamlined formulation obtained by Maltsiniotis [68] of the hypotheses is: F and G must be functors acting between two *Quillen model categories*, F (resp. G) must send *weak equivalences* between *cofibrant objects* (resp. between *fibrant* objects) into *weak equivalences*. Fuks theorem says that the existence of a Quillen theorem holding in **Ban** exactly means an affirmative answer to problem 30.20.

On the other hand, adjunction and duality are not so different (actually, several texts consider adjunction of contravariant functors a form of duality). If we are going to give some credit to that, asking whether $\mathrm{Tor}(X,\cdot) \dashv \mathrm{Ext}_{\mathbf{B}}(X,\cdot)$ is not hugely different from asking whether $\mathrm{Tor}(X,\cdot)$

and $\mathrm{Ext}_{\mathbf{B}}(X,\cdot)$ are duals to each other. Of course the problem here is to assign a reasonable meaning to *dual*. With that purpose in mind, have a look at [24, Section 6] and let us revisit [40, Section 7]: Fuks defined the dual functor $D\mathcal{F}$ of a functor $\mathcal{F}$ in the form:

$$D\mathcal{F} = \blacktriangleleft \mathcal{F}, \widehat{\otimes} \blacktriangleright .$$

With such definition we have:

PROPOSITION 30.21. *$D\widehat{\otimes}_A = \mathfrak{L}_A$ and $D\mathfrak{L}_A = \widehat{\otimes}_A$.*

The proof?

$$D\widehat{\otimes}_A(B) = \blacktriangleleft \widehat{\otimes}_A, \widehat{\otimes}_B \blacktriangleright = \mathfrak{L}(A,B) = \mathfrak{L}_A(B),$$
$$D\mathfrak{L}_A(B) = \blacktriangleleft \mathfrak{L}_A, \widehat{\otimes}_B \blacktriangleright = \widehat{\otimes}_B(A) = \widehat{\otimes}_A(B).$$

In [40, Proposition 7.3] the result is upgraded to the next level: to maintain the similarity, let us write $\mathsf{Q}^{(1)}_X$ to mean the covariant functor $\mathsf{Q}^{(1)}(X,\cdot)$.

PROPOSITION 30.22. *$D\mathsf{Q}^{(1)}_X = \widehat{\otimes}_{\mathrm{co}^{(1)}(X)}$ and $D\widehat{\otimes}_{\mathrm{co}^{(1)}(X)} = \mathsf{Q}^{(1)}_X$*

Just for the sake of clarity, recall that when $\Omega \in \mathsf{Q}^{(1)}(X,Y)$ is a 1-quasilinear map then ϕ_Ω denotes the operator $\phi_\Omega : \mathrm{co}^{(1)}(X) \to Y$ induced by Ω. We can make explicit the identifications:

- We assign to each element $p \in \mathrm{co}^{(1)}(X)\widehat{\otimes}_\pi A$ the natural transformation $\eta_p \in \blacktriangleleft \mathsf{Q}^{(1)}(X,\cdot), \widehat{\otimes}_A \blacktriangleright$ that associates to a space Y the operator $\eta_{pY} : \mathsf{Q}^{(1)}(X,Y) \longrightarrow Y\widehat{\otimes}_\pi A$:

$$\eta_{pY}(\Omega) = \big(\phi_\Omega \widehat{\otimes} 1_A)(p)\big).$$

- We assign to each natural transformation $\eta \in \blacktriangleleft \mathsf{Q}^{(1)}_X, \widehat{\otimes}_A \blacktriangleright$ the element

$$\eta_{\mathrm{co}^{(1)}(X)}(\Omega_X) \in \mathrm{co}^{(1)}(X)\widehat{\otimes}_\pi A.$$

It seems natural again to believe that the introduction of the required equivalence relations will transform the duality between $\mathsf{Q}^{(1)}_X$ and $\widehat{\otimes}_{\mathrm{co}^{(1)}(X)}$ in a duality between $\mathrm{Ext}_{\mathbf{B}}(X,\cdot)$ and $\mathrm{Tor}(X,\cdot)$ (and, subsequently, between $\mathrm{Ext}^{(n)}_{\mathbf{B}}(X,\cdot)$ and $\mathrm{Tor}_n(X,\cdot)$). In support of this believing we can lay down [56, Corollary 4.4]: in the category of finitely generated modules the functors

$\mathrm{Ext}^{(n)}(X,\cdot)$ and $\mathrm{Tor}_n(X,\cdot)$ are duals to each other. The meaning of "dual" in that result stems from [56, Proposition 4.1]: *If* $\mathbf{C}$ *denotes the category of coherent functors acting on finitely generated* A*-modules, then there is a unique functor* $*:\mathbf{C}\to\mathbf{C}$ *which is exact, contravariant, satisfies* $** = id$ *and is such that* $*(\forall_X) = \widehat{\otimes}_X$.

30.6. The 3-space property. A property P of Banach (resp. quasi-Banach) spaces is said to be a 3-space property if whenever one has an exact sequence $0\to Y\to Z\to X\to 0$ of Banach (resp. quasi-Banach) spaces in which Y,X have P then also Z has P. It does not seem necessary to explain how much the author is fond of 3-space problems [35]. From the categorical point of view, there is something very elusive about 3-space problems. The first red flag waving is that while the property of being a quasi-Banach space is an "absolute" 3-space property, namely, given an exact sequence $0\to Y\to Z\to X\to 0$ of topological vector spaces in which both Y,X are quasi-Banach spaces then Z must also be a quasi-Banach space. The reason is that

$$\text{quasi-Banach} = \text{metrizable} + \text{complete} + \text{locally bounded}$$

and those three are 3-space properties (details can be found at [13, 2.3]. However,

$$\text{Banach} = \text{quasi-Banach} + \text{locally convex},$$

and local convexity is not a 3-space property [13, 3.2 and 3.4]. Thus, "being a Banach space" is not a 3-space property in the domain of quasi-Banach spaces. This result is structural and, as we said before, any attempt to understand twisted sums of Banach spaces has to include quasi-Banach spaces. Therefore, it seems likely that attempts to construct a heart for the category of Banach spaces should include quasi-Banach spaces.

Problem 30.23. Identify a Heart of **QBan** .

But a second red flag flamboyantly emerges when exact categories are considered [11, 57]; and it appears under the notion of *Serre* subcategory [84] (see also [57, Definition 2.8. A1], [11, Lemma 10.20]): cheating a bit, a full subcategory $\mathbf{S}$ of (an Abelian) category $\mathbf{A}$ is a Serre category when (skipping details and in our language), being an object of $\mathbf{S}$ is a 3-space property in $\mathbf{A}$. Thus, unfortunately, **Ban** is not a Serre subcategory of **QBan** . And this means problems:

PROBLEM 30.24. Identify the smallest Serre category containing **Ban** [19].

PROBLEM 30.25. Identify the quotient category **QBan** / **Ban** .

31. Eclipse

There is no doubt that the farther one goes on the categorical road, the darker it looks. However, paraphrasing Joe Walsh [20], the darker you play, the searcher you get. It is true that even now categorical Banach space theory remains a rather dark place to stay: categorical results do not find easy accommodation within classical Banach space theory and classical results in Banach space theory do have hard times moving to the categorical stratosphere. But, to quote the magnificent Alan Moore [21]. Once there was only black: We are winning.

Acknowledgements

$ This research has been supported by Project PID2023-146505NB-C21, funded by Ministerio de Ciencia Innovación y Universidades/ AEI/10.13039/5011000110331/FEDER/UE and projects IB24002 and GR24056 funded by Junta de Extremadura.

♡ This paper was written during the long summer/fall of 2024 in which my mother, Irene Castillo (102, 3 months, 3 days), passed away. An echo lingers on: "How can we keep the music playing?/ How can we make it last?/ How can we keep the song from fading/ too fast ?"

♯ People interested in the inner structure of this paper might like to keep at hand: Pink Floyd, *The dark side of the moon*, album EMI (1973); Roger Waters, *The dark side of the moon redux*, album SGB-Cooking (2023); and D. Hall, *Essays after eighty*, Mariner books (2015).

$\Re$ We acknowledge the truly remarkable work of an outstanding referee pointing out the many places where the paper required postproduction.

[19] Proposed by José Navarro.

[20] Guitar of The Eagles; but before that he released in 1973 the album: The smoker you drink, the player you get.

[21] Alan Moore, Gene Ha and Zander Cannon, comic Top 10. The sentence was then used as the closing line of Matthew McConaughey for the first season of True Detective.

References

[1] A.M. Abejón, J.M.F. Castillo, M. González, Classes of operators preserved by extensions or liftings, *J. Math. Anal. Appl.* **462** (2018), 471–482.

[2] M. Auslander, Coherent functors, in "Proc. Conf. Categorical Algebra (La Jolla, 1965)", Springer-Verlag New York, Inc., New York, 1966, 189–231.

[3] A.K. Austin, Miscellanea: Modern Research in Mathematics, *Amer. Math. Monthly* **84** 7 (1977), p. 566

[4] A. Avilés, F. Cabello Sánchez, J.M.F. Castillo, M. González, Y. Moreno, "Separably injective Banach spaces", Lecture Notes in Math. 2132, Springer, 2016.

[5] H-J. Baues, M. Jibladze, A. Tonks, Cohomology of monoids in monoidal categories, in "Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995)", Contemp. Math., 202, American Mathematical Society, Providence, RI, 1997, 137–165.

[6] A.A. Beĭlinson, J. Bernstein, P. Deligne, Faisceaux pervers, in "Analysis and Topology on Singular Spaces, I (Luminy, 1981)", *Astérisque*, 100, Société Mathématique de France, Paris, 1982, 5–71.

[7] J. Bourgain, New classes of $\mathcal{L}_p$-spaces, Lecture Notes in Math., 889, Springer-Verlag, Berlin-New York, 1981.

[8] D.A. Buchsbaum, Exact categories and duality, *Trans. Amer. Math. Soc.* **80** (1955), 1–34.

[9] D.A. Buchsbaum, A note on homology in categories, *Ann. of Math. (2)* **69** (1959), 66–74.

[10] D.A. Buchsbaum, Satellites and universal functors, *Ann. of Math. (2)* **71** (1960), 199–209.

[11] T. Bühler, Exact categories, *Expo. Math.* **28** (2010), 1–69.

[12] T. Bühler, "On the algebraic foundations of bounded cohomology", Mem. Amer. Math. Soc. 214 (2011), n. 1006.

[13] F. Cabello Sánchez, J.M.F. Castillo, "Homological methods in Banach space theory", Cambridge Stud. Adv. Math., 203, Cambridge University Press, Cambridge, 2023.

[14] F. Cabello Sánchez, J.M.F. Castillo, W.H.G. Correa, Higher order derivatives of analytic families of Banach spaces, Studia Math. 272 (2023) 245–293.

[15] F. Cabello Sánchez, J.M.F. Castillo, W.H.G. Corrêa, V. Ferenczi, R. García, On the Ext^2-problem for Hilbert spaces, *J. Funct. Anal.* **280** (2021), Paper No. 108863, 36 pp.

[16] F. Cabello Sánchez, J.M.F. Castillo, R. García, Homological dimensions of Banach spaces, *Mat. Sb.* **212** (2021) 91–112.

[17] F. Cabello Sánchez, J.M.F. Castillo, R. García, Bounded nets of Banach spaces, *preprint*, 2024.

[18] F. Cabello Sánchez, J.M.F. Castillo, N.J. Kalton, Complex interpolation and twisted twisted Hilbert spaces, *Pacific J. Math.* **276** (2015) 287 – 307.

[19] F. Cabello Sánchez, J.M.F. Castillo, Y. Moreno, Trimming the Johnson bonsai, *Banach J. Math. Anal.* **19** (2025), Paper No. 9, 15 pp.

[20] F. Cabello Sánchez, J.M.F. Castillo, A. Salguero, N. Trejo, Derived functors associated to the ideal of compact operators, *preprint*, 2025.

[21] F. Cabello Sánchez, A. Salguero, When Kalton and Peck met Fourier, *Ann. Inst. Fourier (Grenoble)* **73** (2023), 1843 – 1883.

[22] F. Cabello Sánchez, R. García, The Twisted Hilbert space ideal, *Integral Equations Operator Theory* **94** (2022), Paper No. 21, 12 pp.

[23] F. Cabello Sánchez, J. Navarro, On the category of quotient Banach spaces after Wegner, *J. Pure Appl. Algebra* **227** (2023), Paper No. 107153, 20 pp.

[24] J.M.F. Castillo, The Hitchhiker Guide to Categorical Banach Space Theory, Part I, *Extracta Math.* **25** (2010), 103 – 149.

[25] J.M.F. Castillo, The Hitchhiker Guide to Categorical Banach Space Theory. Part II, *Extracta Math.* **37** (2022), 1 – 56.

[26] J.M.F. Castillo, The freewheeling twisting of Hilbert spaces, in "The Mathematical Legacy of Victor Lomonosov–operator yheory" (Edited by Richard M. Aron, Eva A. Gallardo Gutiérrez, Miguel Martin, Dmitry Ryabogin, Ilya M. Spitkovsky and Artem Zvavitch), Adv. Anal. Geom., 2, De Gruyter, Berlin, 2020, 43 – 66.

[27] J.M.F. Castillo, W. Cuellar, V. Ferenczi, Y. Moreno, On disjointly singular centralizers, *Israel J. Math.* **252** (2022), 215 – 241.

[28] J.M.F. Castillo, W. Cuellar, M. González, R. Pino, On symplectic Banach spaces, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM* **117** (2023), Paper No. 56, 22 pp.

[29] J.M.F. Castillo, V. Ferenczi, Group actions on twisted sums of Banach spaces, *Bull. Malays. Math. Sci. Soc.* **46** (2023), Paper No. 135, 47 pp.

[30] J.M.F. Castillo, R. García, Bilinear forms and the Ext^2-problem in Banach spaces, *Linear Algebra Appl.* **566** (2019), 199 – 211.

[31] J.M.F. Castillo, R. García, Bilinear derivation in Banach spaces, *Results Math.* (to appear).

[32] J.M.F. Castillo, R. García, J. Jaramillo, Extension of bilinear forms on Banach spaces, *Proc. Amer. Math. Soc.* **129** (2001), 3647 – 3656.

[33] J.M.F. Castillo, R. García, Y. Moreno, Derivation of operator ideals in Banach spaces, *Ann. Funct. Anal.* (to appear).

[34] J.M.F. Castillo, R. García, J. Suárez, Extensions and liftings of operators and polynomials, *Mediterr. J. Math.* **9** (2012), 767 – 788.

[35] J.M.F. Castillo, M. González, "Three-space problems in Banach space theory", Lecture Notes in Math., 1667, Springer-Verlag, Berlin, 1997.

[36] J.M.F. Castillo, M. González, R. Pino, The structure of Rochberg spaces, *J. Funct. Anal.* **287** 4 (2024), Paper No. 110489, 32 pp.

[37] J.M.F. Castillo, M. González, R. Pino, Operators on the Kalton-Peck space Z_2, *Israel J. Math.* **269** (2025), 185 – 212.

[38] J.M.F. Castillo, Y. Moreno, On isomorphically equivalent extensions of quasi-Banach spaces Recent, in "Recent progress in functional analysis (Valencia, 2000)" (Edited by Klaus D. Bierstedt, José Bonet, Manuel Maestre and Jean Schmets), North-Holland Math. Stud., 189, North-Holland Publishing Co., Amsterdam, 2001, 263 – 272.

[39] J.M.F. Castillo, Y. Moreno, The category of exact sequences on Banach spaces, in "Methods in Banach space theory" (Proceedings of the 5th Conference on Banach Spaces held at the University of Extremadura, Cáceres, September 13-18, 2004. Edited by Jesús M. F. Castillo and William B. Johnson.), London Math. Soc. Lecture Note Ser., 337, Cambridge University Press, Cambridge, 2006, 139 – 158.

[40] J.M.F. Castillo, Y. Moreno, Twisted dualities in Banach spaces theory, in "Banach spaces and their applications in analysis" (Proceedings of the International Conference in honor of Nigel Kalton's 60th birthday held at Miami University, Oxford, OH, May 22–27, 2006. Edited by Beata Randrianantoanina and Narcisse Randrianantoanina), Walter de Gruyter GmbH & Co. KG, Berlin, 2007, 59 – 76.

[41] J.M.F. Castillo, Y. Moreno, All $\mathcal{L}_\infty$-spaces are separably injective, kind of, *RACSAM* (to appear).

[42] J.M.F. Castillo, R. Pino, The Rochberg garden, *Expo. Math.* **41** (2023), 333 – 397.

[43] W.H.G. Corrêa, S. Dantas, D.L. Rodríguez-Vidanes, Twisted Hilbert spaces defined by bi-Lipschitz maps, *Israel J. Math.* (to appear) `arXiv:2408.07827`.

[44] A. Defant, K. Floret, "Tensor norms and operator ideals", North Holland Math. Stud., 176, North-Holland Publishing Co., Amsterdam, 1993.

[45] J. Diestel, H. Jarchow, A. Tonge, "Absolutely summing operators", Cambridge Stud. Adv. Math., 43, Cambridge University Press, Cambridge, 1995.

[46] G. Köthe, G., Hebbare lokalkonvexe Räume, *Math. Ann.* **165** (1966), 181 – 195.

[47] P. Enflo, J. Lindenstrauss, G. Pisier, On the "three space problem", *Math. Scand.* **36** (1975) 199 – 210.

[48] S.J. Farlow, A rebuke of A.B. Smith's paper, 'A note on Piffles', *Int. J. Math. Educ. Sci. Technol.* **11** (1980), 285 – 304.

[49] P. Freyd, Relative homological algebra made absolute, *Proc. Nat. Acad. Sci. U.S.A.* **49** (1963), 19 – 20.

[50] L. Frerick, D. Sieg, Exact categories in functional analysis, *preprint*, 2010, `www.math.uni-trier.de/abteilung/analysis/HomAlg.pdf`.

[51] D.B. Fuks, Eckmann-Hilton duality and theory of functors in the category of topological spaces, *Russian Math. Surveys* **21** (1966), 1 – 33; *Uspehi Mat. Nauk* **21** (2)(128) (1966), 3 – 40).

[52] P. Gabriel, M. Zisman, "Calculus of fractions and homotopy theory", Ergeb. Math. Grenzgeb., Band 35, Springer-Verlag New York, Inc., New York, 1967.

[53] A. Grothendieck, Sur quelques points d'algèbre homologique, *Tohoku Math. J. (2)* **9** (1957) 119 – 221.

[54] The Grothendieck – Serre correspondence, `webusers.imj-prg.fr/ leila.schneps/grothendieckcircle/Letters/GS.pdf`. Also available as Grothendieck-Serre Correspondence: Bilingual Edition, A co-publication of the AMS and the Société Mathématique de France, Pierre Colmez and Jean-Pierre Serre (Eds).

[55] M. Hansen, On tensor products of quasi-Banach spaces, Research Report No. 2010-31 (2010), Seminar für Angewandte Mathematik, Eidgenössische Technische Hochschule, Zürich.

[56] R. Hartshorne, Coherent functors, *Adv. Math.* **140** (1998) 44 – 94.

[57] R. Henrard, S. Kvamme, A-Ch van Roosmalen, Auslander's formula and correspondence for exact categories, *Adv. Math.* **401** (2022), Paper N0. 108296, 65 pp.

[58] E. Hilton, K. Stammbach, "A course in homological algebra", Grad. Texts in Math., Vol. 4, Springer-Verlag, New York-Berlin, 1971.

[59] N. Kalton, An example in the theory of bilinear maps, *Canad. Math. Bull.* **25** (1982), 377 – 379.

[60] N.J. Kalton, The three-space problem for locally bounded F-spaces, *Compositio Math.* **37** (1978) 243 – 276.

[61] N.J. Kalton, Nonlinear commutators in interpolation theory, *Mem. Amer. Math. Soc.* **73** (1988), no. 385.

[62] N.J. Kalton, Differentials of complex interpolation processes for Köthe function spaces, *Trans. Amer. Math. Soc.* **333** (1992), 479 – 529.

[63] N.J. Kalton, N.T. Peck, Twisted sums of sequence spaces and the three space problem, *Trans. Amer. Math. Soc.* **255** (1979) 1 – 30.

[64] N.J. Kalton, J.W. Roberts, Uniformly exhaustive submeasures and nearly additive set functions, *Trans. Amer. Math. Soc.* **278** (1983) 803 – 816.

[65] H. Krause, Deriving Auslander's formula, *Doc. Math.* **20** (2015), 669 – 688.

[66] J. Lindenstrauss, Extension of compact operators, *Mem. Amer. Math. Soc.* **48** (1964), 112 pp.

[67] S. MacLane, "Categories for the working mathematician", Grad. Texts in Math., Vol. 5, Springer-Verlag, New York-Berlin, 1971.

[68] G. Maltsiniotis, Quillen's adjunction theorem for derived functors, revisited, *C.R. Math. Acad. Sci. Paris* **344** (2007), 549 – 552.

[69] A. Martsinkovsky, Deconstructing Auslander's formulas, I. Fundamental sequences associated with additive functors, *Homology Homotopy Appl.* (to appear) `arXiv:2410.14097v2`.

[70] A. Martsinkovsky, J. Russell, Injective stabilization of additive functors I. Preliminaries, *J. Algebra* **530** (2019), 129 – 169.

[71] A. Martsinkovsky, J. Russell, Injective stabilization of additive functors II. (Co)torsion and the Auslander-Gruson-Jensen functor, *J. Algebra* **548** (2020), 53 – 95.

[72] A. Martsinkovsky, J. Russell, Injective stabilization of additive functors III. Asymptotic stabilization of the tensor product, *Algebra Discrete Math.* **31** (2021), 120 – 151.

[73] Y. Moreno, Theory of z-linear maps, Ph.D. Thesis, Univ. Extremadura, Badajoz, 2003.

[74] A. Ortyński, On complemented subspaces of $\ell^p(\Gamma)$ for $0 < p \leq 1$, *Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys.* **26** (1978), 31 – 34.

[75] B. Pachuasvili, Cohomologies and extensions in monoidal categories, *J. Pure Appl. Algebra* **72** (1991), 109 – 147.

[76] B. Paregis, Cohomology of groups in arbitrary categories, *Proc. Amer. Math. Soc.* **15** (1964), 803 – 809.

[77] A. Pietsch, "Operator ideals", North-Holland Math. Library, 20, North-Holland Publishing Co., Amsterdam-New York, 1980.

[78] K. Pothoven, Projective and injective objects in the category of Banach spaces, *Proc. Amer. Math. Soc.* **22** (1969), 437 – 438.

[79] D.G. Quillen, "Homotopical algebra", Lecture Notes in Math., No. 43, Springer-Verlag, Berlin-New York, 1967.

[80] D.G. Quillen, Higher algebraic K-theory. I, in "Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)", Lecture Notes in Math., Vol. 341,Springer-Verlag, Berlin-New York, 1973, 85 – 147.

[81] M. Ribe, Examples for the nonlocally convex three space problem, *Proc. Amer. Math. Soc.* **73** (1979), 351 – 355.

[82] R. Rochberg, G. Weiss, Derivatives of analytic families of Banach spaces, *Ann. of Math. (2)* **118** (1983), 315 – 347.

[83] M. Scott Osborne, "Basic homological algebra", Grad. Texts in Math., 196, Springer-Verlag, New York, 2000.

[84] J.P. Serre, Groupes d'homotopie et classes de groupes abéliens, *Ann. of Math. (2)* **58** (1953), 258 – 294.

[85] P. Turpin, Produits tensoriels d'espaces vectoriels topologiques, *Bull. Soc. Math. France* **110** (1982), 3 – 13.

[86] L. Waelbroeck, "Bornological quotients", Memoir of the Science Section. Collection in-4o. 3rd Series, , VII, Académie Royale de Belgique, Classe des Sciences, Brussels, 2005. With the collaboration of Guy Nöel.

[87] Ch.A. Weibel, "An introduction to homological algebra", Cambridge Stud. Adv. Math., 38, Cambridge University Press, Cambridge, 1994.

[88] K.W. Yang, A note on reflexive Banach spaces, *Proc. Amer. Math. Soc.* **18** (1967), 859 – 861.