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On functional analytic approach for corona and Gleason's problems for holomorphic Lipschitz algebras

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Abstract: We study Lipschitz algebras of *holomorphic* functions of the order k, $0 \leq k \leq \infty$, and the exponent α, $\alpha \in (0, 1]$. The Gel'fand theory and maximal ideal spaces of these algebras are discussed. Further, we solve the corona problem (1962) and Gleason's problem (1964) for these algebras on certain bounded pseudoconvex/poly domains G in $\mathbb{C}^n$ (e.g., the ball and polydisc). As a welcome bonus, we affirmatively solve Fornæss and Øvrelid's problem (1983) for holomorphic Hölder and Lipschitz spaces. In fact, we establish an equivalency between the two problems for these algebras. As an application, we establish the I.J. Schark's theorem for Lipschitz algebras on these G's. Indeed, we extend our recent work on Gleason's problem, based on the functional analytic approach, as well as extend recent results of Clos for these algebras, and apply the usual Banach algebra method.

Key words: Corona problem, Gleason's problem, Gel'fand theory, locally Stein algebras, finitely generated maximal ideal, Lipschitz algebras of holomorphic functions, Gleason A-property.

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1. History of the two problems

Corona problem. In 1942, Kakutani asked the following fundamental question (known as *corona problem*). Let $H^\infty(D)$ be the Banach algebra of all bounded holomorphic functions on the open unit disc D in the complex plane $\mathbb{C}$, equipped with the usual sup-norm, and let $f_j \in H^\infty(D)$, $j = 1, 2, \ldots, n$, with the property that $\sum_{j=1}^n |f_j(z)| > \epsilon$ for some $\epsilon > 0$ and all $z \in D$ (namely, a finite set of corona data on the disc D), do there exist $g_j \in H^\infty(D)$, $j = 1, 2, \ldots, n$, such that $\sum_{j=1}^n f_j g_j = 1$; i.e., $\sum_{j=1}^n f_j(z) g_j(z) = 1$ for all $z \in D$? This latter equation is known as the Bézout's equation [24, Theorem V.1.8].

The equivalent formulation of this question is to ask whether D is dense in the maximal ideal space (aka spectrum) $M(H^\infty(D))$ w.r.t. the Gel'fand (i.e., relatively weak*-) topology (equivalently, whether the corona is empty). Carleson affirmatively solved the problem on the disc D [12].

By a *domain* we mean an open, connected and bounded set G in $\mathbb{C}^n$. At this time there is no known domain in $\mathbb{C}$ on which this problem is failed.

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Since 1962, the corona problem is open for arbitrary domains (including the balls and polydiscs) in $\mathbb{C}^n$, $n > 1$, and is a lot more complicated. There have been investigations of this problem on various types of domains, as well as in various Banach spaces of holomorphic functions on these domains. For example, see some partial results on the corona problem in [11, 34, 35, 51]. Not only this, but a variety of counterexamples to the corona problem in the theory of SCV have been produced in [22, 49]. The extensive exposition of history of the research of the problem is outlined in [19]. One of the two main purposes of this paper is to provide favorable results on the corona problem for $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G, d)$ (defined in Section 2), G a certain pseudoconvex/poly domain (e.g., the ball and polydisc) in $\mathbb{C}^n$ (Theorem 3.4 below). So we also establish the weak corona theorem for these algebras (cf. [3, 4]).

In fact, after the Carleson's solution, most of the subsequent attempts to extend it for various Banach spaces of holomorphic functions on more general domains in $\mathbb{C}^n$ were aimed at solving the Bézout's equation by various classical methods (mostly, by solving the $\bar{\partial}$-equation); see, for instance, [34, 35, 51]. These approaches required hard classical analysis methods and has met serious difficulties in the case when $n > 1$, NOT leading to the solution to the original H^∞ corona problem. In this paper, we discuss a functional analytic approach different from those classical analytic approaches.

Indeed, in Section 3, we first generalize the result of Clos (Theorem 3.2 and Theorem 3.3 below) on the Gleason solvability for $\mathrm{Lip}_H^{k,\alpha}(G, d)$ (resp., $A^k(\overline{G})$) of holomorphic functions on G, where G is a *certain* pseudoconvex domain in $\mathbb{C}^n$ (i.e., not just a pseudoconvex domain for which the $\bar{\partial}$-problem is solvable in L^∞, namely, L^∞-pseudoconvex domain). Then, using this result, we establish the corona theorem with a finite C^k-data (resp., C^k-Lipschitz data). In fact, Theorem 3.4 and Theorem 3.6 below establish an *equivalency* of the corona problem with another significant problem in the theory of SCV, namely, *Gleason's problem* (1964), which we now describe in fuller details as follows.

GLEASON'S PROBLEM. Let $A(G)$ be the Banach algebra of all holomorphic functions on a domain G in $\mathbb{C}^n$ that extend by continuity to the closure of G, equipped with the usual compact open topology given by the sup-norm. Thus, it is a closed subalgebra of $H^\infty(G)$. For our convenience, we may consider $A(G)$ as the Banach algebra of all uniformly continuous, holomorphic functions on G. The Gleason's problem is to decide whether the ideal in $A(G)$ (or $H^\infty(G)$), consisting of functions vanishing at a fix point $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in G$, is algebraically finitely generated by the shifted coordinate functions $z_j - \lambda_j$ for $j = 1, 2, \dots, n$ (see [7] and other references therein for details). This is actually a variant of the original Gleason's problem for $A(G)$, G the unit ball in $\mathbb{C}^2$ and $\lambda = (0, 0)$ [25]. The positive solution in the case of strictly pseudoconvex domains with smooth boundaries was established in [28]. It has been even strengthened in [30] as follows. The continuous linear operators

$$T_j : A(G) \longrightarrow A(G),\ f \longmapsto T_j(f), \qquad j = 1, 2, \dots, n,$$

solving Gleason's problem exist, which means that

$$f(z) - f(\lambda) = \sum_{j=1}^{n} (z_j - \lambda_j)(T_j(f))(z), \qquad z \in G,\ f \in A(G).$$

For polydomains analogous decompositions are easily available by fixing all but one variable method.

This implies, in particular, that the part of the spectrum $M(A(G))$ in the fibers over G is mapped in a bijective way to G by the Gel'fand transform $(\hat{z}^1, \dots, \hat{z}^n)$ of the coordinate functions. By the result of [27], the entire spectrum $M(A(G))$ is mapped bijectively onto $\overline{G}$ where G is a smooth pseudoconvex domain in $\mathbb{C}^n$. It also suffices to consider a pseudoconvex domain G with a Stein neighbourhood basis; see [45]. Hence the Gel'fand topology coincides with the Euclidean topology on $\overline{G}$ and G is open in $M(A(G))$.

Background. This paper contains a continuation of the work begun in [43], and we shall feel free to use the terminology and conventions established there. However, this current work is specifically concerned with the solvability of the two problems for $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k;\alpha}(G, d)$. In [43], we extend the Gleason's result to finitely generated ideals in Fréchet algebras, answering affirmatively a question posed in [13] for Fréchet algebras. As consequences, locally Stein algebras are completely characterized by intrinsic properties within the class of Fréchet algebras, using which an affirmative answer to Gleason's problem for such algebras is provided. In fact, we remark that complex analysts may find this functional analytic approach interesting from applications point of view. For example, see [43, Theorem 4.2, Theorem 4.3 and Corollary 4.4], and references (and their reviews) to [25] in MathSciNet (MR0159241 (28 #2458)). As we will see, this is, indeed, the case.

Banach algebras satisfying [43, Lemma 3.1] (in particular, the polydisc algebra $A(D^n)$, the ball algebra $A(B^n)$ and $H^\infty(G)$) are locally Stein-Banach algebras. In Section 3, we show that Banach subalgebras $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G)$ of $H^\infty(G)$, for which we want to solve Gleason's problem, are also locally Stein-Banach algebras. Further, a domain G has the Gleason A-property if Gleason's problem has an affirmative solution at all points of G for a locally Stein algebra A. Clos calls it the Gleason solvability property of a domain G for $H^\infty(G)$. The Gleason solvability was first proved by Leibenson, albeit informally [46]. The Gleason solvability for holomorphic Bergman spaces, holomorphic (harmonic) mixed-norm spaces, Bergman-Sobolev spaces and holomorphic (harmonic) Bloch spaces were studied in [46, 37, 41, 53] ([29]). The domains considered were initially the unit ball, but then were generalized to (bounded) strongly pseudo(convex) domains with C^2 boundary.

In [43, Corollary 4.4], we show that the dense open subspace has the Gleason A-property, where A is a semi-simple locally Stein algebra. In particular, A can be either $H^\infty(G)$ or $A(G)$, where G is a (strictly or weakly) pseudoconvex domain in $\mathbb{C}^n$ with various boundary conditions and containing the origin. Thus the functional analytic method recaptures the classical results on Gleason's problem obtained by Beatrous Jr. [8], Fornæss and Øvrelid [21], Kerzman and Nagel [32], Lieb [36], Noell [40], and Backlund and Fällström [7] (see other references therein for a list of papers on this problem). This problem is still open for Lipschitz algebras of holomorphic functions. In fact, we affirmatively answer a question of Fornæss and Øvrelid (1983) for holomorphic Hölder and Lipschitz spaces (Remarks E. 2 below).

As we shall see, we find it necessary (mathematically, "sufficient") to consider the Gleason solvability while solving the corona problem. On the other hand, although we have not spelled it out explicitly, but have considered the corona theorem as a hypothesis while solving Gleason's problem [43, Theorem 4.3 and Corollary 4.4]. In fact, we have a pleasant result of Żelazko [52, Proposition 2], showing a possible connection between these two problems through their algebraic formulations. Moreover Kerzman and Nagel used the Koszul complex while solving Gleason's problem for certain function algebras on certain domains in $\mathbb{C}^n$. Now, all this developments stem to think on an equivalency of these two problems, as well as to give a short elegant proof, by using the (maximal) ideal structures of Banach algebras $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G,d)$ alone. That means, we try to keep ourselves away from the analytical aspects such as Gleason parts of the spectra of these algebras.

One of our main goals is to extend [16, Theorem 4 and Theorem 6] in Section 3. In fact, we show that if G is a certain "pseudoconvex" domain in $\mathbb{C}^n$, then it has the Gleason A-property, where A is either $A^k(\overline{G})$ or $\mathrm{Lip}_H^{k,\alpha}(G,d)$

(including the case $k = \infty$ as well), in Theorem 3.2. Then we bring in the result of Arens to show that the Bézout's equation is solvable, establishing the corona problem for these algebras on these domains, including the balls and polydiscs.

In Section 2, we discuss the theory of Lipschitz algebras of *holomorphic* functions of the order k, $0 \leq k \leq \infty$, and the exponent α, $\alpha \in (0, 1]$ (throughout the paper, the exponent α will be in $(0, 1]$ and the order $k \neq \infty$ unless otherwise specified). In particular, the Gel'fand theory and maximal ideal spaces of these algebras are discussed. In Section 4, we give some concluding remarks on our approach, leading to a couple of open problems (including a clear possibility of solving the original H^∞ corona problem).

2. Lipschitz algebras of holomorphic functions

Throughout the paper, "algebra" will mean a complex, commutative algebra with identity. Sherbert studied the Banach algebra $\mathrm{Lip}(X, d')$ of all Lipschitz continuous functions (resp., Banach algebras $\mathrm{lip}^\alpha(X, d')$ for $\alpha \in (0, 1]$) on the metric space (X, d'). Sibony discussed the existence of Stein neighbourhood basis for the closure of weakly pseudoconvex domains G in $\mathbb{C}^n$ and the spectra of $A^k(\overline{G})$ for $0 \leq k \leq \infty$ among other Banach algebras of bounded holomorphic functions (including $H^\infty(G)$) in Section 3 and Section 4, respectively. So we shall here feel free to use the terminology, conventions, a couple of arguments and proofs of theorems from [47, 48, 50] in order to keep our argument short for the new classes of Banach/Fréchet algebras $A^k(\overline{G})$, $\mathrm{Lip}^\alpha_H(G, d)$ and $\mathrm{Lip}^{k,\alpha}_H(G, d)$. As far as we know, Banach/Fréchet algebras from the latter two cases, were never studied, which are, indeed, important objects for study compared to their superalgebras $\mathrm{Lip}^\alpha(G, d)$ for $\alpha \in (0, 1]$ and $\mathrm{Lip}^{k,\alpha}(G, d)$ for $0 \leq k \leq \infty$ and $\alpha \in (0, 1]$. As we shall see in this paper, study of these two classes is essential to further develop the theory of *Banach algebras of holomorphic functions*. So, we study $\mathrm{Lip}^\alpha_H(G, d)$ and $\mathrm{Lip}^{k,\alpha}_H(G, d)$ in view of the two problems (e.g., Gleason's problem, discussed in [43]). Thus our results extend the Sherbert's results (and some results of Sibony's as well). We now examine some basic properties of these Banach algebras, namely, Lipschitz algebras of holomorphic functions.

Gel'fand theory of Lipschitz algebras of holomorphic functions. For each $0 \leq k \leq \infty$, let $A^k(\overline{G})$ denote the collection of all bounded, complex-valued functions in $C^k(\overline{G})$ which are holomorphic on the bounded domain G (see [50, §4] for details). For each $\alpha \in (0,1]$, let $\mathrm{Lip}_H^\alpha(G,d)$ denote the collection of all bounded, complex-valued, α-Lipschitz continuous (aka Hölder continuous), holomorphic functions defined on (G,d) in $\mathbb{C}^n$, where d is the usual metric on $\mathbb{C}^n$ induced by either the max-norm or the Euclidean norm. Also, for each $0 \le k \le \infty$ and for each $\alpha \in (0,1]$, let $\mathrm{Lip}_H^{k,\alpha}(G,d)$ denote $A^k(\overline{G}) \bigcap \mathrm{Lip}_H^\alpha(G,d)$. Thus, $\mathrm{Lip}_H^\alpha(G,d)$ (resp., $\mathrm{Lip}_H^{k,\alpha}(G,d)$) consists of all f defined on (G,d) (resp., on $(\overline{G},d)$) such that both

$$\|f\|_k := \max_{0\le i\le k}\left(\sup\left\{|f^{(i)}(z)| : z \in \overline{G}\right\},\ 0 \le k < \infty\right)$$

and

$$\|f\|_\alpha := \sup\left\{\frac{|f(z)-f(w)|}{\|z-w\|^\alpha} : z,w \in G,\ z \ne w\right\}$$

are finite. It is easy to see that, with the norm $\|\cdot\|_{\mathrm{Lip}^{k,\alpha}}$, defined by $\|f\|_{\mathrm{Lip}^{k,\alpha}} = \|f\|_k + \|f\|_\alpha$, $\mathrm{Lip}_H^{k,\alpha}(G,d)$ is a commutative Banach algebra with identity. Note that another equivalent algebra norms are also possible; e.g., if we take maximum of $\|f\|_k$ and $\|f\|_\alpha$ or if we take

$$\sup\left\{|f^{(i)}(z_0)|,\ \frac{|f(z)-f(w)|}{\|z-w\|^\alpha}\ :\ z_0 \in \overline{G},\ z,w \in G,\ z \ne w,\ 0 \le i \le k\right\}.$$

Also, instead the norm $\|f\|_k$, we may take an equivalent C^k-norm, given by

$$\|f\|_{C^k} = \sum\nolimits_{i=0}^{k} \frac{\|f^{(i)}\|_\infty}{i!}, \qquad 0 \le k < \infty.$$

If $k = \infty$, then $\mathrm{Lip}_H^{\infty,\alpha}(G,d)$ is a commutative Fréchet algebra whose Fréchet algebra topology may be given by an increasing sequence $(\|\cdot\|_{\mathrm{Lip}^{k,\alpha}})$ of norms, where $0 \le k < \infty$; in fact, we have $\mathrm{Lip}_H^{\infty,\alpha}(G,d) = \bigcap_{k=0}^\infty \mathrm{Lip}_H^{k,\alpha}(G,d)$. Also, when $k=0$, we have $\mathrm{Lip}_H^{0,\alpha}(G,d) = \mathrm{Lip}_H^\alpha(G,d)$, with the norm $\|\cdot\|_{\mathrm{Lip}^\alpha}$, defined by $\|f\|_{\mathrm{Lip}^\alpha} = \|f\|_\infty + \|f\|_\alpha$. Further we see that there are several equivalent norms possible on the Banach space $\mathrm{Lip}_H^\alpha(G,d) = H\Lambda(\alpha/n,\infty,\infty) = \Lambda_\alpha = \Lambda_\alpha^{\infty,\infty}$ due to various characterizations discussed in [20, 14, 15, 31]. In fact, it is a closed subalgebra of a Banach algebra $\mathrm{Lip}^\alpha(G,d)$ of α-Lipschitz (aka Hölder) continuous functions (resp., Banach algebra $\mathrm{Lip}(G,d)$ of Lipschitz continuous functions, when $\alpha = 1$). For the exponent $\alpha \in (0,1]$ and a fixed k, it is evident that

$$\mathrm{Lip}_H^k(G) \subset \mathrm{Lip}_H^{k,\alpha}(G) \subsetneq A^k(\overline{G}) \subsetneq H^\infty(G) \subsetneq \mathrm{Hol}(G). \tag{1}$$

For $0 \le l \le k \le \infty$ and a fixed α, we have

$$\mathrm{Lip}_H^{k,\alpha}(G) \subset \mathrm{Lip}_H^{l,\alpha}(G) \subsetneq A^l(\overline{G}) \subsetneq H^\infty(G) \subsetneq \mathrm{Hol}(G), \tag{2}$$

G a domain in $\mathbb{C}^n$. All inclusion mappings in (1) and (2) are continuous.

It will be assumed throughout the paper that (G, d) is a relatively compact metric space in $\mathbb{C}^n$ unless otherwise specified. Since each element of $\mathrm{Lip}_H^\alpha(G, d)$ is uniformly continuous on (G, d), it extends uniquely, continuously and in norm preserving way to an element of $\mathrm{Lip}^\alpha(\overline{G}, d)$. Thus, as Banach algebras, $\mathrm{Lip}_H^\alpha(G, d)$ and its image in $\mathrm{Lip}^\alpha(\overline{G}, d)$ are isometrically isomorphic. So $\mathrm{Lip}_H^\alpha(G, d)$ may be taken as a closed subalgebra of $\mathrm{Lip}^\alpha(\overline{G}, d)$.

Let $(A, \|\cdot\|_A)$ be a commutative semi-simple Banach algebra of *holomorphic functions*, i.e., for every $f \in A$, $\hat{f}$ is holomorphic in $M(A)$ w.r.t. the Gel'fand topology (resp., of *holomorphic functions with extensions to the boundary as C^k-functions*, i.e., for every $f \in A$, $\hat{f}$ is holomorphic in $M(A)$ and is C^k-function on $\mathrm{Bdry}(M(A))$ w.r.t. the Gel'fand topology). Equivalently, $\hat{A}$ has an analytic structure in $M(A)$ for such an A (cf. [43, p. 3]). We may also consider $\hat{f}$ as the restriction of a weak*-holomorphic function on the whole A^* to $M(A)$ (author is indebted to the referee for pointing out an alternative definition of a Banach algebra of holomorphic functions). Note that, by the standard Banach space theory, $M(A)$ lies on the unit sphere of A^*. Thus, as a subset of A^*, $M(A)$ inherits the Gel'fand topology (i.e., restriction of the weak*-topology $w(A^*, A)$ to $M(A)$), and inherits the relative norm or metric topology. The Gel'fand theory of commutative Banach algebras uses the Gel'fand topology. For each $f \in A$, the function $\hat{f}$ is defined on $M(A)$ by $\hat{f}(\phi) = \phi(f), \phi \in M(A)$. The Gel'fand topology is the weakest topology on $M(A)$ such that the family $\{\hat{f} : f \in A\}$ is continuous on $M(A)$. Let $C(M(A))$ will denote the algebra of complex-valued continuous functions on $M(A)$ w.r.t. the Gel'fand topology, given with the sup-norm. Then the Gel'fand mapping $f \to \hat{f}$ is a continuous isomorphism of A into $C(M(A))$.

We now consider the metric topology of $M(A)$. The metric σ on $M(A)$ induced by the norm $\|\cdot\|_A^*$ of the dual space A^*, is defined by

$$\sigma(\phi, \psi) = \|\phi - \psi\|_A^*, \qquad \phi, \psi \in M(A).$$

In terms of the functions $\hat{f}, f \in A$, we may express the metric σ by

$$\sigma(\phi, \psi) = \sup\left\{\left|\hat{f}(\phi) - \hat{f}(\psi)\right| : f \in A,\ \|f\|_A \le 1\right\}, \qquad \phi, \psi \in M(A).$$

The metric topology of $M(A)$ is stronger than the Gel'fand topology. Therefore, since $M(A)$ is closed in A^* w.r.t. the weak*-topology, it is also closed w.r.t. the metric topology. Hence, $(M(A), \sigma)$ is a complete metric space. The metric σ is bounded as $M(A)$ lies on the unit sphere of A^*.

With this metric space $(M(A), \sigma)$ for a fixed exponent $\alpha \in (0, 1]$, we form the Lipschitz algebra $\mathrm{Lip}^\alpha(M(A), \sigma)$ with norm $\|\cdot\|_{\mathrm{Lip}^\alpha}$ defined by

$$\|g\|_{\mathrm{Lip}^\alpha} = \|g\|_\infty + \|g\|_\alpha, \qquad g \in \mathrm{Lip}^\alpha(M(A), \sigma).$$

We now discuss the Gel'fand theory of $\mathrm{Lip}_H^\alpha(G, d)$ (resp., of $\mathrm{Lip}_H^{k,\alpha}(G, d)$). These representations are drawn from the Gel'fand representation. We use the metric topology in place of the Gel'fand topology of $M(A)$. Thus we have a "Lipschitz representation of order α" in the following

Proposition 2.1. *Let A be a unital commutative semi-simple Banach algebra of holomorphic functions. Then the Gel'fand mapping is a continuous isomorphism of A onto a subalgebra of $\mathrm{Lip}^\alpha(M(A), \sigma)$. Furthermore, for each $f \in A$, $\|\hat{f}\|_\alpha \leq \|f\|_A$ and $\|\hat{f}\|_\infty \leq \|f\|_A$.*

Proof. First, we discuss an important subset $\mathrm{lip}^\alpha(M(A), \sigma)$ consisting of all those functions $f \in \mathrm{Lip}^\alpha(M(A), \sigma)$ with the property that

$$\frac{|f(z) - f(w)|}{\sigma(z, w)^\alpha} \to 0 \;\text{ as }\; \sigma(z, w)^\alpha \to 0.$$

It is certainly a closed subalgebra of $\mathrm{Lip}^\alpha(M(A), \sigma)$; this was established by Mirkil [39] for the case $X = [0, 2\pi]$ and $d(x, y)^\alpha = |x - y|^\alpha, 0 < \alpha < 1$, but his proof is valid in general.

Next, it is easy to show that for each $\alpha \in (0, 1)$,

$$\mathrm{Lip}(M(A), \sigma) \subset \mathrm{lip}^\alpha(M(A), \sigma) \subset \mathrm{Lip}^\alpha(M(A), \sigma)$$

and the Banach algebra $\mathrm{lip}^\alpha(M(A), \sigma)$ (and so, $\mathrm{Lip}^\alpha(M(A), \sigma)$ as well) separates the points of $M(A)$ by [48, Proposition 1.6]. Further, these inclusion mappings are continuous. Now, the proof is straightforward by [47, Proposition 2.1]. Moreover, the subalgebra $\hat{A}$ of $\mathrm{Lip}^\alpha(M(A), \sigma)$ consists of functions that are continuous on $(M(A), \sigma)$ and holomorphic in $(M(A), \sigma)$. ∎

Remark A. For a fixed order $0 \leq k < \infty$ and a fixed exponent $\alpha \in (0, 1]$, we form the Lipschitz algebra $\mathrm{Lip}^{k,\alpha}(M(A), \sigma)$ with the norm $\|\cdot\|_{\mathrm{Lip}^{k,\alpha}}$, where

$$\|g\|_{\mathrm{Lip}^{k,\alpha}} := \|g\|_k + \|g\|_\alpha, \qquad g \in \mathrm{Lip}^{k,\alpha}(M(A), \sigma).$$

Then, the Gel'fand mapping takes a commutative, semi-simple Banach algebra A of holomorphic functions with extensions to the boundary as C^k-functions into the Lipschitz algebra $\mathrm{Lip}^{k,\alpha}(M(A),\sigma)$, giving a "Lipschitz representation of order (k,α)" in the above proposition. We omit its proof as it is an obvious variant of Proposition 2.1 (which is the case $k=0$).

Next, we study $M(A)$ of the Banach algebra $A = \mathrm{Lip}_H^{k,\alpha}(G,d)$, $0 \le k < \infty$, $\alpha \in (0,1]$, in view of some important properties that are useful in establishing the corona theorem and the Gleason solvability for this algebra, as discussed in Section 3 below. We first remark that it contains the identity element, namely, $f \equiv 1$ on G. It also separates the points of G as it also contains the coordinate functions, but it is not self-adjoint. So each $z \in \overline{G}$ can be identified with the point evaluation functional ϕ_z in $M(A)$, where $\phi_z(f) = f(z)$. More precisely, the injection mapping $z \to \phi_z$ is one-to-one from $\overline{G}$ to $M(A)$ and we may regard $\overline{G}$ as a subset of $M(A)$. The metric σ on $\overline{G}$ when restricted to $\overline{G}$, can be expressed by

$$\sigma(z,w) = \sup\left\{|f(z)-f(w)| :\ f \in \mathrm{Lip}_H^{k,\alpha}(G,d),\ \|f\|_{\mathrm{Lip}^{k,\alpha}} \le 1\right\}, \quad z,w \in \overline{G}.$$

An algebra B of functions defined on a set X is called inverse-closed if for every function $f \in B$ satisfying $|f(x)| \ge \epsilon > 0$ for all $x \in X$, then $f^{-1} \in B$ as well. In the below lemma, G is a smooth strictly pseudoconvex domain in $\mathbb{C}^n$, or else, G is a pseudoconvex domain with a Stein neighbourhood basis (e.g., Stein domains in $\mathbb{C}^n$). We will consider only such pseudoconvex domains G in $\mathbb{C}^n$ in the sequel, unless otherwise stated.

Lemma 2.2. *Let $M(A)$ be the maximal ideal space of $\mathrm{Lip}_H^{k,\alpha}(G,d)$, where $0 \le k < \infty$ and $\alpha \in (0,1]$. Then $M(A) = \overline{G}$ in the Gel'fand topology provided that G is a pseudoconvex domain in $\mathbb{C}^n$ as above. In particular, if (G,d) is relatively compact in $\mathbb{C}^n$, then the Gel'fand topology coincides with the d-topology of G.*

Proof. We recall that the second inclusion in (1) above is continuous as $\|f\|_k \le \|f\|_{\mathrm{Lip}^{k,\alpha}}$. Since the polynomials in $z_1, z_2, \ldots, z_n$ are dense in $A^k(\overline{G})$ and they are also in $\mathrm{Lip}_H^{k,\alpha}(G,d)$, $\mathrm{Lip}_H^{k,\alpha}(G,d)$ is a dense Banach subalgebra of $A^k(\overline{G})$. It is easy to see that $\mathrm{Lip}_H^{k,\alpha}(G,d)$ is an inverse-closed algebra by following [48, Proposition 1.7]. So, $\mathrm{Lip}_H^{k,\alpha}(G,d)$ is also inverse-closed in $A^k(\overline{G})$. Further, the maximal ideal space of $A^k(\overline{G})$ is the joint spectrum $\mathrm{sp}(z_1, z_2, \ldots, z_n)$. Combining all these arguments, we have that

$M(A) = \mathrm{sp}(z_1, z_2, \ldots, z_n)$ by the several variables analogue of [9, Proposition 1]. Alternatively, as discussed in the proof of Theorem 3.6 below, $\mathrm{Lip}_H^{k,\alpha}(G,d)$ is a semi-simple, finitely generated Banach algebra of power series in $z_1, z_2, \ldots, z_n$. So, $M(A) = \mathrm{sp}(z_1, z_2, \ldots, z_n)$ by the general Banach algebra theory. Further, if G is a smooth strictly pseudoconvex domain in $\mathbb{C}^n$ [50], or else, G is a pseudoconvex domain with a Stein neighbourhood basis [45] (e.g., Stein domains in $\mathbb{C}^n$), then $\mathrm{sp}(z_1, z_2, \ldots, z_n) = \overline{G}$. Hence $M(A) = \overline{G}$ in this case. ∎

Remarks B. 1. There are bounded domains in $\mathbb{C}^n$ such that neither they have a smooth boundary (e.g., Hartogs triangle) nor they have a Stein neighbourhood basis (e.g., worm domain) [18].

2. By the above lemma, we have, indeed, solved the corona problem for commutative semi-simple Banach algebras $\mathrm{Lip}_H^{k,\alpha}(G,d)$ (G can be a relatively compact polydomain in $\mathbb{C}^n$ as well; see Theorem 3.4 below). In fact, in this case, we have a better result in the sense that the relative Gel'fand topology of G is equivalent to the d- and σ-topologies of G (Proposition 2.3, Proposition 2.5 and Corollary 2.8 below).

3. For $k = \infty$ and for each $\alpha \in (0,1]$, the maximal ideal space of a commutative, semi-simple Fréchet algebra $\mathrm{Lip}_H^{\infty,\alpha}(G,d)$ is still $\overline{G}$ by the standard Gel'fand theory of a commutative Fréchet algebra. In fact,

$$M\big(\mathrm{Lip}_H^{\infty,\alpha}(G,d)\big) = \bigcup\nolimits_{k=0}^{\infty} M\big(\mathrm{Lip}_H^{k,\alpha}(G,d)\big) = \bigcup\nolimits_{k=0}^{\infty} \overline{G} = \overline{G}.$$

4. Note that for an equivalency of the Gel'fand topology with the d-topology of G, relative compactness of G is as such not required when $k = 0$. Indeed, we have the following result whose proof we omit (see [47, Proposition 3.3]).

Proposition 2.3. *Let G be a pseudoconvex domain in $\mathbb{C}^n$ as above. Then the relative Gel'fand topology of G and the d-topology of G are equivalent.*

We now turn to the Gel'fand representation of $A = \mathrm{Lip}_H^{\alpha}(G,d)$ (resp., $\mathrm{Lip}_H^{k,\alpha}(G,d)$), where G is as in [50, 45]. Proposition 2.1 (resp., Remark A) tells us that the Gel'fand mapping $f \mapsto \hat{f}$ takes $\mathrm{Lip}_H^{\alpha}(G,d)$ (resp., $\mathrm{Lip}_H^{k,\alpha}(G,d)$) isomorphically into $\mathrm{Lip}^{\alpha}(M(A),\sigma)$ (resp., $\mathrm{Lip}^{k,\alpha}(M(A),\sigma)$), and satisfies $|||\hat{f}|||_\infty \le \|f\|_{\mathrm{Lip}^\alpha}$ (resp., $|||\hat{f}|||_k \le \|f\|_{\mathrm{Lip}^{k,\alpha}}$) and $|||\hat{f}|||_\alpha \le \|f\|_{\mathrm{Lip}^\alpha}$ (resp., $|||\hat{f}|||_\alpha \le \|f\|_{\mathrm{Lip}^{k,\alpha}}$) for all $f \in \mathrm{Lip}_H^{\alpha}(G,d)$ (resp., for all $f \in \mathrm{Lip}_H^{k,\alpha}(G,d)$).

These statements pursue from the general consideration. In the particular case of $A = \mathrm{Lip}^{\alpha}_{H}(G,d)$ (resp., $\mathrm{Lip}^{k,\alpha}_{H}(G,d)$), this can be strengthened in the following

THEOREM 2.4. *The Gel'fand mapping $f \mapsto \hat{f}$ is an isomorphism of $\mathrm{Lip}^{\alpha}_{H}(G,d)$ (resp., of $\mathrm{Lip}^{k,\alpha}_{H}(G,d)$) onto the closed subalgebra of $\mathrm{Lip}^{\alpha}(M(A),\sigma)$ (resp., of $\mathrm{Lip}^{k,\alpha}(M(A),\sigma)$) consisting of those functions in $\mathrm{Lip}^{\alpha}(M(A),\sigma)$ (resp., in $\mathrm{Lip}^{k,\alpha}(M(A),\sigma)$) that are continuous (resp., C^k-functions) on $M(A)$ (resp., on Bdry$(M(A))$) and are holomorphic in $M(A)$ w.r.t. the σ-topology. In particular, for each $\alpha \in (0,1]$ (resp., for each order $0 \le k < \infty$ and for each $\alpha \in (0,1]$), $\mathrm{Lip}^{\alpha}_{H}(G,d)$ (resp., $\mathrm{Lip}^{k,\alpha}_{H}(G,d)$) is a commutative semi-simple Banach algebra.*

Proof. First, we prove the theorem for $A = \mathrm{Lip}^{\alpha}_{H}(G,d)$. If $f \in \mathrm{Lip}^{\alpha}_{H}(G,d)$ and $\|f\|_{\mathrm{Lip}^{\alpha}} \le 1$, then $\|f\|_{\alpha} \le 1$, so $|f(z) - f(w)| \le d(z,w)^{\alpha}$ for all $z, w \in G$. Thus $\sigma(z,w) \le d(z,w)^{\alpha}$ for all $z, w \in G$. Hence for any $f \in \mathrm{Lip}^{\alpha}_{H}(G,d)$, $\|f\|_{\alpha} \le |||\hat{f}|||_{\alpha}$. Since each $\hat{f}, f \in \mathrm{Lip}^{\alpha}_{H}(G,d)$, is continuous on $M(A)$ w.r.t. the Gel'fand topology and since G is dense in $M(A)$ w.r.t. the Gel'fand topology by Lemma 2.2, we have $\|f\|_{\infty} = |||\hat{f}|||_{\infty}$. Thus for all $f \in \mathrm{Lip}^{\alpha}_{H}(G,d)$,

$$\|f\|_{\mathrm{Lip}^{\alpha}} = \|f\|_{\infty} + \|f\|_{\alpha} \le |||\hat{f}|||_{\infty} + |||\hat{f}|||_{\alpha} = |||\hat{f}|||_{\mathrm{Lip}^{\alpha}}.$$

This together with the inequality from [47, Proposition 2.1] gives

$$\|f\|_{\mathrm{Lip}^{\alpha}} \le |||\hat{f}|||_{\mathrm{Lip}^{\alpha}} \le 2\|f\|_{\mathrm{Lip}^{\alpha}}.$$

Hence the mapping $f \mapsto \hat{f}$ is a bicontinuous isomorphism, and the image of $\mathrm{Lip}^{\alpha}_{H}(G,d)$ is therefore a closed subalgebra of $\mathrm{Lip}^{\alpha}(M(A),\sigma)$.

Let $g \in \mathrm{Lip}^{\alpha}(M(A),\sigma)$ be continuous on $M(A)$ and holomorphic in $M(A)$ w.r.t. the Gel'fand topology and let $f = g|G$ denote the restriction of g to G. Then $f \in \mathrm{Lip}^{\alpha}_{H}(G,d)$ as $\sigma(z,w) \le d(z,w)^{\alpha}$ for all $z, w \in G$; and $\hat{f} = g$ since both are continuous on $M(A)$ and holomorphic in $M(A)$ w.r.t. the Gel'fand topology and agree on the dense subset G by Lemma 2.2. Thus those $g \in \mathrm{Lip}^{\alpha}(M(A),\sigma)$ which are continuous on $M(A)$ and holomorphic in $M(A)$ w.r.t. the Gel'fand topology lie in the range of the mapping $f \mapsto \hat{f}$ from $\mathrm{Lip}^{\alpha}_{H}(G,d)$. Since every $\hat{f}$ is continuous on $M(A)$ and holomorphic in $M(A)$ w.r.t. the Gel'fand topology, the image of $\mathrm{Lip}^{\alpha}_{H}(G,d)$ under the mapping $f \mapsto \hat{f}$ is exactly the set of functions in $\mathrm{Lip}^{\alpha}(M(A),\sigma)$ which are continuous on $M(A)$ and holomorphic in $M(A)$ w.r.t. the Gel'fand topology, as well as w.r.t. the σ-topology by Proposition 2.1.

The similar arguments together with norm inequalities in the brackets in the preceding paragraph of Theorem 2.4, apply for $A = \mathrm{Lip}_H^{k,\alpha}(G,d)$, recalling that, for a fixed order $0 \leq k < \infty$ and a fixed exponent $\alpha \in (0,1]$, we have the Lipschitz algebra $\mathrm{Lip}^{k,\alpha}(M(A),\sigma)$ (see Remark A above). ∎

For the remaining part of the section, $k \neq \infty$. As a subset of $M(A)$, where $A = \mathrm{Lip}_H^{k,\alpha}(G,d)$, G inherits the Gel'fand and the metric topologies of $M(A)$. We now compare these inherited topologies of G with its original d-topology. We first see that when $k = 0$, the relative Gel'fand topology of G and the d-topology of G are equivalent by Lemma 2.2 and Proposition 2.3.

We now compare the two metric topologies of G. The next two propositions are concerned with the relation between d and σ on G. We omit the proofs in our case as the arguments are almost same (see [47, §3] for notions, definitions, their interrelations and equivalence properties). Also the results on the homomorphisms and automorphisms, obtained in Section 5 (Theorem 5.1 and Corollary 5.2), hold true in our cases: (1) $\mathrm{Lip}^{k,\alpha}(X,d')$, X a compact Hausdorff space; and (2) $\mathrm{Lip}_H^{k,\alpha}(G,d)$, G relatively compact in $\mathbb{C}^n$.

Since $M(A)$ lies on the unit sphere of the dual space of $A = \mathrm{Lip}_H^{\alpha}(G,d)$ (which is, indeed, the space $H\Lambda^*(\alpha,\infty,\infty)$ by [14] when G is a bounded symmetric domain in $\mathbb{C}^n$), the diameter of $(M(A),\sigma)$ is at most two. Thus σ is always a bounded metric. Moreover, since G is a bounded domain in $\mathbb{C}^n$, the diameter of (G,d) is finite and we have the following

PROPOSITION 2.5. *The metric σ on G is boundedly equivalent to d.*

As a consequence, we have the following

COROLLARY 2.6. *Let d_1 and d_2 be bounded metrics on G. Then $A_1 = \mathrm{Lip}_H^{k,\alpha}(G,d_1)$ and $A_2 = \mathrm{Lip}_H^{k,\alpha}(G,d_2)$ have the same elements if and only if d_1 and d_2 are boundedly equivalent.*

Moreover, we have the following

PROPOSITION 2.7. *Given the metric space (G,d), the Banach algebras $\mathrm{Lip}_H^{k,\alpha}(G,d)$ and $\mathrm{Lip}_H^{k,\alpha}(G,d')$, where $d' = \frac{d}{1+d}$, have the same elements and their norms are equivalent.*

As a consequence, we have the following

COROLLARY 2.8. *The metrics d and σ on G are always uniformly equivalent.*

3. Equivalency of the two problems

As discussed in Section 1, we first show that results analogous to [16, Theorem 5 and Theorem 6] hold true for $\mathrm{Lip}_H^{k,\alpha}(G,d)$ and $A^k(\overline{G})$, where G is a certain "pseudoconvex" domain in $\mathbb{C}^n$ [50, 45] as mentioned in the preceding paragraph of Lemma 2.2. This is possible because of [16, Theorem 5 and Theorem 6] and

Lemma 3.1. *Let G be a "pseudoconvex" domain in $\mathbb{C}^n$. Then, for $0 \leq k \leq \infty$ and for $\alpha \in (0,1]$, $\mathrm{Lip}_H^{k,\alpha}(G,d)$ and $A^k(\overline{G})$ are dense Banach (resp., Fréchet, when $k=\infty$) subspaces of the Bergman space $B(G)$.*

Proof. Recall that the inclusion mappings in (1) in Section 2 are continuous w.r.t. the usual topologies on these spaces (e.g., the second inclusion is continuous, as discussed in the proof of Lemma 2.2). This implies that

$$\mathrm{Lip}_H^{k,\alpha}(G)\bigcap L^2(G) \subset A^k(\overline{G})\bigcap L^2(G) \subset H^\infty(G)\bigcap L^2(G) \subset B(G),$$

where $B(G)$ is the Bergman space of a "pseudoconvex" domain G. The polynomials in $z_1, z_2, \dots, z_n$ are dense in $B(G)$ since $B(G)$ is isometrically isomorphic to the weighted space $\ell^\infty\big((\mathbb{Z}^+)^n, \frac{1}{|N|+1}\big)$ (see [42] for the analogous notion of the Beurling-Banach algebra $\ell^1\big((\mathbb{Z}^+)^n, \omega\big)$ of weight type). Moreover, the polynomials in $z_1, z_2, \dots, z_n$ are also in $\mathrm{Lip}_H^{k,\alpha}(G,d)$. Since each function of $\mathrm{Lip}_H^{k,\alpha}(G,d)$ (resp., of $A^k(\overline{G})$) is uniformly continuous on G, it extends uniquely and in a norm preserving way to a continuous function on $\overline{G}$ (see Section 2), it is obviously square-integrable on $\overline{G}$. Hence $f \in L^2(G)$ whenever $f \in \mathrm{Lip}_H^{k,\alpha}(G,d)$ (resp., $f \in A^k(\overline{G})$), and so, $f \in B(G)$. Thus both $\mathrm{Lip}_H^{k,\alpha}(G,d)$ and $A^k(\overline{G})$ are dense Banach (resp., Fréchet, when $k=\infty$) subspaces of the Bergman space $B(G)$ (certainly, both are not closed subspaces of $B(G)$ for an obvious reason). ∎

Thus we have the following crucial theorem, extending results of [16, 32]. The method of proof will be used again in the proof of Theorem 3.3; also, some parts of the proof share similar trick that we used to get certain closed ideals algebraically finitely generated in the proof of [42, Theorem 3.7].

Theorem 3.2. *Let G be a "pseudoconvex" domain in $\mathbb{C}^n$, and let $0 \leq k \leq \infty$ and $\alpha \in (0,1]$. Suppose $f \in \mathrm{Lip}_H^{k,\alpha}(G,d)$ (resp., $f \in A^k(\overline{G})$) and $f(\lambda) = 0$ for some $\lambda \in G$. Then there exists $f_1, f_2, \dots, f_n$ in $\mathrm{Lip}_H^{k,\alpha}(G,d)$*

(resp., in $A^k(\overline{G})$) so that $f = \sum_{j=1}^{n}(z_j - \lambda_j)f_j$. In particular, G is Gleason solvable for $\mathrm{Lip}_H^{k,\alpha}(G,d)$ and $A^k(\overline{G})$.

Proof. Fix $k = \infty$. By Lemma 3.1, $f \in A^\infty(\overline{G}) \bigcap B(G)$. Also, by Remarks B. 3, M_λ is a maximal ideal in $\mathrm{Lip}_H^{\infty,\alpha}(G,d)$ containing f. It is enough to show that M_λ is algebraically finitely generated maximal ideal, generated by the shifted coordinate functions $z_j - \lambda_j, j = 1, 2, \ldots, n$. By [16, Theorem 6], there exists $f_1, f_2, \ldots, f_n$ in $A^\infty(\overline{G}) \bigcap B(G)$ so that $f = \sum_{j=1}^{n}(z_j - \alpha_j)f_j$. This shows that the maximal ideal M_λ' in $A^\infty(\overline{G})$ is algebraically finitely generated maximal ideal, generated by the functions $z_j - \lambda_j, j = 1, 2, \ldots, n$ (equivalently, this theorem holds true for $A^\infty(\overline{G})$).

The polynomials in $z_1, z_2, \ldots, z_n$ are dense in $A^\infty(\overline{G})$, and are in $\mathrm{Lip}_H^{\infty,\alpha}(G,d)$. Hence, $\mathrm{Lip}_H^{\infty,\alpha}(G,d)$ is a dense Fréchet subalgebra of $A^\infty(\overline{G})$ such that the inclusion mapping is a continuous homomorphism due to (1) in Section 2. So, by [26, Lemma 3.2.5], [50, Theorem 4.1] and Remarks B. 3, the adjoint spectral mapping $\mathrm{in}^* : M\big(A^\infty(\overline{G})\big) = \overline{G} \to M\big(\mathrm{Lip}_H^{\infty,\alpha}(G,d)\big) = \overline{G}$ is injective and continuous w.r.t. the Gel'fand topology, as well as w.r.t. both the metric topologies by Remarks B. 2; it is easy to show that the Gel'fand topology and d-topology on $M\big(A^\infty(\overline{G})\big) = \overline{G}$ are equivalent. Indeed, in^* is bijective (which would suffice here), and so it is a homeomorphism.

Since M_λ' is algebraically finitely generated maximal ideal in $A^\infty(\overline{G})$, and $M_\lambda = M_\lambda' \bigcap \mathrm{Lip}_H^{\infty,\alpha}(G,d)$ is the corresponding element in the above bijection, we see that M_λ is clearly algebraically finitely generated maximal ideal in $\mathrm{Lip}_H^{\infty,\alpha}(G,d)$, generated by the functions $z_j - \lambda_j, j = 1, 2, \ldots, n$. Thus there exists $f_1, f_2, \ldots, f_n$ in $\mathrm{Lip}_H^{\infty,\alpha}(G,d)$ so that $f = \sum_{j=1}^{n}(z_j - \lambda_j)f_j$ whenever $f(\lambda) = 0$ for some $\lambda \in G$. In particular, since λ is arbitrary in G, G is Gleason solvable for $\mathrm{Lip}_H^{\infty,\alpha}(G,d)$.

Now, let $0 \le k < \infty$. We show that the maximal ideal of $\mathrm{Lip}_H^{k,\alpha}(G,d)$ is algebraically finitely generated by the functions $z_j - \lambda_j, j = 1, 2, \ldots, n$. For this, note that by the standard arguments of the Arens-Michael representation of the Fréchet algebra $\mathrm{Lip}_H^{\infty,\alpha}(G,d)$, $\mathrm{Lip}_H^{\infty,\alpha}(G,d)$ is a dense subalgebra of $\mathrm{Lip}_H^{k,\alpha}(G,d)$ such that the inclusion mapping is a continuous homomorphism due to (2) in Section 2. So, by [26, Lemma 3.2.5], Lemma 2.2 and Remarks B. 3, the adjoint spectral mapping $\mathrm{in}^* : M\big(\mathrm{Lip}_H^{k,\alpha}(G,d)\big) = \overline{G} \to M\big(\mathrm{Lip}_H^{\infty,\alpha}(G,d)\big) = \overline{G}$ is a continuous injection w.r.t. the Gel'fand topology, as well as w.r.t. both the metric topologies by Remarks B. 2. It is easy to show that the Gel'fand topology and d-topology on $M\big(\mathrm{Lip}_H^{\infty,\alpha}(G,d)\big) = \overline{G}$ are equivalent by Remarks B. 3. Indeed,as above, in^* is a homeomorphism.

Since the restriction of in* to G is one-to-one over G, the inverse mapping maps G onto a subset G of $\overline{G}$, and so, the inverse image of M_λ, denoted by M_λ again, is also a maximal ideal of $\mathrm{Lip}_H^{k,\alpha}(G,d)$. To show that it is, indeed, finitely generated, we remark that the finitely generated maximal ideal M_λ of $\mathrm{Lip}_H^{\infty,\alpha}(G,d)$ is, in fact, an ideal of $\mathrm{Lip}_H^{k,\alpha}(G,d)$. By the Zorn's lemma, the maximal element of the family $\mathcal{I}$ containing finitely generated ideals of $\mathrm{Lip}_H^{k,\alpha}(G,d)$ consisting of functions in $\mathrm{Lip}_H^{k,\alpha}(G,d)$ vanishing at λ in G (note that $\mathcal{I}$ is non-empty, since $M_\lambda \in \mathcal{I}$), is an algebraically finitely generated maximal ideal. In fact, this maximal element is not only a primary ideal, but is obviously contained in the maximal ideal M_λ of $\mathrm{Lip}_H^{k,\alpha}(G,d)$. From these facts, we conclude that the maximal element is, indeed, the maximal ideal M_λ of $\mathrm{Lip}_H^{k,\alpha}(G,d)$. In particular, G is Gleason solvable for $\mathrm{Lip}_H^{k,\alpha}(G,d)$.

Finally, we show that the maximal ideal of $A^k(\overline{G})$, where $k<\infty$, is algebraically finitely generated by the functions $z_j-\lambda_j, j=1,2,\dots,n$. For this, one repeats the arguments in the above paragraph by noting that the maximal ideal M_λ, $\lambda\in G$, of $A^\infty(\overline{G})$ is algebraically finitely generated, as discussed in the proof above. This completes our proof. ∎

Remark C. In [17], it was not evident what relationship there was between the condition that G be open in $M(A)$ w.r.t. Gel'fand topology, where A is either $H^\infty(G)$ or $A(G)$, and the condition that G be an L^∞-pseudoconvex domain. However, Clos showed the one-way implication (see [16, Theorem 4] and the fact, discussed in [17], that G is open in $M(A)$ if it is Gleason solvable). In the above theorem, we establish the one-way implication for $\mathrm{Lip}_H^{k,\alpha}(G,d)$ and $A^k(\overline{G})$, G a smooth *strictly* pseudoconvex domain in $\mathbb{C}^n$ containing an L^∞-pseudoconvex domain, since Theorem 3.2 extends the result of Clos. In fact, there are smooth pseudoconvex domains in $\mathbb{C}^n$ for which L^∞-estimates for $\bar\partial_b$ do not hold and so, uniform and Lipschitz estimates for $\bar\partial_b$ do not hold; see [10, 22, 50]. Thus, the above theorem is a significant generalization of results analogous to [16, Theorem 4 and Theorem 6] for these algebras. One reason for interest in this discussion stems from an application to the commuting Toeplitz operators problem on Bergman spaces of pseudoconvex domains in $\mathbb{C}^n$ (see [16, p. 2] for his comment).

For a "polydomain" G (including the polydisc in $\mathbb{C}^n$), obtained as a product of smooth strictly pseudoconvex domains, we extend the above theorem a bit in the following

THEOREM 3.3. *Let G be a "polydomain" in $\mathbb{C}^n$. Then G is Gleason solvable for* $\mathrm{Lip}_H^{k,\alpha}(G,d)$ *and* $A^k(\overline{G})$, *where* $0\le k\le\infty$ *and* $\alpha\in(0,1]$.

Proof. Recall that when G is a "polydomain" in $\mathbb{C}^n$, the Gleason solvability is easily available through the analogous decomposition, discussed in Section 1, by fixing all but one variable method. Jakóbczak in [30] discussed the decomposition of operators for $A(G)$, where G is a product of smooth strictly pseudoconvex domains. First, let $0 \leq k < \infty$. Using his result and the method of proof of [43, Lemma 3.1 (iii)], it is easy to derive the same theorem for $\mathrm{Lip}_H^{k,\alpha}(G,d)$ (resp., for $A^k(\overline{G})$), where G is as in [30], as follows. First, we note that the linear operators $T_j, j = 1, 2, \ldots, n$, in [30] are, indeed, the multiplication operators by the shifted coordinate functions $z_j - \lambda_j$, $j = 1, 2, \ldots, n$, on $A(G)$; this is also clear from the method of proof of [43, Lemma 3.1 (iii)]. Moreover, the second inclusion mapping is a continuous linear operator due to (1) in Section 2, as well as $A^k(\overline{G})$ is also continuously and densely embedded in $A(G)$. Now if $f \in \mathrm{Lip}_H^{k,\alpha}(G,d)$, then the functions $T_j^{'}(f)$, $j = 1, 2, \ldots, n$, are also in $\mathrm{Lip}_H^{k,\alpha}(G,d)$ since it is a multiplication operator by the function $z_j - \lambda_j$, and $\mathrm{Lip}_H^{k,\alpha}(G,d)$ is a dense Banach subalgebra in $A(G)$ by (2) in Section 2 and a result analogous to Lemma 3.1 above for these two Banach algebras. Thus we have continuous linear operators $T_j^{'}, j = 1, 2, \ldots, n$, on $\mathrm{Lip}_H^{k,\alpha}(G,d)$. Hence we have the Gleason solvability of "polydomains" in $\mathbb{C}^n$ for $\mathrm{Lip}_H^{k,\alpha}(G,d)$.

To obtain the Gleason solvability of "polydomains" in $\mathbb{C}^n$ for Banach algebras $A^k(\overline{G})$, one repeats the above arguments, with mild changes at a few places as follows. Instead the second inclusion mapping, the first inclusion mapping in (2) in Section 2, but for $A^k(\overline{G})$, is a continuous linear operator. Now if $f \in A^k(\overline{G})$, then the functions $T_j^{'}(f)$, $j = 1, 2, \ldots, n$, are also in $A^k(\overline{G})$. Thus we have continuous linear operators $T_j^{'}$, $j = 1, 2, \ldots, n$, on $A^k(\overline{G})$. Similarly, we also obtain the Gleason solvability of "polydomains" in $\mathbb{C}^n$ for Fréchet algebras $\mathrm{Lip}_H^{\infty,\alpha}(G,d)$ and $A^\infty(\overline{G})$ by using the method of proof of [43, Main Theorem (iii)] and (2) in Section 2. ∎

We recall that we used the *corona theorem* as a hypothesis to solve Gleason's problem for locally Stein algebras in [43]. However, it was not evident what relationship there is between the *corona theorem* and the *Gleason solvability property*; i.e., it was not obvious whether either of these problems implies the other. Also, Krantz and Li in [35] constructed *explicit linear* solutions for the corona problem with Lipschitz data in the polydisc by using iteration of one variable techniques. So we study Lipschitz algebras of holomorphic functions on (G,d), G a certain domain in $\mathbb{C}^n$ (including the unit ball and polydisc) for two reasons: (1) we establish an equivalency between the corona problem and Gleason's problem by first showing that the Gleason solvability, obtained above, establishes the corona theorem below, and then, by showing that the corona theorem implies the Gleason solvability in Theorem 3.6 below; and (2) we show the existence of *non-linear* solutions for the corona problem with C^k-data, uniform data and Lipschitz data (as well as, combination of these data) in $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G,d)$, G a certain bounded domain in $\mathbb{C}^n$, in the below theorem. We use the Banach algebra method.

Also we recall that there are smooth pseudoconvex domains G in $\mathbb{C}^2$ and $\mathbb{C}^3$, for which the corona theorem fails for $H^\infty(G)$ [18, 22, 49]. These domains are of type EL^∞; i.e., they are not H^∞ domain of holomorphy [33, §4]. However, for the same domains G (including the Hartogs triangle T and worm domain W), the corona theorem holds for $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G,d)$, where $0 \le k \le \infty$ and $\alpha \in (0,1]$ by [50, Theorem 4.1] and Remarks B. 3. This exhibits a significant difference between these two classes of algebras and the algebra H^∞. That is, there exists a holomorphic function f in $A^k(\overline{G})$ (resp., in $\mathrm{Lip}_H^{k,\alpha}(G,d)$) which cannot be analytically continued to a larger domain; i.e., these domains are NOT of type EA^k, and are NOT of type $E\,\mathrm{Lip}_H^{k,\alpha}$ (both defined analogously; see [33, §4] and [50, §4]).

THEOREM 3.4. *Let G be either a "pseudoconvex" domain or else a "polydomain" in $\mathbb{C}^n$. Then the corona problem holds true in $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G,d)$, where $0 \le k \le \infty$ and $\alpha \in (0,1]$. That is, the spectrum $M\big(A^k(\overline{G})\big)$ (resp., $M\big(\mathrm{Lip}_H^{k,\alpha}(G,d)\big)$ for each $0 \le k \le \infty$ and for each $\alpha \in (0,1]$) is the Gel'fand closure of (the canonical image of) G.*

Proof. To show that the corona problem has an affirmative solution for G, we use its algebraic formulation, namely, a finite *corona data* from $A^k(\overline{G})$, aka C^k *data* (resp., from $\mathrm{Lip}_H^{k,\alpha}(G,d)$, aka C^k*-Lipschitz data*) implies the existence of the solution of the Bézout equation in $A^k(\overline{G})$ (resp., in $\mathrm{Lip}_H^{k,\alpha}(G,d)$). In fact, we show that if there does not exist a solution of the Bézout equation in $A^k(\overline{G})$ (resp., in $\mathrm{Lip}_H^{k,\alpha}(G,d)$), then there are no finite C^k (resp., C^k-Lipschitz) data available. To this effect, we recall that there is a one-to-one correspondence between the (closed) maximal ideals and the elements of the spectrum of the Banach (Fréchet) algebra A. As an easy consequence of this fact, we further know that the algebraically finitely generated ideal, generated by $f_1, \dots, f_n \in A$, is proper if and only if

$$\big\{\phi \in M(A) :\ \phi(f_j) = 0,\ j = 1, \dots, n\big\} \neq \emptyset. \tag{3}$$

When A is a Fréchet algebra, we apply [26, Corollary 6.1.10]; also, since Fréchet algebras $A^\infty(\overline{G})$ and $\mathrm{Lip}_H^{\infty,\alpha}(G,d)$ are finitely generated Fréchet algebras, generated by the coordinate functions $z_1, z_2, \ldots, z_n$ (see the proof of Theorem 3.6 below for details), the maximal ideals in these algebras are closed by [1].

In our case, since G is Gleason solvable, by Theorem 3.2 and Theorem 3.3, it turns out that M_λ, $\lambda \in G$, is algebraically finitely generated maximal ideal if and only if (3) holds for $A^k(\overline{G})$ (resp., for $\mathrm{Lip}_H^{k,\alpha}(G,d)$) if and only if there are no $g_1, g_2, \ldots, g_n \in A^k(\overline{G})$ (resp., $\mathrm{Lip}_H^{k,\alpha}(G,d)$) such that $\sum_{j=1}^n f_j g_j = 1$, where $f_j, j = 1, 2, \ldots, n$, are the generators of M_λ. Thus there does not exist a solution of the Bézout equation in $A^k(\overline{G})$ (resp., in $\mathrm{Lip}_H^{k,\alpha}(G,d)$) if and only if there are NO finite C^k (resp., C^k-Lipschitz) data available. That means, $\lambda \in G$ is such that for every $\delta > 0$, $\sum_{j=1}^n |f_j(\lambda)| < \delta$, which implies that $\sum_{j=1}^n |f_j(\lambda)| = 0$ if and only if $f_j(\lambda) = 0$ for $j = 1, 2, \ldots, n$. Hence the corona problem holds true in $A^k(\overline{G})$ (resp., in $\mathrm{Lip}_H^{k,\alpha}(G,d)$), where G is either a "pseudoconvex" domain or a "polydomain" in $\mathbb{C}^n$. ∎

Remark D. It is interesting to note that author lately came to know about the IDEAL PROBLEM, explained in the history of the corona problem (see [19, §7]), as well as came to know about the method of proof of [3, Lemma 2.3], establishing the I.J. Schark's theorem for the Banach algebra $H^\infty(B_X)$ (or its closed subalgebra $A_u(B_X)$), B_X the open unit ball of a complex Banach space X. So our Banach-algebraic technique is purely based on some well-known results from the Banach algebra theory as well as author's recent work on Gleason's problem in [43]. However, some more comments are in order.

It is easy to see that our method here is similar to that of Xiao's theorem on the multipliers of the Dirichlet space $\mathcal{D}$ in the unit disc (see [19, p. 22]) where it is discussed that the corona data is necessary for the corona theorem to hold (see also a remark on an alternative proof which also deals with infinitely many generators on [19, p. 23]). Further it is easy to see that we have, in fact, solved the Rubel and Shields' question for the function $\phi \equiv 1$ in $\mathrm{Lip}_H^{k,\alpha}(G,d)$ (and so, in $A^k(\overline{G})$ by (1) in Section 2), where G is as above, including the ball and polydisc in the below corollary (see [19, §7, Questions 1, 2 and 3], but for $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G,d)$; for Questions 2 and 3, we take $h \equiv 1$ on $[0,\infty]$ and $\phi \equiv 1$ in the above theorem; note that, to answer Question 3, we consider a maximal ideal $M_\lambda, \lambda \in G$, generated by $f_1, f_2, \ldots, f_n$ in the above theorem).

Since the ball and polydisc are (essentially the only) basic examples of a pseudoconvex/poly domain in $\mathbb{C}^n$ of our kind, we have the following

COROLLARY 3.5. *Let G be either an open unit ball or a polydisc in $\mathbb{C}^n$. Then the corona problem holds true in $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G,d)$, where $0 \le k \le \infty$ and $\alpha \in (0,1]$.*

We have already shown the Gleason solvability of a certain G for $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G,d)$ in Theorem 3.2 and Theorem 3.3. However, as a part of our goal to establish an equivalency between the two problems, in the converse direction, we have the following

THEOREM 3.6. *Let G be either a relatively compact "pseudoconvex" domain or else a "polydomain" in $\mathbb{C}^n$. Then G is Gleason solvable for $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G,d)$ for $0 \le k \le \infty$ and $\alpha \in (0,1]$, provided that the corona problem holds true in these algebras.*

Proof. First, we refer to the definitions of Banach/Fréchet algebras of power series (briefly, BAPS/FrAPS) in several variables (resp., of finitely generated Fréchet algebras) from [42, §1] (resp., from [26, Chapter 5]). Clearly, by identifying a function with its Taylor series expansion, all the Banach/Fréchet algebras in (1) and (2) in Section 2 are BAPS/FrAPS in several variables, since the coordinate projections are continuous functionals on these algebras by the maximum modulus principle and Cauchy inequalities. Also one follows discussion, given in the preceding paragraph to [43, Proposition 2.3], to claim that all these algebras (except $H^\infty(G)$) are finitely generated by the polynomials in $z_1, z_2, \ldots, z_n$, where some results from [44] and [42, Theorem 3.1] have been used. Thus $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G,d)$ are semi-simple, finitely generated BAPS/FrAPS in $z_1, z_2, \ldots, z_n$ such that the polynomials in $z_1, z_2, \ldots, z_n$ are dense. Now, since they satisfy the hypotheses of the main theorem (resp., of Lemma 3.1 in the Banach case) of [43], by these results, they have an analytic variety at ϕ_λ, $\lambda \in G$, where $M_\lambda = \ker \phi_\lambda$. So the Gel'fand representations of these algebras have analytic structures in each point ϕ_λ, $\lambda \in G$ (for $\mathrm{Lip}_H^{k,\alpha}(G,d)$, we use Theorem 2.4 and Remarks B. 3 above). Then it is easy to show that $A^k(\overline{G})$ (resp., $\mathrm{Lip}_H^{k,\alpha}(G,d)$) is a locally Stein algebra, since the completion of the polynomials (resp., of $\mathrm{Lip}_H^{k,\alpha}(G,d)$) under the C^k-norm(s) is $A^k(\overline{G})$ while applying proof of (ii) implied (i) of [43, Theorem 4.1]. Hence, by [43, Theorem 4.3] (when applied to $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G,d)$, resp.), we see that Gleason's problem is solvable, because G is dense in $M\big(A^k(\overline{G})\big)$ by [50, Theorem 4.1] (resp., in $M\big(\mathrm{Lip}_H^{k,\alpha}(G,d)\big)$ by Lemma 2.2). ∎

Remarks E. 1. From Lemma 2.2, Theorem 3.2 and Theorem 3.3, we see that the solutions of the two prestigious problems in the theory of SCV stand independently. However, from Theorem 3.4 and Theorem 3.6, it is clear that both the problems are equivalent for $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G,d)$. Moreover the corona theorem does imply the I.J. Schark's theorem (see 4.2 below for details on this theorem) for $\mathrm{Lip}_H^{k,\alpha}(G,d)$ and $A^k(\overline{G})$, where G is a certain pseudoconvex/poly domain in $\mathbb{C}^n$, giving results for these algebras (see [23, 38] for the same results for $H^\infty(G)$, G a polydomain and a strongly pseudoconvex domain, resp.).

In fact, the corona problem has a topological nature (i.e., whether the domain G is dense in the spectra of these Banach/Fréchet algebras w.r.t. the Gel'fand topology) whereas the Gleason's problem has an algebraic nature (i.e., whether a maximal ideal $M_\lambda, \lambda \in G$, is algebraically finitely generated in these algebras; equivalently, whether G has the Gleason $A^k(\overline{G})$-property (resp., the Gleason $\mathrm{Lip}_H^{k,\alpha}(G)$-property)). The corona problem also has an algebraic nature (i.e., given a finite C^k (resp., C^k-Lipschitz) data whether the Bézout's equation holds true in these algebras, respectively). In fact, we recall a pleasant result of Żelazko [52, Proposition 2], showing a possible connection between these two problems through their algebraic formulations.

2. [21] is a good reference; in the final paragraph of Section 1, the authors state the main theorem can still be proved by replacing $A(G)$ by various holomorphic Hölder- and Lipschitz-spaces and by replacing the coordinate functions as the generators by arbitrary generators of the maximal ideal in these spaces. In this connection, we consider an application of [43, Theorem 4.3] in the above proof. In Lemma 2.2, we have established the corona theorem in a stronger sense, and as an application of these results, we have here established the Gleason solvability of certain pseudoconvex/poly domains in $\mathbb{C}^n$ for Lipschitz algebras (Theorem 3.2 and Theorem 3.3). Thus this answers affirmatively the question left open by Fornæss and Øvrelid in [21] for the holomorphic Hölder and Lipschitz spaces on pseudoconvex domains of finite type in $\mathbb{C}^2$ (and in particular, such domains with a real analytic boundary).

4. Concluding remarks on our approach and open problems

Above, we solve the corona problem (algebraic formulation) by considering the approach of solving the IDEAL PROBLEM (which is of algebraic nature). Thus this approach clearly shows the possible connection between the corona problem and Gleason's problem in the SCV case (cf. [52, Proposition 2]). However, until we consider this connection for the first time in [43, Theorem 4.3], as far as we know, no analysts did take this route, and it is not a great surprise for this for obvious reasons. Then, our extension of the results analogous to the results of Clos also plays a significant role while filling the gap via the Gleason solvability of certain "pseudoconvex/poly" domains G for $A^k(\overline{G})$ and $\mathrm{Lip}_H^{k,\alpha}(G,d)$ when one wants to take the approach of solving the ideal problem by using the Banach algebra method for these algebras. This actually provides the necessary and sufficient conditions for the ideal problem [19, §7]. Thus we have a very strong connection between the two significant problems in the theory of SCV through this equivalence. Hence this paper may be treated as the test case for solving the most prestigious, long-standing, original corona problem for $H^\infty(G)$, G certain pseudoconvex/poly domains in $\mathbb{C}^n$ (including the ball and polydisc) by applying the Gleason solvability of these domains for $H^\infty(G)$, an equivalency of these two problems and the Banach algebra method. Moreover, the solvability of both the problems for Lipschitz algebras of holomorphic functions may provide us a concrete base for applying a combination of the classical dead-end space method and the interpolation (by bounded holomorphic functions) method to get the solutions of these two problems for various familiar Banach spaces of bounded holomorphic functions (e.g., (Besov-)Lipschitz spaces) on bounded symmetric domains in $\mathbb{C}^n$ (including Hardy-Sobolev spaces, holomorphic mean (Besov-)Lipschitz spaces and weighted Bergman spaces on the unit ball).

Open problems. We now discuss a couple of open problems in view of our approach as follows.

4.1. Throughout the paper, we have considered certain domains G in $\mathbb{C}^n$ only. As commented in Section 1, our functional analytic method from [43] has been applied to deduce the Gleason A-property of a semi-simple locally Stein algebra in Theorem 4.3 or Corollary 4.4, where Y may be a domain (i.e., a (reduced) Stein space) from the complex manifolds (of finite dimension). Moreover the Banach algebra method is also available for Fréchet algebras [26, Corollary 6.1.10]. Hence it is of interest to solve the corona problem for such domains/(reduced) Stein spaces from the complex manifolds of finite dimension for various locally Stein-Banach algebras of bounded holomorphic functions. Recall that if G is a relatively compact domain in a Riemann surface M, then $M(A(G)) = \overline{G}$, and so is true if G is relatively compact such that $\overline{G}$ is an S_δ set (equivalently, G has a Stein neighbourhood basis) in a complex analytic manifold M (see [45, Theorem 2.12]).

4.2. It is worthwhile mentioning some work that is somewhat (tangentially) related to the work given here (see [2, 3, 4, 5, 6, 44] for details). For example, they study the Gleason parts of spectra of these algebras when $X = c_0$ in [5], and study the I.J. Schark's theorem for Banach algebras $A_u(B_X)$ and $H^\infty(B_X)$, where $X = c_0$, ℓ^1, ℓ^2, in [3, 4], and study the same theorem for the Fréchet algebra $H_b(X)$, where $X = \ell^1, \ell^2$ in [4]. We establish the I.J. Schark's theorem for Lipschitz algebras; see Remarks E. 1. To establish a relation between their work with the current work, we below conjecture that the questions about the I.J. Schark's theorem may have an affirmative solution.

In [43, 4.4], we have raised the Gleason solvability/Gleason's problem in the infinite dimensional case for the Banach algebras $H^\infty(B_X)$ and $\mathcal{A}_u(B_X)$ (a closed subalgebra of $H^\infty(B_X)$), where B_X is the open unit ball of a complex Banach space X. It is interesting to further develop our approach (i.e., the functional analytic method and an equivalency of the two problems) so as to solve these problems for Banach/Fréchet algebras $\mathrm{Lip}_H^{k,\alpha}(B_X, d)$ and $A^k(\overline{B_X})$ (defined analogously). One reason for interest in solving the corona problem in the infinite dimensional case stems from the point of view of the I.J. Schark's Cluster Value Theorem in this case. The authors in [3] concentrated only on obtaining several cluster value theorems in this case, leaving to prove this theorem in the case $H^\infty(B_X)$, where $X = \ell^2$ (see below).

Recall that if $f \in H^\infty(B_X)$ and $x \in \overline{B_{X^{**}}}$, the cluster set of f at x is the set $\mathrm{Cl}(f,x) := \{w \in \mathbb{C} : \exists (x_n) \subset B_X \text{that converges weak-star to } x, f(x_n) \to w\}$. Then the *Cluster Value Theorem* is: For f and x as above,

$$\mathrm{Cl}(f,x) = \{\phi(f) : \pi(\phi) = x\} = \hat{f}(M_x),$$

where $\pi : M\big(H^\infty(B_X)\big) \to \mathbb{C}$ and the fiber of $M\big(H^\infty(B_X)\big)$ over $x \in \overline{B_{X^{**}}}$ to be $M_x = \pi^{-1}(x)$. In [3], this theorem is proved for $X = c_0$. We conjecture that the theorem holds true for the case $X = \ell^2$ as well (note that they could not prove it because the crucial lemma, proved for the case $X = c_0$, fails in this case). This theorem is also known as the "poor man's/weak corona theorem", since if the corona theorem holds true, then the I.J. Schark's theorem also holds, but the converse may not hold. Thus, based on the results, obtained here, we now have a road map to establish the I.J. Schark's theorem for $\mathrm{Lip}_H^{k,\alpha}(B_X, d)$ and $A^k(\overline{B_X})$ as follows. One first solves the two questions from [43, 4.4] for these algebras, and then, as an application, one establishes the corona theorem, using the Banach algebra method, which would also establish the I.J. Schark's theorems. Then these algebras may be considered as test cases to obtain the I.J. Schark's theorems for $H^\infty(B_X)$, where $X = \ell^1, \ell^2$, for $\mathcal{A}_u(B_X)$, where $X = \ell^1$, and for the Fréchet algebra $H_b(X)$, where $X = c_0, \ell^2$, by following our approach.

In connection to Theorem 3.4 and Theorem 3.6, it is natural to ask whether the I.J. Schark's theorem implies the Gleason solvability for $\mathrm{Lip}_H^{k,\alpha}(G, d)$ and $A^k(\overline{G})$, where G is as in those theorems. The reason is: McDonald used the Gleason solvability [32, p. 215 (4)] while proving the I.J. Schark's theorem for $H^\infty(G)$, where G is a smooth strongly pseudoconvex domain in $\mathbb{C}^n$.

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